Delay and Information Aggregation in Stopping Games with Private Information

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Abstract

We consider a timing game with private information about a common values payoff parameter. Information is only transmitted through the stopping decisions and therefore the model is one of observational learning. We characterize the symmetric equilibria of the game and we show that even in large games where pooled information is sufficiently accurate for first best decisions, aggregate randomness in outcomes persists. Furthermore, the best symmetric equilibrium induces delay relative to the first best.

1 Introduction

We analyze a game of timing where the players are privately informed about the optimal time to stop the game. The stopping decision may, for example, relate to irreversible investment, which is the case analyzed in the real options literature. Our point of departure from that literature is in the nature of uncertainty. Rather than assuming exogenous uncertainty in a publicly observable payoff parameter such as the market price, we consider the case of dispersed private information on the common profitability of the investment. We assume that information is only transmitted through observed actions. In other words, our model is one of observational learning, where communication between players is not allowed.

The key question in our paper is how the individual players balance the benefits from observing other players' actions with the costs of delaying their stopping decision beyond

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what is optimal based on their own private information. Observational learning is potentially socially valuable, because it allows common values information to spread across players. However, when choosing their optimal timing decisions, the players disregard the informational benefits that their decisions have for the other players. This informational externality leads to too late stopping decisions from the perspective of effective information transmission, and this delay dissipates most of the potential informational benefit from the observational learning. Our main findings are: i) The most informative symmetric equilibrium of the game involves delay, ii) the delay persists even if the number of players is large, iii) information aggregates in random bursts of action, and iv) aggregate uncertainty remains even when aggregate information in the model is sufficiently accurate to determine the optimal investment time.

In our model, the first-best time to invest is common to all players and depends on a single state variable ω. Without loss of generality, we identify ω directly as the first-best optimal time to invest. Since all players have information on ω, their observed actions contain valuable information as long as the actions depend on the players' private information.

The informational setting of the game is otherwise standard for social learning models: The players' private signals are assumed to be conditionally i.i.d. given ω and to satisfy the monotone likelihood property. The payoffs are assumed to be quasi-supermodular in ω and the stopping time t. Given these assumptions, the equilibria in our game are in monotone strategies such that a higher signal implies a later investment decision. Our main characterization result describes a simple way to calculate the optimal stopping moment for each player in the most informative symmetric equilibrium of the game. The optimal investment time is exactly the optimal moment calculated based on the knowledge that other (active) players are not stopping. The game has also less informative equilibria, where all the players, irrespective of their signals, stop immediately because other players stop as well. These equilibria bear some resemblance to the non-informative equilibria in voting games with common values as in Feddersen & Pesendorfer (1997), and also herding equilibria in the literature on observational learning as in Smith & Sorensen (2000).

In order to avoid complicated limiting procedures, we model the stopping game directly as a continuous-time model with multiple stages. Each stage is to be understood as the time interval between two consecutive stopping actions. At the beginning of each stage, all remaining players choose the time to stop, and the stage ends at the first of these stopping times. The stopping time and the identity of the player(s) to stop are publicly observed, and the remaining players update their beliefs with this new information and start immediately the next stage. This gives us a dynamic recursive game with finitely many stages (since the number of players is finite). Since the stage game strategies are simply functions from the type space to non-negative real numbers, the game and its payoffs are well defined.

The most informative equilibrium path involves two qualitatively different phases. When a stage lasts for a strictly positive amount of time, we say that the game is in the *waiting phase*. Since the equilibrium strategies are monotonic in signals, the fact that no players are currently stopping implies that their signals must be above some cutoff level. This in turn implies that it is more likely that the true state is higher, i.e. the first-best optimal stopping time is later. Thus, during the waiting phase all players update their beliefs gradually upwards. Eventually the waiting phase comes to an end as some player stops. At that point, the remaining players learn that the signal of the stopping player is the lowest possible consistent with equilibrium play, and by monotone likelihood ratio property they update their belief about the state discretely downwards. As a result, a positive measure of player types will find it optimal to stop immediately. If such players exist, the following stage ends at time zero, and the game moves immediately to the next stage, where again a positive measure of types stop at time zero. As long as there are consecutive stages that end at time zero, we say that the game is in the *stopping phase*. This phase ends when the game reaches a stage where no player stops immediately. The game alternates between these two phases until all players have stopped. Notice that information accumulates in an asymmetric manner. Positive information (low signals indicating early optimal action) arrives in quick bursts, while pessimistic information indicating higher signals and the need to delay accumulates gradually.

To understand the source of delay in our model, it is useful to point out an inherent asymmetry in learning in stopping games. That is, while the players can always revise their stopping decisions forward in time in response to new information, they can not go backward in time if they learn to be too late. In equilibrium, every player stops at the optimal time based on her information at the time of stopping. As a consequence, if at *any* moment during the game the current estimate of the stopping player is too high in comparison to the true state realization, then all the remaining players end up stopping too late. In contrast, errors in the direction of too early stopping times tend to be corrected as new information becomes available.

