Notes on auctions

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Consider a standard independent private value setting. Values are drawn independently from the same uniform distribution on $[0,1]$. In such a setting, a player who gets a draw v_i also gets an idea about how his value compares to others. For example, if v_i is close to 0, then bidder i is almost sure that he is the lowest bidder.

This seems to be an artefact of the model: there is no reason to believe that learning about one's own valuation should tell us something about how it **compares** to others' valuations. We may have learned over time what is our chance of being the highest valuation bidder, (and in a symmetric model, this chance ought to be $1/n$), but, we face problems with substantial variation, it is hard to believe that there is a strong connection between the realization v_i and the chance of being the highest valuation bidder;

One objective of this paper is to provide a model where bidders cannot use v_i to make inferences about their chances of being the highest valuation bidder.

Consider the following alternative specification:

$$v_i = \alpha + x_i$$

where x_i are, as before, drawn independently from a uniform distribution on $[0,1]$, and where α is drawn from a uniform distribution with large support, and agents observe only v_i..

Then, except near the boundaries of the distribution, learning v_i tells nothing about how v_i compares with v_j. For most values of v_i, any player i seeing his value v_i must think he has a chance $1/n$ of being the highest valuation player.[1]

As we shall see, in the first price auction, and for most realizations of v_i, the optimal bid must take the following simple form:

$$b_i = v_i - a$$

Comments on the philosophy of the exercise.

(a) The auction literature has noticed that even the independent private value model embodies a special relationship between the value realization v_i and the chance to be the highest valuation bidder, and that literature has noted that other assumptions on the joint distribution of valuations would imply other relationships between the value realization v_i and the chance that one is the highest value bidder. What we propose goes further: we propose a model where there is no connection to be made.

[1] Note that this particular specification implies that the particular realization v_i does not convey information about the dispersion of values across bidders. This assumption may be questionnable and a specification of the type: $v_i = \alpha x_i$ with $x_i \in [0.9, 1.1]$, say, might be more realistic.

(b) One would think that when bidding, a bidder should pay attention to how much the object is worth to him, say v_i, and to his chances of being the highest valuation bidder, say π_i: a bidder with only small chance of being the highest valuation bidder should presumably be cautious in not shading his bid by too much.

In a standard Bayesian model with private values, values are drawn from a joint distribution, and each bidder's strategy is a function of his value, i.e. $b_i(v_i)$. Seemingly, a bidder thus does not worry about his chance of bidding. The reason is as follows. In equilibrium, a bidder's equilibrium bid is optimal given the joint distribution over values and others' strategies. We do not necessarily mean to assume that the bidder knows the joint distribution nor the strategies of the others, but, playing optimally, he ends up acting as if he knew it, and correctly shades his bid to take into account his chance of winning. But the actual assessment that a player might have about his chances of winning plays no overt role but is buried into the bid function $b_i(v_i)$.

(c) To be more precise, consider the standard private value model and define $\pi_i(v_i) = \Pr\{\max_{j \neq i} v_j < v_i \mid v_i\}$ as the probability that he has the highest valuation for the object given v_i. Assume that in addition to v_i, player i gets an imperfect signal $y_i \in \{0, 1\}$ about $\pi_i(v_i)$, assuming for example that $y_i = 1$ if and only if $\pi_i(v_i) > \pi^*$. A priori, one would think that the estimate y_i ought to play a role. In equilibrium however, bidders optimal strategy will be independent of y_i. The fine dependence on v_i that the bidder's strategy $b_i(v_i)$ allows for is enough to make signal y_i useless: the signal y_i brings no new information about rank than v_i already does.

The story would be different if one restricted attention to, say, the following class of strategies:

$$b_i(v_i) = v_i - a - cy_i$$

where the bidder tries to find the values of a and c that are optimal (across realizations of v_i and y_i).

In other words, there is a link between the set of strategies that are allowed and what we implicitly assume about what agents know about the distributions: by allowing all possible strategies, a player who plays optimally ends up bidding as if he perfectly knew his chance of winning for each possible realization of v_i (thus making signal y_i useless). With constraints on the strategy set, this is no longer the case, and signal y_i, if correlated with π_i, may end up being useful.

(d) The classic view that bidders would actually know the joint distribution over draws does not seem tenable. How would one learn about the distributions of valuations that are never observed? And if equilibrium behavior relies on learning, how then one would learn about optimal behavior if the space of strategies is the set of all functions?

(e) Why do we assume that there are no connection to be made between valuations and the probability to be the highest bidder?

Our aim is a parsimonious model of auctions, and we begin with informational assumptions that sound *minimal*. A model that implicitly assume that bidders have a fine knowledge of their chance of having the highest valuation for each possible realization of their own valuation does not sound parsimonious.

We see two (connected) ways of achieving this parsimony.

- Either by making distributional assumptions such as the one described above that preclude any relationship between valuations and ranking;

- Or by adding constraints on the strategy set. By focusing on the set of strategies of the form $b_i = v_i - a$, bidders find an optimal response that does not depend in a fine way on the actual relationship between valuation and ranking.

We do not mean to suggest that information about rank never arise. We have in mind that actual value estimation should be a poor instrument to estimate rank in many contexts, and that as a first step, it may be a better modelling strategy to ignore the relationship between valuation and rank altogether. As a second step however, one could investigate a model in which players do attempt to take into account signals about how their valuation ranks compared to other, and such a model could take the form outlined above, namely:

$$b_i = v_i - a - cy_i$$

(f) When modelling long run interactions, we often rely on infinite horizon games. We do not believe that the horizon is actually infinite, but as explained by Rubinstein (comments the interpretation of game theory, *Econometrica*), we believe that the infinite horizon is a better model of a strategic situation in which the date at which the game ends game is not known for sure: we do not want the outcome to depend on the fine details of how or when the game will end.

In a similar vein, we believe it is difficult for bidders to get a clear idea of their probability of winning as a function their valuation v_i and the way they shade their bid, and we do not want the outcome to depend on the fine knowledge of this relationship between value and ranking.

1 The basic model.

As explained above, the basic model considers the following specification:

$$v_i = \alpha + x_i$$

where x_i's are drawn independently from a distribution on $[\underline{x}, \bar{x}]$ (let f denote the density), and where α is drawn from a uniform distribution with large support. Bidder i learns v_i, but not α or x_i.

We consider the first price auction. Possible strategies are assumed to be of the form

$$b_i = v_i - a$$

This is a restriction on the strategy set. Yet given the assumptions made on the distribution of valuations, then, even if there were no restrictions on bid functions, bidding in this way would be optimal for all realizations but those that fall near the boundary of α. (See the note below).

We now look for a symmetric equilibrium of the first price auction. Assume that all bidders except i shade their bids by a, while bidder i shades his bid by $a+y$. Denote by $u_i(y,a)$ the expected payoff that player i derives in that case, and denote by $\phi(y)$ the probability that x_i exceeds $\max_{j\neq i} x_j$ by more than y, that is,

$$\phi(y) = \Pr(x_i - \max_{j\neq i} x_j > y).$$

We have

$$u_i(y,a) = (a+y)\phi(y)$$

The first order condition for a symmetric equilibrium thus writes as

$$a\phi'(0) + \phi(0) = 0.$$

Since the x_i 's are drawn from the same distribution, we have $\phi(0) = \frac{1}{n}$, hence

$$a = \frac{1}{-\phi'(0)n}$$

When x_i is distributed according to a density f, with cumulative distribution F, we have:

$$-\phi'(0) = \frac{d}{dy}\Pr(x_i + y > x_j \text{ for all } j)\bigg|_{y=0} = \frac{d}{dy}\int f(x_i)[F(x_i+y)]^{n-1}dx_i\bigg|_{y=0}$$

which implies:

$$-\phi'(0) = \int (n-1)(f(x_i))^2[F(x_i)]^{n-2}dx_i. \tag{1}$$

In the case that the distribution is uniform on the interval $[\underline{x}, \bar{x}]$, we have: $\phi'(0) = -1/(\bar{x} - \underline{x})$, hence

$$a = \frac{\bar{x} - \underline{x}}{n}.$$

Shading thus depends on the number of bidders and dispersion of valuations: the more dispersed are valuations, the less risky it is to shade one's bid.

Comments:

1. In comparison with the standard independent private value model, in which shading is contingent on v_i, shading is here independent of v_i.

In general, we should expect shading to be contingent on the chance of being the highest bidder and the dispersion of values. In the alternative model proposed, both the chances of being the highest bidder and the dispersion of values are essentially independent of the particular realization v_i. Not surprisingly, then, shading is constant across v_i.

2. Computations have been made under the assumption that bidders look for the optimal strategy among strategies of the form $b_i = v_i - a$. If the distribution over α is flat on a large interval, then, even if bidders look for the optimal

strategy among all possible bid functions, then, except near the boundary of the distribution over v_i, the computations made are exact, because learning v_i is not informative about x_i. Formally, call g the distribution over α, assumed to be flat over $[\underline{\alpha}, \overline{\alpha}]$, then for any $v_i \in [\underline{\alpha} + \overline{x}, \overline{\alpha} - \underline{x}]$ we have:

$$f(x_i \mid \alpha + x_i = v_i) = \frac{f(x_i)g(v_i - x_i)}{\int_{y_i} f(y_i)g(v_i - y_i)dy_i} = f(x_i).$$

3. We did not check above for *second order conditions.* We have to ensure that there are no profitable large deviations. The first order condition can be written as

$$-a = y + \frac{\phi(y)}{\phi'(y)}. \tag{2}$$

If $y+\frac{\phi(y)}{\phi'(y)}$ *is increasing in* y, then the solution to (2) is unique, so at $a = -\frac{\phi(0)}{\phi'(0)}$, $y = 0$ is the unique interior optimum. The solution must be a maximum since gains are 0 when $y = -a$ (because then $b_i = v_i$) and when $y = \underline{x} - a$ (because then i has no chance of winning then), and are strictly positive for values of y between.

