Information-based trade[1]

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Abstract

We study the possibility of trade for purely informational reasons. We depart from previous analyses (e.g. Grossman and Stiglitz 1980 and Milgrom and Stokey 1982) by allowing the final payoff of the asset being traded to depend on an action taken by its eventual owner. We characterize conditions under which equilibria with trade exist. We demonstrate that our model also applies to a portfolio allocation setting, and use our results to show that trade is possible whenever there is sufficient uncertainty about market betas.

1 Introduction

Following Grossman and Stiglitz (1980) and Milgrom and Stokey (1982), economists have reached a consensus that under many circumstances it is impossible for an individual to profit from superior information.[1] This result is often described as the "no trade" or "no speculation" theorem. The underlying argument is, at heart, straightforward. If a buyer is prepared to buy an asset from a seller for price p, then the buyer must believe that, conditional on the seller agreeing to the trade, the asset value exceeds p in expectation. But conversely, knowing this the seller is at least as well off keeping the asset.

This insight has had enormous consequences for financial economics. Almost all observers of financial markets regard trade for informational reasons — *information-based trade* — as a key motive for trade. To generate information-based trade, the vast majority of papers studying financial markets introduce "noise traders" who trade for (typically exogenous) non-informational reasons.[2] Provided strategic agents are unable to observe the volume of noise trader activity information-based trade is possible. However, the modeling device of noise traders has often been criticized, as is well-illustrated by Dow and Gorton's (2008) survey. Moreover, a significant amount of trade takes place directly between relatively sophisticated parties — a setting that lies outwith the standard noise-trader framework.[3]

[1] See also Kreps (1977), Tirole (1982), Holmström and Myerson (1983) and Fudenberg and Levine (2005).

[2] See, for example, Kyle (1985), and Glosten and Milgrom (1985).

[3] For example, many trades occur in "upstairs" markets, i.e., are trades in which "buyers and sellers negotiate in the 'upstairs' trading rooms of brokerage firms" (Booth *et al*, 2002). Identifying upstairs trades is relatively hard, but using detailed data from Finland Booth *et al* report that upstairs trades account for 50% of total volume. In the last few years "dark liquidity pools" (Liquidnet and Pipeline are well-known examples) have captured a significant share of trade volume, particularly for midcap stocks, and as is the case for upstairs markets are used only by relatively sophisticated traders.

In this paper we develop a distinct and hitherto neglected reason for trade between differentially informed parties. In many cases the holder of the asset must make a decision that affects its value. If better information leads to superior decisions, then the information released in trade is socially valuable. This possibility, which is implicitly ruled out in Milgrom and Stokey's otherwise general framework, is enough to generate trade even without noise traders.

An example

The intuition for our results is best illustrated by an example. A risk neutral agent (the *seller*) owns an asset that he can potentially trade with a second risk neutral agent (the *buyer*). The asset's payoff depends on two factors: an underlying but currently unobservable fundamental $\theta \in \{a, b\}$, and what the eventual asset owner chooses to do with the asset. The best action for the asset owner to take depends on θ. If $\theta = a$ the best action is A, and the asset is worth 2 if this action is taken. If $\theta = b$ the best action is B, and the asset is worth 1 if it is taken. The asset is valueless if any action other than the (θ-contingent) best action is taken. The buyer and seller have the same "skill" in taking actions A and B, so that the action-contingent asset payoffs for *both* parties are as given above. We discuss various interpretations below; an immediate one is that the asset is a debt claim and the action is a restructuring decision (e.g., liquidation vs. reorganization).

The unconditional probability of fundamental a is $1/2$. Both the buyer and seller receive partially informative signals about the true fundamental θ. Conditional on the fundamental the signals are distributed independently and identically. Specifi-

Our paper will provide one explanation for trade between two sophisticated parties. Another explanation would be that at least one of the parties is trading on behalf of retail investors, and the associated agency problems affect trading behavior. (For models of the effect of delegation on trading and prices see, e.g., Dow and Gorton 1997b and Dasgupta and Prat 2007.)

cally, if the true fundamental is a (respectively, b) then each party observes signal s^a (respectively, s^b) with probability $3/4$.

Consider the following trading game: after observing his signal, the buyer decides whether or not to offer to buy the asset, and if so, the price p at which he offers to buy. The seller either accepts or rejects the offer. We claim the following is an equilibrium: the buyer offers to buy the asset for $p = 0.8$ independent of his signal, and the seller accepts if and only if he observes signal s^b.

First, consider the situation faced by the seller. If he ends up with the asset, he must decide what to do using only his own information. As such, if he sees signal s^a and does not sell, his expected payoff is $3/2$, while if he sees signal s^b and does not sell his expected payoff is $3/4$.[4] Consequently, after signal s^b the seller prefers to sell at a price $p = 0.8$ rather than keep the asset; and after signal s^a, prefers to keep the asset rather than sell at this price.

Next, we show the buyer is prepared to buy at price $p = 0.8$. Note that in equilibrium the buyer learns the seller's signal when he acquires the asset, since in equilibrium the seller only accepts the buyer's offer when he observes signal s^b. So on the one hand, if the buyer observes signal s^a he regards $\theta = a$ and $\theta = b$ as equally likely, since he knows the seller saw s^b. Consequently he will choose action A, giving an expected payoff of $2 \times 1/2 = 1$. On the other hand, if the buyer observes signal s^b, then given the seller also observed signal s^b the buyer's probability assessment that $\theta = b$ is $9/10$.[5] Given this, he chooses action B, yielding an expected payoff of $1 \times 9/10 = 9/10$. In both cases, the buyer's expected payoff exceeds the price

[4]Note that since $3/4 \times 1 > 1/4 \times 2$, action B is the better action to take if the only information available is that one of the signals is s^b.

[5]Specifically, the buyer's posterior belief is given by:

$$\Pr\left(b|s^b s^b\right) = \frac{\Pr(b)\Pr\left(s^b|b\right)^2}{\Pr(a)\Pr\left(s^b|a\right)^2 + \Pr(b)\Pr\left(s^b|b\right)^2} = \frac{\left(\frac{3}{4}\right)^2}{\left(\frac{1}{4}\right)^2 + \left(\frac{3}{4}\right)^2} = \frac{9}{10}.$$

$p = 0.8$. As such, the behavior described is indeed an equilibrium.[6]

In this example both parties are strictly better off under the trade. Moreover, they are both better off even after conditioning on any information they acquire in equilibrium. The reason this is possible is that the asset value endogenously depends on the information possessed by its owner. In the example, trade transfers the asset from the seller when he observes signal s^b to the buyer. Trade creates value because it leads to a better decision after the signal pair $s^b s^a$. Specifically, after these signals the seller would take action B because he observes only signal s^b; while the value-weighted best action is A, and the buyer takes this action. In essence, trade transfers the asset from an agent who is likely to make the wrong decision to one who is more likely to make the right decision. In contrast, in Grossman and Stiglitz (1980) and Milgrom and Stokey (1982) asset holders have no decision to make since the final asset payoffs are exogenous to the information possessed by its owner.

Applications

A number of different situations are captured by this model:

1. Most directly, the asset is a controlling equity stake in a firm; or (as noted above) a debt claim that needs restructuring.

2. The asset is a large but non-controlling block of shares in a firm with an upcoming shareholder vote.[7]

[6] There exist out-of-equilibrium beliefs for the seller such that the buyer is unable to profitably deviate to any offer other than 0.8. Details are contained in an earlier draft.

[7] While private benefits such as synergies can also explain trade of a controlling equity stake, this explanation is less readily applicable in the case of non-controlling blocks. That is, while the owners of such blocks can affect a firm's decisions by choosing how to vote, it is less clear how they can derive substantial private benefits (at least without engaging in self-dealing).

3. The asset is one of several debt claims in a firm with an upcoming bankruptcy vote.

4. (Arguably of widest applicability:) The asset is an equity or debt claim with no direct decision rights, but the holder must still decide how to allocate the remainder of his portfolio. Specifically, suppose now that the seller and buyer are risk averse, and that the asset's market beta differs across fundamentals a and b. Depending on his beliefs about the fundamental the asset holder chooses different portfolio allocations. Thus in place of an action directly affecting the asset's payoff this setting features an action (the portfolio choice) that affects the asset holder's utility. We return to this application in much greater detail in Section 5 below.

Paper outline

We describe our relation to the existing literature immediately below. In Section 2 we present our general model, which closely resembles the example above but with the binary action set and signal space replaced with an arbitrary action set and continuous signal space. In Section 3 we derive conditions that are required for trade to occur *regardless* of the trading mechanism used. In Section 4 we examine two simple trading mechanisms. By doing so, we are able to give a simple necessary and sufficient condition for trade. In Section 5 we apply our results to the case in which agents trade an asset over which they have no direct control, but instead choose portfolio allocations.We relate our previously established trade conditions to standard measures of asset risk — in particular, to the market beta of the asset. Section 6 concludes.

Related literature

A key ingredient of our model is that the economic agent who decides how to use an asset is able to infer useful information from the trading process. The notion that prices reveal information that is useful for real decisions is an old one in economics. Nonetheless, it is only comparatively recently that researchers have constructed formal models in which, for example, managers learn from the share price. The key difficulty, of course, is that if share prices affect decisions, those decisions in turn affect share prices. Contributions to this so-called "feedback effect" literature include Khanna, Slezak, and Bradley (1994), Dow and Gorton (1997a), Subrahmanyam and Titman (1999), Dye and Sridhar (2002), Dow and Rahi (2003), Goldstein and Guembel (2005), and Dow, Goldstein and Guembel (2006). Chen, Goldstein and Jiang (2005) and Durnev, Mork and Yeung (2004) both present empirical evidence that managers are indeed able to make better decisions as a result of information obtained from stock prices. In more general terms, our paper belongs to a growing literature that seeks to combine insights from corporate finance with those from the distinct market microstructure and asset pricing literatures.

A number of classic papers (notably, Hirshleifer 1971) note the distinction between information in an exchange economy and information in a production economy. However, the subsequent literature on the possibility of trade between differentially and privately informed parties has focused almost exclusively on information in an exchange economy. In particular, the seminal papers of Grossman and Stiglitz (1980) and Milgrom and Stokey (1982) show that under many circumstances trade is impossible in such an environment. Milgrom and Stokey's "no trade" or "no speculation" result rests on two assumptions: Pareto optimality of the initial allocation, and concordancy of beliefs, in the sense that agents agree on how to interpret future information. A subsequent literature has explored conditions under which the "no trade" conclusion does not hold. The literature is too large to adequately survey.

Representative approaches include departing from the common prior assumption, as in Morris (1994) and Biais and Bossaerts (1998), and thus breaking belief concordancy; departing from Pareto optimality, as in Dow and Gorton (1995), who assume that some agents can trade only a subset of assets; and introducing multiple trading rounds, as Grundy and McNichols (1998)[8] do when they show that both belief concordancy and Pareto optimality may fail at the intermediate date of a three-period model.[9]

None of the above papers study the possibility of trade for purely informational reasons in an economy in which asset owners must decide how to use their assets. To the best of our knowledge the only previous consideration of this case is a chapter of Diamond's (1980) dissertation.[10] He derives conditions under which a rational expectations equilibrium (REE) with trade exists when there are two types of agents: one type is uninformed, while the other type observes a noisy signal. The main differences between our paper and his are that (i) we study trade between agents who both possess information, (ii) we show that as a consequence, information is never fully revealed, and (iii) instead of restricting attention to the competitive (REE) outcome, in the spirit of Milgrom and Stokey (1982) we allow for all possible trading mechanisms. Moreover, Diamond's assumption that one side of the trade is completely uninformed means that assets always flow from the less to the more informed party.[11] In contrast, when both parties to the trade have some information, assets can flow to

[8]Related, see also Coury and Easley (2006).

[9]One can also avoid the no-trade conclusion by using non-standard preferences: see, e.g., Halevy (2004).

[10]Less closely related is a recent working paper of Tetlock and Hahn (2007), who show that a decision maker would be willing to trade and act as a loss-making market maker in "weather" securities (or more generally, securities whose value is exogenous to the decision).

[11]Diamond does consider an equilibrium in which uninformed agents end up holding the asset. However, to support the equilibrium he must assume that uninformed agents learn only from the price at which the trade takes place, and not from the volume of trade.

the party with lower quality information.

In this paper we analyze the degree to which efficiency gains arising from additional information make information-based trade possible. However, before proceeding to the details of our analysis, we wish to make the following clear: we are *not* arguing that trade is a superior mechanism relative to other alternatives. Instead, we view trade as a particular information-sharing mechanism that deserves focused attention: it is widely observed, has long interested economists, and has many appealing features.

