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An Equilibrium Theory of Learning, Search and Wages

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Abstract

We construct an equilibrium theory of learning from search in the labor market, which addresses the search behavior of workers, the creation of jobs, and the wage distribution as functions of unemployment duration. In the model, each worker has incomplete information about his job-finding ability and learns about it from his search outcomes. The theory formalizes a notion akin to that of discouragement: over the unemployment spell, unemployed workers update their beliefs about their job-finding abilities downward and reduce their desired wages. One contribution of the paper is to integrate learning from search into an equilibrium framework. We show that the equilibrium exhibits wage dispersion among homogeneous workers, and that workers with longer unemployment spells have lower permanent incomes. Another contribution is to apply lattice-theoretic techniques to analyze learning from experience, which is useful because learning generates convex value functions and, in principle, multiple solutions to a worker's optimization problem.

JEL classifications: E24, D83, J64.
Keywords: Learning; Wages; Unemployment; Directed search; Supermodularity.

1. Introduction

The estimated costs of unemployment to workers in terms of future wage losses are substantial, and longer unemployment spells are associated with significantly larger losses (e.g., Nickell et al., 2002). In this paper, we construct a theory of *learning from search* and use it to explain how this evidence can be consistent with a labor market equilibrium. The main assumption is that each unemployed worker does not have precise knowledge of his job-finding ability and, therefore, learns about this ability from search histories. We embed this assumption into a model of directed search to characterize the search behavior of workers, the creation of jobs, and the wage distribution. Employing lattice-theoretic techniques, we show that in the labor market equilibrium, each unemployed worker reduces the wage he searches for as his unemployment duration increases.

Learning from search captures a notion akin to discouragement. Search outcomes contain useful information when workers have unknown search ability that determines how likely they will be a good match to a random firm. After a worker fails to find a match, he updates his beliefs about his own ability downward and, hence, becomes more pessimistic about his probability of finding a job. In subsequent searches, the worker will lower his desired wage in order to increase the probability of getting a match. Despite the intuitive appeal of this mechanism, it has not been incorporated into an equilibrium analysis; instead, the learning literature has focused on a single agent's optimal stopping problem. In the absence of an equilibrium framework, it is difficult to assess whether learning can occur in the equilibrium or learning from search is a robust explanation for the dependence of observed wage outcomes on workers' unemployment spells. The objective of this paper is to examine the interactions between search decisions, job creation and wage offers in order to understand the equilibrium consequences of learning from search.

The main difficulty in analyzing learning from search in an equilibrium framework stems from the need to address the interaction between individuals' learning and aggregate prices. To appreciate why, it is useful to consider the work of Burdett and Vishwanath (1988), where workers learn about the unknown distribution of wages. In their model, each worker receives a wage offer from an exogenous distribution and then decides whether to accept the offer or to reject it and continue to search. A worker who receives an offer lower than expected revises his beliefs about the wage distribution downward. Workers with longer unemployment spells are precisely the workers who have drawn and rejected relatively lower wages in the past, and so they perceive the jobs available to them as jobs offering low wages. As a result, reservation wages are negatively related to unemployment spells. While this

is an interesting result, Burdett and Vishwanath (1988) examine only the workers' side of the labor market. In equilibrium, however, learning by the market participants will affect the wage distribution by affecting firms' decisions on wage offers.

To provide a tractable analysis of an equilibrium with learning from search, we change two modeling aspects. First, we propose that the source of incomplete information is a worker characteristic. We model this characteristic as a worker's exogenous ability that affects his efficiency units in search. The advantage of this modeling assumption is that this fundamental characteristic does not change with learning. In contrast, when agents learn about the wage distribution, the distribution itself changes as firms update their beliefs and change their wage offers. Second, we model search as a directed and competitive process.[1] Similar to Acemoglu and Shimer (1999), this search process allows individuals to sort themselves into submarkets, which makes the equilibrium analysis tractable by eliminating the dependence of individuals' decisions on the wage distribution.[2]

In more detail, our model is as follows. Each worker has an unknown permanent ability, which is either high or low. Ability determines the worker's efficiency units in search but, once matched, all workers have the same productivity. The process of competitive search is as follows. There is a continuum of submarkets, each of which is associated with a wage and tightness. Firms and workers observe all such pairs and choose which submarket to enter, understanding that a submarket with a higher wage has relatively fewer vacancies. We refer to this choice as the agent's search decision and the wage in the chosen submarket as the worker's desired wage. In each submarket, the number of matches is given by a function of the number of vacancies and the total efficiency units of searching workers in that submarket. In any given submarket, a worker with higher ability has a higher matching probability than a worker with lower ability. However, this matching probability remains unknown because an individual does not know his ability. Information is incomplete but symmetric. In particular, all search histories are public information.

Search outcomes convey useful information about a worker's type. When an unemployed worker searches and fails to find employment, the worker views this search outcome as bad news and revises his beliefs downward. Subsequently, he will choose to search for

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[1]Peters (1984, 1991) formulates directed search as a strategic problem. Burdett et al. (2001) and Shi (2001), among others, have further explored this strategic formulation. Competitive search is analyzed by Moen (1997) and Acemoglu and Shimer (1999). In a large market, the two formulations often generate the same equilibrium outcome.

[2]Acemoglu and Shimer (1999) analyze sorting according to workers' wealth, unemployment benefits and/or risk aversion. However, their result that competitive directed search induces sorting is more general than their model's specifics. We explore sorting according to workers' beliefs about their own matching abilities. Shi (2001) analyzes sorting according to workers' skills.

jobs that will be easier to get. Those jobs will necessarily come with lower wages as part of the equilibrium tradeoff between wages and market tightness. Thus, learning from search induces not only reservation wages, but also desired wages, to decline with unemployment duration. As firms cater to the workers with different beliefs, there is a non-degenerate distribution of equilibrium wages among *ex ante* identical workers (between submarkets).[3]

In addition to an equilibrium formulation of learning from search, we provide an analytical procedure for resolving a main theoretical problem in the analysis of optimal learning from experience. This problem is caused by convexity of the value function. Because search outcomes generate variations in a worker's posterior beliefs about his ability, search conveys valuable information only if these variations in beliefs are valuable to the worker, that is, if the worker's nonlinear value function is convex in beliefs. Although the literature (e.g., Easley and Kiefer, 1988) recognizes that such convexity is likely to lead to multiple solutions and to render the first-order conditions inapplicable, previous work has either ignored the difficulty or focused on corner solutions (e.g. Balvers and Cosimano, 1993). We resolve this difficulty by exploiting a connection between convexity of the value function and the property of *supermodularity* in lattice-theoretic techniques.

This connection is not obvious at first glance and, to our knowledge, has not been examined. In our model, neither a worker's current payoff nor his objective function is supermodular as is often required in applications (see Topkis, 1998, and Milgrom and Shannon, 1994). We proceed in two steps. First, we use convexity of a worker's value function to show that a particular monotone transformation makes the worker's objective function supermodular. This approach differs considerably from other applications of lattice-theoretic techniques to dynamic programming (e.g., Amir et al., 1991, Mirman et al., 2007), which assume the value function to be concave. Second, we establish that workers' optimal decisions (i.e., desired wages) are strictly decreasing with unemployment duration. Generally, lattice-theoretic techniques establish only weak monotonicity. For strict monotonicity, the literature has required strong assumptions on differentiability, e.g., Amir (1996) and Edlin and Shannon (1998). Because such assumptions can be violated here, we establish strict monotonicity in an alternative way, again by exploiting convexity of the value function.

In our model, learning from search induces sorting of workers according to their beliefs about matching ability. Because beliefs are deteriorating during unemployment, workers choose to sort themselves out according to unemployment duration even though they

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[3]Workers who differ in *ex post* beliefs may also choose different levels of search intensity and labor market participation. Although our analysis can shed light on such differences, we choose to abstract from them in the interest of simplicity.

have the same productivity once employed. This sorting mechanism is different from the common argument that workers with lower productivity tend to have longer unemployment spells (Lockwood, 1991) or that workers' skills deteriorate during unemployment (Pissarides, 1992). Also, Blanchard and Diamond (1994) examine sorting among workers of the same productivity by showing that unemployment duration can act as a device for equilibrium selection. Specifically, they use a random-matching model to show that there is an equilibrium in which firms give employment priority to workers with lower employment durations even though there is no fundamental reason for such a bias. In their model, the distribution of wages is degenerate, and the length of the unemployment spell has no effect on a worker's future wage. In contrast to this mechanism, sorting in our model is a result of learning driven by the workers' willingness to accept lower wages as their beliefs about their ability deteriorate during unemployment. Finally, learning from search generates the dependence of a worker's job-finding probability on unemployment duration, which we will discuss in Section 6.

2. The Model

2.1. Agents, Markets and Matching

Time is discrete and all agents discount the future at a rate $r > 0$. There are large numbers of workers and firms. A worker is either employed or unemployed. When employed, a worker produces $y > 0$ units of goods. When unemployed, a worker searches for a job, and the utility of leisure is normalized to zero. To focus on learning from search, we assume that employment is an absorbing state.[4] In this environment, the steady state distribution of workers is non-trivial only if there is a flow into unemployment. For this reason, we assume that the labor force grows at a constant rate $n > 0$. Thus, if L is the labor force at the beginning of period t, a mass nL of new workers enters the labor market in period t through unemployment. The number of firms is determined endogenously by free entry, as described later.

Each worker has unknown ability, a, that is equal to either a_H or a_L, where $a_H > a_L > 0$. A worker's ability determines the worker's effectiveness in the matching process, as specified later.[5] This unknown ability is a permanent characteristic, determined at the time when

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[4]If a worker's ability, described below, is a new draw every time a worker enters unemployment, then it is straightforward to incorporate workers' re-entry into unemployment through job separation.

[5]We are very grateful to Daron Acemoglu and the referees for directing us toward this formulation. In a previous version of the paper (Gonzalez and Shi, 2007), we formulated the problem as one of incomplete information about the characteristics of local markets rather than individuals.

the worker enters the labor market. A new worker has ability a_H with probability p, and a_L with probability $(1-p)$, where $p \in (0, 1)$. We call a worker with a_H a high-ability worker, and a worker with a_L a low-ability worker. We will also use the notation $a_i \in \{a_H, a_L\}$.