4. *An example where second order conditions fail.*

Consider two bidders, and assume that for each player, x_i is with probability p concentrated and uniformly distributed on $[-\varepsilon, \varepsilon]$, and with probability $(1-p)$ dispersed and uniformly distributed on $[-\overline{x}, \overline{x}]$. Then

$$-\phi'(0) \geq p^2 \frac{1}{2\varepsilon}$$

hence the first order condition holds for some a that satisfies

$$a \leq \frac{\varepsilon}{p^2}$$

If both players were to follow that strategy, they would thus earn an expected payoff no larger than $\frac{\varepsilon}{2p^2}$. If one player decides to choose $a = \overline{x}/2$, then he would certainly win in all events where $x_j < -\overline{x}/2$, hence he would win with probability at least equal to $\frac{1-p}{4}$ hence make a profit at least equal to to $\frac{\overline{x}(1-p)}{8}$. It follows that if

$$\varepsilon < \overline{x}p^2(1-p)/8,$$

the second order condition cannot be satisfied, and the candidate equilibrium obtained by looking at the first order condition is not an equilibrium.

Intuitively, because the distribution is somewhat concentrated around 0, in a candidate pure strategy symmetric equilibrium, changing slightly one's bid has a strong effect on the chances of winning. This Bertrand competition effect induces bidders to shade their bids very little, implying that expected payoffs must be very small in such a candidate equilibrium. But then there exists a force toward large shading: you can take a chance for a large benefit, even if it is at the risk of having little chance of winning.

In standard private auctions, one typically assumes *affiliation*, and existence of a pure strategy equilibrium is then assured. Our example above is not in contradiction with this existence result. Though preferences exhibit a relatively simple form of positive correlation, affiliation does not necessarily hold.

To see why, consider two bidders with valuations $v_1 \in \{v, v+d\}$ and $v_2 \in \{v-d, v\}$. Call $h(.)$ the distribution over $x_1 - x_2$. Affiliation requires that

$$f(v, v-d)f(v, v+d) > f(v, v)f(v+d, v-d).$$

or equivalently, given our flat support assumption on α,

$$h(d)h(-d) > h(0)h(2d)$$

When the x_i's are concentrated, and for a choice of d that is neither too small nor too large, that inequality typically does not hold.

There are examples of non existence of pure strategy equilibria in the literature. These examples rely on the fact that first order conditions do not imply a monotone bid function. Here the bid function that is obtained from the first order conditions *is* monotone. Some large deviations however turn out to be profitable.

5. Construction of a symmetric mixed strategy equilibrium

As pointed out above, pure strategy equilibria may not always exist. We construct below a simple example of a symmetric mixed strategy equilibrium.

We consider two bidders and define ϕ as above. We shall look for a mixed strategy equilibrium in which each player randomizes between two values of a, say $\bar{a}$ and $\underline{a}$, and choose $\bar{a}$ with probability η. Denote by σ that strategy. We have:

$$V(a, \sigma) = a[\eta\phi(a - \bar{a}) + (1 - \eta)\phi(a - \underline{a})]$$

which gives the following f.o.c.:

$$a = \frac{\eta\phi(a - \bar{a}) + (1 - \eta)\phi(a - \underline{a})}{\mid \eta\phi'(a - \bar{a}) + (1 - \eta)\phi'(a - \underline{a}) \mid} \text{ for } a = \bar{a}, \underline{a}.$$

These two conditions, along with the equality

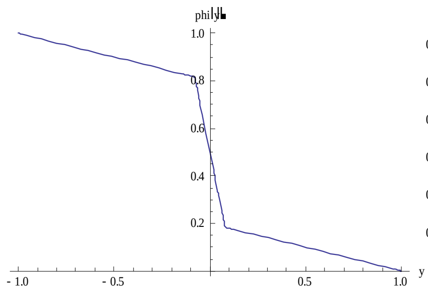
$$V(\bar{a}, \sigma) = V(\underline{a}, \sigma)$$

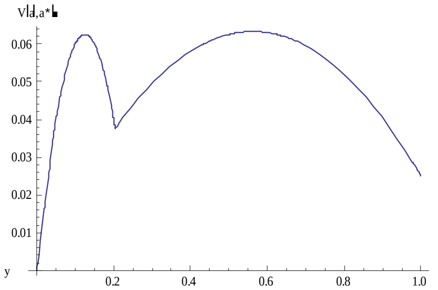
can be used to find a candidate mixed strategy equilibrium. We now check in an example that this can be done.

Assume that the difference $z = x_1 - x_2$ is distributed on $[-1, 1]$ according to the density $h(.)$ where $h(z) = h_0(> 1/2)$ if $\mid z \mid < \varepsilon$ and $h(z) = h_1(< 1/2)$ otherwise.[2] Note that by definition, we have $\phi(y) = \int_y^1 h(z)dz$. Also note that the only candidate symmetric pure strategy equilibrium must satisfy $a^* = \frac{1}{2h_0}$, with an expected gain equal to $a^*/2 = 1/(4h_0)$.

[2] Note that h_0 and h_1 determine ε.

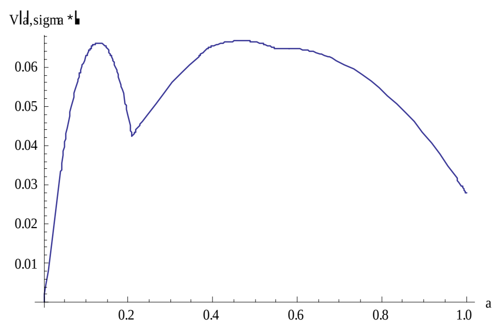
For $h_0 = 4$ and $h_1 = 0.2$, the following figures plot $\phi(.)$ and the expected gain $V(a, a^*) = a\phi(a - a^*)$:





The figure on the right shows the expected payoff to bidder i when the other bidder bids by shading his value by $a^* = \frac{1}{8}$ and bidder 1 shades by a. There is a local maximum at $a = \frac{1}{8}$, but this is not a global maximum. There is a second local maximum at about .55, and while it is difficult to tell from the graph, $V(.55, a^*) > V(\frac{1}{8}, a^*)$, and consequently there is no pure strategy equilibrium.

Still for $h_0 = 4$ and $h_1 = 0.2$, the first order conditions yield the following candidate mixed strategy equilibrium σ^*: play $\bar{a} = 0.47$, with probability 0.027 and $\underline{a} = 0.13$ otherwise. The following figure plots the expected gain $V(a, \sigma^*)$ as a function of a:



Here, $V(.47, \sigma^*) = V(.13, \sigma^*) = \max_a V(a, \sigma^*)$, thus confirming that σ^* is a mixed strategy equilibrium.

6. *Risk aversion.*

Assume players are risk averse, and call $u(a)$ the utility that player i derives from getting a payoff equal to a. The expected utility from shading by $a + y$ when others shade by a becomes $u(a + y)\phi(y)$, so the first order condition for a symmetric equilibrium thus writes as

$$u(a)\phi'(0) + u'(a)\phi(0) = 0.$$

With $u(a) = a^\gamma$ with $\gamma < 1$ for example, we get:

$$a = \gamma \frac{1}{-\phi'(0)}$$

7. Releasing information about the dispersion of valuations

We assume that the distribution over x_i may be more or less dispersed, and we wish to compare the case where bidders have access to information on the dispersion of valuations, to the case where they don't.

We assume that there are K states of the world, and that in state k, the x_i's are drawn from $f(. \mid k)$. We define $\phi_k(y) = \Pr\{x_i > \max_{j \neq i} x_j \mid k\}$. When bidders know k, the equilibrium shading is:

$$a_k^* = \frac{1}{-n\phi_k'(0)}$$

When bidders do not know k, and the equilibrium shading becomes:

$$a^* = \frac{1}{-nE\phi_k'(0)}$$

Since $y \to 1/y$ is a convex function, $\frac{1}{-E_k\phi_k'(0)} < E_k \frac{1}{-\phi_k'(0)}$, so

$$a^* < Ea_k^*$$

bidders bid more aggressively on average when they do not know k.

8. Increasing number of bidders.

We evaluate the effect of more bidders on shading. We now write $\phi_n(y)$ to indicate the number of bidders, and throughout we assume that $y + \frac{\phi_n(y)}{\phi_n'(y)}$ is increasing in y for all n. We denote by a_n^* the level of equilibrium shading when there are n bidders present. We have the following proposition:

> **Proposition 1.** *Assume the* x_i 's are drawn independently from a density f *that is centered, symmetric, say around* 0*, and single peaked. Then*
>
> $$a_n^* = \frac{1}{n\beta_n}$$
>
> *where* β_n *is a decreasing sequence. The sequence* β_n *may decrease to* 0 *if* $f(\bar{x}) = 0$*. For* n *large however* $n\beta_n$ *increases without bound.*

Intuitively, when the number of bidders increases, the winning bidder tends to have a higher realization of x_i. Since f is single-peaked, the distribution over other bidders' valuations tends to be more dispersed on average, and β_n decreases.