2 The model

Our model is closely related to the opening example. As in the example, there are two risk neutral agents,[12] who we refer to as a seller (agent 1) and a buyer (agent 2). The seller owns an asset. The payoff from the asset depends on the combination of the action taken by the asset-owner and the realization of some fundamental $\theta \in \{a, b\}$. Neither agent directly observes the fundamental θ, but before meeting, both agents $i = 1, 2$ receive noisy and partially informative signals s_i. Whereas in the example signals were binary, in our main model they have full support in $\mathbb{R}$.

The eventual asset owner must decide what action to take. Regardless of whether the asset-owner is agent 1 or 2, the range of available actions is given by a compact set $\mathcal{X}$, with a typical element denoted by X. (In the opening example, $\mathcal{X}$ is simply the binary set $\{A, B\}$.) We write $v(X, \theta)$ for the payoff when action X is taken and the fundamental is θ, where $v(\cdot, \theta)$ is continuous as a function of X. We emphasize that the asset payoff is independent of the identity of the asset-owner — both agents 1 and 2 are equally capable of executing all actions in $\mathcal{X}$.

[12] Our analysis also covers the case of agents with constant absolute risk aversion preferences. We consider this case in Section 5 below.

Pre-trade information

The information structure of the economy is described by a probability measure space $(\Omega, \mathcal{F}, \mu)$, where $\Omega = \{a, b\} \times \mathbb{R}^2$ and $\mathcal{F}$ is the σ-algebra $\{\{a\}, \{b\}, \{a, b\}\} \times \mathcal{B}^2$. (Throughout, we denote the Borel algebras of $\mathbb{R}$ and $\mathbb{R}^2$ by $\mathcal{B}$ and $\mathcal{B}^2$ respectively.) We write a typical state as $\omega = (\theta, s_1, s_2)$, where θ is the fundamental, s_1 is the signal observed by the seller (agent 1) and s_2 is the signal observed by the buyer (agent 2). For $i = 1, 2$ and $\theta = a, b$ let $\eta_i^\theta : \mathcal{B} \to \mathbb{R}$ be the conditional distribution of s_i given θ. We write F_i^θ for the associated distribution functions, and make the following distributional assumptions. (I) The signals s_1 and s_2 are conditionally independent given θ. (II) For $i = 1, 2$ and $\theta = a, b$ the conditional distribution has full support. (III) For $i = 1, 2$ and $\theta = a, b$ the conditional distribution η_i^θ has a density, which we denote f_i^θ. (IV) For $i = 1, 2$ signal s_i satisfies the strict monotone ratio likelihood property (MLRP), i.e., $L_i(s_i) \equiv \frac{f_i^a(s_i)}{f_i^b(s_i)}$ is strictly increasing in s_i. Moreover, we assume that the likelihood ratio is unbounded, i.e.,

$$L_i(s_i) \to \pm\infty \text{ as } s_i \to \pm\infty. \tag{1}$$

That is, there are extreme realizations of each agent's signal that are very informative — even if an agent's signal is generally uninformative. (We stress that none of the results of Section 3 depend on either the existence of densities or the assumption of unbounded likelihood ratios. See also the discussion on page 16.)

Agent i directly observes only his own signal. Formally, the information of agents $i = 1, 2$ before trade is given by the sub σ-algebras $\mathcal{F}_1 = \{a, b\} \times \mathcal{B} \times \mathbb{R}$ and $\mathcal{F}_2 = \{a, b\} \times \mathbb{R} \times \mathcal{B}$.

Trade

An allocation in our economy is a pair of mappings $\kappa : \Omega \to \{1, 2\}$ and $\pi : \Omega \to \mathbb{R}$ where κ specifies which agent owns the asset, and π specifies a transfer from agent 2 to agent 1. Since neither agent observes the fundamental θ both κ and π must be measurable with respect to the σ-algebra $\{a, b\} \times \mathcal{B}^2$. Let $(\hat{\kappa}, \hat{\pi})$ denote the initial allocation, in which agent 1 owns the asset and no transfer takes place: $(\hat{\kappa}, \hat{\pi}) \equiv (1, 0)$.

A *trade* is an allocation (κ, π) with $\kappa(\omega) = 2$ with strictly positive probability. To rule out trades in which both parties are exactly indifferent between trading and not trading the asset we assume that whenever the asset changes hands its final value is reduced by $\delta > 0$.

Post-trade information

After trade, agent i's information is given by a σ-algebra $\mathcal{F}_i^{\kappa,\pi} \subset \mathcal{F}$, where $\mathcal{F}_i \subset \mathcal{F}_i^{\kappa,\pi} \subset \{a, b\} \times \mathcal{B}^2$. That is, each agent remembers his own signal, and learns at most the other agent's signal.

Each agent observes the outcome of the trade, and updates his information accordingly. Formally, κ and π are $\mathcal{F}_i^{\kappa,\pi}$-measurable. Moreover, in principle it is possible that the trade mechanism entails the release of additional information to agent i. In this case, the σ-algebra generated by (κ, π) would be a strict sub-algebra of $\mathcal{F}_i^{\kappa,\pi}$.

An important object in our analysis is the probability that an agent attaches to fundamental a (or b) conditional on some information. Notationally, for any σ-algebra $\mathcal{G}$ let $Q(\omega; \mathcal{G})$ denote the conditional probability of $\{a\} \times \mathbb{R}^2$ in state ω relative to $\mathcal{G}$.

Endogenous asset values

The eventual asset owner must select an action $X \in \mathcal{X}$ without knowing the realization of fundamental θ. For each candidate action X he can evaluate the expected payoff under that action. We denote this expected payoff by $V(q; X) \equiv qv(X, a) + (1 - q) v(X, b)$, where q denotes the probability the agent places on fundamental a. Since the asset owner chooses the action with the highest expected payoff, his valuation of the asset is given by

$$V(q) \equiv \max_{X \in \mathcal{X}} V(q; X). \tag{2}$$

Note that V is continuous over $[0, 1]$, and hence bounded.[13]

Ex post individually rational trade

Our primary goal is to characterize when trade can — and cannot — occur for purely informational reasons. The answer to this question clearly depends to some extent on the institutional environment. However, it is also clear that we want our results to be as independent as possible of *a priori* assumptions about the trading environment.

To meet these objectives, we begin by establishing necessary conditions for trade to occur in a very wide class of trading mechanisms. The only condition we impose is that trades must be *ex post individually rational*. That is, both agents 1 and 2 must prefer the post-trade outcome to the original allocation (in which agent 1 owns the asset), even after conditioning on any information they acquire in equilibrium. This condition must be met state-by-state. We adopt this requirement for two

[13] We have assumed that the fundamental θ is binary-valued. The significance of this assumption is that it allows us to define the asset value V as a function of a one-dimensional summary statistic, namely the probability q that the fundamental is a. That is, uncertainty is unidimensional. Unidimensionality greatly facilitates the derivation of sufficient conditions for trade in Section 4. We conjecture the necessary conditions of Section 3 would extend to more general state spaces.

It should also be noted that it is possible to obtain a similarly tractable unidimensional framework with a richer set of fundamentals, though at the cost of introducing more assumptions on the asset payoff functions $v(X, \theta)$. For example, one could allow the fundamental θ to be drawn from an arbitrary subset of $\mathbb{R}$, but restrict the asset payoff to take the form $v(X, \theta) = K(X) + M(X)\theta$ for an arbitrary pair of continuous functions K and M. In this case, the expected asset payoff given action X is a linear function of the expected value of θ, and so one can define an analogous function to V that depends only on a one-dimensional variable (i.e., the expected value of θ as opposed to the probability of a).

reasons. First, it is a demanding condition to satisfy, and so biases our analysis against generating trade. Second, it is used in many prior analyses. In particular, it is equivalent to Milgrom and Stokey's (1982) requirement of common knowledge of gains from trade;[14] and is part of the definition of a rational expectations equilibrium.

Formally, a trade (κ, π) is *ex post* individually rational (IR) if

$$\pi(\omega) \geq V\left(Q\left(\omega; \mathcal{F}_1^{\kappa,\pi}\right)\right) \tag{3}$$

$$V\left(Q\left(\omega; \mathcal{F}_2^{\kappa,\pi}\right)\right) - \delta - \pi(\omega) \geq 0 \tag{4}$$

almost always when the buyer gets the asset, i.e., for almost all ω such that $\kappa(\omega) = 2$. Note that the information used by agent i to evaluate the trade is $\mathcal{F}_i^{\kappa,\pi}$, i.e., the information of agent i after trade. The analogous conditions for states ω in which no trade occurs are that $\pi(\omega) \geq 0$ and $0 \geq \pi(\omega)$ almost always when the seller keeps the asset, i.e., $\kappa(\omega) = 1$. It follows trivially that *ex post* IR implies that no money changes hands in almost all states ω in which the seller keeps the asset, i.e., $\pi(\omega) = 0$ when $\kappa(\omega) = 1$.

For any allocation let Ω^T denote the states in which the buyer acquires the asset (i.e., $\kappa(\omega) = 2$) and the *ex post* IR conditions (3) and (4) hold. Note that $\mu\left(\Omega^T\right) > 0$ in any *ex post* IR trade.

[14]In Milgrom and Stokey, agent i evaluates the trade according to "his information at the time of trading, including whatever he can infer from prices or from the behavior of other traders" (page 19). We take this information to include at least the information revealed by the post-trade allocation. In Milgrom and Stokey's framework, there would still be no trade even if one instead assumed that agent i possessed coarser information. In contrast, in our model coarsening the information that agent i uses to evaluate the trade will generally enhance trade opportunities, since it weakens the *ex post* individual rationality condition.

Pareto optimality of the original allocation

Milgrom and Stokey's "no speculation" theorem establishes that trade cannot occur purely for information-based reasons. Of course, this in no way affects the possibility of trade for risk-sharing reasons. As such, Milgrom and Stokey's result is predicated on the Pareto optimality of the pre-trade state-contingent allocation.

In our setting, both agents are risk neutral, and are equally capable of executing any action $X \in \mathcal{X}$. As such, the *only* possible motivation for trade is the differential information of the two parties. Formally, since risk-sharing motivations are absent, any state-contingent allocation is Pareto optimal. Of course, this ignores the fact that agents 1 and 2 potentially have different information, and so take different actions. However, trade motivated by such considerations is precisely information-based trade, and is the main object of our analysis.

3 Necessary conditions for trade

In this section we establish necessary conditions for trade to take place, all of which must hold *regardless* of the trading mechanism employed. First:

Proposition 1. *No* ex post *IR trade exists if V is monotone.*

All proofs are in the appendix. The intuition is most easily understood by considering again the opening example, which is displayed graphically in Figure 1. Recall that in the example trade occurs whenever the seller sees signal s^b. The buyer's valuation when trade occurs is driven by the probability he places on fundamental a, i.e., $\Pr\left(a|s^b s^a\right)$ or $\Pr\left(a|s^b s^b\right)$, depending on his own signal realization. Trade is possible because the value of the asset given these probabilities exceeds the value of the asset when the probability of fundamental a is $\Pr\left(a|s^b\right)$, which is the information the seller has. Since $\Pr\left(a|s^b s^a\right) > \Pr\left(a|s^b\right) > \Pr\left(a|s^b s^b\right)$, trade is clearly only possible in this