We obtain the sharpest results for games with a large number of players. First, in the large game limit, almost all the players stop too late relative to the first-best stopping time (except in the case where the state is the highest possible and the first-best stopping time is the last admissible stopping time). The intuition for this result is straight-forward. With a large number of players the pooled information contained in the players' signals is precise. If a non-negligible fraction of players were to stop too early, this would reveal the true state. But then it would be optimal for all players to delay, and this would contradict the presumption of too early stopping. Second, we show that almost all players stop at the same instant of real time (even though they may stop in different stages) where the game also ends. This is because in the informative equilibrium, all observed stopping decisions are informative. With a large number of players, most of the players thus have precise information about state when they stop. But as explained above, information can not aggregate before first-best stopping time, which means that players become aware of the true state too late. This leads to a collapse where all the remaining players stop together fully aware of being too late. Finally, we show that even if we condition on the true state, the time at which the players stop remains stochastic.

Our paper is related to the literature on herding. The paper closest to ours is the model of entry by Chamley & Gale (1994).[1] The main difference to our paper is that in that model it is either optimal to invest immediately or never. We allow a more general payoff structure that allows the state of nature to determine the optimal timing to invest, but which also captures Chamley & Gale (1994) as a special case. This turns out to be important for the model properties. With the payoff structure used in Chamley & Gale (1994), uncertainty is resolved immediately but incompletely at the start of the game. In contrast, our model features gradual information aggregation over time. The information revelation in our model is closely related to our previous paper Murto & Välimäki (2009). In that paper learning over time generates dispersed information about the optimal stopping point, and information aggregates in sudden bursts of action. Moscarini & Squintani (2008) analyze a R&D race, where the inference of common values information is similar to our model, but as their focus is on the interplay between informational and payoff externalities, they have only two players. Our focus, in contrast, is on the aggregation of information that is dispersed within a potentially large population.

It is also instructive to contrast the information aggregation results in our context with those in the auctions literature. In a k^{th} price auction with common values, Pesendorfer & Swinkels (1997) show that information aggregates efficiently as the number of object grows with the number of bidders. Kremer (2002) further analyzes informational properties of large common values auctions of various forms. In our model, in contrast, the only link between the players is through the informational externality, and that is not enough to eliminate the inefficiencies. The persistent delay in our model indicates failure of information aggregation even for large economies. On the other hand, Bulow & Klemperer

[1] See also Levin & J.Peck (2008), which extends such a model to allow private information on opportunity costs.

(1994) analyzes an auction model that features "frenzies" that resemble our bursts of actions. In Bulow & Klemperer (1994) those are generated by direct payoff externalities arising from scarcity, while in our case they are purely informational.

The paper is structured as follows. Section 2 introduces the basic model. Section 3 establishes the existence of a symmetric equilibrium. Section 4 discusses the properties of the game with a large number of players. In section 5 we illustrates the model by Monte-Carlo simulations. Section 6 concludes with a comparison of our results to the most closely related literature.

2 Model

2.1 Payoffs and signals

N players consider investing in a project. The payoff from an investment at time t_i of each firm depends on the state ω:

$$v(t_i, \omega)$$

For simplicity, we may take

$$\omega = \arg\max_t v(t, \omega).$$

The players share a common prior $p^0(\omega)$ on Ω.

We assume that the payoff function v is quasi-supermodular in t_i, ω:

Assumption 1

$$v_i(t_i, \omega) - v_i(t_i', \omega)$$

is strictly single crossing in ω and

$$v_i(t_i, \omega') - v_i(t_i, \omega)$$

is strictly single crossing in t_i.

Because we allow the players to stop at infinity, we use the topology generated by the one-point compactification of $\mathbb{R}_+ \cup \infty$. We assume throughout that $v(t_i, \omega)$ is continuous in t_i in this topology. [2] Under this assumption, individual stopping problems have maximal solutions.

Players are initially privately informed about ω. Player i observes privately a signal θ_i from a joint distribution $G(\theta, \omega)$ on $[\underline{\theta}, \overline{\theta}] \times \Omega$. We assume that the distribution is

[2]This assumption holds e.g. under bounded payoff functions and discounting.

symmetric across i, and that signals are conditionally i.i.d. Furthermore, we assume that the conditional distributions $G(\theta \mid \omega)$ and corresponding densities $g(\theta \mid \omega)$ are well defined and they have full supports $[\underline{\theta}, \overline{\theta}]$ independent of ω.

We assume that the signals satisfy monotone likelihood property (MLRP).

Assumption 2 *For all i, $\theta' > \theta$, and $\omega' > \omega$,*

$$\frac{g(\theta' \mid \omega')}{g(\theta \mid \omega')} > \frac{g(\theta' \mid \omega)}{g(\theta \mid \omega)}.$$

This assumption allows us to conclude that optimal individual stopping times for player i, $t_i(\theta)$ is monotonic in other players' types: For all j,

$$\frac{\partial t_i(\theta)}{\partial \theta_j} \geq 0.$$

Assumption 2 also guarantees that the pooled information in the game becomes arbitrarily revealing of the state as N is increased towards infinity.