The timing of the events in a period is as follows. First, new workers enter the labor market through unemployment and nature determines the ability of each new worker, while the ability of a worker who was born in the past remains the same as before. Second, all firms and unemployed workers make their search decisions. Third, after the matching process is completed in the period, matched firms and workers exit the search process to produce permanently, while unmatched workers remain unemployed.

Search is competitive, as follows. There is a continuum of *submarkets* indexed by x, which will be related to matching rates in that submarket. The domain of x is $X = [0, 1/a_H]$. A submarket x is characterized by a wage level, $W(x)$, and a tightness, $\lambda(x)$. The functions $W(.)$ and $\lambda(.)$ are public information, taken as given by agents and determined in the equilibrium. In each period, a worker's or a firm's search decision is to choose which submarket to enter, i.e., x. Search is directed in the sense that an agent's choice of a submarket involves a tradeoff between the wage and the tightness, because a submarket with a high wage has relatively fewer vacancies per worker in the equilibrium. Note that in line with the formulations by Moen (1997) and Acemoglu and Shimer (1999), a firm does not directly set wages. Instead, by choosing a submarket, a firm chooses a pair (w, λ) from the menu $\{(w(x), \lambda(x)) : x \in X\}$.[6]

Since workers differ in their search effectiveness, we define the total efficiency units of workers searching in submarket x as

$$u_e(x) = a_H u_H(x) + a_L u_L(x), \tag{2.1}$$

where $u_i(x)$, for $i \in \{L, H\}$, denotes the mass of unemployed workers with ability a_i who choose submarket x. Let $v(x)$ be the number of vacancies created in submarket x, and define the effective tightness in this submarket as $\lambda(x) = v(x)/u_e(x)$. We assume that the number of matches in submarket x is given by a function, $F(u_e(x), v(x))$. The index x is the matching rate for each efficiency unit of a worker in submarket x; that is,

$$x = \frac{F(u_e(x), v(x))}{u_e(x)}.$$

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[6]Although the submarkets can alternatively be indexed by workers' unemployment duration, τ, it is more convenient to index them by x. First, all the effects of τ on agents' decisions eventually go through x because a worker's tradeoff is between the wage and the matching probability. Second, in contrast to the discrete variable τ, x is a continuous variable which allows agents to make a continuous tradeoff between w and x.

For a worker with ability a_i, the matching probability in submarket x is $a_i x$. Thus, given x, the lower a worker's ability, the lower his matching probability.

We impose the following standard assumption on the matching function F:

Assumption 1. *The function $F(u_e, v)$: (i) is strictly increasing, strictly concave and twice differentiable in each argument, (ii) is linearly homogeneous, and (iii) has $F(u, 0) = 0$ and $F(u, \infty) > 1/a_H$ for all $u \in (0, \infty)$.*

One can determine the tightness $\lambda(x)$ by $F(1, \lambda(x)) = x$ and verify that $\lambda(x)$ satisfies:

$$\lambda'(x) > \frac{\lambda(x)}{x} > 0, \ \lambda''(x) > 0, \text{ for all } x \in (0, 1/a_H]. \tag{2.2}$$

Note that for a vacancy in submarket x, the matching probability is $F/v = x/\lambda(x)$. The above properties of λ imply that a firm's recruiting probability decreases in x. That is, if it is easy for a worker to find a job at x, it must be difficult for a firm to recruit at x.

The following examples of the matching function satisfy Assumption 1 and will be considered in various parts of our analysis:

Example 2.1. *(i) One example of F is the CES function: $F(u_e, v) = [(1 - \alpha) u_e^\rho + \alpha v^\rho]^{1/\rho}$, where $\rho < 1$ and $\alpha \in (0, 1)$. In this case, $\lambda(x) = [1 + (x^\rho - 1)/\alpha]^{1/\rho}$. A special case of this example is the Cobb-Douglas function, where $\rho = 0$, which leads to $\lambda(x) = x^{1/\alpha}$. (ii) Another example is the urn-ball matching function: $F(u_e, v) = v(1 - e^{-u_e/v})$. In this case, $\lambda(x)$ is implicitly defined by $\lambda(x)[1 - e^{-1/\lambda(x)}] = x$.*

The key feature of the model is the incomplete information about worker ability in the matching process. Because the matching probability is less than one regardless of the worker's ability, failure to match does not automatically imply that the worker has low ability. Thus, a worker who fails to obtain a match faces a signal extraction problem. Search histories are informative because low-ability workers are more likely to fail to get matches in any given period. In contrast to workers, firms do not face a signal extraction problem. Given a firm's choice of the submarket, the firm's matching probability and expected profit are determined independently of any particular worker's ability.

Note that our formulation implies that there is no need for individual agents to learn about the composition of workers' abilities in a submarket. To see this, consider an arbitrary submarket x where the total efficiency units of workers are $u_e(x)$. Free entry of firms into the submarket will ensure that the effective tightness will indeed be $\lambda(x)$, and so a worker's or a firm's matching probability and expected payoffs in the submarket will be

determined independently of the level and the composition of $u_e(x)$. This feature clearly depends on the use of directed search, but it also depends on the matching function. If we specified the matching function instead as $F(u_H(x), u_L(x), v(x))$ where the first two variables are not perfect substitutes, then an agent's matching probability and the wage offered in the submarket would depend on the composition, $(u_H(x), u_L(x))$. In this case, a worker would use unemployment duration to make inference on this composition in addition to learning about his own ability. An equilibrium analysis would be complicated as the composition evolves with unemployment duration. Although this role of unemployment duration in the inference of the ability composition in each submarket is interesting, it is not a necessary part of a worker's learning about his own ability. By eliminating this role, our formulation maintains tractability.

In the interest of clarity, let us emphasize the information structure. Information is symmetric in the sense that all workers' search histories and all statistics in each submarket are common knowledge. In particular, every agent observes the number of vacancies, the number of matches, and the composition of workers in each submarket. Despite the availability of such public information, individual workers can still learn from their private search histories.

2.2. Learning from Search

Workers update their beliefs about ability after observing whether or not they have a match. The updating depends on the particular submarket into which the worker just searched. To describe the updating process, it is convenient to express a worker's belief in terms of his expected type. Let the initial prior expectation of a for a worker who has not yet searched be $\mu_0 \in (a_L, a_H)$. From the distribution of new workers across ability types, we can calculate: $\mu_0 = pa_H + (1-p)a_L$, where $p \in (0,1)$. This mean belief is common to all new workers and it is public information.

Consider the updating process for an arbitrary worker. Let P_i be the prior probability with which $a = a_i$, where $a_i \in \{a_H, a_L\}$. Let μ be the expected value of a according to this prior belief and refer to μ simply as the belief. Note that the prior distribution of a is Bernoulli, with mean μ. From the definition of μ, we can solve P_i in terms of μ:

$$P_H = \frac{\mu - a_L}{a_H - a_L}, \qquad P_L = \frac{a_H - \mu}{a_H - a_L}. \tag{2.3}$$

Let $k \in \{0, 1\}$ be the matching outcome in the current period, where $k = 0$ if the worker

fails to get a match and $k = 1$ if the worker succeeds in getting a match. Then,

$$P(a_i|x, \; k = 1) = \frac{a_i}{\mu} P_i, \qquad P(a_i|x, \; k = 0) = \frac{1 - xa_i}{1 - x\mu} P_i. \tag{2.4}$$

The conditional distribution of a is Bernoulli with mean $\mathbb{E}(a|x, k) = a_H P(a_H|x, k) + a_L [1 - P(a_H|x, k)]$. Substituting $P(a_H|x, k)$ from (2.4) and P_H from (2.3), we have:

$$\begin{aligned}
\mathbb{E}(a|x, k = 1) &= a_H + a_L - a_H a_L / \mu; \\
\mathbb{E}(a|x, k = 0) &= a_H - \tfrac{1-xa_L}{1-x\mu}(a_H - \mu).
\end{aligned} \tag{2.5}$$

Note that if the initial mean belief μ_0 exceeds a_L, $\mathbb{E}(a|x, k) > a_L$ for both $k = 0$ and $k = 1$.

The updating process above has two preliminary properties. First, the sequence of mean beliefs is a Markov process. Second, a worker's mean belief, μ, is a sufficient statistic for the worker's unemployment history. Denote the domain of μ as $M = [a_L, a_H]$.

The value of x measures the informativeness of search. Search in a market with relatively higher x is more informative in the sense of Blackwell (1951). Consider the information revealed by search in two different submarkets, with $x > x'$. Let K and K' be the random number of matches associated with x and x'. Intuitively, one can construct the random variable K' by "adding noise" to K as follows. First, let the worker randomize with probability of success ax, with $a \in \{a_L, a_H\}$; then, whenever the realization is a success, randomize again with success probability x'/x. The result is a Bernoulli trial with probability of success equal to ax'. In other words, if $x > x'$, the random variable, or experiment, K is sufficient for K' (see DeGroot, 1970, pp.433-439).

The informational content of x is asymmetric with respect to the matching outcome. After a successful match, a worker's posterior, $P(a|x, k = 1)$, and the posterior mean belief, $\mathbb{E}(a|x, k = 1)$, are independent of x, because the probability of getting a match is linear in x regardless of the worker's type. However, after a match failure, a worker's posterior and the posterior mean belief both depend on x.

Because our focus is on unemployed workers' decisions, it is useful to explicitly write the posterior belief of a worker who fails to find a job as $\mathbb{E}(a|x, k = 0) = H(x, \mu)$, where

$$H(x, \mu) \equiv a_H - \frac{1 - xa_L}{1 - x\mu}(a_H - \mu). \tag{2.6}$$

One can verify the following properties (the verification is omitted here):

Lemma 2.2. The function $H(x, \mu)$ satisfies: (i) $H_1 < 0$; (ii) $H_2 > 0$, (iii) $H_{11} = \frac{2\mu}{1-x\mu} H_1 < 0$ and $H_{22} = \frac{2x}{1-x\mu} H_2 > 0$; (iv) $\mu(1 - x\mu)H_{12} - H_1 - \mu^2 H_2 = -a_H a_L$.