Proof: We already know that $\beta_n = -\phi'(0)$. Integrating by parts the expression on the right hand side of (1) and observing that $f'(x) = f'(-x)$ and $F(-x) = 1 - F(x)$, we obtain:

$$\beta_n = f(\bar{x}) + \int_{\underline{x}}^{0} f'(x)([1 - F(x)]^{n-1} - [F(x)]^{n-1})dx$$

Now for any given $p < 1/2$, define $\Delta_n = (1-p)^n - p^n$, and $k_n = \Delta_n/(1-2p)$. We have $k_1 = 1$. Since

$$\Delta_{n+1} = (1-p)\Delta_n + (1-2p)p^n = (1-2p)((1-p)k_n + pp^{n-1})$$

This directly implies $k_n > p^{n-1}$ for all n, which further implies that k_{n+1} is an average of k_n and p^{n-1}, hence k_n is a strictly decreasing sequence. Applying this observation to each $p = F(x)$, we obtain that β_n is also a strictly decreasing sequence. To check the last assertion, assuming for example that $f'(\underline{x}) > 0$, it is sufficient to take a Taylor expansion of $F(x)$ close to $\underline{x}$ and then observe that $\beta_n \geq O(1/n)^{1/2}$.

9. *Stochastic number of bidders.*

We assume that the actual number of bidders is stochastic. We follow Matthews (1987) and McAfee and McMillan (1987) and compare the case where the seller commits to reveal the number of actual bidders, to the case where he commits to kept is secret.

Bidders are drawn from a pool of size N .The realized number of participants is denoted n. We assume that n is always at least equal to 2, and that conditional on n each bidder has equal chance of being a participant. For bidder i, the value of shading by $a + y$ when others shade by a can be written

$$V^i(y, a) = (a + y)E_n^i \phi_n(y)$$

where E_n^i indicates the expectation over n conditional on i participating. Since $E_n^i \phi_n(0) = E_n^i(1/n)$, the equilibrium shading writes as:

$$a^* = \frac{E_n^i(1/n)}{-E_n^i \phi_n'(0)} = \frac{E_n^i(1/n)}{E_n^i(\beta_n)},$$

If, upon participating, bidder i were told the number of participants, he would shade by $a_n^* = 1/(n\beta_n)$, hence in expectation, he would shade by $E_n^i a_n^*$. As we shall explain (see comment below), a^* may be larger than $E_n^i a_n^*$, hence, conditional on participating, bidder i would bid more aggressively on average if he were told the number of bidders.

$>$From the point of view of the seller, however, a^* should be compared to the unconditional expectation $E_n a_n^*$. Observing that for any random variable x_n, $E_n^i x_n = \frac{E_n n x_n}{E_n n}$,[3] we may rewrite a^* as:

$$a^* = \frac{1}{E_n n \beta_n}$$

Since $f(y) = 1/y$ is a convex function, $\frac{1}{E_n n\beta_n} < E_n \frac{1}{n\beta_n}$, hence we obtain:

Proposition 2: $E_n a_n^* > a^*$.

[3] This is because conditional on n, bidder i has a chance n/N of being a participant.

As in the case examined earlier where information about the dispersion of valuations could be revealed (and by a similar argument), the policy of committing to not revealing such information increases the revenue of the seller;

Comment: Under the conditions of proposition 1, and if $n\beta_n$ is an increasing sequence, then $a^* > E^i_n a^*_n$.[4] Thus conditional on participating, bidder i is bids more aggressively on average if he is told the number of bidders (that is $a^* > E^i_n a^*_n$). However, because participating is more likely when the number of bidders is larger, and because a^* is computed taking that into account, bidders bid more aggressively when they are **not** told the number of bidders ($a^* < E_n a^*_n$).

10. Further existence issues

With a stochastic number of bidders, existence of a pure strategy equilibrium for each realization of n with n known to bidders does not guarantee existence of a pure strategy equilibrium in the uncertain case.

To illustrate, consider a case where x_i is drawn uniformly on $[0, 1]$, and where n may take two values, either $n = 2$ or $n = n_0$, with n_0 assumed to be large. Let $p = \Pr\{n = 2 \mid i \text{ participates}\}$. For the uniform distribution, $\beta_{n_0} = \beta_2 = 1$, so the only candidate pure symmetric equilibrium is

$$a^* = E\phi_n(0) = \frac{1}{n_0} + p(\frac{1}{2} - \frac{1}{n_0})$$

and it yields an expected payoff equal to $a^* E\phi_n(0) = (a^*)^2$.

Consider now the deviation that consists in choosing a as if only 2 players where present, that is, $a = \frac{1}{2}$. With that deviation, the bidder wins with probability at least $p/2$, hence the deviation yields an expected payoff at least equal to $p/4$. If

$$\frac{p}{4} > (\frac{1}{n_0} + p(\frac{1}{2} - \frac{1}{n_0}))^2$$

then the deviation is profitable. It is easy to see that this inequality holds for $p = \frac{2}{n_0}$ and $n_0 > 7$, in which case a pure strategy symmetric equilibrium fails to exist.

This failure of existence is not specific to our model specification. Consider the standard independent private value model where each bidder who participates gets a draw v_i from the uniform distribution. The only candidate pure strategy equilibrium has bidder i choose bid $b_i = v_i(1 - a^*)$ where a^* is defined as above, thus yielding bidder i with valuation v_i an expected payoff equal to $v_i(a^*)^2$. Bidder i however could choose to bid $v_i/2$. He would win with probability at least $pv_i/2$, hence he would obtain an expected payoff at least equal $p(v_i)^2/2$. So whenever $pv_i/2 > (a^*)^2$, the deviation is profitable. For $p = \frac{2}{n_0}$ and $n_0 > 7$ in particular, all types $v_i > 1/2$ find the deviation profitable.

11. Revenue equivalence.

[4] β_n and $1/(n\beta_n)$ are then decreasing sequences, so we have: $E^i_n(\beta_n)E^i_n[1/(n\beta_n)] < E^i_n(1/n)$, which implies $a^* > E^i_n a^*_n$.

Under the alternative model, equivalence no longer holds: with a uniform distribution for example, the first price auction generates less revenue than the second price auction (see details below).

There are also examples of distributions where the first price dominates however. To see this, consider again the case of two bidders examined earlier where for each bidder i, x_i is with probability p concentrated and uniformly distributed on $[-\varepsilon, \varepsilon]$, and with probability $(1-p)$ dispersed and uniformly distributed on $[-\overline{x}, \overline{x}]$. Choose ε small, but not too small, so that a symmetric equilibrium in pure strategies exists. Then, because of a Bertrand competition effect, shading in the first price will be very small, and one can show that there are values of ε and p for which the revenues are larger in the first price auction.

Intuitively, bidders do not shade their bid much in the first price auction, so for any realization of (v_1, v_2), revenues are close to $\max v_i$. This is true even in events where one of the bidder gets a draw x_i drawn from the dispersed distribution (distributed on $[-\overline{x}, \overline{x}]$). Under that event, however, revenues in the second price auction are typically far away from $\max v_i$.

While it is not surprising that equivalence would not hold here (because valuations are correlated), and while it is not surprising that the second price auction does not always generate more revenues (because affiliation does not necessarily hold), it suggests a fragility of the revenue equivalence or revenue ranking results: it hinges on the assumption that there is a special relationship between one's own valuation and one's belief about how one's own valuation compares to others, and on the ability of players to correctly take into account this relationship through their bidding strategy: when bidders use simple bid functions, the above comparison applies (whatever the distribution on α and whether the realized α is known or not).

2 Incorporating information about rank

We motivated our model by arguing that in many contexts, a bidder's own valuation is a poor tool for estimating how his value compares to others' valuations. We do not mean to suggest, however, that bidders cannot form predictions about rank stemming from signals about others that they might receive.

In this Section, we illustrate how information about rank can be incorporated in our basic model. Of course, such information could also be introduced in the standard model. But the technical difficulty there is then that bidders have a two dimensional type, and equilibria are then difficult to characterize. In contrast, we will add a signal that reflects coarse information about rank, taking only two possible values $\theta_i \in \{0, 1\}$, and a strategy for bidder i will thus consist of a pair (a_i^0, a_i^1).

Formally, we assume that in addition of v_i, each player receives a signal $\theta_i \in \{0, 1\}$ that is correlated with the ranking over valuations. Specifically, define H_i to be the event that i has highest valuation. We assume that there exists p such that for each valuation vector $v = (v_i)_i$,

$$\begin{aligned}\Pr\{\theta_i &= 1 \mid v \in H_i\} = p \text{ and} \\ \Pr\{\theta_i &= 1 \mid v_i \notin H_i\} = 1 - p\end{aligned}$$

So when $p = 1/2$, the signal is uninformative, while for $p = 1$, it is perfectly informative of whether i has the highest valuation.

We thus define a standard Bayesian game, in which each bidder i chooses a shading strategy contingent on the signal θ_i, hence a pair (a_i^0, a_i^1). Clearly, if the signal is uninformative, we should expect to get back the equilibrium found in the previous Section. We are interested here in analyzing how information about rank affects bidding.

For simplicity, we focus on the case of two bidders, and start by examining the case where $p = 1$. We also let $\psi(a_0) \equiv \arg\max a\phi(a - a_0)$. We have:

Proposition 3: $\underline{a} < a^*$ and $\bar{a} = \psi(\underline{a})$.

Since in general ψ is a decreasing function,[5] that is, more aggressive behavior from the other bidder triggers more aggressive bidder, the auction with information about rank is more competitive.

Intuitively, when players get information about rank, a player receiving a bad signal must bid more aggressively to have a chance to win, and this adversely affects the bidder who receives a good signal. So despite knowing that he can afford to bid less aggressively than the other player to have a significant chance to win, he ends up bidding more aggressively than if neither bidder had had information about rank.