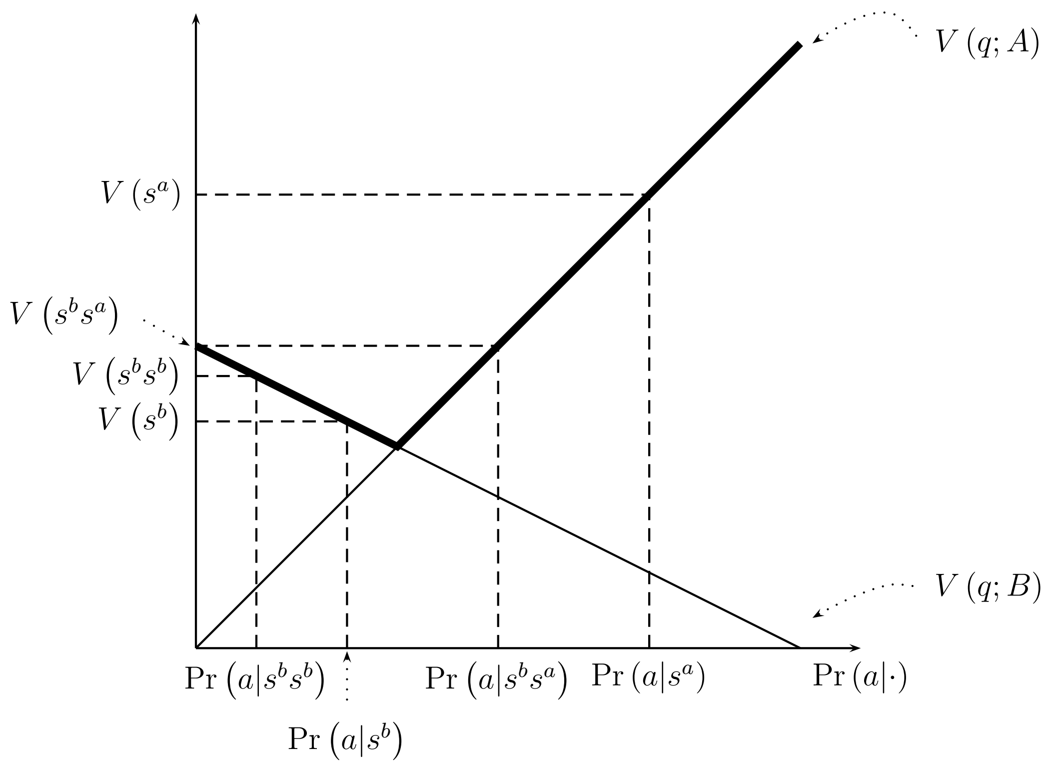

Figure 1: The graph displays $V(q;X)$ for the opening example: the action set is $\mathcal{X} = \{A, B\}$ and both the buyer and seller observe signals drawn from $\{s^a, s^b\}$. The bold line is the upper envelope of these two functions, and corresponds to the function $V(q)$.

example if V is non-monotone. Proposition 1 extends this observation to our main model, and to any trading mechanism.

As an immediate corollary of Proposition 1 we obtain:

Corollary 1. *Trade is possible only if (i) there is no dominant action, i.e.,* $\nexists X \in \mathcal{X}$ *such that* $v(X, \theta) \geq v(X', \theta)$ *for all* $X' \neq X$ *and* $\theta = a, b$*; and (ii) there is no dominant fundamental, i.e.,* $\nexists \theta \in \{a, b\}$ *such that* $v(X, \theta) \geq v(X, \theta')$ *for all* $X \in \mathcal{X}$ *and* $\theta' \neq \theta$*.*

A second key property of the trade equilibrium in the opening example is that the buyer learns the seller's signal when trade occurs. The fact that the buyer learns everything about the seller's signal is an artifact of the binary nature of signals in the example. In general, however, a necessary condition for trade is that the buyer learns *something* about the seller's signal when trade occurs:

Proposition 2. *There is no* ex post *IR trade in which the buyer learns nothing whenever he acquires the asset.*

Proposition 2 says that trade is not possible if it does not convey some information to the buyer. This conclusion is very much in line with the existing no-trade literature. At the same time, and as our opening example makes clear, trade is at least sometimes possible if it enables the buyer to learn the seller's signal.

To understand Proposition 2, it again helps to reconsider the opening example. Suppose that trade occurred in this example *without* the buyer learning anything about the seller's signal. For specificity, suppose further that trade only occurs when the buyer sees signal s^a.[15] Recall that agents always learn at least the information revealed by the trade allocation. Consequently, for the buyer not to learn anything trade must occur after both signal pairs $s^a s^a$ and $s^b s^a$. So when trade occurs, the

[15] Similiar arguments apply for the cases of trade following signal s^b, and trade after both buyer signal realizations.

buyer places probability $\Pr(a|s^a)$ on the fundamental being a; while the seller places probability $\Pr(a|s^a s^a)$ or $\Pr\left(a|s^b s^a\right)$ on the fundamental being a, depending on his own signal realization.

To see why this information is inconsistent with trade, look again at Figure 1. As a function of the probability q of fundamental a, the asset value V is V-shaped. Since the buyer does not learn the seller's signal, at one state in which trade occurs the seller places a higher probability on fundamental a than does the buyer, while in another state the seller places a lower probability on fundamental a. Specifically, $\Pr(a|s^a s^a) > \Pr(a|s^a) > \Pr\left(a|s^b s^a\right)$. Given the V-shape of V it follows that at least one of $V(\Pr(a|s^a s^a))$ and $V\left(\Pr\left(a|s^b s^a\right)\right)$ exceeds $V(\Pr(a|s^a))$. But in words, this comparison says that at at least one of $s^a s^a$ and $s^b s^a$ the seller's valuation exceeds the buyer's valuation — contradicting the trade conditions.

The key step in this argument is the shape of the V function. Since V is the upper envelope of functions $V(q;X)$, each of which is linear, we obtain:

Lemma 1. *V is a convex function.*

The proof of Proposition 2 follows from the convexity of V, and is along the same lines as the above discussion of the example. The main complication in the formal proof is the need to form conditional probabilities for arbitrary information possessed by the seller. At the same time, the proof is simplified somewhat by our assumption of unbounded likelihood ratios (see (1)). We emphasize, however, that (as the example illustrates) this property is not essential for the result, and a proof for the case of bounded likelihood ratios is contained in an earlier working paper.

For our next two results, it is useful to separate the benefits and costs of trade. *Ex post* IR implies

$$\int_{\Omega^T} \left(V\left(Q\left(\omega;\mathcal{F}_2^{\kappa,\pi}\right)\right) - V\left(Q\left(\omega;\mathcal{F}_1^{\kappa,\pi}\right)\right)\right)\mu(d\omega) \geq \mu\left(\Omega^T\right)\delta. \tag{5}$$

The lefthand side is the benefit of trade. Since the buyer's information in state ω

is different from the seller's, he potentially takes a different action. This causes the value of the asset when owned by the buyer to potentially diverge from the value of the asset owned by the seller, in spite of their equal ability to execute all actions $X \in \mathcal{X}$. The righthand side is the direct cost of trade, i.e., the trade cost δ multiplied by the probability of trade occurring.

An almost immediate consequence of (5) is:

Proposition 3. *Suppose an* ex post *IR trade exists. Then there exists a non-null subset of the trade set Ω^T in which the buyer's action differs from the action the seller would take if he controlled the asset in the same state.*

Note that Proposition 3 is also a corollary of Milgrom and Stokey's main result.

Proposition 3 says that trade is associated with a change in action. Two possible applications include the role of vulture investors in debt restructuring, and corporate raiders. With regard to the former, it is widely perceived that vulture investors' behavior in restructuring negotiations differs from that of the original creditors (see, e.g., Morris 2002). With regard to the latter, there is evidence that large scale layoffs and divestitures follow takeovers (see, e.g., Bhagat *et al* 1990).

A second implication of inequality (5) is that as the seller's information becomes infinitely accurate the probability of trade converges to zero. The reason is that as the seller's information quality grows his own signal almost perfectly reveals the fundamental θ in most states, and so the seller takes the full-information optimal action. This effectively eliminates the gains from trade. Since the direct costs of trade are fixed by the parameter δ, the probability of trade must approach zero. Formally, the seller's signal is high quality if the likelihood ratio L_1 of the signal is either very low or very high with high probability:

Proposition 4. *Consider a sequence of economies, indexed by n, that are identical apart from the conditional distribution of the seller's signal, $\eta^{\theta}_{1(n)}$, along with a corresponding sequence of* ex post *IR trade sets $\Omega^T_{(n)}$. Suppose the quality of the seller's* *signal becomes arbitrarily good as $n \to \infty$, in the sense that for any $\varepsilon > 0$ and $\theta = a, b$*

$$\eta^{\theta}_{1(n)} \left(\left\{ s_1 : L_{1(n)} \left(s_1 \right) \in [\varepsilon, 1/\varepsilon] \right\} \right) \to 0.$$

Then the probability of trade converges to zero, i.e., $\mu \left(\Omega^{T}_{(n)} \right) \to 0$.

Similar to Proposition 4, one can show that if the buyer's signal becomes arbitrarily uninformative then the probability of trade likewise converges to zero.

Recall that Proposition 2 says that the buyer must learn something about the realization of the seller's signal if trade is to occur. However, the buyer must have information of his own to complement information he acquires from the seller. That is, if instead the seller's information is much more informative than the buyer's, the buyer's information adds almost nothing and the above observations imply that the probability of trade is very low.

4 Sufficient conditions for trade

Proposition 1 establishes that trade is possible only if the asset value V is non-monotone in the probability of fundamental a. We next show that this necessary condition is also sufficient for at least one simple trading mechanism.

Third-party posted price

In many trading environments both buyers and sellers take the price as exogenous. This is the case, for example, when traders submit market orders; in upstairs trades, in which the upstairs broker proposes the price; and in crossing networks (POSIT is a well-known example) in which the price is determined elsewhere. Formally, we consider a simple trading mechanism in which: (1) a non-strategic third-party — a "broker" — sets a price p, and then (2) the buyer and seller simultaneously and publicly[16] announce whether they wish to trade at price p. We show that this mechanism allows trade whenever V is non-monotone. Combined with Proposition 1, this implies that non-monotonicity of V is both *necessary* and *sufficient* for trade.

Proposition 5. *Suppose V is non-monotone and the third-party posted price mechanism is used. Choose any price $p \in (\min V, \min\{V(0), V(1)\} - \delta)$. There exists an equilibrium of the following form: the seller offers to sell when he sees a signal $s_1 \in S_1^T \equiv [\underline{s}_1, \bar{s}_1]$, and the buyer offers to buy if he sees a signal $s_2 \in S_2^T \equiv \mathbb{R} \backslash (\underline{s}_2, \bar{s}_2)$. The* ex post *IR constraints are satisfied in equilibrium.*

In the equilibrium of Proposition 5, the buyer offers to buy whenever his signal is either high or low, that is, when it is relatively informative of the fundamental. Given that V is non-monotone and convex (see Figure 1), the buyer's valuation of the asset is relatively high at such signals. Similarly, the seller offers to sell when he sees an intermediate signal, that is, a signal that is relatively uninformative about the fundamental. Given the shape of V the seller's valuation is relatively low at such signals.

In equilibrium, trade transfers control of the asset from an agent who has received an uninformative signal to one who has received an informative signal. Moreover, because of its contingent nature trade also reveals information about the agents' signals to each other. Specifically, when the seller retains the asset he learns whether or not the buyer's signal is in S_2^T; and when the buyer acquires the asset, he learns

[16] Clearly the seller learns whether the buyer offers to buy if he offers to sell. Likewise, the buyer learns whether the seller offers to sell if he offers to buy. It is irrelevant what the buyer learns if he does not offer to buy. As such, the only remaining question is whether the seller learns whether the buyer offers to buy if he *does not* offer to sell.

For many upstairs trades it is reasonable to suppose that the seller learns the buyer's announcement when he does not offer to sell. That is, for many stocks only a few individuals hold a large block, and so the holder of a block learns whether or not there is buying interest from whether or not an upstairs broker contacts him.

that the seller's signal is in S_1^T.

Proposition 5 establishes the existence of a continuum of equilibria, indexed by the trade price p. Comparing the lowest price $p = \min V$ to the highest price $p = \min\{V(0), V(1)\} - \delta$, the buyer's demand (i.e., the probability of accepting the price) decreases (from 1 to 0), while the seller's supply increases (from 0 to 1). That is, the comparative static across equilibria generates an downwards sloping demand curve and an upwards sloping supply curve.[17]

An important implication of the no-trade theorems established in the existing literature is that economic agents would not spend resources to acquire information. In contrast, our next result shows that this is not true in our model. The key reason is, of course, that information is valuable. The non-trivial aspect of the result consists of showing that an agent's information is valuable above-and-beyond the information he acquires from the other agent in the course of trade.

Proposition 6. *Suppose the buyer and seller must each incur a cost k in order to observe their signals. Fix any price $p \in (\min V, \min\{V(0), V(1)\} - \delta)$. Provided the information acquisition cost k is sufficiently small there exists an equilibrium of the third-party posted price mechanism in which both the buyer and seller acquire their signals and trade occurs with positive probability.*

Buyer proposes a price

The third-party price mechanism above approximates various real-world trading arrangements, and facilitates the analysis of when trade occurs by avoiding the issue of strategic price setting. However, in at least some situations it is the trading parties themselves who set the price. Accordingly, we consider a simple example of such a

[17] Because of the interdependency between the buyer and seller, more conditions would be required to establish that the demand (respectively, supply) curve is monotonically downwards (respectively, upwards) sloping.

mechanism: (1) the buyer proposes a price $p \in P$, where P is finite set of possible offers,[18] and (2) the seller accepts or rejects. We assume that P contains at least one element lying between $\min V$ and $\min \{V(0), V(1)\} - \delta$; and moreover that $0 \in P$, so that the buyer can effectively abstain from making an offer by offering the zero price.