Furthermore we make the assumption that the information content in individual signals is bounded.

Assumption 3 *There is a constant $\kappa > 0$ such that*

$$\forall \theta, \omega, \qquad \frac{1}{\kappa} > g(\theta, \omega) > \kappa.$$

Finally, we assume that signal densities are continuous in θ:

Assumption 4 *For all ω, $g(\theta \mid \omega)$ is continuous in θ within $[\underline{\theta}, \overline{\theta}]$.*

2.2 Strategies and information

The game has consists of a random number of stages with partially observable actions. In stage 0, all players choose their investment time $\tau_i^0(\theta_i) \in [0, \infty]$ depending on their signal θ. The stage ends at $t^0 = \min_i \tau_i^0$. At that point, the set of players $\mathcal{S}^0 = \{i : \tau_i^0(\theta_i) = t\}$ is announced, but the actions of the other players are not observed. The public history after stage 0 and at the beginning of stage 1 is then $h^1 = (t^0, \mathcal{S}^0)$. The vector of signals θ and the stage game strategy profile $\tau(\theta) = (\tau_1(\theta_1), ..., \tau_N(\theta_N))$ induce a probability distribution on the set of histories H^1.The public posterior on Ω (conditional on the public history only) at the end of stage 0 is given by Bayes' rule:

$$p^1\left(\omega \left| h^1 \right.\right) = \frac{p^0(\omega) \Pr\left(h^1 \left| \omega \right.\right)}{\int_\Omega p^0(\omega) \Pr\left(h^1 \left| \omega \right.\right)}.$$

In each stage k, players that have not yet invested choose an investment time τ_i^k. The public history available to the players is

$$h^k = h^{k-1} \cup \left(t^{k-1}, \mathcal{S}^{k-1}\right).$$

A strategy in the game is a callection of stage game stategies

$$\begin{aligned}
\tau_i \quad &: \quad = \{\tau_i^k\}_{k=0}^K \\
\tau_i^k \quad &: \quad H^k \times [\underline{\theta}, \overline{\theta}] \to \Delta[0, \infty].
\end{aligned}$$

A stretegy profile $\tau = (\tau_i, ..., \tau_N)$ is a Perfect Bayesian Equilibrium of the game if for all i and all θ_i and h^k, τ_i^k is a best response to $\tau_{-i.}$

It is useful to record the real time elapse at the start of stage k:

$$T^k = \sum_{i=0}^{k-1} t^k.$$

3 Monotone Symmetric Equilibrium

In this section, we analyze symmetric equilibria in monotone pure strategies.

Definition 5 *A strategy σ_i is monotonic if for all k, h^k and for all $\theta_i > \theta_i'$,*

$$\sigma_i^k \left(h^k, \theta_i\right) > \sigma_i^k \left(h^k, \theta_i'\right).$$

For each t^k let

$$\theta^k := \max\left\{\theta \left| \tau^k (\theta) = t^k \right.\right\}.$$

In any symmetric equilibrium in monotone pure strategies, public history h^k then implies that $\theta_j > \theta\left(t^{k-1}\right)$ for all players j that have not invested yet. Furthermore, if stage k has not been stopped before t , then each player knows that for all active players j,

$$\theta_j \geq \theta^k (t),$$

where $\theta^k (\cdot)$ is the inverse function of $\tau^k (\cdot)$, i.e.

$$\tau^k \left(\theta^k (t)\right) = t.$$

3.1 Informative Equilibrium

In this section, we find the symmetric equilibium that maximizes information transmission in the set of symmetric monotone pure strategy equilibria. In order to acheive the characterization, it will be useful to consider the following myopic stopping time:

$$\tau_*^k(\theta) := \min\left\{t \geq 0 \left| \mathbb{E}\left[v\left(t+T^k,\omega\right) \left| h^k, \theta^k(t) = \theta\right.\right]\right.\right.$$
$$\left.\geq \mathbb{E}\left[v\left(t'+T^k,\omega\right) \left| h^k, \theta^k(t) = \theta\right.\right] \text{ for all } t' \geq t\right\}. \tag{1}$$

Note that (1) allows $\tau_*^k(\theta) = \infty$. In words, $\tau_*^k(\theta)$ simply indicates the first time at which a player with with signal θ wants to stop the game given the information at the beginning of the stage h^k and given the information that for all active playesr j, $\theta_j \geq \theta$.The following Lemma states that $\tau_*^k(\theta)$ is increasing in θ (strictly so when $0 < \tau_*^k(\theta) < \infty$), and therefore defines a symmetric monotonic strategy profile.