Proof: Define:

$$\bar{\phi}(y) = \Pr\{x_i > x_j + y \mid x_i > x_j\} \text{ and } \underline{\phi}(y) = \Pr\{x_i > x_j + y \mid x_i < x_j\}$$

We have $\bar{\phi}(y) = 2\phi(y)$ for $y \geq 0$, $\bar{\phi}(y) = 1$ for $y \leq 0$, and by symmetry $\underline{\phi}(y) = 1 - \phi(-y)$. We look for a symmetric equilibrium $\sigma = (a(\theta))_{\theta=0,1} = (\underline{a}, \bar{a})$. For each bidder, the value from bidding $\bar{a} + y$ in event $\theta_i = 1$ is:

$$V^1(y, \sigma) = (\bar{a} + y)[\bar{\phi}(\bar{a} - \underline{a} + y)],$$

and the value from bidding $\underline{a} + y$ in event $\theta_i = 0$ is:

$$V^0(y, \sigma) = (\underline{a} + y)[\underline{\phi}(\underline{a} - \bar{a} + y)].$$

Defining $z = \bar{a} - \underline{a}$, the first order conditions yield:

$$\bar{a} = \frac{\phi(z)}{-\phi'(z)} \text{ and } \underline{a} = \frac{1 - 2\phi(z)}{-2\phi'(z)}.$$

The equilibrium difference $z^* = \bar{a} - \underline{a}$ thus solves:

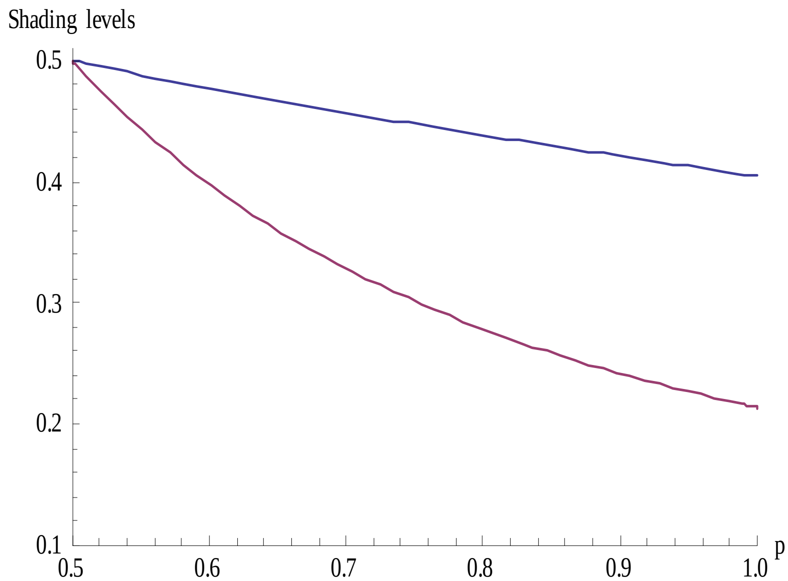
$$z^* = \frac{4\phi(z^*) - 1}{-2\phi'(z^*)}.$$

[5] In case the x_i are drawn from uniform distributions for example, ψ is a decreasing function.

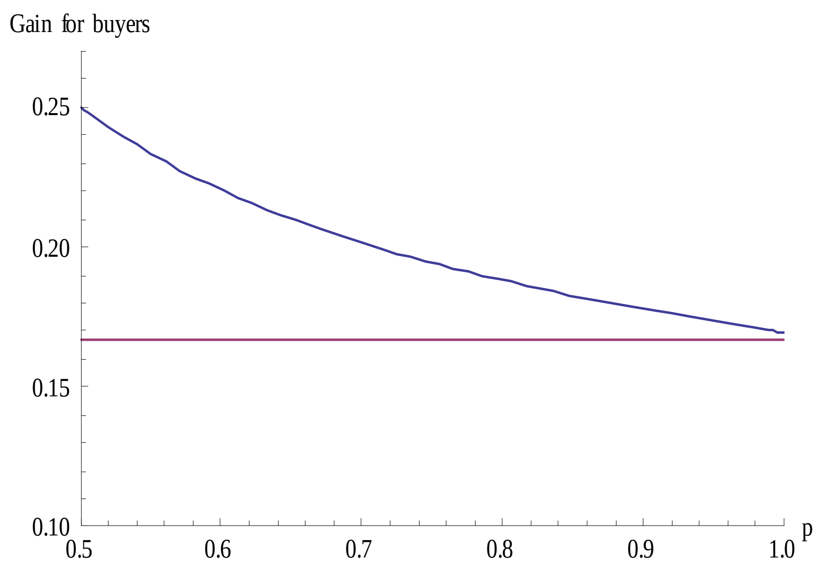
So z^* is strictly positive. Besides, compared to the case where no information about rank is provided, and where $a^* = -\frac{\phi(0)}{\phi'(0)}$, and since $y + \frac{\phi(y)}{\phi'(y)}$ is assumed to be increasing, we have $\underline{a} < a^*$, and $\bar{a} = \psi(\underline{a})$ follows immediately from the definition of V^1. QED

Similar analysis can be performed for the case where the signal is less perfect (see Appendix).

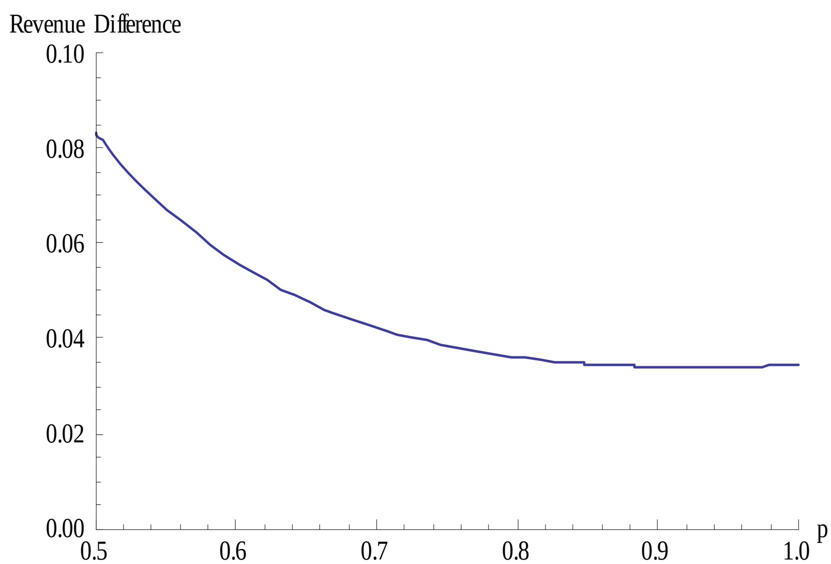
Assuming that the x_i are drawn from uniform distributions, we plot equilibrium shading as a function of the precision of the signals.



As the precision of the signals increases, bidding is thus more competitive, and this translates into lower gains for buyers as the following figure illustrates (the lower line represents the gain under a second price auction).



Under the distributional assumptions made, buyers' gains are thus reduced to those of the second price auction. Note however that the signal about rank induces inefficiencies, as there are events where $0 < x_1 - x_2 < z^*$ and $\theta_1 = 1$ and $\theta_2 = 0$, hence, events where bidder 1 has higher valuation and yet the object is allocated to bidder 2. As a result, the seller does not fully benefit from this increase in competition. The following figure shows the difference in revenues from a second and first price auction.



3 Noisy valuations

We extend our basic model to accommodate the possibility that bidders only have an imperfect estimate of their valuations.

As before we assume that

$$v_i = \alpha + x_i$$

where x_i are drawn independently from a distribution on $[\underline{x}, \bar{x}]$ (call f the density), and where α is drawn from a uniform distribution with large support. We assume that bidders do not observe their valuation v_i. Rather, they observe an estimate

$$y_i = v_i + \varepsilon_i$$

where ε_i are drawn independently (from one another and from the v_i). Again, possible strategies are assumed to be of the form

$$b_i = y_i - a$$

We look for a symmetric equilibrium of the first price auction. Denote by $\phi_\varepsilon(y)$ the probability that $x_i + \varepsilon_i$ exceeds $\max_{j \neq i} x_j + \varepsilon_j$ by more than y, that is,

$$\phi_\varepsilon(y) = \Pr(x_i + \varepsilon_i - \max_{j \neq i} x_j + \varepsilon_j > y).$$

Define

$$H(y) = E[\varepsilon_i \mid x_i + \varepsilon_i - \max_{j \neq i} x_j + \varepsilon_j > y]$$

We have the following Proposition:

> **Proposition 4**: *In a symmetric equilibrium, bidders shade their bid by*
>
> $$a^*_\varepsilon = H(0) + \frac{1 - H'(0)}{-n\phi'_\varepsilon(0)}.$$
>
> *The equilibrium payoff is:*
>
> $$V^* = \frac{1 - H'(0)}{-n^2\phi'_\varepsilon(0)}.$$

To prove proposition 4, it is convenient to denote by $h(\varepsilon_i, z)$ the joint distribution over ε_i and $z \equiv x_i + \varepsilon_i - \max_{j \neq i} x_j + \varepsilon_j$. Assume that all bidders except i shade their bids by a, while bidder i shades his bid by $a + y$. Denote by $V(y, a)$ the expected payoff that player i derives in that case. When player i wins, he obtains a payoff equal to $\alpha + x_i - (\alpha + x_i + \varepsilon_i - a - y) = a + y - \varepsilon_i$, so we have:

$$V(y, a) = \int_{z \geq y} (a + y - \varepsilon_i) h(\varepsilon_i, z) d\varepsilon_i dz = \phi_\varepsilon(y)(a + y - H(y)).$$

The first order condition thus yields:

$$(a - H(0))\phi'_\varepsilon(0) + \phi_\varepsilon(0)(1 - H'(0)) = 0,$$

hence the proof of the Proposition.