Property (i) states that a higher x reduces the worker's posterior beliefs after the worker fails to find a match, as discussed above. In particular, property (i) implies that $H(x, \mu) < \mu$ for all $x > 0$ and $\mu > a_L$. Thus, a worker's beliefs about his ability decrease over time as the number of search failures increases. Of course, if a worker's beliefs have reached a_L, there is no further updating; that is, $H(x, a_L) = a_L$ for all x. Property (ii) states that, for any given x, a worker with higher prior beliefs will also have higher posterior beliefs. Properties (iii) and (iv) will be useful later.

2.3. The Value of Search

Consider an unemployed worker who enters a period with belief μ. Let $V(\mu)$ be his value function. If he chooses to search in submarket x, the expected probability of finding a match is $x\mu$. In principle, some workers may have incentive to engage in the following "experimentation": searching during a period solely to gather information and, thus, refusing to enter a match once they learn that a match has occurred. This may occur because a worker who found a match in submarket x will revise his belief upward (see (2.5)). We do not think that this form of experimentation is important in practice, unless it is associated with heterogeneous matches, which is not the case here. Thus, we rule out such experimentation by focusing on the case in which search is sufficiently costly that a worker always prefers to accept a match that he searches for.

Assumption 2. *Labor productivity satisfies:* $y/c \geq (1 + r) a_H \lambda (1/a_H)$.

This sufficient condition implies that a worker prefers getting the lowest equilibrium wage every period starting now to remaining unemployed in the current period and then getting the full surplus from a match every period starting with next period. Intuitively, the condition requires that the opportunity cost of rejecting a match, as reflected by y, should be sufficiently high to a worker.[7] Stronger than necessary, this condition significantly simplifies the analysis and the exposition of our main results. As in Burdett and Vishwanath (1988), one can relax the condition by introducing a constant cost of search per period, which further increases a worker's opportunity cost of rejecting an offer. For simplicity, however, we have not included such a cost of search.

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[7]The discount rate in Assumption 2 reflects both workers' and firms' discount rate. For a worker, a higher discount rate lowers the benefit from experimentation for any given wage. However, when firms discount future at a higher rate, the present value of a filled job falls, and the wage rate in every submarket must be lower in order to induce firms to enter, which implies that the loss of the current wage from experimentation falls. With a common discount rate, the effect through firms' discount rate dominates.

Because employment is permanent, the present value of the job is $W(x)/r$. If the worker does not find a job in the current period, he will revise the beliefs to $H(x, \mu)$ and continue to search in the next period. In this case, the expected value from the next period onward will be $V(H(x, \mu))$. Thus, the expected payoff of searching in a submarket x is:

$$R(x, \mu) \equiv x\mu \frac{W(x)}{r} + (1 - x\mu)\frac{V(H(x, \mu))}{1 + r}. \tag{2.7}$$

Under Assumption 2, the value of search under beliefs μ is given by:

$$V(\mu) = \max_{x \in X} R(x, \mu). \tag{2.8}$$

Denote the set of optimal decisions as $G(\mu) = \arg\max_{x \in X} R(x, \mu)$ and a selection from $G(\mu)$ as $g(\mu)$. A worker's *desired wage* is $w(\mu) = W(g(\mu))$. In contrast, a worker's *reservation wage* can be defined in the conventional way as the lowest permanent income that a worker will accept to forego search. This is given by $rV(\mu)$.

2.4. Free Entry of Firms and the Equilibrium Definition

There is free entry of firms into the market. After incurring a vacancy cost $c \in (0, y)$, a firm can post a vacancy for a period in any one of the submarkets. Let J be the net value of a vacancy. With free entry, $J = 0$. Recall that the matching probability of a firm in any submarket x is $x/\lambda(x)$ and that employment is permanent. Thus, a firm's optimal choice of the submarket solves the following problem:

$$J = \max_{x \in X} \left[-c + \frac{x}{\lambda(x)} \frac{y - W(x)}{r}\right]. \tag{2.9}$$

As said earlier, the firm faces no signal extraction problem, because all elements in the above problem are public information.

The first-order condition of the above problem is a differential equation for the wage function. Without an initial condition, this equation has a continuum of solutions, which means that there is a continuum of choices of x that are optimal for the firm. That is, a firm is willing to enter any submarket, provided that the wage in the submarket is consistent with the free-entry condition, $J = 0$.

With (2.9), the free-entry condition yields the following wage function:

$$W(x) = y - rc\frac{\lambda(x)}{x}. \tag{2.10}$$

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Recall that $x \leq 1/a_H$ and that $\lambda(x)/x$ is increasing in x. For future reference, it is useful to note that, for all $x \in X$, the function $W(x)$ is twice continuously differentiable, and that it has the following properties:

$$(i)\ 0 < W(x) \leq y; \quad (ii)\ W'(x) < 0, \quad (iii)\ 2W'(x) + xW''(x) < 0. \tag{2.11}$$

Assumption 2 ensures (i), while (2.2) implies (ii) and (iii). Part (ii) says that a higher employment probability occurs together with a lower wage. This negative relationship is necessary for providing a meaningful tradeoff between the two variables in directed search. As such, part (ii) is necessary for inducing firms to enter the submarket. Part (iii) is implied by $\lambda''(x) > 0$, and it says that the function $xW(x)$ is strictly concave in x.

Focus on stationary symmetric equilibria. An *equilibrium* consists of workers' choices of x, firms' choices of x, and a wage function $W(x)$, that meet the following requirements. (i) Given the wage function, all workers with the same belief μ at the beginning of a period use the same search policy $x = g(\mu)$. (ii) A firm's choice solves the problem in (2.9). (iii) Conditional on unsuccessful search, a worker's beliefs are updated according to $H(g(\mu), \mu)$. (iv) Consistency: for every submarket x with positive entry, the mass of all firms who choose x divided by the efficiency units of workers who choose x is equal to $\lambda(x)$. (v) Free-entry: for each submarket, the wage function W satisfies (2.10).

In the above definition, we have left out the steady-state conditions on worker flows and the wage distribution, which will be characterized in Section 6. We deliberately do so in order to emphasize the model's feature that individuals' decisions and matching probabilities can be analyzed without any reference to the wage distribution. Instead, all that is required for such an analysis is the wage function $W(.)$ and the tightness function $\lambda(.)$, which are determined by firms' free-entry condition and the matching function. This feature makes the analysis tractable by reducing the dimensionality of the state variables for individuals' decision problems significantly.[8] In contrast, if search were undirected, an individual's search decision would depend on the wage distribution which, in turn, would evolve as individuals learn about his ability.

3. Learning in Competitive Search Equilibrium

Let us analyze a worker's optimization problem, (2.8). When choosing a submarket x, the worker faces two considerations. One is the familiar tradeoff between wages and the matching probability in models of directed search. That is, a submarket with a higher x

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[8]See Shi (2006) for an exploration of this feature in the context of wage-tenure contracts.

has a lower wage and a higher job-finding probability. Another consideration is learning from the search outcome. As discussed earlier, search in a submarket with a high x (i.e., a low wage) is more informative than search in a submarket with a low x. To see how the model captures the value of search, we examine the value function.

It is easy to see that the mapping defined by the right-hand side of (2.8) is a contraction. Using the features in (2.11), standard arguments show that a unique value function V exists, which is positive, bounded and continuous on $M = [a_L, a_H]$ (see Theorem 4.6 in Stokey and Lucas, 1989, p.79). Moreover, the set of maximizers, G, is nonempty, closed, and upper-hemicontinuous. Existence of the optimal decision, together with the characterization of the steady-state distribution in section 6, establishes existence of an equilibrium.

Lemma 3.1. *Under Assumption 2, there exists an equilibrium where all matches are accepted.*

The following lemma describes additional properties of the value function (see Appendix B for a proof):

Lemma 3.2. *V is strictly increasing, strictly convex, and almost everywhere differentiable.*

Monotonicity of the value function determines the behavior of reservation wages, defined as $rV(\mu)$. Because $V(\mu)$ is strictly increasing, the reservation wage strictly falls with a worker's unemployment duration as his beliefs about his own ability deteriorate. Put differently, a worker's permanent income strictly declines over unemployment spells. Similarly, with strict monotonicity of V, (2.8) implies that a worker's reservation wage is always strictly lower than the desired wage, i.e., $rV(\mu) < W(g(\mu))$ for all $\mu > a_L$.

In contrast to reservation wages, desired wages are much more difficult to analyze since they depend on optimal learning from search. Search generates information by creating variations in the worker's posterior beliefs. As is well known in the learning literature, such variations are valuable to the individual if the value function is strictly convex in beliefs. The information content of search depends on both the search choice, x, and the prior belief, μ. A higher x generates more information in the sense that, for fixed μ, a higher x causes a mean-preserving spread in the distribution of the posterior expectation $\mathbb{E}(a|x, k)$. To see this, note that the search outcome k is a random variable, where $k = 1$ with probability ax and $k = 0$ with probability $(1 - ax)$. Using (2.5), one can verify that the expectation of $\mathbb{E}(a|x, k)$ with respect to k and a is equal to μ, which is independent of x. The variance is:

$$Var(\mathbb{E}(a|x, k)) = \mathbb{E}\left[(\mathbb{E}(a|x, k))^2\right] - \mu^2 = \frac{x(a_H - \mu)^2(\mu - a_L)^2}{\mu(1 - x\mu)}.$$

This variance increases in x, for any fixed μ. However, given x, the above variance is not monotone in μ. Instead, the variance is maximal at some intermediate value of μ, becoming zero at the boundary points, a_L and a_H. This feature reflects the fact that information is relatively less useful when prior beliefs are already extreme. Consequently, the value of information becomes negligible as the prior approaches $\mu = a_L$ or $\mu = a_H$.

Despite such non-monotonicity of the value of information in μ, the optimal choice, $g(\mu)$, is monotone in μ, as we will establish in the next section. The reason is that the opportunity cost of learning from search also depends on the beliefs, μ. To see this, note that a worker with beliefs μ will search in submarket x rather than $x' < x$ whenever the value of information gained by doing so exceeds the opportunity cost of learning. The opportunity cost of searching in submarket x as opposed to $x' < x$ is $\mu \left[x'W(x') - xW(x) \right] / r$. As beliefs deteriorate with unemployment duration, this opportunity cost decreases, making it possible that the optimal choice of x increases with unemployment duration. As it will become clear in the next section, monotonicity of the optimal search decision relies crucially on convexity of the value function.