Lemma 6 (Monotonicity of $\tau_*^k(\theta)$) *Let $\tau_*^k(\theta)$ denote the stopping strategy defined in (1).*

- *If $0 < \tau_*^k(\theta) < \infty$ for some $\theta \in (\underline{\theta}, \overline{\theta})$, then for all $\theta' \in [\underline{\theta}, \theta)$ and $\theta'' \in (\theta, \overline{\theta}]$, we have*
$$\tau_*^k(\theta') < \tau_*^k(\theta) < \tau_*^k(\theta'').$$
- *If $\tau_*^k(\theta) = 0$ for some $\theta \in (\underline{\theta}, \overline{\theta})$, then for all $\theta' \in [\underline{\theta}, \theta)$ we have $\tau_*^k(\theta') = 0$.*
- *If $\tau_*^k(\theta) = \infty$ or some $\theta \in (\underline{\theta}, \overline{\theta})$, then for all $\theta'' \in (\theta, \overline{\theta}]$ we have $\tau_*^k(\theta'') = \infty$.*

Proof. These follow directly from the Assumptions 1 and 2. ■

The next Theorem states that this profile is an equilibrium. The proof makes use of the one-step deviation principle and the assumption of MLRP. We call this profile the informative equilibrium of the game.

Theorem 7 (Informative equilibrium) *The game has a symmetric equilibrium, where every player adopts at stage k the strategy $\tau_*^k(\theta)$ defined in (1).*

Proof. Assume that all players i use strategies given by (1) in each stage k. It is clear that no player can benefit by deviating to $\tau_i < \tau_*^k(\theta_i)$. Let $\widehat{\tau}_i(\theta_i) > \tau_*^k(\theta_i)$ be the best deviation for player i of type θ_i in stage k. Let $\widehat{\theta}_i$ be the type of player i that solves

$$\tau_*^k\left(\widehat{\theta}_i\right) = \widehat{\tau}_i(\theta_i).$$

By Assumptions 1 and 2, we know that $\widehat{\theta}_i > \theta_i$, and also that

$$\mathbb{E}\left[v\left(t+T^k,\omega\right)\middle|h^k,\theta^k\left(t\right)=\widehat{\theta}_i,\theta_i\right]$$

is decreasing in t at $t=\tau_*^k\left(\widehat{\theta}_i\right)$. But this contradicts the optimality of the deviation to $\widehat{\tau}_i\left(\theta_i\right)$.

Since there are no profitable deviations in a single stage for any type of player i, the claim is proved by the one-shot deviation principle. ■

Let us next turn to the properties of the informative equilibrium. The equilibrium stopping strategy $\tau_*^k\left(\theta\right)$ defines a time dependent cutoff level $\theta^k\left(t\right)$ for all $t\geq 0$ as follows:

$$\theta_*^k\left(t\right)\equiv\begin{cases}\theta^k \text{ if } 0\leq t<\tau_*^k\left(\theta^k\right)\\ \overline{\theta} \text{ if } t>\tau_*^k\left(\overline{\theta}\right)\\ \max\left\{\theta\mid\tau_*^k\left(\theta\right)\leq t\right\} \text{ if } \tau_*^k\left(\theta^k\right)\leq t\leq\tau_*^k\left(\overline{\theta}\right)\end{cases}. \tag{2}$$

In words, $\theta_*^k\left(t\right)$ is the highest type that stops at time t in equilibrium. The key properties of $\theta_*^k\left(t\right)$ for the characterization of equilibrium are given in Proposition 9 below. Before that, we note that the equilibrium stopping strategy is left-continuous in θ:

Lemma 8 (Left-continuity of $\tau_*^k\left(\theta\right)$) *Let $\tau_*^k\left(\theta\right)$ denote the informative equilibrium stopping strategy defined in (1). For all $\theta\in\left(\underline{\theta},\overline{\theta}\right)$,*

$$\lim_{\theta'\uparrow\theta}\tau_*^k\left(\theta'\right)=\tau_*^k\left(\theta\right).$$

Proof. Assume on the contrary that for some θ, we have $\tau_*^k\left(\theta\right)-\lim_{\theta'\uparrow\theta}\tau_*^k\left(\theta'\right)>0$ (Lemma 6 guarantees that we can not have $\tau_*^k\left(\theta\right)-\lim_{\theta'\uparrow\theta}\tau_*^k\left(\theta'\right)<0$). Denote $\Delta t=t''-t'$, where $t''=\tau_*^k\left(\theta\right)$ and $t'=\lim_{\theta'\uparrow\theta}\tau_*^k\left(\theta'\right)$. Denote $u\left(t,\theta\right)=\mathbb{E}\left[v\left(t,\omega\right)\middle|h^k,\theta_j\geq\theta\right]$. By definition of $\tau_*^k\left(\theta\right)$, we have then $u\left(t'',\theta\right)>u\left(t,\theta\right)$ for all $t\in\left[t'-\delta,t'+\delta\right]$ for any $0<\delta<\Delta t$.