Our first result shows that an equilibrium with meaningful trade is necessarily more complicated when the buyer chooses the price than when the price is simply imposed non-strategically. To see this, start by noting that in the equilibrium of Proposition 5 both the buyer and the seller make zero profit at the boundaries of their trade sets S_1^T and S_2^T. This is a consequence of continuity: for example, for the buyer, when $s_2 \in S_2^T$ the buyer's valuation of the asset exceeds p, while when $s_2 \notin S_2^T$ the price p exceeds the buyer's valuation.

Because the buyer has zero profits at the boundary signals $\underline{s}_2$ and $\bar{s}_2$ of S_2^T, he faces a strong temptation to offer a lower price after observing these signals. In fact, regardless of the seller's out-of-equilibrium beliefs the buyer could make strictly positive profits at at least one of $\underline{s}_2$ and $\bar{s}_2$ by offering $\tilde{p}$ between p and $\min V$. The reason is that the seller's response to $\tilde{p}$ either increases or decreases the buyer's belief that the fundamental is a relative to his equilibrium belief, and given the convex and non-monotone shape of V this raises the buyer's valuation at one of $\underline{s}_2$ and $\bar{s}_2$.

It follows that the only possibility for an equilibrium with trade at just one price entails trade at the lowest price in P that still exceeds $\min V$, i.e.,

$$p^* \equiv \min \{p \in P : p > \min V\}.$$

However, the seller only accepts this low price when he sees a signal that places his valuation between $\min V$ and p^*. When p^* is close to $\min V$ this can occur only rarely. Formally:

[18]The assumption that the offer set P is finite ensures that the action set is finite. As is well-known, equilibrium existence is not guaranteed in games with infinite action spaces.

Proposition 7. *Suppose V is non-monotone, never flat,*[19] *and the buyer-posted price mechanism is used. Suppose an equilibrium exists in which trade occurs at only one price. Then (I) the trade price is p^*, and (II) the probability of trade approaches zero as $p^* \to \min V$.*

Proposition 7 implies that when the buyer chooses the price trade can occur with a meaningful probability only if the buyer makes different offers after different signals, and the seller accepts multiple offers with different probabilities. Characterizing such an equilibrium is challenging. Formally, our environment is close to a bargaining game with interdependent values and two-sided asymmetric information,[20] and we are not aware of any paper to consider such a game with more than two types.[21]

In order to illustrate the possibilities for trade when the buyer proposes the price, we focus on the simplest environment in which strategic offers by the buyer are possible, namely that in which V is non-monotone and the offer set is $P = \left\{p^C, p^D, 0\right\}$, where $\min\left\{V(0), V(1)\right\} - \delta > p^C > p^D > \min V$. As we explain below, even here the fact that there are a continuum of buyer "types" necessitates a numerical simulation in order to verify the incentive constraints.

A trade equilibrium with trade at both prices p^C and p^D is characterized by signal sets S_2^C and S_2^D in which the buyer offers the prices p^C and p^D respectively, with corresponding signal sets S_1^C and S_1^D in which the seller accepts these offers. Given that V is convex and non-monotone the seller's acceptance sets S_1^C and S_1^D must be of the form $\left(\underline{s}_1^C, \bar{s}_1^C\right)$ and $\left(\underline{s}_1^D, \bar{s}_1^D\right)$. As in the equilibrium of Proposition 5, the seller

[19] That is, there is no interval over which V is constant.

[20] However, our environment is *not* a special case of such a bargaining game because the value of the asset is endogenous.

[21] Schweizer (1989) analyzes the case of two types. In general, the literature on bargaining with interdependent values is small and focuses on the case of one-sided asymmetric information: see Evans (1989), Vincent (1989), Deneckere and Liang (2006), Dal Bo and Powell (2007).

has zero profits at the boundaries of S_1^C and S_1^D:

$$V\left(\Pr\left(a|\underline{s}_1^j, s_2 \in S_2^j\right)\right) = V\left(\Pr\left(a|\bar{s}_1^j, s_2 \in S_2^j\right)\right) = p^j \text{ for } j = C, D. \tag{6}$$

Since both p^C and p^D exceed $\min V$, there must exist a non-empty subset of signals at which the buyer prefers to make no offer. Again by the convexity and non-monotonicity of V, this no-offer set is an interval of the form $\left[\underline{s}_2^D, \bar{s}_2^D\right]$. Thus the equilibrium conditions for the buyer are that for $s_2 \in S_2^C$

$$\begin{aligned} &\Pr\left(s_1 \in S_1^C|s_2\right)\left(V\left(\Pr\left(a|s_1 \in S_1^C, s_2\right)\right) - \delta - p^C\right) \\ &\geq \max\left\{0, \Pr\left(s_1 \in S_1^D|s_2\right)\left(V\left(\Pr\left(a|s_1 \in S_1^D, s_2\right)\right) - \delta - p^D\right)\right\}; \end{aligned} \tag{7}$$

while for $s_2 \in S_2^D$,

$$\begin{aligned} &\Pr\left(s_1 \in S_1^D|s_2\right)\left(V\left(\Pr\left(a|s_1 \in S_1^D, s_2\right)\right) - \delta - p^D\right) \\ &\geq \max\left\{0, \Pr\left(s_1 \in S_1^C|s_2\right)\left(V\left(\Pr\left(a|s_1 \in S_1^C, s_2\right)\right) - \delta - p^C\right)\right\}; \end{aligned} \tag{8}$$

and if $s_2 \in \left[\underline{s}_2^D, \bar{s}_2^D\right]$,

$$0 \geq \max\left\{V\left(\Pr\left(a|s_1 \in S_1^C, s_2\right)\right) - \delta - p^C, V\left(\Pr\left(a|s_1 \in S_1^D, s_2\right)\right) - \delta - p^D\right\}.$$

Finally, in light of the form the equilibrium takes in the third-party mechanism, along with the non-monotonicity of V, a natural conjecture for the form of the buyer's trade sets S_2^C and S_2^D is as follows. The buyer makes the higher offer p^C only when his signal is relatively informative, that is, S_2^C is of the form $\mathbb{R}\backslash\left[\underline{s}_2^C, \bar{s}_2^C\right]$. The buyer then makes the lower offer p^D after the remaining signals $\mathbb{R}\backslash\left(S_2^C \cup \left[\underline{s}_2^D, \bar{s}_2^D\right]\right)$, so that $S_2^D = [\underline{s}_2^C, \underline{s}_2^D) \cup (\bar{s}_2^D, \bar{s}_2^C]$. By continuity, the following equalities are then *necessary* for an equilibrium of this type to exist:

$$V\left(\Pr\left(a|s_1 \in S_1^D, s_2\right)\right) - \delta = p^D \tag{9}$$

at $s_2 = \underline{s}_2^D, \bar{s}_2^D$; while at $s_2 = \underline{s}_2^C, \bar{s}_2^C$,

$$\begin{aligned} &\Pr\left(s_1 \in S_1^C|s_2\right)\left(V\left(\Pr\left(a|s_1 \in S_1^C, s_2\right)\right) - \delta - p^C\right) \\ &= \Pr\left(s_1 \in S_1^D|s_2\right)\left(V\left(\Pr\left(a|s_1 \in S_1^D, s_2\right)\right) - \delta - p^D\right). \end{aligned} \tag{10}$$

Together, equations (6), (9) and (10) constitute eight equations in the eight parameters $\left\{\underline{s}_1^C, \bar{s}_1^C, \underline{s}_1^D, \bar{s}_1^D, \underline{s}_2^C, \bar{s}_2^C, \underline{s}_2^D, \bar{s}_2^D\right\}$ that characterize the equilibrium of the type described.

Figure 2 displays an example of such an equilibrium for a specific set of parameter values.[22] The figure plots the buyer's expected profits from each of the offers p^C and p^D as a function of his signal s_2. The vertical lines are drawn at the boundaries of the sets S_2^C and S_2^D, i.e., at $\underline{s}_2^C, \underline{s}_2^D, \bar{s}_2^C, \bar{s}_2^D$ respectively. The figure makes clear that the solution to equations (6), (9) and (10) defines an equilibrium in this case: whenever $s_2 \in S_2^C$ the buyer prefers offering p^C to p^D, and whenever $s_2 \in S_2^D$ the buyer prefers offering p^D to p^C. The advantage of the higher offer p^C is that it is accepted more often; the disadvantage is, of course, that the buyer pays more.

The significance of this example relative to Proposition 7 is that it shows that trade can occur with significant probability when the buyer proposes different prices after different signals. That is, although the probability of trade at the lower price p^D is relatively low, the buyer sometimes offers the higher price p^C, and the seller's acceptance probability at this price is higher. Moreover, our focus on an equilibrium with trade at two prices is solely for tractability, and we conjecture that with a finer offer set P there exist equilibria with trade at a large number of different prices. We leave the further exploration of this bargaining framework for future research.

5 Portfolio selection

In our basic model agents choose an action that directly affects the payoff produced by the asset. Clearly in many circumstances agents trade assets over which they have little or no direct control. Even in such settings, however, an agent who owns an asset

[22] Unfortunately, we have not been able to characterize the conditions under which equations (6), (9) and (10) have a solution; and even if we were to do so, one would still need to check that the equilibrium conditions (7) and (8) hold away from the boundaries of the S_2^C and S_2^D.

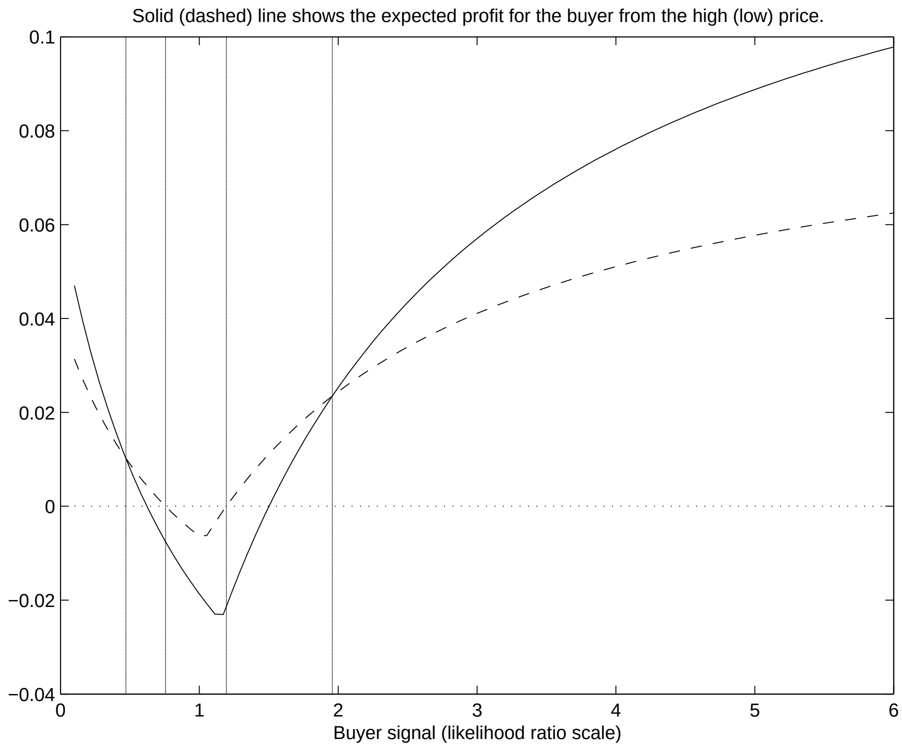

Figure 2: The asset value is as in the opening example. The prices are $p^C = 14/18$ and $p^D = 13/18$, and $\min V = 12/18$. Also as in the opening example the *ex ante* probability of fundamentals a is $1/2$. Both the buyer and seller observe a normally distributed signal with standard deviation 1 (for both fundamentals a, b) and a mean of 0 (respectively, 1) when the fundamental is b (respectively, a). The solid (dashed) line shows the buyer's expected profit from making the offer p^C (respectively, p^D). The vertical lines are drawn at $\underline{s}_2^C, \underline{s}_2^D, \bar{s}_2^D, \bar{s}_2^C$ respectively.

must still decide how to allocate the remainder of his portfolio, and if the agent is risk averse this decision affects the agent's utility from holding the asset. When agents are unsure about how asset returns are distributed, gains from trade arise as in our basic model, for the same reasons. In this section we study when portfolio decisions of this type generate trade, even in the absence of direct control over an asset.