4. Monotonicity of Workers' Desired Wages

In this section, we establish the central result that a worker's desired wage, $w(\mu)$, increases with the worker's beliefs and, hence, decreases with the worker's unemployment duration. Because $w(\mu) = W(g(\mu))$, where $W(.)$ is decreasing, monotonicity of $w(\mu)$ is equivalent to the feature that the worker's optimal choice of the submarket, $x = g(\mu)$, is decreasing in μ. Let us define $z = -x$ and refer to z, rather than x, as the worker's search decision. This transformation will be useful for what follows, and it enables us to attach the label *monotone decisions* naturally to the feature that z increases in the beliefs. After the transformation, the objective function in (2.8) becomes $R(-z, \mu)$, and the feasible set of choices is $-X \ni z$. The domain of μ is $M = [a_L, a_H]$.

Before undertaking the task, note that strict convexity of the value function implies that the objective function $R(-z, \mu)$ is not necessarily single-valued or differentiable. Thus, the optimal decision is not necessarily unique or interior, and the first-order condition may not be applicable.[9] Although these difficulties are well known in the literature on optimal learning (e.g., Easley and Kiefer, 1988), this literature has either ignored them or focused on corner solutions (e.g., Balvers and Cosimano, 1993). We need to examine all

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[9]In different modeling environments, there are techniques to generate smooth optimal choices and differentiable value functions, e.g., Santos (1991). However, those techniques require the value function to be concave, which is violated here.

solutions in order to characterize how optimal search behavior and desired wages depend on unemployment duration.

In the presence of the above difficulties, a natural way to establish monotonicity of the workers' search decision is to use supermodularity in lattice-theoretic techniques (see Topkis, 1998). In our model, supermodularity of the objective function is equivalent to the feature of increasing differences in (z, μ), because these variables lie in closed intervals of the real line.[10] However, the connection between supermodularity and the dynamic programming problem in (2.8) is far from obvious. In (2.8), the value function is strictly convex and the current payoff function, $-\mu z W(-z)/r$, is not supermodular in (μ, z). The opposite features are often required in applications of supermodularity to dynamic programming, which use concavity of the value function and supermodularity of the current payoff function, recursively via the Bellman equation, to establish that the objective function is supermodular.[11]

However, monotonicity of optimal choice is invariant to transformations of the objective function that are monotone in the choice variables. Although the objective function $R(-z, \mu)$ is unlikely to be supermodular in (μ, z), it is possible to transform the objective function into a supermodular function. To that end, we transform (2.8) as

$$V(\mu) = \mu \max_{z \in -X} \hat{R}(z, \mu),$$

where $\hat{R}$ is defined as follows:

$$\hat{R}(z, \mu) \equiv \frac{1}{\mu} R(-z, \mu) = -z \frac{W(-z)}{r} + \left(z + \frac{1}{\mu} \right) \frac{V(H(-z, \mu))}{1 + r}. \tag{4.1}$$

Denote $Z(\mu) = \arg\max_{z \in -X} \hat{R}(z, \mu)$ and $z(\mu) \in Z(\mu)$. Clearly, the set of optimal choices for x is $G(\mu) = -Z(\mu)$, and a typical selection is $g(\mu) = -z(\mu)$. Denote the greatest selection of $Z(\mu)$ as $\bar{z}(\mu)$ and the least selection as $\underline{z}(\mu)$.

The following theorem states the result on monotonicity (see Appendix C for a proof):

Theorem 4.1. *Let $z \in -X$ and $\mu \in M$. The function $\hat{R}(z, \mu)$ is strictly supermodular in (z, μ). Thus, every selection $z(\mu)$ is an increasing function. Similarly, every selection $g(\mu)$ from $G(\mu)$ is a decreasing function, and the wage $w(\mu)$ is an increasing function.*

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[10]Let $z \in Z$ and $\mu \in M$, where Z and M are partially ordered sets. A function $f(z, \mu)$ has increasing differences in (z, μ) if $f(z_1, \mu_1) - f(z_1, \mu_2) \geq f(z_2, \mu_1) - f(z_2, \mu_2)$ for all $z_1 > z_2$ and $\mu_1 > \mu_2$. If the inequality is strict, then f has strictly increasing differences. In our model, Z, M and $Z \times M$ (under the product order) are all lattices. In this case, the feature of increasing differences implies supermodularity (see Topkis, 1998, p.45).

[11]Examples include Amir et al. (1991), Mirman et al. (2007), and Becker and Boyd (1997, pp. 277-284).

The main task in the proof of this theorem is to establish strict supermodularity of $\hat{R}$, after which monotonicity of $z(\mu)$ follows from Topkis (1998, p.79) and Milgrom and Shannon (1994). Three aspects of the proof are worth noting. First, every solution for z is an increasing function of the beliefs. This strong result comes from the feature that $\hat{R}(z,\mu)$ is strictly supermodular. Second, strict convexity of the value function plays an important role for strict supermodularity of $\hat{R}$ and, hence, for monotone optimal choices. Third, strict supermodularity of $\hat{R}$ relies only on the properties of the value function, V, and the updating function, H, not directly on those of the wage function, W. In particular, strict supermodularity of $\hat{R}$ does not require the current payoff function, $-\mu z W(-z)/r$, to be supermodular.

Monotonicity of optimal choices is a general result that holds even when optimal choices are corner solutions and multiple solutions. However, as a general result, the above theorem allows the possibility that a solution $z(\mu)$ is only a weakly increasing function. In the next section, we address strict monotonicity and other issues.

5. Strict Monotonicity and Uniqueness of the Optimal Path

In this section, we answer two further questions. First, when are desired wages *strictly* declining with unemployment duration? I.e., when are optimal choices, $z(\mu)$, a strictly increasing function of beliefs? Second, if optimal choices are not unique, is there any discipline on the set of paths of optimal choices? To answer these questions, we impose the following assumption, which ensures that optimal choices are interior solutions:

Assumption 3. *Labor productivity satisfies:* $y/c < (r + a_L/a_H)\,\lambda'\left(a_H^{-1}\right) - a_L\lambda\left(a_H^{-1}\right).$

As shown in Appendix D, this assumption ensures that the objective function, $\hat{R}(z, a_L)$, is strictly increasing in z at $z = -1/a_H$, so that even a worker with beliefs $\mu = a_L$ will find it optimal to choose $z > -1/a_H$. Since $z(\mu)$ is increasing, this assumption is sufficient for all workers' choices to be interior.

It should be noted that Assumptions 2 and 3 can hold simultaneously. For instance, a sufficient condition for this to be the case is that labor productivity satisfies:

$$(r + a_L/a_H)\,\lambda'\left(a_H^{-1}\right) - a_L\lambda\left(a_H^{-1}\right) > y/c \geq (1+r)a_H\lambda\left(a_H^{-1}\right).$$

For the above interval of y/c to be non-empty, it is sufficient that

$$r + a_L/a_H > \left[\frac{\lambda'\left(1/a_H\right)}{a_H\lambda\left(1/a_H\right)} - 1\right]^{-1}.$$

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The right-hand side of the above inequality is positive, because $\lambda'(x) > \lambda(x)/x$ for all $x \in (0, 1/a_H]$ (see (2.2)). With each of the two matching functions in Example 2.1, one can find a non-empty parameter region in which the above condition is satisfied. In particular, when the matching function is Cobb-Douglas, the above condition becomes $r + a_L/a_H > \alpha/(1-\alpha)$, which is satisfied when α is small.

In Appendix D, we establish the following lemma.

Lemma 5.1. *Under Assumption 3, an unemployed worker's optimal choices are interior. Moreover, the derivative $V'(H(-z(\mu), \mu))$ exists for all $z(\mu) \in Z(\mu)$. Thus, optimal choices obey the first-order condition, $\hat{R}_1(z(\mu), \mu) = 0$, where*

$$\hat{R}_1(z(\mu), \mu) = \frac{z(\mu)W'(-z(\mu)) - W(-z(\mu))}{r} + \frac{V(H(-z(\mu), \mu))}{1+r}$$
$$- \left(\frac{1}{\mu} + z(\mu)\right) \frac{V'(H(-z(\mu), \mu))}{1+r} H_1(-z(\mu), \mu).$$

In addition to ensuring interior solutions, this lemma describes a limited sense of differentiability of the value function: the value function is differentiable in future periods at the posterior beliefs induced by optimal choices, i.e., along the path of optimal choices. Despite the fact that the value function may still fail to be differentiable in the first period and at beliefs off the optimal paths, the limited sense of differentiability is enough for the first-order condition to be applicable in every period. In turn, the first-order condition enables us to establish strict monotonicity of optimal choices, as stated in the following theorem (see Appendix E for a proof):

Theorem 5.2. *Under Assumption 3, every selection of optimal choices, $z(\mu)$, is a strictly increasing function. Therefore, along every path of optimal choices, desired wages are strictly declining with unemployment duration.*

As it is the case with supermodularity, strict monotonicity of optimal choices relies on the properties of $V(.)$ and $H(.)$, but not those of the wage function W directly. Not surprisingly, strict convexity of the value function plays a critical role for strict monotonicity of optimal choices. It is worth noting that Amir (1996) and Edlin and Shannon (1998) also establish strict monotonicity of optimal choices but, in our model, their methods would require the value function to be continuously differentiable.[12] We do not rely on this requirement because it does not hold in our model.

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[12]In particular, Edlin and Shannon (1998) assume that the objective function, $\hat{R}(z, \mu)$, has increasing marginal differences. To compute marginal differences, $\hat{R}(z, \mu)$ must be continuously differentiable with respect to z. Because $\hat{R}$ depends on z through the future value function, as well as W, it is differentiable with respect to z only if the value function is so.