Because signal densities are continuous in θ, $u\left(t,\theta\right)$ must be continuous in θ. This means that there must be some $\varepsilon>0$ such that $u\left(t'',\theta'\right)>u\left(t,\theta'\right)$ for all $t\in\left[t'-\delta,t'+\delta\right]$ and for all $\theta'\in\left[\theta-\varepsilon,\theta\right]$. But on the other hand $\lim_{\theta'\uparrow\theta}\tau_*^k\left(\theta'\right)=t'$ implies that $\tau_*^k\left(\theta'\right)\in\left[t'-\delta,t'+\delta\right]$ if θ' is chosen sufficiently close to θ. By definition of $\tau_*^k\left(\theta'\right)$ this means that $u\left(\tau_*^k\left(\theta'\right)\right)\geq u\left(t'',\theta'\right)$, and we have a contradiction. We can conclude that for all θ, $\lim_{\theta'\uparrow\theta}\tau_*^k\left(\theta'\right)=\tau_*^k\left(\theta\right)$. ■

The next proposition allows us to characterize the key properties of the informative equilibrium. It says that $\theta^k\left(t\right)$ is continuous, which means that at each $t>0$, only a single type exits, and hence the probability of more than one player stopping simultaneously is zero for $t>0$. In addition, the Proposition says that along equilibrium path, $\theta^k\left(0\right)>\underline{\theta}^k$

for all stages except possibly the first one. This means that at the beginning of each stage there is a strictly positive probability that many players stop simultaneously.

Proposition 9 $\theta_*^k(t) : [0, \infty) \to \left[\underline{\theta}^k, \overline{\theta}\right]$ *defined in (2) is continuous, (weakly) increasing, and along the path of the informative equilibrium* $\theta_*^k(0) > \theta^k$ *for* $k \geq 1$.

Proof. Continuity and monotonicity of $\theta^k(t)$ follow from definition (2) and the properties of $\tau_*^k(\theta)$ given in Lemmas 6 and 8.

Take any stage $k \geq 1$ along the informative equilibrium path. To see that we must have $\theta^k(0) > \underline{\theta}^k$, consider how information of the marginal player changes at time t^{k-1}. If $t^{k-1} = 0$, the player with signal $\theta_+^{k-1} = \underline{\theta}^k$ was willing to stop at $t^{k-1} = 0$ conditional on being the lowest type within the remaining players. However, since the stage ended at $t^{k-1} = 0$, at least one player had a signal within $\left(\theta_i^{k-1}, \theta_+^{k-1}\right]$. By MLRP and quasi-supermodularity, this additional information updates the beliefs of the remaining players discretely downwards. Therefore, $\tau_*^k(\theta) = 0$ for all $\theta \in \left[\underline{\theta}^k, \underline{\theta}^k + \varepsilon\right]$ for some $\varepsilon > 0$, which by (2) means that $\theta^k(0) > \underline{\theta}^k$.

On the other hand, if $t^{k-1} > 0$, the lowest signal within the remaining players in stage $k-1$ was $\theta_+^{k-1} = \underline{\theta}^k$. The player with this signal stopped optimally under the information that all the remaining players have signals within $\left[\underline{\theta}^k, \overline{\theta}\right]$. But as soon this player stops and the game moves to stage k, the other players update on the information that one of the players remaining in the game in stage $k-1$ had the lowest possible signal value amongst the remaining players. Again, by MLRP and quasi-supermodularity, the marginal cutoff moves discretely upwards, and we have $\theta^k(0) > \underline{\theta}^k$. ■

To understand the equilibrium dynamics, note that as real time moves forward, the cutoff $\theta^k(t)$ moves upward, thus shrinking from left the interval within which the signals of the remaining players lie. By MLRP and quasi-modularity this new information works towards delaying optimal stopping time for all the remaining players. At the same time, keeping information fixed, the passage of time brings forth the optimal stopping time for additional types. In equilibrium, $\theta^k(t)$ moves at a rate that exactly balances these two effects keeping the marginal type indifferent. As soon as the stage ends at $t^k > 0$, the expected value from staying in the game drops by a discrete amount for the remaining players (again by MLRP and quasi-supermodularity). This means that the marginal cutoff moves discretely upwards and thus $\theta^{k+1}(0) > \theta^k\left(t^k\right) = \underline{\theta}^{k+1}$, and at the beginning of the new stage there is thus a mass point of immediate exits. If at least one player stops, the game moves immediately to stage $k+2$ with another mass point of exits, and this continues as long as there are consecutive stages in which at least one player stops at $t = 0$. Thus, the equilibrium path alternates between "stopping phases", i.e.

consecutive stages that end at $t = 0$ and result with multiple simultaneous exits, and "waiting phases", i.e. stages that continue for a strictly positive time.

Note that the random time at which stage k ends,

$$t^k = \tau_*^k \left(\min_{i \in \mathcal{N}^k} \theta_i \right),$$

is directly linked to the first order statistic of the player types remaining in the game at the beginning of stage k. If we had a result stating that for all k, $\tau^k(\theta_i)$ is strictly increasing in θ_i, then the description of the equilibrium path would be equivalent to characterizing the sequence of lowest order statistics where the realization of all previous statistics is known. Unfortunately this is not the case, since for all stages except the very first one there is a strictly positive mass of types that stop immediately at $t = 0$, which means that the signals of those players will be revealed only to the extent that they lie within a given interval. However, in Section 4.3 we will show that in the limit where the number of players is increased towards infinity, learning in equilibrium is equivalent to learning sequentially the exact order statistics of the signals.