Shading thus has two components. The term $H_\varepsilon(0)$ takes into account the fact that more optimistic bidders tend to win the auction (i.e., the winner's curse), and rational bidders should correct for that; the second term equals $a^* - H(0)$, and exactly corresponds to what a winner gets on average when he wins.

This second term equals $(\frac{1}{n})\frac{1}{-\phi'_\varepsilon(0)}(1 - H'(0))$, and the two first factors are analogous to those that appear in the no noise case: $(\frac{1}{n})$ corresponds to the expected probability of winning, and $\frac{1}{-\phi'_\varepsilon(0)}$ captures how bidders take advantage of the dispersion in valuations. This term is typically larger when noise is larger. Finally the last term results from the fact that when a bidder slightly increases shading, he typically reduces the winner's curse effect (with symmetric distributions for example, $H'(0) \geq 0$).

Illustrations and comments

(1) With ε taking two values, $H'(0) = 0$. So

$$a^*_\varepsilon = H(0) + \frac{1}{-n\phi'_\varepsilon(0)} \text{ and } V^* = \frac{1}{-n^2\phi'_\varepsilon(0)}.$$

Compared to the case with noise, bidders are thus unambiguously better off, as $\mid \phi'_\varepsilon(0) \mid$ decreases with noise. Intuitively, when estimates are noisier, dispersion of valuations increases and bidders can take advantage of that by shading their bids more.

That insight typically does not arise in the standard model, because in the standard model, noisier estimates translate into **less** dispersed expected valuations (by a regression to the mean effect). For example, in a standard independent value model with noisy estimate, $y_i = v_i + \varepsilon_i$, each bidder would compute $E(v_i \mid y_i)$ and these expectations have smaller support when the noise increases. As a result, noisier estimates induce less dispersed valuations, hence, (surprisingly) stronger competition.

The regression to the mean effect does not arise in our model because α has a flat density. The ε_i are estimation errors, and learning y_i tells nothing about the estimation error. In contrast, in the standard model, learning the estimate y_i would allow the agent, through knowledge of the distribution and Bayesian updating, to learn about the estimation error.

(2) Assume that ε may take two values, $\bar{\varepsilon}$ (with probability p) or $\underline{\varepsilon}$, so that bidders are either optimistic or pessimistic, and that $\bar{\varepsilon} - \underline{\varepsilon} > \Delta = \bar{x} - \underline{x}$. The assumption $\bar{\varepsilon} - \underline{\varepsilon} > \Delta$ implies that in a symmetric equilibrium, bidder i may only win when he is optimistic ($\varepsilon_i = \bar{\varepsilon}$) or when all bidders are pessimistic ($\varepsilon_j = \underline{\varepsilon}$ for all j). Clearly $H'_\varepsilon(0) = 0$, so the equilibrium shading can be written:

$$a^* = E[\varepsilon \mid \varepsilon_i = \max_j \varepsilon_j] + \frac{1}{-n\phi'_\varepsilon(0)}.$$

We thus need to derive $\phi'_\varepsilon(0)$. To fix ideas we assume that x_i is distributed uniformly on $[\underline{x}, \bar{x}]$, so that for any given number of bidders n, $\phi'_n(0) = 1/\Delta$. Define $\widetilde{n}$ as the random variable that gives the number of bidders who have a chance to win in the event $(\varepsilon_i)_i$, that is:

$$\widetilde{n} = \#\{i, \varepsilon_i = \max_j \varepsilon_j\}$$

We have:

Proposition 5: *In equilibrium, bidders shade their bid by*

$$a^* = E[\varepsilon \mid \varepsilon_i = \max_j \varepsilon_j] + \Delta E \frac{1}{\widetilde{n}} \tag{3}$$

Besides, letting $\rho = (1-p)^n/p$, we have:

$$E[\varepsilon \mid \varepsilon_i = \max_j \varepsilon_j] = \bar{\varepsilon} - \frac{\rho}{1+\rho}(\bar{\varepsilon} - \underline{\varepsilon}) \textit{and } E\frac{1}{\widetilde{n}} = \frac{1}{pn(1+\rho)}$$

Intuitively, the first term corresponds to the expected "optimism" of the bidder conditional on winning, and that term gets close to $\bar{\varepsilon}$ when n increases.

The second term describes how bidders further shade their bids. Compared to the case without noise where they would shade by Δ/n, bidders shade more because they are facing less intense competition: because only optimistic bidders may win (except in the event all are pessimistic), a bidder is endogenously facing fewer competitors.

Proof: We need only compute $\phi'_\varepsilon(0)$. Observe that

$$\phi_\varepsilon(y) = p \sum_{n_0} \Pr(\widetilde{n} = n_0 \mid \varepsilon_i = \bar{\varepsilon})\phi_{n_0}(y) + (1-p)(1-p)^{n-1}\phi_n(y).$$

Thus, given that $\phi'_n(0) = -1/\Delta$ for all n, we have:

$$\begin{aligned} \phi'_\varepsilon(0) &= -[p \sum_{n_0} \Pr(\widetilde{n} = n_0 \mid \varepsilon_i = \bar{\varepsilon}) + (1-p)(1-p)^{n-1}]/\Delta \\ &= -(p + (1-p)(1-p)^{n-1})/\Delta. \end{aligned}$$

Now observe that by symmetry each bidder has a chance $1/n$ of winning, and that conditional on $\widetilde{n} = n_0$ and $\{\varepsilon_i = \max \varepsilon_j\}$, bidder i has a chance $\frac{1}{n_0}$ of winning. This implies

$$\frac{1}{n} = \sum_{n_0=1}^{n} \frac{1}{n_0}[p \Pr(\widetilde{n} = n_0 \mid \varepsilon_i = \bar{\varepsilon}) + (1-p)\frac{1}{n}(1-p)^{n-1}],$$

Consequently,

$$\begin{aligned} \frac{1}{-n\phi'_\varepsilon(0)} &= \Delta \frac{\sum_{n_0=1}^{n} \frac{1}{n_0} p \Pr(\widetilde{n} = n_0 \mid \varepsilon_i = \bar{\varepsilon})] + (1-p)\frac{1}{n}(1-p)^{n-1}]}{\sum_{n_0=1}^{n} p \Pr(\widetilde{n} = n_0 \mid \varepsilon_i = \bar{\varepsilon}) + (1-p)(1-p)^{n-1}} \\ &= \Delta E[\frac{1}{\widetilde{n}} \mid \varepsilon_i = \max \varepsilon_j] = \Delta E[\frac{1}{\widetilde{n}}]. \qquad QED \end{aligned}$$

(3) Winning conveys information about $x_i + \varepsilon_i$, hence about the estimation error ε_i and the common value element α. The bid adjustment $H(0)$ however reflects only the extent of learning about the estimation error ε_i (rather than learning about the common value element α).

4 The Buyer/seller case

We consider a seller with a value v_1 for the object to be sold, and a buyer with a value v_2 for the object. As before, we assume that

$$v_i = \alpha + x_i.$$

We are interested in understanding which player gains from making a take-it-or-leave offer to the other player, and compare these selling mechanisms to the split-the-difference mechanism.

In what follows, we define

$$\phi(y) = \Pr\{x_2 - x_1 \geq y\}$$

and let $S(y)$ denote the expected surplus that results when transactions take place if and only if $x_2 > x_1 + y$, that is:

$$S(y) = \int_{z>y} -z\phi'(z)dz = \phi(y)E[x_2 - x_1 \mid x_2 - x_1 > y].$$

When the seller makes an offer equal to $p = v_1 + a_1$, the buyer accepts iff $x_2 - x_1 \geq a$, hence he obtains an expected payoff equal to

$$G_S = v_1 + a_1\phi(a_1).$$

When the buyer makes an offer $p = v_2 - a_2$, the seller accepts if $p \geq v_1$, that is, if $x_2 - a_2 \geq x_1$, hence he obtains an expected payoff equal to

$$G_B = a_1\phi(a_1).$$

The optimal values of a_1 and a_2 are thus the same, and we call this value $a^* = \arg\max a\phi(a)$, and denote by G_S^* and G_B^* the corresponding gains for the seller and the buyer. Note that the expected surplus to be shared is the same whether the seller or the buyer makes the offer, and it is equal to $S(a^*)$. Who makes the offer thus only affects how the expected surplus is shared.

To see how the expected surplus $S(a^*)$ is shared, observe that when the buyer makes the offer, the seller obtains:

$$\begin{aligned} R_S &= v_1 + E[\max(x_2 - x_1 - a^*, 0)] \\ &= v_1 + S(a^*) - a^*\phi(a^*). \end{aligned}$$

So the seller prefers to make the offer when $G_S^* > R_S$, that is when:

$$S(a^*) < 2a^*\phi(a^*).$$

Since $S(y) = \int_{z>y} -z\phi'(z)dz = y\phi(y) + \int_{z>y} \phi(z)dz$, we obtain the following Proposition:

Proposition: Let $a^* = \arg\max y\phi(y)$. *The seller prefers to let the buyer make the offer if and only if*

$$\int_{y>a^*} \phi(y)dy > a^*\phi(a^*)$$

Examples:

(1) If the x_i's are drawn independently from the uniform distribution on $[0,1]$, then for $z>0$, $\phi(y) = (1-y)^2/2$. The player who makes an offer chooses a^* that maximizes $\max y\phi(y)$, hence $a^* = 1/3$, which yields an expected gain of $2/27 = 6/81$. In contrast, the expected surplus to be shared is $S(a^*) = 10/81$, so the seller prefers to make the offer.