Let us now turn to the set of optimal paths. When there are multiple solutions, optimal choices may evolve over time in many ways. One case is that multiple choices occur in every period, in which case the path of optimal choices branches out. Another case is that multiplicity occurs only in the first period. Clearly, the path of optimal choices is more predictable in the second case than in the first case. To know more about the set of paths of optimal choices, we establish a link between multiplicity of optimal choices and differentiability of the value function at all possible beliefs. The following lemma states the link (see Appendix F for a proof):

Lemma 5.3. *Maintain Assumption 3. For each μ_a in the interior of (a_L, a_H), let μ_a^+ denote the limit to μ_a from the right (above) and μ_a^- the limit from the left (below). Then, $V'(\mu_a^+) = R_2(-\bar{z}(\mu_a), \mu_a)$ and $V'(\mu_a^-) = R_2(-\underline{z}(\mu_a), \mu_a)$. Moreover, $V'(\mu_a^+) \geq V'(\mu_a^-)$, where the inequality is strict if and only if $\bar{z}(\mu_a) > \underline{z}(\mu_a)$.*

This lemma says that, at arbitrary beliefs $\mu \in (a_L, a_H)$, the value function is differentiable if and only if the beliefs induce a unique choice to be optimal. If multiple choices are optimal at particular beliefs, then the right derivative of the value function is strictly greater than the left derivative. Denote the set of such beliefs as

$$N = \{\mu \in (a_L, a_H) : \bar{z}(\mu) > \underline{z}(\mu)\}.$$

Because V is almost everywhere differentiable, the set N has measure zero in M.

For any μ_0 in the interior of M, let $\{\mu_\tau\}_{\tau=0}^\infty$ be a path of beliefs generated by optimal choices; i.e., $\mu_\tau = H(-z(\mu_{\tau-1}), \mu_{\tau-1})$ with $z(\mu_{\tau-1}) \in Z(\mu_{\tau-1})$, for $\tau = 1, 2, ...$ For arbitrary initial beliefs, μ_0, the following theorem characterizes the entire set of paths of beliefs and optimal choices (see Appendix F for a proof):

Theorem 5.4. *For any $\mu_0 \in (a_L, a_H)$, let $z(\mu_0)$ be an arbitrary selection from $Z(\mu_0)$ and let $\mu_1 = H(-z(\mu_0), \mu_0)$ be the posterior beliefs induced by $z(\mu_0)$. Given μ_1, μ_τ is unique, $Z(\mu_\tau)$ is a singleton, and $V'(\mu_\tau)$ exists for all $\tau = 1, 2, ...$ If $\mu_0 \notin N$, then $Z(\mu_0)$ is also a singleton, in which case the entire path $\{\mu_\tau\}_{\tau=0}^\infty$ is unique and $V'(\mu_\tau)$ exists for all $\tau = 0, 1, 2, ...$ If $\mu_0 \in N$, then $\bar{z}(\mu_0) > \underline{z}(\mu_0)$, $H(-\bar{z}(\mu_0), \mu_0) > H(-\underline{z}(\mu_0), \mu_0)$ and $V'(\mu_0^+) > V'(\mu_0^-)$.*

This theorem states that the paths of optimal choices and induced beliefs are unique almost everywhere. The only case of non-uniqueness occurs when the worker's initial prior belief lies in the set N, which has measure zero. Even in this case, non-uniqueness occurs

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only in the first period of search. Given any optimal choice in the first period and the induced posterior, future paths of optimal choices and induced beliefs are unique from that point onward. Thus, no matter where initial beliefs lie, the worker will choose search decisions optimally to keep the beliefs out of the set N from the second period onward. More precisely, whenever the search decision will induce the posterior beliefs to be close to a particular level in the set N, it is optimal to modify the decision so as to keep the posterior beliefs above that level. This result is a consequence of the value of learning, as captured by strict convexity of the value function.

To understand why a worker chooses optimally to avoid the set N in future periods, suppose counterfactually that the worker's choice in some period τ induces the posterior beliefs to lie in N; that is, $\mu_{\tau+1} = H(-z_\tau, \mu_\tau) \in N$ for some $\tau \geq 0$, where $z_\tau = z(\mu_\tau)$. In this case, multiple choices will be optimal in period $(\tau + 1)$, which induce the left derivative of $V(\mu_{\tau+1})$ to be lower than its right derivative. The derivative of the future value function captures an implicit (opportunity) cost of learning bad news. Thus, the discrete fall in $V'(\mu_{\tau+1})$ from the right side of $\mu_{\tau+1}$ to the left side implies that learning slightly more about one's ability in the current period increases the cost of learning by a discrete amount. The worker can avoid this discretely larger cost by choosing z_τ slightly above $z(\mu_\tau)$, which will keep the posterior slightly above $\mu_{\tau+1}$. In contrast to this discrete increase in the benefit, the increase in the cost of z_τ is a marginal reduction in the matching probability. Thus, the net gain from increasing z_τ slightly above $z(\mu_\tau)$ is positive. This contradicts the optimality of z_τ.

6. Steady State Distributions

We now analyze the aggregate characteristics of the market. One purpose of this analysis is to clarify that the learning process in previous sections is consistent with aggregation. The other purpose is to distinguish the duration dependence at an individual's level from the aggregate duration dependence.

Let $\hat{U}$, E and L denote the economy-wide unemployment, employment and labor force at the beginning of period t. The aggregate number of searchers in period t, denoted as U, includes both $\hat{U}$ and the number of newborns, nL. Let f denote the average job-finding rate in the economy. Denoting next period's variables with a prime, we have:

$$U = \hat{U} + nL, \quad L = \hat{U} + E, \quad E' = E + fU, \quad \hat{U}' = (1 - f)U.$$

Use lowercase letters to denote the ratios of these variables to the labor force, with $u =$

$U/\left[(1+n)\,L\right]$, $\hat{u} = \hat{U}/L$, and $e = E/L$. Focus on the equilibrium in which these ratios are stationary. The above equations yield:

$$u = \frac{n}{n+f}, \quad e = \frac{(1+n)f}{n+f} \quad \hat{u} = \frac{(1-f)\,n}{n+f}.$$

Aggregation works out as expected (e.g., Acemoglu and Shimer, 1999). To be more specific, let τ denote a worker's unemployment duration and μ_τ denote the worker's belief after τ periods of search. Define $Q_i(\tau)$ as the probability with which a worker with ability $a_i \in \{a_L, a_H\}$ fails to find a match in τ consecutive periods. Then,

$$Q_i(\tau) = \prod_{s=0}^{\tau-1} \left[1 - a_i g\left(\mu_s\right)\right].$$

Note that the ratio $Q_H(\tau)/Q_L(\tau)$ decreases in τ:

$$\frac{Q_H(\tau)}{Q_L(\tau)} = \left[\frac{Q_H(\tau-1)}{Q_L(\tau-1)}\right]\left[\frac{1 - a_H g\left(\mu_{\tau-1}\right)}{1 - a_L g\left(\mu_{\tau-1}\right)}\right] < \frac{Q_H(\tau-1)}{Q_L(\tau-1)}.$$

Define $L_{H,t-\tau} = pL_{t-\tau}$ and $L_{L,t-\tau} = (1-p)L_{t-\tau}$, where $nL_{t-\tau} = nL_t/\left(1+n\right)^\tau$ is the mass of new workers who entered the labor market τ periods before t, and p is the proportion of new workers who are endowed with ability a_H. At the beginning of period t, the mass of unemployed workers with ability a_i who have already searched for τ periods, denoted as $\hat{U}_{i,t}(\tau)$, is $\hat{U}_{i,t}(\tau) = nL_{i,t-\tau}Q_i(\tau)$, and the mass of all unemployed workers who have already searched for τ periods, denoted as $\hat{U}_t(\tau)$, is $\hat{U}_t(\tau) = \hat{U}_{H,t}(\tau) + \hat{U}_{L,t}(\tau)$. Total unemployment in period t is:

$$U_t = \sum_{\tau=1}^{\infty} \hat{U}_t(\tau) + nL_t,$$

and the aggregate unemployment rate is:

$$u = \frac{U_t}{(1+n)\,L_t} = \sum_{\tau=1}^{\infty} \frac{\hat{U}_t(\tau)}{(1+n)\,L_t} + \frac{n}{1+n}.$$

An implication of our analysis is that learning from search generates unambiguously negative duration dependence in desired wages: workers lower their desired wages over their unemployment spell because lower-paid jobs are easier to get. Consequently, positive duration dependence in job-finding probabilities, controlling for the worker's ability, is the equilibrium counterpart of negative duration dependence in desired wages. However, without controlling for the worker's ability, the job-finding probability may either increase or decrease with unemployment duration. This ambiguity arises from the fact that the

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ability composition of workers in the market changes endogenously with unemployment duration. Workers with longer unemployment durations are precisely those who have failed to find a match previously and, at every duration, they are more likely to be workers with low job-finding ability. To see this, note that the average job-finding probability is given by the ratio of new matches, $g\left(\mu_\tau\right)\{a_H[\hat{U}_t(\tau)-\hat{U}_{L,t}(\tau)]+a_L\hat{U}_{L,t}(\tau)\}$, to total unemployment, $\hat{U}_t(\tau)$. Although $g\left(\mu_\tau\right)$ increases with τ, the proportion of low-ability unemployed workers increases in τ as given below:

$$\frac{\hat{U}_{L,t}(\tau)}{\hat{U}_t(\tau)} = \frac{n(1-p)L_{t-\tau}Q_L(\tau)}{pnL_{t-\tau}Q_H(\tau)+(1-p)nL_{t-\tau}Q_L(\tau)} = \left[1+\left(\frac{p}{1-p}\right)\frac{Q_H(\tau)}{Q_L(\tau)}\right]^{-1}.$$

$\hat{U}_{L,t}(\tau)/\hat{U}_t(\tau)$ increases in τ, because $Q_H\left(\tau\right)/Q_L\left(\tau\right)$ decreases in τ, as shown earlier. As a result, the average efficiency of search decreases in τ.

The ambiguous dependence of the average job-finding probability on unemployment duration could help explain the lack of evidence of negative duration dependence after controlling for unobserved heterogeneity (Devine and Kiefer, 1991, Machin and Manning, 1999).[13] As a related piece of evidence, Shimer (2004) finds that average search intensity tends to increase during recessions, when the returns to search are expected to be relatively low. He points out that this finding presents a problem to standard search models of the labor market, which build in a complementarity between search intensity and the returns to search. Although the context is different here, choosing a submarket with a higher x has the same effect as choosing higher search intensity, since both increase the worker's job-finding probability.

7. Conclusion

In this paper, we have proposed an equilibrium theory of learning from search in the labor market, which addresses the search behavior of workers, the creation of jobs, and the wage distribution as functions of unemployment duration. The main assumption is that unemployed workers have incomplete information about their job-finding abilities and, therefore, learn about the abilities from their search outcomes. The theory formalizes a notion akin to that of discouragement. That is, over the unemployment spell, unemployed workers update their beliefs about their job-finding abilities downward and reduce not only reservation wages, but also desired wages.