3.2 Uninformative equilibria

While the model always admits the existence of the informative symmetric equilibrium defined above, some stage games also allow the possibility of an additional symmetric equilibrium, where all players stop at the beginning of the stage irrespective of their signals. We call these uninformative equilibria.

To understand when such uninformative equilibria exist, consider the optimal stopping time of a player who has private signal θ, conditions on all information s^k obtained in all stages $k' < k$, but who does not obtain any new information in stage k. Denote the optimal stopping time of such a player by $\overline{\tau}^k(\theta)$:

$$\overline{\tau}^k(\theta) \equiv \min \left\{ t \geq 0 \left| \mathbb{E}\left[v\left(t + T^k, \omega\right) \left| s^k, \theta \right.\right] \geq \mathbb{E}\left[v\left(t' + T^k, \omega\right) \left| s^k, \theta \right.\right] \text{ for all } t' \geq t \right.\right\}.$$

If $\overline{\tau}^k(\theta) > 0$ for some $\theta \in [\underline{\theta}, \overline{\theta}]$, then an uninformative equilibrium cannot exist: it is a strictly dominant action for that player to continue beyond $t = 0$. But if $\overline{\tau}^k(\theta) = 0$ for all players, then an uninformative equilibrium indeed exists: If all players stop at $t = 0$ then they learn nothing from each other. And if they learn nothing from each other, then $t = 0$ is their optimal action.

Since $\overline{\tau}^k(\theta)$ is clearly increasing in θ, the existence of uninformative equilibria depends simply on whether $\overline{\tau}^k(\overline{\theta})$ is zero:

Proposition 10 *If at stage k we have $\overline{\tau}^k\left(\overline{\theta}\right) = 0$, then the game has a symmetric equilibrium, where at stage k all active players stop at time $\tau^k = 0$ irrespective of their signals.*

The equilibrium, where all the active players choose $\tau^k = 0$ in *all* stages with $\overline{\tau}^k\left(\overline{\theta}\right) = 0$, is the *least informative equilibrium* of the game. There are of course also intermediate equilibria between the informative and least informative equilibria, where at some stages with $\overline{\tau}^k\left(\overline{\theta}\right) = 0$ players choose $\tau^k(\theta)$ defined in (1), and in others they choose $\tau = 0$.

Note that there are also stages where the informative equilibrium commands all players to stop at $t = 0$. This happens if the remaining players are so much convinced that they have already passed the optimal stopping time that even finding out that all of them have signals $\theta = \overline{\theta}$ would not make them think otherwise. In that case $\tau^k(\theta) = 0$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$, where $\tau^k(\theta)$ is defined in (1).

It is easy to rank the symmetric equilibria of the game. The informative equilibrium is payoff dominant in the class of all symmetric equilibria of the game. The option of stopping the game is always present for all players in the game, and as a result, not stopping must give at least the same payoff.

4 Informative Equilibrium in Large Games

In this section we study the limiting properties of the model, when we increase the number of players towards infinity. In subsection 4.1 we show that the informative equilibrium exhibits delay and randomness. In subsection 4.2 we discuss the effect on the players' payoffs of the observational learning. In subsection 4.3 we analyze the information of the players in equilibrium, and derive a simple algorithm for simulating the equilibrium path directly in the large game limit.

4.1 Delay in Equilibrium

We state here a theorem that characterizes the equilibrium behavior in the informative equilibrium for the model with a general state space Ω in the limit $N \to \infty$. Let $T_N(\theta, \omega)$ denote the random exit time (in real time) in the informative equilibrium of a player with signal θ when the state is ω and the number of players at the start of the game is N. We will be particularly interested in the behavior of $T_N(\theta, \omega)$ as N grows and we define

$$T(\omega, \theta) \equiv \lim_{N\to\infty} T_N(\omega, \theta),$$

where the convergence is to be understood in the sense of weak convergence.[3] Since we have assumed Ω to be compact, we know that the sequence $T_N(\theta, \omega)$ has a convergent subsequence. For now, we take $T(\omega, \theta)$ to be the limit of any such subsequence. Along the way, we shall prove that this is also the limit of the original sequence.

The real time instant when the last player with signal $\overline{\theta}$ stops is given by $T_N(\omega, \overline{\theta})$ and we let

$$T_N(\omega) \equiv T_N(\omega, \overline{\theta}) \text{ and } T(\omega) \equiv \lim_{N\to\infty} T_N(\omega).$$

We let $F(t \mid \omega)$ denote the distribution of $T(\omega)$, or in other words,

$$F(t \mid \omega) = \Pr\{T(\omega) \leq t\},$$

and use $f(t \mid \omega)$ to refer to the corresponding probability density function. The following Theorem characterizes the asymptotic behavior of the informative equilibrium as the number of players becomes large.