(2) Define $h(z) = -\phi'(z)$. h corresponds to the density over $z = x_2 - x_1$. Assume that $h(z) = \frac{k}{z^2}$ over the interval $[1,K]$. Set $k = 1/(1-1/K)$ so that $\int_1^K h(y) = 1$. We have $\phi(1) = 1$, and for any $a > 1$,

$$\begin{aligned} a\phi(a) &= a\int_a^K h(z)dz = a(\frac{k}{a} - \frac{k}{K}) = k(1-1/K) - (a-1)k/K \\ &= 1-(a-1)k/K < 1. \end{aligned}$$

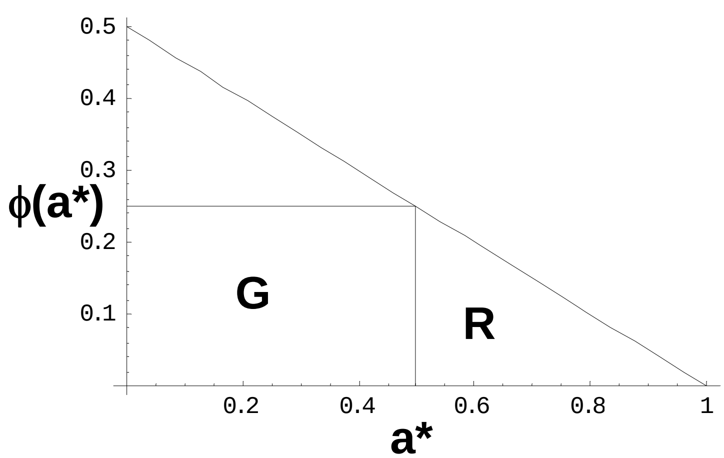
It is thus optimal for the seller to set $a^* = 1$. Now observe that when $a^* = 1$,

$$S(a^*) = \int_1^K zh(z)dz = k\ln K = \frac{\ln K}{1-1/K}$$

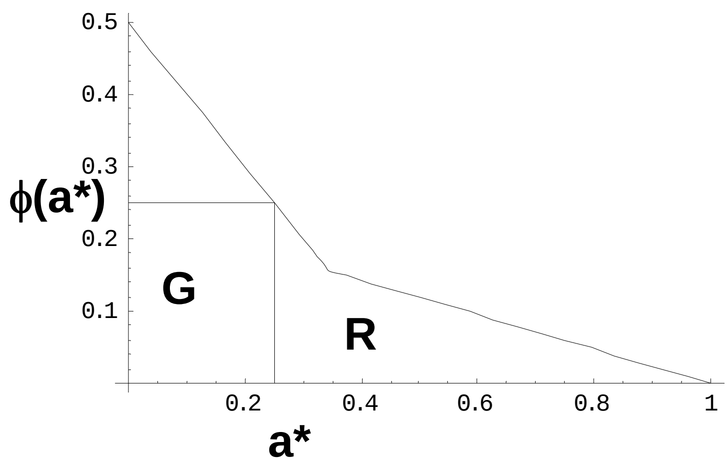
which is larger than $2a^*\phi(a^*) = 2$ when K is sufficiently large ($K \geq 4$,or so...). So the seller obtains less than half the surplus if he makes the offer, while he gets more than half the surplus if he lets the buyer make an offer.

(3) The following figures compare the gain G obtained by the seller when he makes the offer to the revenue R that he obtains when he let the buyer make the offer under three different assumptions about how $y = x_2 - x_1$ is distributed.

(a) y is distributed uniformly over $[-1,1]$: $G > R$



(b) y is either distributed uniformly over $[-1/3, 1/3]$ or uniform over $[-1, 1]$. The figure below has been drawn for the case $p < 1/2$ with p close to $1/2$. For $p > 1/2$, $a^* = 1/2$ and $G > R$ (as in example (a) above). For p very small, we are also in a situation analogous to the uniform distribution examined above (with $a^* = 1/6$). As p increases, R increases, and for p not too small and yet below $1/2$, $G < R$ because the seller still finds it optimal to pick a relatively small value of a (a^* is close to $1/4$ when p is close to $1/2$). An alternative good candidate for optimal a would be to choose $a = 1/2$, but given that $1-p > 1/2$, the seller finds it too risky to take a chance to miss the sale.



Comparison with the split the difference mechanism.

Under the split the difference mechanism (see Chatterjee and Samuelson (1983)), the buyer and seller simultaneously offer respectively prices p_1 and p_2. In the event $p_2 - p_1 \geq 0$, the transaction takes place at price $p = (p_1 + p_2)/2$, otherwise it does not take place.

We assume that the seller chooses a_1 and offers a price $p_1 = v_1 + a_1$, while the buyer chooses a_2 and offers a price $p_2 = v_2 - a_2$. The transaction takes place in the event $p_2 - p_1 \geq 0$, that is in the event $x_2 - x_1 \geq a_1 + a_2$, so the expected gain of the seller writes as:

$$\int_{y \geq a_1 + a_2} \frac{y + a_1 - a_2}{2} \phi(y) dy.$$

and similarly, the expected gain for the buyer can be written

$$\int_{y \geq a_1 + a_2} \frac{y + a_2 - a_1}{2} \phi(y) dy.$$

Let a^* as defined earlier. We verify that $a_1^* = a_2^* = a^*/2$ is an equilibrium.

Assume 1 chooses $b_1 = a^*/2 + \delta$. Then he obtains a payoff $H(\delta)$ equal to:

$$H(\delta) = \frac{1}{2}\int_{y \geq a^*+\delta}(y+\delta)\phi(y)dy = \frac{1}{2}(\int_{y \geq a^*+\delta}(y-a^*)\phi(y)dy + G(a^*+\delta)).$$

Each of the terms on the right hand side is maximum for $\delta = 0$. So $a_1^* = a_2^* = a^*/2$ is an equilibrium.

In other words, that mechanism generates the same surplus as the two others. Among these three mechanisms, **and whether one considers the buyer's perspective or the seller's perspective,** splitting the difference cannot the most preferred mechanism**.**

5 Auctions with asymmetries

Our objective here is to illustrate that our basic model delivers reasonable insights in asymmetric contexts. We consider various forms of asymmetries. We mostly restrict attention to the two bidder case, though similar analysis could be performed with many bidders.

5.1 Asymmetries in valuations.

We consider a setting where bidder 1 has a value advantage over the other bidder. We assume that

$$v_i = \alpha + x_i + \gamma_i$$

where the x_i's are drawn independently from a distribution on $[\underline{x}, \bar{x}]$ (let f denote the density), where α is drawn from a uniform distribution with large support. We also assume that $\gamma_1 \geq \gamma_2 = 0$. As before, possible strategies are assumed to be of the form

$$b_i = y_i - a_i.$$

We look for an equilibrium of the first price auction in pure strategies where bidder i chooses a_i. The function $\phi(y)$ is defined as before as the probability that x_i exceeds $y + \max_{j \neq i} x_j$. Player 1 wins when $x_1 + \gamma - a_1 \geq x_2 - a_2$, so we have:

$$V_1(a_1, a_2) = a_1\phi(a_1 - a_2 - \gamma)$$

and

$$V_2(a_1, a_2) = a_2(1 - \phi(a_1 - a_2 - \gamma)).$$

Denote by z the difference $a_1 - a_2$. We have:

Proposition: *In equilibrium* $z = a_1 - a_2 \in (0, \gamma)$. *Besides,* $\frac{dz}{d\gamma}\mid_{\gamma=0} = \frac{2}{3}$, *and* $da_1/d\gamma = 1/3$.

As one expects, the bidder with a value advantage shades his bid more than his opponent.

Proof: The two first order conditions can be written as

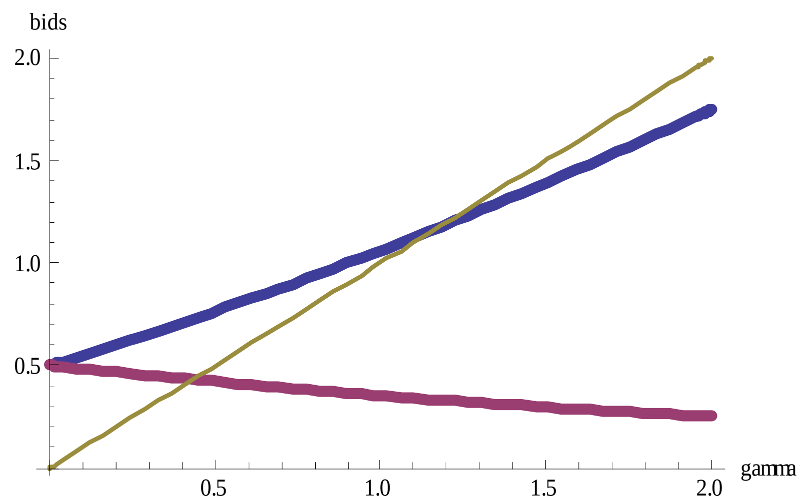
$$\begin{aligned} -a_1\phi'(z-\gamma) &= \phi(z-\gamma) \\ -a_2\phi'(z-\gamma) &= 1-\phi(z-\gamma) \end{aligned} \tag{4}$$

which implies:

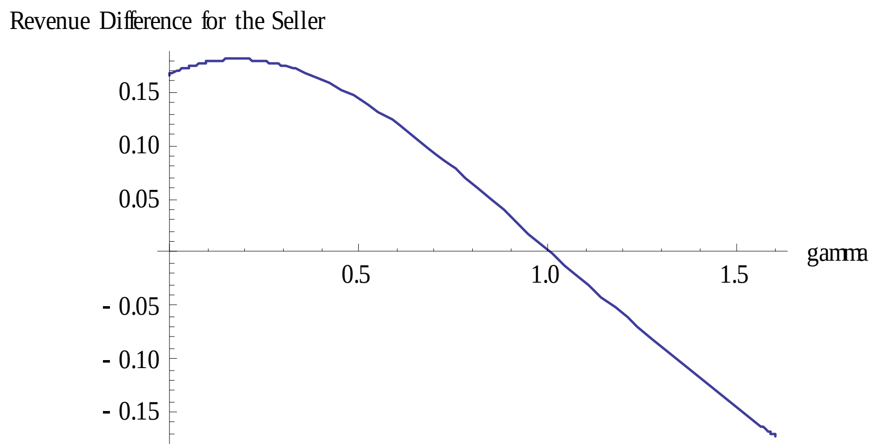
$$z = \frac{2\phi(z-\gamma)-1}{-\phi'(z-\gamma)}. \tag{5}$$

Since $\phi(0) = 1/2$ and $\phi' < 0$, we immediately obtain that $z \in (0, \gamma)$ when $\gamma > 0$. Besides, since $\phi(z) = 1/2 - \phi(-z)$, we have $\phi''(0) = 0$ so, differentiating (5) and (4) yields $dz/d\gamma = 2/3$. QED

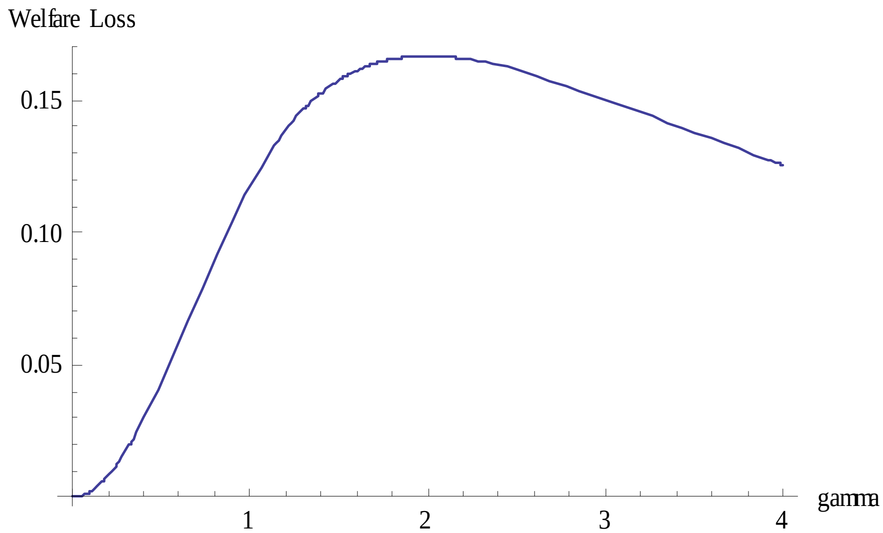
We plot below equilibrium shading as a function of γ for each player in case the x_i are drawn from uniform distributions on $[0, 1]$.



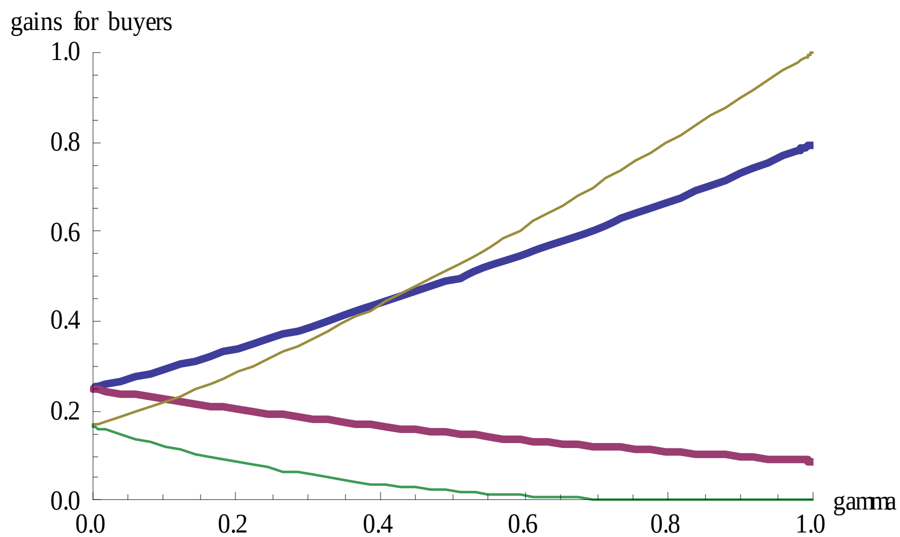
When $\gamma = 0$, both bidders shade by 0.5. As γ increases, bidder 1 gets more cautious as he fears losing against bidder 2 would have a good draw: at the limit when γ becomes very large, he shades by an amount close to $\gamma - 1$ and wins with probability close to 1. This cautiousness explains why the first price auction generates more revenues to the seller when γ gets large, as the following figure (which plots the difference in revenues between the second and first price auctions) confirms:



Since shading differs across bidders, inefficiencies result, though these must disappear at the limit where γ is very large. The following figure plots the welfare loss as a function of γ:



Finally, we show below the extent to which bidders win or lose between first and second price auctions (thick lines are for the first price auction):



So the weaker player always prefers the first price auction, while the stronger prefers the first price only if the asymmetry is not too large.

5.2 Asymmetries in information about rank.

We investigate a case where one player, say player 1, has information about his rank (i.e., whether $x_1 > x_2$) while the other has no such information. Our candidate equilibrium σ thus consists of a triplet $\sigma = (\bar{a}, \underline{a}, a^{**})$ where $(\bar{a}, \underline{a})$ stands for the strategy of player 1 as a function of his information and a^{**} the strategy of player 2.

We show the following:

> **Proposition:** *In equilibrium,* $\underline{a} < a^{**} < \bar{a} < a^*$*, where* a^* *is the equilibrium value of shading when no information about rank is available.*

Intuitively, bidder 1 can exploit his information, which reduces bidder 2's chance to win the object. This induces bidder 2 to bid more aggressively, which in equilibrium makes bidder 1 more aggressive whether he gets good or bad news about ranking.

Proof: As in Section 2, we define:

$$\bar{\phi}(y) = \Pr\{x_i > x_j + y \mid x_i > x_j\} \text{ and } \underline{\phi}(y) = \Pr\{x_i > x_j + y \mid x_i < x_j\}$$

Recall that $\bar{\phi}(y) = 2\phi(y)$ for $y \geq 0$, $\bar{\phi}(y) = 1$ for $y \leq 0$, and by symmetry $\underline{\phi}(y) = 1 - \phi(-y)$.

Our candidate equilibrium σ consists of a triplet $\sigma = (\bar{a}, \underline{a}, a^{**})$. For bidder 1, the value from bidding a in the event $x_1 > x_2$ is

$$V^1(a, \sigma) = a\bar{\phi}(a - a^{**})$$

and the value from bidding a in the event $x_1 < x_2$ is

$$V^0(a, \sigma) = a\underline{\phi}(a - a^{**}).$$

Since $\bar{\phi}(0) = 1$ and $\underline{\phi}(0) = 0$, we must have $\bar{a} \geq a^{**}$ and $\underline{a} < a^{**}$. Defining $z_1 = \bar{a} - a^{**}$ and $z_2 = a^{**} - \underline{a}$, the first order conditions for bidder 1 are:

$$\bar{a} = \frac{\phi(z_1)}{-\phi'(z_1)} \text{ and } \underline{a} = \frac{1 - 2\phi(z_2)}{-2\phi'(z_2)}. \tag{6}$$

Consider now bidder 2. We wish to show that bidder 2 must choose $a^{**} < \bar{a}$. The value from bidding a is:[6]

$$V^2(a, \sigma) = a(\frac{1}{2}\bar{\phi}(a - \underline{a}) + \frac{1}{2}\underline{\phi}(a - \bar{a})),$$

which gives the following first order condition:

$$a^{**} = \frac{1 + 2\phi(z_2) - 2\phi(z_1)}{-2(\phi'(z_2) + \phi'(z_1))}. \tag{7}$$

Assume by contradiction that $z_1 = 0$, that is, $a^{**} = \bar{a}$. Equation (7) becomes

$$a^{**} = \frac{\phi(z_2)}{-(\phi'(z_2) + \phi'(0))} < \frac{\phi(0)}{-\phi'(0)} = a^*.$$

But then from (6), we must have $\bar{a} = \frac{\phi(0)}{-\phi'(0)} = a^*$, contradiction. QED

5.3 Asymmetries in precision of information.

We investigate a case where one player, say player 1, would be perfectly informed of his valuation, while the other would only have an imperfect estimate of his valuation. So player 1 has an informational advantage over player 2, and we wish to examine how this informational advantage affects the bidding strategy.

Formally, we assume as before that

$$v_i = \alpha + x_i,$$

where the x_i's are drawn from the same distribution, but we also assume that player 1 observes v_1 while player 2 only observes an imperfect estimate y_2 of v_2, that is

$$y_2 = v_2 + \varepsilon_2$$

where ε_i is assumed to be symmetric and centered around 0.