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[13]There is some evidence of an effect of unemployment benefits on the job search behavior. This different mechanism is considered in Burdett (1977) and Mortensen (1977).

One contribution of the paper has been to integrate learning from search into an equilibrium framework where search is a directed process. By severing the direct dependence of search behavior on the wage distribution, the directed search framework has simplified the task of addressing jointly the workers' search behavior, the incentives to create jobs, and the wage distribution in equilibrium. Another contribution has been to explore a connection between convexity of a worker's value function in beliefs and the property of supermodularity. This connection enabled us to employ monotone comparative statics to establish the properties of desired wages over unemployment despite the potential presence of non-differentiable value functions and multiple solutions to a worker's optimization problem. This connection is likely to be useful in many other learning problems, because convexity of the value function in beliefs is a natural consequence of learning.

By providing an equilibrium theory of learning from search and discouragement, our model has also provided a novel mechanism for generating wage dispersion. The learning process turns *ex ante* identical workers into *ex post* heterogeneous workers who differ in posterior beliefs about their job-finding probabilities. This mechanism implies that differences in unemployment duration among homogeneous workers may be a factor underlying wage dispersion, particularly among workers who earn relatively low wages. This explanation contrasts with others that also build on search frictions (e.g. Burdett and Judd, 1983, and Burdett and Mortensen, 1998). An empirical investigation of the contribution of this explanation to the large wage dispersion that is observed among similar workers (e.g., Mortensen, 2003) is one avenue for future research.

Other extensions of our model may consider workers' labor force participation, job destruction and on-the-job search. These theoretical extensions do not change the nature of our analysis, but they may provide a useful structural framework for empirical studies of the wage distribution and the distribution of unemployment durations.

Appendix

A. Proof of Lemma 3.1

Suppose that the worker chooses to reject a match in submarket x and continue to search in the next period. Then, his expected payoff of entering a submarket x to search is:

$$R^e(x, \mu) \equiv x\mu \frac{V(a_H + a_L - a_H a_L/\mu)}{1 + r} + (1 - x\mu)\frac{V(H(x, \mu))}{1 + r}. \tag{A.1}$$

Given the analysis leading to Lemma 3.1, it suffices to show that Assumption 2 is sufficient for $V(\mu) = \max_{x \in X} R(x, \mu) \geq \max_{x \in X} R^e(x, \mu)$ for all $\mu \in M$, where $M = [a_L, a_H]$ and $X = [0, 1/a_H]$. A sufficient condition for this is that $W\left(a_H^{-1}\right) \geq y/(1 + r)$. Using the definition of W, this condition amounts to Assumption 2. **QED**

B. Proof of Lemma 3.2

Let $TV(\mu)$ denote the right-hand side of (2.8). The value function, V, is a fixed point of the mapping T. Let $C_1(M)$ be the set containing all bounded, continuous and increasing functions on M. Let $C_1^s(M)$ be the subset of $C_1(M)$ that contains all strictly increasing functions. Similarly, let $C_2(M)$ be the subset of $C_1(M)$ that contains all convex functions, and $C_2^s(M)$ be the subset of $C_2(M)$ that contains all strictly convex functions. We need to show that $V \in C_1^s(M) \cap C_2^s(M)$.

To show that $V \in C_1^s(M)$, it suffices to show that $T : C_1(M) \to C_1^s(M)$, which will be accomplished by Lemma B.1 below. By the argument of contraction mapping, the fixed point of T is strictly increasing. Similarly, to prove $V \in C_2^s(M)$, it suffices to show that $T : C_2(M) \to C_2^s(M)$, which will be accomplished by the last two lemmas in this proof. Because a convex function is almost everywhere differentiable (see Royden, 1988, pp.113-114), V is almost everywhere differentiable.

Let $G(\mu) = \arg\max_{x \in X} R(x, \mu)$, where R is defined by (2.7). Let $g(\mu) \in G(\mu)$.

Lemma B.1. $T : C_1(M) \to C_1^s(M)$.

Proof. Suppose that $V \in C_1(M)$ in the definition of R. Pick any $\mu_a, \mu_b \in M$ with $\mu_a > \mu_b$. Denote $g_i = g(\mu_i)$, where $i \in \{a, b\}$. We need to show that $TV(\mu_a) > TV(\mu_b)$.

First, note that $g_b > 0$: if $g_b = 0$, instead, then $R(g_b, \mu_b) = V(\mu_b)/(1 + r)$, and $V(\mu_b) = TV(\mu_b)$ implies that $V(\mu_b) = 0$, which contradicts the optimality of g_b because $R(x, \mu_b) > 0$ for any $x \in (0, 1/a_H]$ and any $\mu_b \in [a_L, a_H]$. With $g_b > 0$, we have:

$$
\begin{aligned}
R(g_a, \mu_a) - R(g_b, \mu_b) &\geq R(g_b, \mu_a) - R(g_b, \mu_b) \\
&\geq g_b(\mu_a - \mu_b)\left[\frac{W(g_b)}{r} - \frac{1}{1+r}V(H(g_b, \mu_b))\right] \\
&> 0.
\end{aligned}
\tag{B.1}
$$

The first inequality comes from the fact that $g_i \in \arg\max_x R(x, \mu_i)$ and the second one from the fact that $V(H(g_b, \mu_a)) \geq V(H(g_b, \mu_b))$. The last inequality comes from Assumption

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2, the result $g_b > 0$, and the fact that $V(H(g_b, \mu_b)) \leq V(a_H) < y/r$. Hence, $TV(\mu_a) > TV(\mu_b)$. **QED**

Lemma B.2. *If* $V \in C_2(M)$, *then* $R(x, \mu)$ *defined by (2.7) is convex in* μ *for any given* x. *If* $V \in C_2^s(M)$, *then* $R(x, \mu)$ *is strictly convex in* μ.

Proof. We prove the second part of the lemma first. Let V be a strictly convex function. Let μ_a and μ_b be two arbitrarily values in M, with $\mu_a > \mu_b$. Let $\theta \in (0, 1)$ be a number. Denote $\mu_\theta = \theta\mu_a + (1 - \theta)\mu_b$. We show that

$$R(x, \mu_\theta) < \theta R(x, \mu_a) + (1 - \theta)R(x, \mu_b).$$

Denote $H_i = H(x, \mu_i)$, where $i \in \{a, b, \theta\}$. Since $\partial H/\partial \mu > 0$, then $H_a > H_\theta > H_b$. Let $\sigma = (H_\theta - H_b)/(H_a - H_b)$. Note that $\sigma \in (0, 1)$ and $\sigma H_a + (1 - \sigma)H_b = H_\theta$. If V is strictly convex, then

$$V(H_\theta) < \sigma V(H_a) + (1 - \sigma)V(H_b). \tag{B.2}$$

With this result and the definition of R in (2.7), we have:

$$\begin{aligned} R(x, \mu_\theta) &< \mu_\theta x \frac{W(x)}{r} + \frac{1 - \mu_\theta x}{1 + r}[\sigma V(H_a) + (1 - \sigma)V(H_b)] \\ &= \theta R(x, \mu_a) + (1 - \theta)R(x, \mu_b) + \frac{V(H_a)}{1 + r}\Delta_a + \frac{V(H_b)}{1 + r}\Delta_b, \end{aligned}$$

where

$$\begin{aligned} \Delta_a &= (1 - \mu_\theta x)\sigma - \theta(1 - \mu_a x), \\ \Delta_b &= (1 - \mu_\theta x)(1 - \sigma) - (1 - \theta)(1 - \mu_b x). \end{aligned}$$

For $i, j \in \{a, b, \theta\}$, we use (2.6) to compute:

$$\sigma = \frac{(\mu_\theta - \mu_b)(1 - \mu_a x)}{(\mu_a - \mu_b)(1 - \mu_\theta x)} = \frac{\theta(1 - \mu_a x)}{(1 - \mu_\theta x)}.$$

Now it is easy to see that $\Delta_a = 0 = \Delta_b$. Therefore, R is strictly convex.

If V is convex rather than strictly convex, then (B.2) holds as " $\leq$ " instead of " $<$ ". The rest of the proof can be easily adapted to show that $R(x, \mu)$ is convex in μ. **QED**

Lemma B.3. $T : C_2(M) \to C_2^s(M)$.

Proof. Pick any $V_0 \in C_2(M)$. Denote $V_1(\mu) = TV_0(\mu)$. Let μ_a and μ_b be two arbitrary values in M, with $\mu_a > \mu_b$. Let $\theta \in (0, 1)$ be a number. Denote $\mu_\theta = \theta\mu_a + (1 - \theta)\mu_b$. We need to show that

$$V_1(\mu_\theta) < \theta V_1(\mu_a) + (1 - \theta)V_1(\mu_b).$$

We divide the proof in two cases: the case where V_0 is strictly convex and the case where V_0 has linear segments.

Case 1: $V_0 \in C_2^s(M)$. In this case, the previous lemma implies that $R(x, \mu)$ is strictly convex in μ for any given x. Shorten the notation $g(\mu_i)$ to g_i, where $g(\mu_i) \in$

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$\arg\max_x R(x, \mu_i)$ and $i \in \{a, b, \theta\}$. Then, $V_1(\mu_i) = R(g_i, \mu_i)$, with V in (2.7) being replaced with V_0. Strict convexity of V is proven below:

$$V_1(\mu_\theta) = R(g_\theta, \mu_\theta) \quad \begin{aligned} &< \theta R(g_\theta, \mu_a) + (1 - \theta) R(g_\theta, \mu_b) \\ &\leq \theta R(g_a, \mu_a) + (1 - \theta) R(g_b, \mu_b) \\ &= \theta V_1(\mu_a) + (1 - \theta) V_1(\mu_b). \end{aligned} \tag{B.3}$$

The first inequality comes from the fact that R is strictly convex in μ, and the second inequality from the fact that $R(x, \mu_i) \leq R(g_i, \mu_i)$ for all x.