Theorem 11 *In the informative equilibrium of the game, we have for all $\omega < \overline{\omega}$,*

1. *$\operatorname{supp} f(t \mid \omega) = [\max\{t(\underline{\theta}), \omega\}, \overline{\omega}]$.*
2. *For all $\theta, \theta' \in (\underline{\theta}, \overline{\theta}]$,*

$$\lim_{N\to\infty} \Pr\{T_N(\omega, \theta) = T_N(\omega, \theta')\} = 1.$$

Proof. In a symmetric equilibrium, no information is transmitted before the first exit. By monotonicity of the equilibrium strategies, a lower bound for all exit times and hence also for $T_N(\omega)$ for all N is $t(\underline{\theta})$.

Consider next an arbitrary $\theta' > \underline{\theta}$. By the law of large numbers, we have for all ω :

$$\frac{\#\{i \in \{1, ..., N\} \, | \theta_i < \theta'\}}{N} \to G(\theta' \, |\omega).$$

By Assumption 3, and the law of large numbers, for each θ' there is a $\theta'' < \theta'$ such that for all $\omega < \overline{\omega}$ and all $t < \omega$

$$\lim_{N\to\infty} \Pr\{\exists k \text{ such that } \theta_+^k < \theta'' < \theta' < \theta_+^{k+1}\} = 0.$$

This follows from the fact that

$$\lim_{\theta''\to 0} \frac{G(\theta'' \, |\omega)}{G(\theta' \, |\omega)} = 0,$$

[3]In our setting, this is also equivalent to convergence in distribution.

and the fact that by Assumption 2, for all $\omega' \neq \overline{\omega}$,

$$\lim_{N\to\infty}\left(\frac{G\left(\theta' \left|\omega'\right.\right)}{G\left(\theta' \left|\overline{\omega}\right.\right)}\right)^N = 0.$$

Consider therefore the stage k' where a player with the signal θ' stops. Then $\theta'' < \theta_+^{k'-1} < \theta'$, and the player with signal θ'' knows

$$\frac{\#\{i \in \{1, ..., N\} \left| \theta_i < \theta_+^{k'-1} \right.\}}{N}.$$

By the law of large numbers, this is sufficient to identify ω. This implies part 2 of the Theorem and also that $\mathrm{supp} f(t \mid \omega) \subset [\max\{t(\underline{\theta}), \omega\}, \overline{\omega}]$.

The lower bound of the support is by the argument above $\max\{t(\underline{\theta}), \omega\}$, and the remaining task is to argue that the upper bound of the support is $\overline{\omega}$. This follows easily from the fact that if $\Pr\{T_N(\omega) < t\} \to 1$ for some $t < \overline{\omega}$, then the first exit must take place before t with probability 1 but this is inconsistent with symmetric informative equilibrium in monotonic strategies. To see this, let $t' \leq t$ be the smallest instant such that

$$\lim_{N\to\infty} \Pr\{\exists i \in \{1, ..., N\} : \tau_*^1(\theta_i) \leq t'\} = 1.$$

By Assumption 2, conditional on no exit by t', the posterior probability on Ω converges to a point mass on $\overline{\omega}$. ■

4.2 Payoffs in equilibrium

We turn next to the effect of observational learning on the players' payoffs. To be precise about this, we define three ex-ante payoff functions. First, we denote by V^0 the ex-ante value of a player whose belief on the state is given by the prior:

$$V^0 = \sum_{\omega\in\Omega} \pi^0(\omega)\, v\left(T^0, \omega\right),$$

where T^0 is the optimal timing based on the prior only:

$$T^0 = \arg\max_t \sum_{\omega\in\Omega} \pi^0(\omega)\, v(t, \omega).$$

Second, consider a player who has a private signal but does not observe other players. The ex-ante value of such an "isolated" player is:

$$V^1 = \sum_{\omega\in\Omega} \left[\pi^0(\omega) \int_{\underline{\theta}}^{\overline{\theta}} g(\theta \mid \omega) v\left(T^\theta, \omega\right) d\theta\right],$$

where T^{θ} is the optimal stopping time with signal θ and $\pi^{\theta}(\omega)$ is the corresponding posterior:

$$\begin{aligned} T^{\theta} &\equiv \arg\max_{t} \sum_{\omega\in\Omega} \pi^{\theta}(\omega)\, v(t,\omega), \\ \pi^{\theta}(\omega) &\equiv \frac{\pi^{0}(\omega)\, g(\theta \mid \omega)}{\sum_{\omega\in\Omega} \pi^{0}(\omega)\, g(\theta \mid \omega)}. \end{aligned}$$

Third, consider a player in the informative equilibrium of the game. We assume that N is very large, which by Theorem 11 means that almost all players stop at the same random time $T(\omega)$ (the moment of collapse). From an ex-ante point of view, the equilibrium payoff is determined by its probability distribution $f(t \mid \omega)$. The ex-ante equilibrium payoff is thus:

$$V^{*} = \sum_{\omega\in\Omega} \left[\pi^{0}(\omega) \int_{0}^{\infty} f(t \mid \omega)\, v(t,\omega)\, dt \right]. \tag{3}$$