A general framework

Our goal in this section is to show that portfolio allocation concerns can motivate information-based trade, and that, as such, our framework is capable of accounting for trade even in cases where asset-holders exercise no direct control. To avoid moving too far from our basic model we consider the following environment. There are $n+2$ securities in total. Security 1 is a security such that the optimal portfolio choice of an agent holding it depends on the fundamental θ.[23] Security 2 is a standard risk free security. Finally, securities $3, \ldots, n+2$ are risky securities whose attractiveness does not differ across fundamentals a and b. Specifically, the fundamental θ has no effect on how an investor without security 1 would divide his portfolio between the risk free security and securities $3, \ldots, n+2$. (We give a formal condition for this in (12) below.) All investors in the economy are free to take any position in the risk free security, and any position $\psi \in \Psi \subset \mathbb{R}^n$ in securities $3, \ldots, n+2$.

Since the fundamental θ only affects the portfolio decision of an investor holding security 1, this security is analogous to the asset traded in our basic model. To keep as close as possible to our basic model, we assume that only one owner, agent 1, of security 1 is interested in selling it. (Multiple sellers would not fundamentally change our analysis.) From our analysis of necessary conditions for trade, we know that

[23]More generally, our framework could accommodate multiple securities with this property. The assumption that security 1 is the only such security keeps the analysis of this section close to our basic model.

the only potential buyers are investors who observe an informative signal about the fundamental θ. Again, to keep our analysis close to the basic model we assume there is just one such investor, agent 2.[24] We characterize when agent 2 buys security 1 from agent 1. Importantly, agents 1 and 2 are free to rebalance their portfolios by trading the remaining $n+1$ securities *after* they decide whether to trade security 1.

Although our assumptions regarding the ownership of security 1 are made primarily for illustrative purposes, one situation where they are approximately satisfied is that in which security 1 is a share in a private company.[25]

All investors have a common utility function u. The initial wealth of agents 1 and 2 is W_1 and W_2 respectively. As in the basic model, we typically refer to agent 1 as the seller and agent 2 as the buyer. We normalize both the return on the risk free security and the price of each risky security $3, \ldots, n+2$ to unity. We denote the payoff of security 1 by R, and the n-vector of excess returns (over the risk free security) of securities $3, \ldots, n+2$ by r. Both R and r are stochastic, with distributions that potentially differ across fundamentals a and b. The expected utility of an agent with wealth W who owns security 1 and believes that the probability of fundamental a is q is given by

$$U(W,q) \equiv \max_{\psi \in \Psi} E\left[u\left(W + R + \psi r\right)|q\right].$$

Likewise, the expected utility of an agent who does not own security 1 is

$$\bar{U}(W,q) \equiv \max_{\psi \in \Psi} E\left[u\left(W + \psi r\right)|q\right].$$

[24] As with the seller, allowing for multiple buyers would not fundamentally change our analysis.

[25] We also assume that security 1 is indivisible. Various complications would arise if instead it were divisible. First, while a Pareto optimal allocation of the indivisible security necessarily entails a single agent holding the security, with divisibility the Pareto optimal allocation might entail both agents holding a strictly positive quantity of the security. Second, and related, if both agents hold the security initially one would have to consider the *direction* of trade. Third, one would also have to consider the *quantity* of trade. We leave an analysis of these issues for future research.

Exactly as in our basic model, an allocation is described by the pair of mappings κ and π. We continue to assume a small cost δ is associated with trading security 1. Analogous to before, the *ex post* IR conditions are

$$\bar{U}\left(W_1+\pi\left(\omega\right),Q\left(\omega;\mathcal{F}_1^{\kappa,\pi}\right)\right)\geq U\left(W_1,Q\left(\omega;\mathcal{F}_1^{\kappa,\pi}\right)\right)$$
$$U\left(W_2-\pi\left(\omega\right)-\delta,Q\left(\omega;\mathcal{F}_2^{\kappa,\pi}\right)\right)\geq \bar{U}\left(W_2,Q\left(\omega;\mathcal{F}_2^{\kappa,\pi}\right)\right)$$

in almost all states in which the buyer gets the security (i.e., ω such that $\kappa\left(\omega\right)=2$), and

$$U\left(W_1+\pi\left(\omega\right),Q\left(\omega;\mathcal{F}_1^{\kappa,\pi}\right)\right)\geq U\left(W_1,Q\left(\omega;\mathcal{F}_1^{\kappa,\pi}\right)\right)$$
$$\bar{U}\left(W_2-\pi\left(\omega\right),Q\left(\omega;\mathcal{F}_2^{\kappa,\pi}\right)\right)\geq \bar{U}\left(W_2,Q\left(\omega;\mathcal{F}_2^{\kappa,\pi}\right)\right)$$

in almost all states in which the seller keeps the security (i.e., ω such that $\kappa\left(\omega\right)=1$). Note that exactly as in the basic model, this second pair of inequalities implies that $\pi\left(\omega\right)$ for almost all states in which the seller keeps the security.

One complication of this framework relative to our basic model is that wealth effects may lead the buyer and seller to choose different portfolios, even if they have exactly the same information.[26] In contrast, in our basic model agents with the same information always choose the same action. To avoid this complication we assume that both agents share the same constant absolute risk aversion (CARA) utility, $u\left(x\right)=-e^{-\gamma x}$, where $\gamma>0$. In this case, there exist negative-valued functions v and $\bar{v}$ such that $U\left(W,q\right)=e^{-\gamma W}v\left(q\right)$ and $\bar{U}\left(W,q\right)=e^{-\gamma W}\bar{v}\left(q\right)$. After substitution, the *ex post* IR conditions for states in which the buyer gets the security become

$$e^{-\gamma\pi(\omega)}\leq\frac{v\left(Q\left(\omega;\mathcal{F}_1^{\kappa,\pi}\right)\right)}{\bar{v}\left(Q\left(\omega;\mathcal{F}_1^{\kappa,\pi}\right)\right)}$$
$$\frac{v\left(Q\left(\omega;\mathcal{F}_2^{\kappa,\pi}\right)\right)}{\bar{v}\left(Q\left(\omega;\mathcal{F}_2^{\kappa,\pi}\right)\right)}\leq e^{-\gamma(\pi(\omega)+\delta)}.$$

[26] Even if $W_1=W_2$, so that the two agents have the same initial wealth, the buyer's wealth when he acquires the security is lower.

Taking logs and defining $\mathcal{V} \equiv -\frac{1}{\gamma} \ln \frac{v}{\bar{v}}$ then generates precisely equations (3) and (4), with V replaced by $\mathcal{V}$.

Trade and market betas

From above, when agents have CARA preferences with the same degree of risk aversion the model of trade prior to portfolio choice is isomorphic to our basic model of trade prior to a decision that directly affects asset payoffs. We next specialize this framework to one in which we can relate trade to standard measures of risk — specifically, return standard deviations and market betas.

The version of the model we consider is one with just three securities: security 1, the risk free security, and a risky security we term the *market*. The return distributions of security 1 and market are jointly normal. Let μ_θ and σ_θ denote the expected (excess) return and standard deviation of returns for the market when the fundamental is θ, while ν_θ and ζ_θ denote the expected return and standard deviation of returns for security 1. Let ρ_θ be the correlation between the market and security 1. The market beta of security 1 in fundamental θ is $\beta_\theta \equiv \rho_\theta \zeta_\theta / \sigma_\theta$.

In the CARA-normal framework an agent who knows the fundamental is θ and does not own security 1 invests $\mu_\theta / \gamma \sigma_\theta^2$ in the market.[27] Similarly, an agent who owns security 1 chooses a portfolio

$$\psi_\theta \equiv \frac{\mu_\theta}{\gamma \sigma_\theta^2} - \beta_\theta. \tag{11}$$

That is, the agent picks a market position that combines the position he would choose if he did not own security 1 with the variance-minimizing hedge of that security, i.e., $-\beta_\theta$. Recall that we assumed the fundamental θ only affects the portfolio decision

[27] This follows from the fact that $E\left[e^{-\gamma r \psi} | \theta\right] = \exp\left(-\gamma\left(\mu_\theta \psi - \frac{\gamma}{2} \sigma_\theta^2 \psi^2\right)\right)$. Likewise, the fact that $E\left[e^{-\gamma(r\psi(q)+R)} | \theta\right] = \exp\left(-\gamma\left(\nu_\theta + \mu_\theta \psi - \frac{\gamma}{2}\left(\zeta_\theta^2 + \psi^2 \sigma_\theta^2 + 2\beta_\theta \sigma_\theta^2 \psi\right)\right)\right)$ implies (11).

of an agent holding security 1, and so

$$\mu_a/\sigma_a^2 = \mu_b/\sigma_b^2. \tag{12}$$

Our framework assumes that there is a fixed price at which all investors can buy/sell the market. An important implication of equality (12) is that it justifies this assumption: holding the price fixed, equality (12) implies that aggregate demand for the market is independent of the fundamental,[28] thereby justifying the assumption of a fixed price.

In our basic model, *ex post* IR trade of the asset is possible if and only if V is non-monotone (see Propositions 1 and 5). Analogously, *ex post* IR trade of security 1 is possible if and only if $\mathcal{V}$ is non-monotone. Our next result states when $\mathcal{V}$ is non-monotone in terms of the distribution parameters of security 1 and the market:

Proposition 8. *Without loss, assume that $\psi_b \leq \psi_a$.* Ex post *IR trade of security* 1 *is possible if and only if*

$$\begin{aligned}
&\psi_b\gamma\left(\sigma_a^2\psi_a - \sigma_b^2\psi_b\right) + \psi_b^2\frac{\gamma}{2}\left(\sigma_b^2 - \sigma_a^2\right) \\
&< \left(\nu_b - \frac{\gamma}{2}\zeta_b^2\right) - \left(\nu_a - \frac{\gamma}{2}\zeta_a^2\right) + \frac{1}{2\gamma}\left(\frac{\mu_a^2}{\sigma_a^2} - \frac{\mu_b^2}{\sigma_b^2}\right) \\
&< \psi_a\gamma\left(\sigma_a^2\psi_a - \sigma_b^2\psi_b\right) + \psi_a^2\frac{\gamma}{2}\left(\sigma_b^2 - \sigma_a^2\right).
\end{aligned} \tag{13}$$

Intuitively, trade of security 1 is possible only if the fundamental affects the portfolio decision of an agent holding it. That is, trade requires that $\psi_a \neq \psi_b$, or equivalently (given (12)) that the market betas β_a and β_b differ. However, the formal proof of this result is not quite immediate since while v and $\bar{v}$ are both monotone in q if $\beta_a = \beta_b$, by itself this does not eliminate the possibility that $v/\bar{v}$ is non-monotone. Nonetheless, this result does follow as a corollary of Proposition 8:

Corollary 2. Ex post *IR trade of security* 1 *is possible only if $\beta_a \neq \beta_b$.*

[28] Assuming, that is, that agents 1 and 2 are "small" compared to the total population of investors.

Corollary 2 says that the market beta of security 1 must differ across fundamentals in order for trade to occur. Our next result gives a simple sufficient condition for how much beta must differ. It concerns the case in which only the correlation between security 1 and the market differs across fundamentals, so that the only determinant of an agent's utility from holding security 1 is the ease with which it can be hedged.

Corollary 3. *Suppose $\nu_a = \nu_b$, $\zeta_a = \zeta_b$, $\mu_a = \mu_b$, and $\sigma_a = \sigma_b$. Then* ex post *IR trade of security* 1 *is possible if and only if ρ_a and ρ_b are sufficiently different that ψ_a and ψ_b have different signs, i.e., an investor with security* 1 *who knows the fundamental is a is long the market while an investor with security* 1 *who knows the fundamental is b is short the market, or vice versa.*

Corollary 3 implies the existence of a trade equilibrium of the following type. Without loss, assume $\beta_b > \beta_a$, with the difference in market betas large enough that $\psi_a > 0 > \psi_b$. Suppose the initial owner of security 1 (the seller) is unsure whether the fundamental is a or b, i.e., has observed an intermediate signal s_1. Under these circumstances, if he takes a long market position he exposes himself to excessive risk if the true fundamental is b; but if he shorts the market, he gives up expected return and fails to gain an effective hedge if the true fundamental is a. Given his uninformative signal realization he seeks to sell security 1. Meanwhile, a potential buyer who has good evidence that the fundamental is either a or b is prepared to buy: if he thinks the fundamental is a he can combine security 1 with a long market position without taking on excessive risk, while if he thinks the fundamental is b he can effectively hedge security 1 by taking a short market position.