Case 2: V_0 is convex and has some linear segments. If any two of the elements, $V_0(H(g_\theta, \mu_\theta))$, $V_0(H(g_\theta, \mu_a))$ and $V_0(H(g_\theta, \mu_b))$, do not lie on the same linear segment of V_0, then the first inequality in (B.3) is still strict and V_1 is strictly convex. Suppose that all three elements lie on the same linear segment of V_0. Temporarily denote this linear segment as $V_0(H)/(1 + r) = A + BH$, with $B > 0$ (because V is strictly increasing). Using (2.6), we can compute:

$$(1 - \mu x) V_0(H)/(1 + r) = (1 - \mu x)(A + Ba_H) - B(1 - a_L x)(a_H - \mu).$$

This is linear and differentiable in (μ, x). Restrict μ to be such that $V_0(H(x, \mu))$ lies on the linear segment described above. In this case, $R(x, \mu)$ is strictly concave in x iff $[xW(x)]$ is so. The latter property is implied by (2.11). Thus, the solution g is unique and satisfies the following first-order condition:

$$0 = R_1(x, \mu) = \mu \left[(W + xW')/r - A - Ba_H \right] + Ba_L(a_H - \mu).$$

Differentiating this first-order condition and using (2.11), we find that the solution $g(\mu)$ satisfies $g'(\mu) < 0$. Thus, $g_a \neq g_\theta$ and $g_b \neq g_\theta$. Because the solutions are unique in this case, then $R(g_\theta, \mu_b) < R(g_b, \mu_b)$ and $R(g_\theta, \mu_a) < R(g_a, \mu_a)$. The second inequality in (B.3) is strict, and so V_1 is strictly convex. This completes the proofs of the current lemma and Lemma 3.2. **QED**

C. Proof of Theorem 4.1

Take arbitrary $z_a, z_b \in -X$ and arbitrary $\mu_a, \mu_b \in M$, with $z_a > z_b$ and $\mu_a > \mu_b$. Denote:

$$D = \left[\hat{R}(z_a, \mu_a) - \hat{R}(z_a, \mu_b) \right] - \left[\hat{R}(z_b, \mu_a) - \hat{R}(z_b, \mu_b) \right].$$

We need to show $D > 0$. Temporarily denote $H_{ij} = H(-z_i, \mu_j)$ and $V_{ij} = V(H_{ij})$, where $i, j \in \{a, b\}$. Computing D, we have:

$$\frac{(1 + r)D}{z_a - z_b} = \left(\frac{1}{\mu_a} + z_b \right) \frac{V_{aa} - V_{ba}}{z_a - z_b} - \left(\frac{1}{\mu_b} + z_b \right) \frac{V_{ab} - V_{bb}}{z_a - z_b} + (V_{aa} - V_{ab}).$$

There are two cases to consider: $\mu_b = a_L$ and $\mu_b > a_L$. First, suppose that $\mu_b = a_L$. Then $V_{aa} - V_{ba} \geq 0$, with equality if and only if $\mu_a = a_H$; $V_{ab} - V_{bb} = 0$; and $V_{aa} - V_{ab} > 0$. Hence, $D > 0$ in this case. Next, consider the second case, where $\mu_b > a_L$. Here there

are also two subcases to consider: $\mu_a = a_H$ and $\mu_a < a_H$. We start with the second subcase. Suppose that $\mu_a < a_H$. Because $H_1(-z, \mu) < 0$ and $H_2(-z, \mu) > 0$, then $H_{aa} > \max\{H_{ab}, H_{ba}\} \geq \min\{H_{ab}, H_{ba}\} > H_{bb}$. Strict convexity of $V(H)$ implies (see Royden, 1988, p.113):

$$\frac{V_{aa} - V_{ba}}{H_{aa} - H_{ba}} > \frac{V_{ab} - V_{bb}}{H_{ab} - H_{bb}} \quad \text{and} \quad \frac{V_{aa} - V_{ab}}{H_{aa} - H_{ab}} > \frac{V_{ab} - V_{bb}}{H_{ab} - H_{bb}}. \tag{C.1}$$

Thus, the following (strict) inequality holds:

$$\frac{H_{ab} - H_{bb}}{V_{ab} - V_{bb}} \left(\frac{(1+r)D}{z_a - z_b} \right)$$
$$> \left(\frac{1}{\mu_a} + z_b \right) \frac{H_{aa} - H_{ba}}{z_a - z_b} - \left(\frac{1}{\mu_b} + z_b \right) \frac{H_{ab} - H_{bb}}{z_a - z_b} + (H_{aa} - H_{ab}).$$

Substituting H_{ij}, we get:

$$\frac{H_{ab} - H_{bb}}{V_{ab} - V_{bb}} \left(\frac{(1+r)D}{z_a - z_b} \right)$$
$$> \left(\frac{1}{\mu_a} + z_b \right) \frac{a_H - \mu_a}{z_a - z_b} \left(\frac{1 + z_b a_L}{1 + z_b \mu_a} - \frac{1 + z_a a_L}{1 + z_a \mu_a} \right)$$
$$- \left(\frac{1}{\mu_b} + z_b \right) \frac{a_H - \mu_b}{z_a - z_b} \left(\frac{1 + z_b a_L}{1 + z_b \mu_b} - \frac{1 + z_a a_L}{1 + z_a \mu_b} \right) + (1 + z_a a_L) \left(\frac{a_H - \mu_b}{1 + z_a \mu_b} - \frac{a_H - \mu_a}{1 + z_a \mu_a} \right)$$
$$= \frac{(a_H - \mu_a)(\mu_a - a_L)}{\mu_a (1 + z_a \mu_a)} - \frac{(a_H - \mu_b)(\mu_b - a_L)}{\mu_b (1 + z_a \mu_b)} + (1 + z_a a_L) \left(\frac{a_H - \mu_b}{1 + z_a \mu_b} - \frac{a_H - \mu_a}{1 + z_a \mu_a} \right)$$
$$= a_H a_L \frac{\mu_a - \mu_b}{\mu_a \mu_b} > 0.$$

The first equality comes from collecting terms according to $(a_H - \mu_i)$. Hence, $D > 0$ in this subcase as well. Finally, if $\mu_b > a_L$ and $\mu_a = a_H$, the last string of inequalities becomes:

$$\frac{H_{ab} - H_{bb}}{V_{ab} - V_{bb}} \left(\frac{(1+r)D}{z_a - z_b} \right)$$
$$> -\left(\frac{1}{\mu_b} + z_b \right) \frac{a_H - \mu_b}{z_a - z_b} \left(\frac{1 + z_b a_L}{1 + z_b \mu_b} - \frac{1 + z_a a_L}{1 + z_a \mu_b} \right) + (1 + z_a a_L) \left(\frac{a_H - \mu_b}{1 + z_a \mu_b} \right)$$
$$= -\frac{(a_H - \mu_b)(\mu_b - a_L)}{\mu_b (1 + z_a \mu_b)} + (1 + z_a a_L) \left(\frac{a_H - \mu_b}{1 + z_a \mu_b} \right)$$
$$= \frac{a_L (a_H - \mu_b)}{\mu_b} > 0.$$

Thus, the function $\hat{R}(z, \mu)$ is strictly supermodular. Because $-X$ is a lattice, the monotone selection theorem in Topkis (1998, Theorem 2.8.4, p.79) implies that every selection from $Z(\mu)$ is increasing. As a result, every selection $g(\mu)$ from $G(\mu)$ is decreasing, and $w(\mu) = W(g(\mu))$ is increasing. **QED**

D. Proof of Lemma 5.1

First, we show that optimal choices are interior under Assumption 3. Consider the corner, $z = 0$. For any prior beliefs, μ, the choice $z = 0$ yields zero expected wage in the period and the posterior beliefs $H(0, \mu) = \mu$. The value of this choice is $\hat{R}(0, \mu) = 0$, which can be increased by any choice $z < 0$. Thus, the choice $z = 0$ is never optimal.

Now consider the other corner, $z = -1/a_H$. Since optimal choices, $z(\mu)$, are increasing in μ, a sufficient condition for $z > -1/a_H$ is $z(a_L) > -1/a_H$. This condition is guaranteed

if the objective function $\hat{R}(z, \mu)$ (defined in (4.1)) is strictly increasing in z at $z = -1/a_H$. In turn, a sufficient condition for the latter is that $\hat{R}(z, \mu)$ is strictly increasing in z at $z = -1/a_H$ when H is fixed at $H(1/a_H, \mu)$, because $V(H(-z, \mu))$ is increasing in z. Noting that $H(1/a_H, \mu) = a_L$, we can write this sufficient condition as

$$\frac{r}{1+r}V(a_L) - W(1/a_H) - \frac{W'(1/a_H)}{a_H} > 0. \tag{D.1}$$

Because the choice $z = -1/a_H$ is always feasible, $V(a_L) \geq \mu\hat{R}(-1/a_H, \mu)$, i.e.,

$$V(a_L) \geq \frac{1+r}{r}\left(\frac{a_L/a_H}{r + a_L/a_H}\right)W(1/a_H).$$

Substituting this inequality for $V(a_L)$, (2.10) for W, and computing W', we can verify that Assumption 3 is sufficient for (D.1).