Player 1 chooses a bid $b_1 = v_1 - a_1$. Player 2 chooses a bid $b_2 = v_2 - a_2$. As before, we define ϕ_ε as

$$\phi_\varepsilon(y) = \Pr\{x_1 - x_2 \geq \varepsilon_2 + y\}$$

[6] Note that in the event say $\{x_2 < x_1\}$, which has probability 1/2, player 2 knows that player 1 has received that information and thus shades by $\bar{a}$.

and

$$H(y) = E[\varepsilon_2 \mid x_2 + \varepsilon_2 > x_1 + y].$$

Player 1's expected gain from the strategy profile (a_1, a_2) is

$$V_1(a_1, a_2) = a_1 \phi_\varepsilon(a_1 - a_2).$$

Player 2's expected gain can be written as

$$V_2(a_1, a_2) = (1 - \phi_\varepsilon(a_1 - a_2))(a_2 - H(a_2 - a_1)).$$

Let $z = a_2 - a_1$, we obtain the following first order conditions:

$$\begin{aligned} a_2 - H(z) &= \frac{1 - \phi_\varepsilon(-z)}{-\phi'_\varepsilon(-z)}(1 - H'(z)) \\ a_1 &= \frac{\phi_\varepsilon(-z)}{-\phi'_\varepsilon(-z)}. \end{aligned}$$

Note that by symmetry, $H'(0) = 0$ and $\phi_\varepsilon(0) = 1/2$, while $H(0) > 0$, so in equilibrium, we must have $z > 0$. Compared to the case without noise, there are two essential differences. First player 2 has to compensate for the fact that he tends to win when he is optimistic, hence the term $H(z)$. Then, players exploit the dispersion of bids. Since this dispersion is higher with noise, the resulting bids are shaded above the winner's curse effect by an amount that may be even greater than that which would obtain under no noise, as the following examples illustrate.

Illustrations.

(1) x_i is uniform on $[0, 1]$, while ε is uniform on $[-\bar{\varepsilon}, \bar{\varepsilon}]$. TO BE ADDED.

(2) x_i is uniform on $[0, 1]$ and ε_2 may take two values, $\bar{\varepsilon}$ (with probability 1/2) or $\underline{\varepsilon} = -\bar{\varepsilon}$ and we assume that $\bar{\varepsilon} > 1$. Without noise, shading would be equal to $a^* = \frac{1}{2}$ for both bidders. We show below that

$$a_2 = \bar{\varepsilon} + 7/4 \text{ and } a_1 = 9/4.$$

So, bidder 2 shades his bid by an amount that not only compensate for the winner's curse, but goes well beyond the shading a^* that would obtain under no noise, shading which player 1 exploits. Competition is weak because it is only probability 1/2 that player 1 faces a reasonably strong opponent (the optimistic bidder 2).

To compute the equilibrium, observe first that for any $z \geq 0$ bidder 2 will only win in the event $\varepsilon_2 = \bar{\varepsilon}$, so $H(z) = \bar{\varepsilon}$, $H'(z) = 0$, and $\phi_\varepsilon(y) = \frac{1}{2} + \frac{1}{2}\phi(\bar{\varepsilon} + a_1 - a_2)$. Let $\widehat{a}_2 = a_2 - \bar{\varepsilon}$, and $\widehat{z} = a_1 - \widehat{a}_2$. The first order conditions give:

$$a_1 = \frac{1 + \phi(\widehat{z})}{-\phi'(\widehat{z})} \text{ and } \widehat{a}_2 = \frac{1 - \phi(\widehat{z})}{-\phi'(\widehat{z})},$$

from which we conclude that $\widehat{z} > 0$. Since x_i is uniform, $\phi(z) = (1 - z)^2/2$ and we obtain $\widehat{z} = -\frac{1}{2}$, hence $a_1 = \frac{1+1/8}{1/2} = 9/4$, $a_2 = \bar{\varepsilon} + 7/4$. QED

6 Sequential Auctions

Our aim here is to show that our basic model can be used to deal with issues that are in general difficult to handle in the standard model. We consider two heterogenous objects I and II sold sequentially, and assume that there are n bidders each having unitary demands. For object I, preferences are given as before by:

$$v_i^I = \alpha^I + x_i$$

For object II, preferences are given by

$$v_i^{II} = \alpha^{II} + y_i$$

where α^{II} and y_i are drawn independently of α^I and x_i, and where α^I,α^{II} have flat support.

This is in general a difficult problem because bidders information in two dimensional: their preferences are described by a vector (v_i^I, v_i^{II}).

We look for a symmetric equilibrium with bid functions of the form $b_i^I = v_i^I - a^I$ and $b_i^{II} = v_i^{II} - a^{II}$. We are interested in understanding equilibrium bids, as well as which object the seller should sell first, (or whether he should ask for simultaneous bids and maintain uncertainty as to which object he will allocate first).

In what follows, we define as before $\phi_n(y) = \Pr\{x_i \geq \max_{j \neq i} x_j + y\}$ and $\psi_n(y) = \Pr\{y_i \geq \max_{j \neq i} y_j + y\}$. It will also be convenient to let $\beta_n = -\phi_n'(0)$ and $\gamma_n = -\psi_n'(0)$. The parameters β_n and γ_n characterize the intensity of competition in each auction.

When the second object is sold, there are $n-1$ bidders left, hence following the analysis of the basic model, bidders shade their bid by

$$a^{II} = \frac{1}{(n-1)\gamma_{n-1}}.$$

Consider now the first object. If say bidder i does not get the first object, he still has a chance to get the second one (with probability $1/n-1$) and gain $V_*^{II} = a^{II}/(n-1)$. In the first auction, the value of bidding $a+y$ when others bid a is thus:

$$V(y, a) = (a+y)(\phi_n(y)) + (1 - \phi_n(y))V_*^{II}$$

thus yielding

$$\begin{aligned} a^I &= \frac{1}{n\beta_n} + V_*^{II} \\ &= \frac{1}{n\beta_n} + \frac{1}{(n-1)^2\gamma_{n-1}}. \end{aligned}$$

As expected, bidding in the first auction takes into account the option value of losing the first object and possibly getting the second.

Revenue comparisons.

For each bidder, the equilibrium expected payoff is:

$$\begin{aligned} G_{I,II} &= \frac{1}{n}a^I + (1-\frac{1}{n})\frac{a^{II}}{n-1} = \frac{a^I + a^{II}}{n} \\ &= \frac{1}{n^2\beta_n} + \frac{1}{(n-1)^2\gamma_{n-1}}. \end{aligned}$$

This expression can be used to assess how bidders benefit or lose from changing the order in which goods are sold.

To illustrate, assume that the x_i and y_i are drawn from uniform distributions, so $\gamma_n = \gamma$ and $\beta_n = \beta$ for all n, but that the x_i are more dispersed, hence $\gamma > \beta$. Under this assumption

$$G_{I,II} = \frac{1}{n^2\beta} + \frac{1}{(n-1)^2\gamma} < \frac{1}{n^2\gamma} + \frac{1}{(n-1)^2\beta} = G_{II,I}$$

so the bidders are worse off when the good with more dispersed valuation is sold first.

For the seller, one also has to check the welfare changes induced by the change in the order.

Define

$$\Delta_n^I = E_n \max_i x_i - E_{n-1} \max_i x_i$$

as the welfare gain from one additional bidder in the first auction. Starting by selling object I is better for welfare if and only if

$$\Delta_n^I > \Delta_n^{II}.$$

In the case of the uniform distribution examined above, the more dispersed distribution induces a higher welfare gain. So in that case,

$$\Delta_n^I - \Delta_n^{II} > 0 > G_{I,II} - G_{II,I};$$

there is thus no conflict between revenue and welfare maximization.

For general distributions however, there may exist a conflict between welfare maximization and revenue maximization. We give an illustration below, driven by the fact that the extent of competition among bidders has to do with how local properties of the functions ϕ and ψ, while the welfare differences do not depend on such local properties.

Example: To BE ADDED

References

TO BE COMPLETED

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Appendix

A1. Noisy signal about rank.

For bidder 1, the value from bidding a in event $\theta_i = 0$ is:

$$V^1(a,\sigma) = a[p^2\bar{\phi}(a-\underline{a}) + p(1-p)\bar{\phi}(a-\bar{a}) + p(1-p)\underline{\phi}(a-\bar{a}) + (1-p)^2\underline{\phi}(a-\underline{a})],$$

and the value from bidding a in event $\theta_i = 0$ is:

$$V^0(a,\sigma) = a[p^2\underline{\phi}(a-\bar{a}) + p(1-p)\underline{\phi}(a-\underline{a}) + (1-p)p\bar{\phi}(a-\underline{a}) + (1-p)^2\bar{\phi}(a-\bar{a}))].$$

Defining $z = \bar{a} - \underline{a}$, the first order conditions become:

$$\begin{aligned} \bar{a} &= \frac{2p^2\phi(z) + p(1-p)}{-[2p^2\phi'(z) + 2p(1-p)\phi'(0)]} \text{ and} \\ \underline{a} &= \frac{p^2(1-2\phi(z)) + 1 - p}{-[2p^2\phi'(z) + 2(1-p)p\phi'(0)]} \end{aligned}$$

The equilibrium difference $z^* = \bar{a} - \underline{a}$ thus solves:

$$z^* = \frac{4p^2\phi(z^*) - (p^2 + (1-p)^2)}{-[2p^2\phi'(z^*) + 2(1-p)p\phi'(0)]}$$