Volatility and volume

Empirically, stock return volatility and trading volume are positively correlated, both at an aggregate level and at the level of individual stocks.[29] Our next corollary of Proposition 8 illustrates how our framework can be used to deliver a positive correlation between volatility and trade volume:

Corollary 4. *Suppose $\nu_a = \nu_b$, $\rho_a = \rho_b > 0$, $\mu_a = \mu_b$, $\sigma_a = \sigma_b$, $\zeta_a = \zeta - \varepsilon$ and $\zeta_b = \zeta + \varepsilon$ for some ζ and $\varepsilon \in [0, \zeta)$. Then* ex post *IR trade of security* 1 *is possible if and only if ε is sufficiently large.*

Corollary 4 characterizes when trade is possible if the only difference between fundamentals is that security 1 is riskier in one than in the other, i.e., $\zeta_b > \zeta_a$. The basic idea is similar to Corollary 3. If the volatility of security 1 differs a lot across fundamentals (i.e., ε is high), and so there is significant uncertainty about the market beta of security 1, then a seller with an uninformative signal realization is unable to hedge security 1 effectively. In these circumstances there are gains from trading the asset to the buyer if the latter has seen a more informative signal, and is thus in a better position to incorporate security 1 into his portfolio.

Provided the unconditional probabilities of fundamentals a and b are approximately equal, an increase in ε in Corollary 4 increases the unconditional variance of security 1. As such, Corollary 4 predicts a positive correlation between the implied volatility of security 1 and the volume of trade in this security.

[29] See, e.g., the survey by Karpoff (1987). The typical study in this literature relates volume to volatility measured over a trailing window. Our model predicts correlation between volume and perceived future volatility. To the extent to which volatility is persistent the two correlations will be similar. Moreover, *implied* aggregate volatility from options markets is also correlated with aggregate volume (details are available upon request from the authors).

6 Conclusion

In this paper we have shown that if asset payoffs are endogenously determined by the actions of agents, then trade based purely on informational differences is possible. This conclusion stands in sharp contrast to the existing literature, which takes asset values as exogenous. Trade transfers control of the asset from an agent who has received an uninformative signal to one who has received an informative signal. Even without the presence of noise traders, agents in our model would be prepared to spend resources to acquire information; and this information is subsequently partially revealed by trade.

Our analysis generates a number of empirical implications. One general implication is that trade affects the action taken: when the buyer acquires the asset, he (at least sometimes) takes an action that is different from the one the seller would have taken. As we discussed, this implication is consistent with both firm policy after takeovers and with different creditor behaviors in debt restructuring, though other explanations are certainly possible. Other implications depend more on the specific application. In particular, the application of our model to the trade of non-controlling shares implies that trade volume may increase with volatility.

Clearly many avenues for future research exist. A fuller analysis of price negotiation between the buyer and seller is one important topic. Another is the extension of our portfolio allocation application from information-based trade in one asset (security 1) between a designated buyer and seller to information-based trade in multiple securities between arbitrary market participants.

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A Appendix

Proof of Proposition 1: Without loss, suppose V is weakly increasing, and suppose that contrary to the claimed result the set trade set Ω^T is non-null. Let P be the set of prices at which trade occurs, and for each $p \in P$ let $\Omega^T(p)$ be the subset in which trade occurs at price p and the *ex post* IR conditions hold, so $\Omega^T = \cup_{p \in P} \Omega^T(p)$.

We claim that for some $p \in P$ there exists $\omega^* \in \Omega^T(p)$ such that $Q(\omega^*; \mathcal{F}_2^{\kappa,\pi}) \leq Q(\omega^*; \mathcal{F}_1^{\kappa,\pi})$. Since V is weakly increasing, this claim implies that

$$V\left(Q\left(\omega^*; \mathcal{F}_2^{\kappa,\pi}\right)\right) \leq V\left(Q\left(\omega^*; \mathcal{F}_1^{\kappa,\pi}\right)\right).$$

However, *ex post* IR implies

$$V\left(Q\left(\omega^*; \mathcal{F}_1^{\kappa,\pi}\right)\right) \leq p < p + \delta \leq V\left(Q\left(\omega^*; \mathcal{F}_2^{\kappa,\pi}\right)\right),$$

giving the required contradiction.

To prove the claim, suppose to the contrary that $Q(\omega; \mathcal{F}_1^{\kappa,\pi}) < Q(\omega; \mathcal{F}_2^{\kappa,\pi})$ for all $p \in P$ and $\omega \in \Omega^T(p)$. By the definition of conditional probability, for $i = 1, 2$,

$$\int_{\Omega^T} Q(\omega; \mathcal{F}_i^{\kappa,\pi}) \, \mu(d\omega) = \mu\left(\Omega^T \cap \{a\} \times \mathbb{R}^2\right),$$

and so

$$\int_{\Omega^T} \left(Q(\omega; \mathcal{F}_2^{\kappa,\pi}) - Q(\omega; \mathcal{F}_1^{\kappa,\pi})\right) \mu(d\omega) = 0.$$

This gives a contradiction, since by supposition $Q(\omega; \mathcal{F}_2^{\kappa,\pi}) - Q(\omega; \mathcal{F}_1^{\kappa,\pi}) > 0$ and Ω^T has strictly positive measure. ∎

Proof of Proposition 2: We establish Proposition 2 by contradiction. Suppose to the contrary that an *ex post* IR trade (κ, π) exists in which the buyer learns nothing whenever he acquires the asset. That is, trade occurs over Ω^T, where $\mu\left(\Omega^T\right) > 0$, and $\left\{F \cap \Omega^T : F \in \mathcal{F}_2^{\kappa,\pi}\right\} = \left\{F \cap \Omega^T : F \in \mathcal{F}_2\right\}$.

Choose an integer n such that $\Omega_n^T \equiv \Omega^T \cap \{a, b\} \times \mathbb{R} \times [n, n+1]$ has strictly positive mass. Since the buyer learns nothing when he acquires the asset, $Q((\theta, s_1, s_2); \mathcal{F}_2^{\kappa,\pi}) = Q((\theta, s_1, s_2); \mathcal{F}_2) = \Pr(a|s_2)$ for all $(\theta, s_1, s_2) \in \Omega^T$. Let $\underline{q} = \Pr(a|s_2 = n)$ and $\bar{q} = \Pr(a|s_2 = n+1)$, so that $Q(\omega; \mathcal{F}_2^{\kappa,\pi}) \in \left[\underline{q}, \bar{q}\right]$ for $\omega \in \Omega_n^T$.

We claim that

$$\inf_{\omega \in \Omega_n^T} Q(\omega; \mathcal{F}_1^{\kappa,\pi}) < \underline{q} < \bar{q} < \sup_{\omega \in \Omega_n^T} Q(\omega; \mathcal{F}_1^{\kappa,\pi}). \tag{14}$$

This implies the result, as follows. *Ex post* IR for the buyer and convexity of V (see Lemma 1) together imply that

$$\pi(\omega) \leq V\left(Q(\omega; \mathcal{F}_2^{\kappa,\pi})\right) - \delta \leq \max\left\{V\left(\underline{q}\right), V(\bar{q})\right\} - \delta$$

for all $\omega \in \Omega_n^T$. Convexity of V and (14) imply that

$$\max\left\{V\left(\inf_{\omega \in \Omega_n^T} Q(\omega; \mathcal{F}_1^{\kappa,\pi})\right), V\left(\sup_{\omega \in \Omega_n^T} Q(\omega; \mathcal{F}_1^{\kappa,\pi})\right)\right\} \geq \max\left\{V\left(\underline{q}\right), V(\bar{q})\right\}.$$

But then since V is continuous there must exist $\omega \in \Omega_n^T$ such that $V\left(Q\left(\omega; \mathcal{F}_1^{\kappa,\pi}\right)\right) > \pi(\omega)$, contradicting *ex post* IR for the seller.

To complete the proof we must establish (14). Suppose that contrary to (14), $\inf_{\omega \in \Omega_n^T} Q\left(\omega; \mathcal{F}_1^{\kappa,\pi}\right) \geq \underline{q}$. From the definition of conditional probability, for any s_1

$$\int_{\Omega^T \cap (\{a,b\} \times (-\infty, s_1) \times \mathbb{R})} Q\left(\omega; \mathcal{F}_1^{\kappa,\pi}\right) \mu(d\omega) = \mu\left(\Omega^T \cap (\{a\} \times (-\infty, s_1) \times \mathbb{R})\right).$$

Since $\Omega_n^T \subset \Omega^T$ and by supposition $Q\left(\omega; \mathcal{F}_1^{\kappa,\pi}\right) \geq \underline{q}$ over Ω_n^T,

$$\begin{aligned}
&\int_{\Omega^T \cap (\{a,b\} \times (-\infty, s_1) \times \mathbb{R})} Q\left(\omega; \mathcal{F}_1^{\kappa,\pi}\right) \mu(d\omega) \\
&\geq \int_{\Omega_n^T \cap (\{a,b\} \times (-\infty, s_1) \times \mathbb{R})} Q\left(\omega; \mathcal{F}_1^{\kappa,\pi}\right) \mu(d\omega) \\
&\geq \underline{q}\mu\left(\Omega_n^T \cap (\{a,b\} \times (-\infty, s_1) \times \mathbb{R})\right).
\end{aligned}$$

So

$$\underline{q} \leq \frac{\mu\left(\Omega^T \cap (\{a\} \times (-\infty, s_1) \times \mathbb{R})\right)}{\mu\left(\Omega_n^T \cap (\{a,b\} \times (-\infty, s_1) \times \mathbb{R})\right)}.$$

Since after trade the buyer learns nothing, $\Omega^T \in \mathcal{F}_2$ and so is of the form $\{a,b\} \times \mathbb{R} \times S_2^T$, where $S_2^T \in \mathcal{B}$. Note that $\eta_1^\theta\left(S_2^T\right)$ and $\eta_1^\theta\left(S_2^T \cap [n, n+1]\right)$ are both strictly positive for $\theta = a, b$, since $\mu\left(\Omega^T\right) > 0$. So the last inequality rewrites to

$$\underline{q} \leq \frac{\Pr(a)\, F_1^a(s_1)\, \eta_1^a\left(S_2^T\right)}{\Pr(a)\, F_1^a(s_1)\, \eta_1^a\left(S_2^T \cap [n, n+1]\right) + \Pr(b)\, F_1^b(s_1)\, \eta_1^b\left(S_2^T \cap [n, n+1]\right)}.$$

But since the likelihood ratio is unbounded (see (1)) the righthand side converges to 0 as $s_1 \to -\infty$, giving a contradiction and thus showing $\inf_{\omega \in \Omega_n^T} Q\left(\omega; \mathcal{F}_1^{\kappa,\pi}\right) < \underline{q}$. A parallel argument implies $\bar{q} < \sup_{\omega \in \Omega_n^T} Q\left(\omega; \mathcal{F}_1^{\kappa,\pi}\right)$, completing the proof. ■

Proof of Proposition 3: Suppose to the contrary that for almost all $\omega \in \Omega^T$ the buyer takes the same action the seller would take if he controls the asset in state ω. Then $V\left(Q\left(\omega; \mathcal{F}_2^{\kappa,\pi}\right)\right) = V\left(Q\left(\omega; \mathcal{F}_1^{\kappa,\pi}\right)\right)$ for almost all $\omega \in \Omega^T$, which violates (5) and gives a contradiction. ■

Proof of Proposition 4: Recall that if an agent has information given by the σ-algebra $\mathcal{F}$ he knows the true realization of the fundamental. As such,

$$\int_{\Omega^T} \left(V \left(Q \left(\omega; \mathcal{F}_2^{\kappa,\pi} \right) \right) \right) \mu \left(d\omega \right) \leq \int_{\Omega^T} \left(V \left(Q \left(\omega; \mathcal{F} \right) \right) \right) \mu \left(d\omega \right),$$

i.e., the buyer's valuation of asset is less than the value of the asset to a perfectly informed agent. Moreover, since the seller's information becomes arbitrarily good,

$$\int_{\Omega^T} \left(V \left(Q \left(\omega; \mathcal{F}_1^{\kappa,\pi} \right) \right) \right) \mu \left(d\omega \right) \to \int_{\Omega^T} \left(V \left(Q \left(\omega; \mathcal{F} \right) \right) \right) \mu \left(d\omega \right).$$