Next, we show that $V'(H(-z(\mu), \mu))$ exists. For any real number r, define $r^- = \lim_{\varepsilon \downarrow 0}(r - \varepsilon)$ and $r^+ = \lim_{\varepsilon \downarrow 0}(r + \varepsilon)$. Fix $\mu \in (a_L, a_H)$. Under Assumption 3, the optimal choice $z(\mu)$ is interior. Optimality requires $\hat{R}_1(z^-(\mu), \mu) \geq \hat{R}_1(z^+(\mu), \mu)$. Note that a continuous, convex function has left and right derivatives. Because $W(-z)$ is continuous, V is continuous and convex, and H is continuously differentiable, then

$$\hat{R}_1(z^+(\mu), \mu) = \frac{z(\mu)W'(-z(\mu)) - W(-z(\mu))}{r} + \frac{V(H(-z(\mu), \mu))}{1+r}$$
$$- \left(\frac{1}{\mu} + z(\mu)\right)\frac{V'(H^+(-z(\mu), \mu))}{1+r}H_1(-z(\mu), \mu),$$

$$\hat{R}_1(z^-(\mu), \mu) = \frac{z(\mu)W'(-z(\mu)) - W(-z(\mu))}{r} + \frac{V(H(-z(\mu), \mu))}{1+r}$$
$$- \left(\frac{1}{\mu} + z(\mu)\right)\frac{V'(H^-(-z(\mu), \mu))}{1+r}H_1(-z(\mu), \mu).$$

Recall that H_1 denotes the derivative of $H(-z, \mu)$ with respect to the first argument, rather than to z. Since $H_1 < 0$, the optimality condition, $\hat{R}_1(z^-(\mu), \mu) \geq \hat{R}_1(z^+(\mu), \mu)$, implies:

$$V'(H^-(-z(\mu), \mu)) \geq V'(H^+(-z(\mu), \mu)).$$

Because V is convex, the reversed inequality holds. Thus,

$$V'(H^-(-z(\mu), \mu)) = V'(H^+(-z(\mu), \mu)) = V'(H(-z(\mu), \mu)).$$

This implies that optimal choices in every period satisfy the first-order conditions. **QED**

E. Proof of Theorem 5.2

It suffices to show that the case $z(\mu_a) \neq z(\mu_b)$ cannot occur for any pair (μ_a, μ_b) with $\mu_a > \mu_b$. Suppose to the contrary that $z(\mu_a) = z(\mu_b)$. Denote this common value as z^*. By Lemma 5.1, $z(\mu_a)$ and $z(\mu_b)$ are interior and satisfy first-order conditions. That is,

$$\hat{R}_1(z^*, \mu_a) = 0 = \hat{R}_1(z^*, \mu_b).$$

Shorten the notation $H(-z^*, \mu_i)$ to H_i, where $i \in \{a, b\}$. Substituting $\hat{R}_1(z^*, \mu_i)$, we have:

$$(1+r)\left[\hat{R}_1(z^*, \mu_a) - \hat{R}_1(z^*, \mu_b)\right]$$
$$= V(H_a) - V(H_b) - \left(\tfrac{1}{\mu_a} + z^*\right)V'(H_a)H_1(-z^*, \mu_a) + \left(\tfrac{1}{\mu_b} + z^*\right)V'(H_b)H_1(-z^*, \mu_b).$$

Because $V(H)$ is continuous and strictly convex, and because $V'(H_b)$ exists by Lemma 5.1, we have: $V(H_a) - V(H_b) > V'(H_b)(H_a - H_b)$. Then,

$$(1+r)\left[\frac{\hat{R}_1(z^*, \mu_a) - \hat{R}_1(z^*, \mu_b)}{V'(H_b)}\right]$$
$$\geq (H_a - H_b) - \left(\tfrac{1}{\mu_a} + z^*\right)\frac{V'(H_a)}{V'(H_b)}H_1(-z^*, \mu_a) + \left(\tfrac{1}{\mu_b} + z^*\right)H_1(-z^*, \mu_b)$$
$$> (H_a - H_b) - \left(\tfrac{1}{\mu_a} + z^*\right)H_1(-z^*, \mu_a) + \left(\tfrac{1}{\mu_b} + z^*\right)H_1(-z^*, \mu_b).$$

The first inequality comes from substituting the inequality between the V's and the fact that $V' > 0$; the second (strict) inequality comes from the facts that V is strictly convex, $H_a > H_b$, and $H_1 < 0$. Denote the last expression temporarily as $f(\mu_a)$. Because f is differentiable, we can compute:

$$f'(\mu_a) = H_1(-z^*, \mu_a) + \tfrac{1}{(\mu_a)^2}H_1(-z^*, \mu_a) - \left(\tfrac{1}{\mu_a} + z^*\right)H_{12}(-z^*, \mu_a)$$
$$= \tfrac{a_H a_L}{(\mu_a)^2} > 0.$$

The second equality comes from property (iv) in Lemma 2.2. Because $\mu_a > \mu_b$, then $f(\mu_a) > f(\mu_b) = 0$. That is, $\hat{R}_1(z^*, \mu_a) > \hat{R}_1(z^*, \mu_b)$. This result contradicts the supposition that $z(\mu_a) = z(\mu_b)$. **QED**

F. Proofs of Lemma 5.3 and Theorem 5.4

First, we prove the following lemma (which does require optimal choices to be interior):

Lemma F.1. $\bar{z}(\mu)$ *is right-continuous and* $\underline{z}(\mu)$ *is left-continuous at each* $\mu \in M$.

    Proof. Pick an arbitrary $\mu \in M$. Let $\{\mu_i\}$ be a sequence with $\mu_i \to \mu$ and $\mu_i \geq \mu_{i+1} \geq \mu$ for all i. Because $\bar{z}(\mu)$ is an increasing function, then $\{\bar{z}(\mu_i)\}$ is a decreasing sequence and $\bar{z}(\mu_i) \geq \bar{z}(\mu)$ for all i. Thus, $\bar{z}(\mu_i) \downarrow A$ for some $A \geq \bar{z}(\mu)$. On the other hand, the Theorem of the Maximum (see Stokey and Lucas, 1989) implies that the correspondence $Z(\mu)$ is upper hemicontinuous (uhc). Because $\mu_i \to \mu$, and $\bar{z}(\mu_i) \in Z(\mu_i)$ for each i, uhc of Z implies that there is a subsequence of $\{\bar{z}(\mu_i)\}$ that converges to an element in $Z(\mu)$. This element must be A, because all convergent subsequences of a convergent sequence must have the same limit. Thus, $A \in Z(\mu)$, and so $A \leq \max Z(\mu) = \bar{z}(\mu)$. Therefore, $\bar{z}(\mu_i) \downarrow A = \bar{z}(\mu)$, which shows that $\bar{z}(\mu)$ is right-continuous.
    Similarly, by examining the sequence $\{\mu_i\}$ with $\mu_i \to \mu$ and $\mu \geq \mu_{i+1} \geq \mu_i$ for all i, we can show that $\underline{z}$ is left-continuous. This completes the proof of Lemma F.1.

Next, we prove Lemma 5.3. Fix $\mu_a \in (a_L, a_H)$. Because $\bar{z}(\mu)$ maximizes $R(-z, \mu)$ for each given μ, then

$$V(\mu) = R(-\bar{z}(\mu), \mu) \geq R(-\bar{z}(\mu_a), \mu)$$
$$V(\mu_a) = R(-\bar{z}(\mu_a), \mu_a) \geq R(-\bar{z}(\mu), \mu_a).$$

Taking $\mu > \mu_a$, where $\mu_a < a_H$, and dividing the above inequalities by $(\mu - \mu_a)$, we obtain:

$$\frac{R(-\bar{z}(\mu_a), \mu) - R(-\bar{z}(\mu_a), \mu_a)}{\mu - \mu_a} \leq \frac{V(\mu) - V(\mu_a)}{\mu - \mu_a} \leq \frac{R(-\bar{z}(\mu), \mu) - R(-\bar{z}(\mu), \mu_a)}{\mu - \mu_a}.$$

Take the limit $\mu \downarrow \mu_a$. Under Assumption 3, $V'(H(-\bar{z}(\mu_a), \mu_a))$ exists for each μ (see Lemma 5.1). Because $\bar{z}(\mu)$ is right-continuous, then $R_2(-\bar{z}(\mu_a), \mu_a)$ exists. The limits of the first and last ratios are both $R_2(-\bar{z}(\mu_a), \mu_a)$. Thus, $V'(\mu_a^+) = R_2(-\bar{z}(\mu_a), \mu_a)$.

Now conduct the above exercise with $\underline{z}$ replacing $\bar{z}$. For $\mu < \mu_a$ and $\mu_a > a_L$, we have:

$$\frac{R(-\underline{z}(\mu_a), \mu) - R(-\underline{z}(\mu_a), \mu_a)}{\mu - \mu_a} \geq \frac{V(\mu) - V(\mu_a)}{\mu - \mu_a} \geq \frac{R(-\underline{z}(\mu), \mu) - R(-\underline{z}(\mu), \mu_a)}{\mu - \mu_a}.$$

Take the limit $\mu \uparrow \mu_a$. Because $\underline{z}(\mu)$ is left-continuous and interior, then $V'(\mu_a^-) = R_2(-\underline{z}(\mu_a), \mu_a)$.

To establish the inequality between the left- and right-derivatives of V, use the definition $R(-z, \mu) = \mu \hat{R}(z, \mu)$ to compute:

$$R_2(-z(\mu), \mu) = \hat{R}(z(\mu), \mu) + \mu \hat{R}_2(z(\mu), \mu) = V(\mu)/\mu + \mu \hat{R}_2(z(\mu), \mu).$$

Because $\hat{R}(z, \mu)$ is strictly supermodular, $\hat{R}_2(\bar{z}(\mu_a), \mu_a) \geq \hat{R}_2(\underline{z}(\mu_a), \mu_a)$, where the inequality is strict if and only if $\bar{z}(\mu_a) > \underline{z}(\mu_a)$. Therefore, $V'(\mu_a^+) \geq V'(\mu_a^-)$, where the inequality is strict if and only if $\bar{z}(\mu_a) > \underline{z}(\mu_a)$. This completes the proof of Lemma 5.3.

Finally, we prove Theorem 5.4. Given any selection $z(\mu_0) \in Z(\mu_0)$ and the induced beliefs $\mu_1 = H(-z(\mu_0), \mu_0)$, Lemma 5.1 implies that $V'(\mu_1)$ exists. Then, Lemma 5.3 implies $\bar{z}(\mu_1) = \underline{z}(\mu_1)$. That is, $Z(\mu_1) = \{z(\mu_1)\}$ is a singleton. So, the posterior belief induced by $Z(\mu_1)$ is unique and is given by $\mu_2 = H(-z(\mu_1), \mu_1)$. Again, Lemma 5.1 implies that $V'(\mu_2)$ exists and Lemma 5.3 implies that $\bar{z}(\mu_2) = \underline{z}(\mu_2)$. Repeating this argument shows that μ_τ is unique, $Z(\mu_\tau)$ is a singleton, and $V'(\mu_\tau)$ exists for all $\tau = 1, 2, \dots$.

If $\mu_0 \notin N$, then $\bar{z}(\mu_0) = \underline{z}(\mu_0)$ by the definition of N. In this case, the posterior belief $\mu_1 = H(z(\mu_0), \mu_0)$ is unique. Also, Lemma 5.3 implies that $V'(\mu_0)$ exists. If $\mu_0 \in N$, again, the results stated in Theorem 5.4 follow from Lemma 5.3. **QED**

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