It follows that the limit supremum of the lefthand side of (5) is weakly negative, giving the result. ∎

Proof of Proposition 5: We start with some preliminaries. Recall that $L_i \left(s_i \right)$ denotes the likelihood ratio of signal s_i; likewise, for any set S such that $\eta_i^a \left(S \right) > 0$, we let $L_i \left(S \right)$ denote the likelihood ratio $\eta_i^a \left(S \right) / \eta_i^b \left(S \right)$. The asset value V is defined as a function of q, the probability the asset holder attaches to fundamental a. In the trade equilibria under consideration, for the seller the conditional probability q is of the form $\Pr \left(a | s_1, s_2 \notin S_2^T \right)$, while for the buyer it is of the form $\Pr \left(a | s_1 \in S_1^T, s_2 \right)$. It is convenient to rewrite these probabilities as

$$\Pr \left(a | s_1, s_2 \notin S_2^T \right) = \frac{\frac{\Pr(a)}{\Pr(b)} L_1 \left(s_1 \right) L_2 \left(\mathbb{R} \backslash S_2^T \right)}{\frac{\Pr(a)}{\Pr(b)} L_1 \left(s_1 \right) L_2 \left(\mathbb{R} \backslash S_2^T \right) + 1}$$

$$\Pr \left(a | s_1 \in S_1^T, s_2 \right) = \frac{\frac{\Pr(a)}{\Pr(b)} L_1 \left(S_1^T \right) L_2 \left(s_2 \right)}{\frac{\Pr(a)}{\Pr(b)} L_1 \left(S_1^T \right) L_2 \left(s_2 \right) + 1}.$$

Next, define a mapping from likelihood ratios to probabilities by

$$q \left(L \right) \equiv \frac{\frac{\Pr(a)}{\Pr(b)} L}{\frac{\Pr(a)}{\Pr(b)} L + 1} \text{ for any } L \in [0, \infty),$$

along with a transformation V^ℓ of V that takes a likelihood ratio L as its argument, i.e., $V^\ell \equiv V \circ q$. The function V is convex, and by hypothesis is non-monotone. As such, there exist probabilities q^* and $q^{**} \geq q^*$ such that V is strictly decreasing over $[0, q^*]$, flat over $[q^*, q^{**}]$, and strictly increasing over $[q^{**}, 1]$. Since q is strictly increasing in L, $L^* = q^{-1}(q^*)$ and $L^{**} = q^{-1}(q^{**})$ are well-defined, and V^ℓ is strictly decreasing over $[0, L^*]$, flat over $[L^*, L^{**}]$, and strictly increasing over $[L^{**}, \infty)$.

We show there exists an equilibrium of the type described, i.e., $S_1^T \equiv [\underline{s}_1, \bar{s}_1]$ and $S_2^T \equiv \mathbb{R} \backslash (\underline{s}_2, \bar{s}_2)$. If the seller sees signal s_1 his payoff from offering to sell is

$$\Pr\left(s_2 \in S_2^T | s_1\right) p + \Pr\left(s_2 \notin S_2^T | s_1\right) V^\ell\left(L_1(s_1) L_2\left(\mathbb{R} \backslash S_2^T\right)\right),$$

while his payoff from not offering to sell is

$$\Pr\left(s_2 \in S_2^T | s_1\right) V^\ell\left(L_1(s_1) L_2\left(S_2^T\right)\right) + \Pr\left(s_2 \notin S_2^T | s_1\right) V^\ell\left(L_1(s_1) L_2\left(\mathbb{R} \backslash S_2^T\right)\right).$$

Thus it is a best response for the seller to offer to sell whenever $s_1 \in S_1^T$ if and only if

$$\begin{aligned} V^\ell\left(L_1(s_1) L_2\left(S_2^T\right)\right) &\leq p \text{ for all } s_1 \in S_1^T \\ V^\ell\left(L_1(s_1) L_2\left(S_2^T\right)\right) &\geq p \text{ for all } s_1 \notin S_1^T. \end{aligned}$$

By continuity and the shape of V^ℓ, these conditions are satisfied if and only if

$$V^\ell\left(L_1(\underline{s}_1) L_2\left(S_2^T\right)\right) = V^\ell\left(L_1(\bar{s}_1) L_2\left(S_2^T\right)\right) = p. \tag{15}$$

Likewise, in order for the buyer to offer to buy whenever $s_2 \in S_2^T$,

$$\begin{aligned} V^\ell\left(L_1\left(S_1^T\right) L_2(s_2)\right) - \delta &\geq p \text{ for all } s_2 \in S_2^T \\ V^\ell\left(L_1\left(S_1^T\right) L_2(s_2)\right) - \delta &\leq p \text{ for all } s_2 \notin S_2^T, \end{aligned}$$

and these conditions are satisfied if and only if

$$V^\ell\left(L_1\left(S_1^T\right) L_2(\underline{s}_2)\right) - \delta = V^\ell\left(L_1\left(S_1^T\right) L_2(\bar{s}_2)\right) - \delta = p. \tag{16}$$

Thus a trade equilibrium exists if and only if there exist $\underline{s}_1$, $\bar{s}_1 \neq \underline{s}_1$, $\underline{s}_2$, $\bar{s}_2 \neq \underline{s}_2$ such that (15) and (16) hold.

From the shape of V^ℓ, there exists a unique quadruple $\underline{L}_1, \bar{L}_1, \underline{L}_2, \bar{L}_2$ such that $\underline{L}_i < L^* \le L^{**} < \bar{L}_i$ for $i = 1, 2$, and

$$V^\ell\left(\underline{L}_1\right) = V^\ell\left(\bar{L}_1\right) = p$$
$$V^\ell\left(\underline{L}_2\right) - \delta = V^\ell\left(\bar{L}_2\right) - \delta = p.$$

Consequently, a trade equilibrium of the type described exists if and only if there exist $\underline{s}_1$, $\bar{s}_1 \neq \underline{s}_1$, $\underline{s}_2$, $\bar{s}_2 \neq \underline{s}_2$ satisfying the following system of four equations:

$$L_1\left(\underline{s}_1\right) L_2(\mathbb{R}\backslash\left(\underline{s}_2, \bar{s}_2\right)) = \underline{L}_1 \tag{17}$$
$$L_1\left(\bar{s}_1\right) L_2(\mathbb{R}\backslash\left(\underline{s}_2, \bar{s}_2\right)) = \bar{L}_1 \tag{18}$$
$$L_1\left(\left[\underline{s}_1, \bar{s}_1\right]\right) L_2(\underline{s}_2) = \underline{L}_2 \tag{19}$$
$$L_1\left(\left[\underline{s}_1, \bar{s}_1\right]\right) L_2(\bar{s}_2) = \bar{L}_2. \tag{20}$$

To complete the proof we show that such a quadruple does exist. First note that (17) and (18) imply

$$\frac{L_1(\underline{s}_1)}{L_1(\bar{s}_1)} = \frac{\underline{L}_1}{\bar{L}_1} < 1 \tag{21}$$

and (19) and (20) imply

$$\frac{L_2(\underline{s}_2)}{L_2(\bar{s}_2)} = \frac{\underline{L}_2}{\bar{L}_2} < 1. \tag{22}$$

Fix $\underline{s}_1$ and solve for $\bar{s}_1(\underline{s}_1) > \underline{s}_1$ from (21). Similarly solve for $\bar{s}_2(\underline{s}_2) > \underline{s}_2$ from (22). Substituting for $\bar{s}_1(\underline{s}_1)$ and $\bar{s}_2(\underline{s}_2)$, rewrite (17) and (19) as

$$L_1\left(\underline{s}_1\right) L_2(\mathbb{R}\backslash\left(\underline{s}_2, \bar{s}_2(\underline{s}_2)\right)) = \underline{L}_1 \tag{23}$$

$$L_1\left(\left[\underline{s}_1, \bar{s}_1(\underline{s}_1)\right]\right) L_2(\underline{s}_2) = \underline{L}_2. \tag{24}$$

Observe that $\bar{s}_1\left(\underline{s}_1\right) \to \pm\infty$ as $\underline{s}_1 \to \pm\infty$. Consequently $L_1([\underline{s}_1, \bar{s}_1(\underline{s}_1)]) \to \infty$ as $\underline{s}_1 \to \infty$ and $L_1([\underline{s}_1, \bar{s}_1(\underline{s}_1)]) \to 0$ as $\underline{s}_1 \to -\infty$. Thus from (24) define $\underline{s}_2\left(\underline{s}_1\right)$, and note that $\underline{s}_2\left(\underline{s}_1\right) \to \mp\infty$ as $\underline{s}_1 \to \pm\infty$.

Also observe that $\bar{s}_2\left(\underline{s}_2\right) \to \pm\infty$ as $\underline{s}_2 \to \pm\infty$, and so $L_2\left(\mathbb{R}\backslash\left(\underline{s}_2, \bar{s}_2\right)\right) \to 1$ as $\underline{s}_2 \to \pm\infty$. So substituting in for $\underline{s}_2\left(\underline{s}_1\right)$, the lefthand side of (23) approaches 0 as

$\underline{s}_1 \to -\infty$ and grows without bound as $\underline{s}_1 \to \infty$. By continuity it follows that there exists some $\underline{s}_1$ such that

$$L_1 \left(\underline{s}_1\right) L_2(\mathbb{R}\backslash \left(\underline{s}_2 \left(\underline{s}_1\right), \bar{s}_2(\underline{s}_2 \left(\underline{s}_1\right))\right)) = \underline{L}_1,$$

completing the proof. ∎

Proof of Proposition 6: The key to the proof is the following observation: for any set $S_2 \subset \mathbb{R}$, Jensen's inequality, the convexity and non-monotonicty of V, and unbounded MLRP together imply

$$\begin{aligned}
& E\left[V\left(\Pr\left(a|\left\{s_1\right\} \times S_2\right)\right)|s_2 \in S_2\right] \\
= & \sum_{\theta=a,b} \Pr\left(\theta|s_2 \in S_2\right) \int_{-\infty}^{\infty} V\left(\Pr\left(a|\left\{s_1\right\} \times S_2\right)\right) f_1^{\theta}\left(s_1\right) ds_1 \\
> & \sum_{\theta=a,b} \Pr\left(\theta|s_2 \in S_2\right) V\left(\Pr\left(a|S_2\right)\right) = V\left(\Pr\left(a|S_2\right)\right).
\end{aligned}$$

Likewise, for any $S_1 \subset \mathbb{R}$, $E\left[V\left(\Pr\left(a|S_1 \times \left\{s_2\right\}\right)\right)|s_1 \in S_1\right] > V\left(\Pr\left(a|S_1\right)\right)$.

We show that the equilibrium established in Proposition 5 remains an equilibrium when the buyer and seller must pay k to acquire their signals. For an information acquisition cost of $k = 0$, the seller's equilibrium utility is

$$\begin{aligned}
& p \Pr\left(s_1 \in S_1^T, s_2 \in S_2^T\right) \\
& + E\left[V\left(\Pr\left(a|\left\{s_1\right\} \times S_2^T\right)\right)|s_1 \notin S_1^T, s_2 \in S_2^T\right] \Pr\left(s_1 \notin S_1^T, s_2 \in S_2^T\right) \\
& + E\left[V\left(\Pr\left(a|\left\{s_1\right\} \times \mathbb{R}\backslash S_2^T\right)\right)\right] \Pr\left(s_2 \notin S_2^T\right).
\end{aligned} \tag{25}$$

Because the seller could instead always offer to sell, this quantity exceeds

$$p \Pr\left(s_2 \in S_2^T\right) + E\left[V\left(\Pr\left(a|\left\{s_1\right\} \times \mathbb{R}\backslash S_2^T\right)\right)|s_2 \notin S_2^T\right] \Pr\left(s_2 \notin S_2^T\right),$$

which by above is in turn strictly greater than

$$p \Pr\left(s_2 \in S_2^T\right) + V\left(\Pr\left(a|\mathbb{R}\backslash S_2^T\right)\right) \Pr\left(s_2 \notin S_2^T\right).$$

This last expression equals the seller's payoff under the deviation in which he does not buy his signal and always trades. Similarly, the seller's equilibrium utility (25) is greater than his payoff from never trading,

$$\begin{aligned}
&E\left[V\left(\Pr\left(a|\left\{s_1\right\}\times S_2^T\right)\right)|s_2\in S_2^T\right]\Pr\left(s_2\in S_2^T\right)\\
&+E\left[V\left(\Pr\left(a|\left\{s_1\right\}\times\mathbb{R}\backslash S_2^T\right)\right)|s_2\notin S_2^T\right]\Pr\left(s_2\notin S_2^T\right)
\end{aligned}$$

which is in turn strictly greater than

$$V\left(\Pr\left(a|S_2^T\right)\right)\Pr\left(s_2\in S_2^T\right)+V\left(\Pr\left(a|\mathbb{R}\backslash S_2^T\right)\right)\Pr\left(s_2\notin S_2^T\right),$$

the value of the asset to the seller if he observes only the buyer's announcement of whether or not he is prepared to buy. So for all information acquisition costs k that are sufficiently low the seller chooses to buy his information.

For an information acquisition cost of $k=0$, the buyer's equilibrium utility is

$$E\left[V\left(\Pr\left(a|S_1^T\times\left\{s_2\right\}\right)\right)-p-\delta|s_1\in S_1^T,s_2\in S_2^T\right]\Pr\left(s_1\in S_1^T,s_2\in S_2^T\right).$$

Because the buyer could instead always offer to buy, this exceeds

$$E\left[V\left(\Pr\left(a|S_1^T\times\left\{s_2\right\}\right)\right)-p-\delta|s_1\in S_1^T\right]\Pr\left(S_1^T\right),$$

which in turn strictly exceeds

$$\left(V\left(\Pr\left(a|S_1^T\right)\right)-p-\delta\right)\Pr\left(S_1^T\right),$$

the buyer's payoff under the deviation in which he does not buy his signal and always trades. Finally, if the buyer deviates to not buying the signal and never trading, his payoff is simply zero, and which is strictly less than his equilibrium utility. Again, for all information acquisition costs k that are sufficiently low the buyer chooses to buy his information. ∎

Proof of Proposition 7: Let p denote the unique buyer's offer that is accepted, and let S_1 denote the set of signals at which the seller accepts this offer and sells. It follows that the buyer offers p if and only if

$$V^{\ell}\left(L_1\left(S_1\right)L_2\left(s_2\right)\right) - \delta \geq p,$$

where the likelihood ratios L_i and function V^{ℓ} are as defined in the proof of Proposition 5. From Proposition 1 and the proof of Proposition 5, we know that there exists L^* such that V^{ℓ} is strictly decreasing over $[0, L^*)$ and strictly increasing over (L^*, ∞). Since the seller accepts the offer p with strictly positive probability, $p > \min V^{\ell}$. It follows that there exist $\underline{s}_2$ and $\bar{s}_2$ such that the buyer offers p if $s_2 \in (-\infty, \underline{s}_2) \cup (\bar{s}_2, \infty)$, with

$$V^{\ell}\left(L_1\left(S_1\right)L_2\left(\underline{s}_2\right)\right) = V^{\ell}\left(L_1\left(S_1\right)L_2\left(\bar{s}_2\right)\right) = p + \delta.$$

So the buyer makes zero profits when he sees $s_2 \in \{\underline{s}_2, \bar{s}_2\}$, regardless of whether or not he offers p at these signals. Moreover, observe that V^{ℓ} is decreasing at $L_1\left(S_1\right)L_2\left(\underline{s}_2\right)$ and increasing at $L_1\left(S_1\right)L_2\left(\bar{s}_2\right)$.

To prove (I), suppose to the contrary that $p > p^*$, and consider the deviation in which the buyer offers p^*. We show that this deviation is strictly profitable for the buyer at at least one of the signals $\underline{s}_2, \bar{s}_2$. The seller's beliefs are completely summarized by $\tilde{L}_2$. The seller then sells if $s_1 \in \tilde{S}_1$. If $L_1\left(\tilde{S}_1\right) > L_1\left(S_1^T\right)$ the buyer has a profitable deviation at $\bar{s}_2$. If $L_1\left(\tilde{S}_1\right) < L_1\left(S_1^T\right)$ the buyer has a profitable deviation at $\underline{s}_2$. If $L_1\left(\tilde{S}_1\right) = L_1\left(S_1^T\right)$ the buyer has a profitable deviation at both $\underline{s}_2$ and $\bar{s}_2$. In each case we have shown that p is not a best response for the buyer, giving a contradiction. To complete the proof, simply note that if $p < p^*$ then $p \leq \min V$ and the seller accepts the offer with zero probability.

To prove (II), let S_2^T denote the set of buyer signals at which the buyer offers p, and suppose to the contrary that the trade probability remains bounded away from 0 as $p^* \to \min V$. It follows that $L_2\left(S_2^T\right)$ is also bounded away from 0. The seller sells at signal s_1 only if $p \geq V^\ell\left(L_1(s_1)\, L_2\left(S_2^T\right)\right)$. This occurs only if $L_1(s_1)\, L_2\left(S_2^T\right) \in (L^* - \varepsilon, L^* + \varepsilon)$ for some ε, where $\varepsilon \to 0$ as $p^* \to \min V$. Since $L_2\left(S_2^T\right)$ is bounded away from 0 it follows that $\Pr\left(s_1 \in S_1^T\right)$ converges to 0 as $p^* \to \min V$, which gives a contradiction and completes the proof. ∎

Proof of Proposition 8: The proof consists of showing that condition (13) is a necessary and sufficient condition for the non-monotonicity of $\mathcal{V}$. The only complication in the application of Propositions 1 and 5 is that in our basic model V is convex (see Lemma 1), while $\mathcal{V}$ is not necessarily convex. However, we show below that if $\mathcal{V}(q)$ is strictly increasing (respectively, decreasing) for some q, the same is true for all higher (respectively, lower) q. That is, if $\mathcal{V}$ is non-monotone it is decreasing then increasing. Proposition 1 does not use the convexity of V, while in Proposition 5 convexity is used only to show that V has this property when it is non-monotone.

As noted in the main text, an agent without security 1 allocates $\bar{\psi} \equiv \mu_a/\gamma\sigma_a^2 = \mu_b/\gamma\sigma_b^2$ to the market regardless of his beliefs about the fundamental. Let $\psi(q)$ be the optimal market allocation of an investor who holds the special asset and attaches a probability q to fundamental a. Hence

$$v(q) = qE\left[-e^{-\gamma(r\psi(q)+R)}|a\right] + (1-q)\,E\left[-e^{-\gamma(r\psi(q)+R)}|b\right]$$
$$\bar{v}(q) = qE\left[-e^{-\gamma r\bar{\psi}}|a\right] + (1-q)\,E\left[-e^{-\gamma r\bar{\psi}}|b\right].$$

Since $\mathcal{V}'(q) \geq 0$ is equivalent to $\frac{-v'}{-v} \leq \frac{-\bar{v}'}{-\bar{v}}$ (recall that v and $\bar{v}$ are both negative-valued), by the envelope theorem $\mathcal{V}$ is weakly increasing if and only if

$$\frac{E\left[e^{-\gamma(r\psi(q)+R)}|a\right] - E\left[e^{-\gamma(r\psi(q)+R)}|b\right]}{qE\left[e^{-\gamma(r\psi(q)+R)}|a\right] + (1-q)\,E\left[e^{-\gamma(r\psi(q)+R)}|b\right]}$$
$$\leq \frac{E\left[e^{-\gamma r\bar{\psi}}|a\right] - E\left[e^{-\gamma r\bar{\psi}}|b\right]}{qE\left[e^{-\gamma r\bar{\psi}}|a\right] + (1-q)\,E\left[e^{-\gamma r\bar{\psi}}|b\right]}.$$

By straightforward algebra, this inequality is itself equivalent to

$$\frac{E\left[e^{-\gamma(r\psi(q)+R)}|a\right]}{E\left[e^{-\gamma(r\psi(q)+R)}|b\right]} \leq \frac{E\left[e^{-\gamma r\bar{\psi}}|a\right]}{E\left[e^{-\gamma r\bar{\psi}}|b\right]}. \tag{26}$$

To evaluate the righthand side of (26), note that by normality of the market return r,

$$E\left[e^{-\gamma r\bar{\psi}}|a\right] = \exp\left(-\gamma\left(\mu_a\bar{\psi} - \frac{\gamma}{2}\sigma_a^2\bar{\psi}^2\right)\right) = \exp\left(-\left(\frac{\mu_a^2}{\sigma_a^2} - \frac{1}{2}\frac{\mu_a^2}{\sigma_a^2}\right)\right) = \exp\left(-\frac{\mu_a^2}{2\sigma_a^2}\right).$$

So $\mathcal{V}'(q) \geq 0$ is equivalent to

$$E\left[e^{-\gamma(r\psi(q)+R)}|a\right] \leq E\left[e^{-\gamma(r\psi(q)+R)}|b\right]\exp\left(-\gamma\left(\frac{\mu_a^2}{2\gamma\sigma_a^2} - \frac{\mu_b^2}{2\gamma\sigma_b^2}\right)\right).$$

Applying the joint normality of r and R, taking logs and dividing by $-\gamma$ implies that $\mathcal{V}'(q) \geq 0$ is equivalent to

$$\begin{aligned} &\nu_a + \psi(q)\mu_a - \frac{\gamma}{2}\left(\zeta_a^2 + \psi(q)^2\sigma_a^2 + 2\beta_a\sigma_a^2\psi(q)\right) \\ &\geq \nu_b + \psi(q)\mu_b - \frac{\gamma}{2}\left(\zeta_b^2 + \psi(q)^2\sigma_b^2 + 2\beta_b\sigma_b^2\psi(q)\right) + \frac{1}{2\gamma}\left(\frac{\mu_a^2}{\sigma_a^2} - \frac{\mu_b^2}{\sigma_b^2}\right). \end{aligned}$$

Recall that $\psi_\theta = \frac{\mu_\theta}{\gamma\sigma_\theta^2} - \beta_\theta$ for $\theta = a, b$, and define the function

$$\begin{aligned} G(\psi) \equiv &\left(\nu_a - \frac{\gamma}{2}\zeta_a^2\right) - \left(\nu_b - \frac{\gamma}{2}\zeta_b^2\right) + \frac{1}{2\gamma}\left(\frac{\mu_b^2}{\sigma_b^2} - \frac{\mu_a^2}{\sigma_a^2}\right) \\ &+\psi\gamma\left(\sigma_a^2\psi_a - \sigma_b^2\psi_b\right) + \psi^2\frac{\gamma}{2}\left(\sigma_b^2 - \sigma_a^2\right). \end{aligned}$$

So $\mathcal{V}'(q) \geq 0$ is equivalent to $G(\psi(q)) \geq 0$.

As the probability q varies from 0 to 1 the allocation $\psi(q)$ increases monotonically and continuously from ψ_b to ψ_a. Differentiation of G yields $G'(\psi_a) = \gamma\sigma_b^2(\psi_a - \psi_b) \geq 0$ and $G'(\psi_b) = \gamma\sigma_a^2(\psi_a - \psi_b) \geq 0$. Since G is quadratic in ψ it follows that G is increasing over the interval $[\psi_b, \psi_a]$. So $\mathcal{V}(q)$ is non-monotonic over $[0,1]$ if and only if $G(\psi_b) < 0 < G(\psi_a)$ (in which case $\mathcal{V}$ is decreasing then increasing), which is equivalent to (13), completing the proof. ∎

Proof of Corollary 3: Under the conditions given, (13) simplifies to $\psi_b(\psi_a - \psi_b) < 0 < \psi_a(\psi_a - \psi_b)$, which holds if and only if $\psi_b < 0 < \psi_a$. (Recall that Proposition 8 assumes without loss that $\psi_b \leq \psi_a$.) ∎

Proof of Corollary 4: Under the conditions given, (13) simplifies to

$$\psi_b \left(\psi_a - \psi_b\right) \sigma_a^2 < \frac{1}{2}\left(\zeta_a^2 - \zeta_b^2\right) < \psi_a \left(\psi_a - \psi_b\right) \sigma_a^2.$$

Substituting in for $\psi_a - \psi_b = \frac{\rho_a}{\sigma_a}\left(\zeta_b - \zeta_a\right)$ and dividing by $\zeta_b - \zeta_a$ gives

$$\psi_b \frac{\rho_a}{\sigma_a} \sigma_a^2 < -\frac{\gamma}{2}\left(\zeta_a + \zeta_b\right) < \psi_a \frac{\rho_a}{\sigma_a} \sigma_a^2.$$

As ε increases, $\zeta_a + \zeta_b$ stays fixed, ψ_a increases, and ψ_b decreases. So condition (13) is satisfied if and only if ε is sufficiently large. ∎