It is clear that additional learning can never reduce the ex-ante value, and therefore we must have:

$$V^{*} \geq V^{1} \geq V^{0}.$$

We call $V^{P} \equiv V^{1} - V^{0}$ the value of private learning, and $V^{S} \equiv V^{1} - V^{*}$ the value of social learning. In Section 5 we demonstrate numerically that V^{S} and V^{P} are closely related to each other. In particular, the value of social information increases as the value of private information is increased. We can also derive analytically an upper bound for V^{S}, which shows that whenever the individual private signals are non-informative in the sense that V^{P} is very small, then also V^{S} must be small (this holds even if the pooled information is still arbitrarily informative).

An important effect of observational learning is that it increases the sensitivity of players' payoffs to the realized state of nature. We will demonstrate this effect numerically in Section 5. We can also define value functions conditional on realized signal:

$$\begin{aligned} V^{1}(\theta) &= \sum_{\omega\in\Omega} \pi^{\theta}(\omega)\, v\left(T^{\theta},\omega\right), \\ V^{*}(\theta) &= \sum_{\omega\in\Omega} \left[\pi^{\theta}(\omega)\, V^{*}(\omega)\right]. \end{aligned}$$

We conjecture that $V^{S}(\theta) \equiv V^{*}(\theta) - V^{1}(\theta)$ is increasing in θ, that is, the additional value of observational learning is more valuable to players who have obtained a high signal. The intuition runs as follows. If the true state is low, a player with a high signal benefits a lot from the information released by the other players who have low signals (since they will act before her). But if the true state is high, a player with a low signal will learn nothing from the other players that have higher signals (because those players will act after her). The computations in Section 5 support this conjecture.

It is clear that the player with the lowest possible signal cannot benefit from observational learning at all (she must be indifferent between following her own signal and following an equilibrium strategy), and we must therefore have

$$V^1(\underline{\theta}) = V^*(\underline{\theta}).$$

4.3 Information in equilibrium

The properties of the informative equilibrium rely on the statistical properties of the order statistics of the players' signals. In this subsection we analyze the information content in those order statistics in the limit $N \to \infty$.

Denote the n:th order statistic in the game with N players by

$$\widetilde{\theta}_n^N \equiv \min\left\{\theta \in \left[\underline{\theta}, \overline{\theta}\right] \mid \#\left\{i \in \mathcal{N} \mid \theta_i \leq \theta\right\} = n\right\}. \tag{4}$$

It is clear that if we now increase N towards infinity while keeping n fixed, $\widetilde{\theta}_n^N$ converges to $\underline{\theta}$ in probability. Therefore, it is more convenient to work with random variable

$$Y_n^N \equiv \left(\widetilde{\theta}_n^N - \underline{\theta}\right) \cdot N. \tag{5}$$

Note that Y_n^N has the same information content as $\widetilde{\theta}_n^N$, but as we will show below, it will converge in distribution to a non-degenerate random variable. This limit distribution, therefore, captures the information content of $\widetilde{\theta}_n^N$ in the limit. Let us also define

$$\Delta Y_n^N \equiv Y_n^N - Y_{n-1}^N = \left(\widetilde{\theta}_n^N - \widetilde{\theta}_{n-1}^N\right) \cdot N, \tag{6}$$

where by convention we let $\theta_0^N \equiv \underline{\theta}$ and $Y_0^N \equiv 0$. Furthermore, let $[\Delta Y_1^\infty, ..., \Delta Y_n^\infty]$ be a vector of n independent exponentially distributed random variables with parameter $g(\underline{\theta} \mid \omega)$:

$$\Pr\left(\Delta Y_1^\infty \leq x_1, ..., \Delta Y_n^\infty \leq x_n\right) = e^{-g(\underline{\theta}|\omega)\cdot x_1} \cdot ... \cdot e^{-g(\underline{\theta}|\omega)\cdot x_n}.$$

Proposition 12 *Fix $n \in \mathbb{N}_+$ Consider the sequence of random variables $\left\{\left[\Delta Y_1^N, ..., \Delta Y_n^N\right]\right\}_{N=n}^\infty$, where for each N the random variables ΔY_i^N are defined by (4) - (6). As $N \to \infty$, we have:*

$$\left[\Delta Y_1^N, ..., \Delta Y_n^N\right] \xrightarrow{\mathcal{D}} \left[\Delta Y_1^\infty, ..., \Delta Y_n^\infty\right],$$

where $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution.

Proof. The probability distribution of ΔY_n^N, conditional on Y_{n-1}^N is given by:

$$\begin{aligned}\Pr\left(\Delta Y_n^N \leq x \mid Y_{n-1}^N\right) &= \Pr\left(\left(\widetilde{\theta}_n^N - \widetilde{\theta}_{n-1}^N\right) \cdot N \leq x \mid \widetilde{\theta}_{n-1}^N\right) \\ &= \Pr\left(\widetilde{\theta}_n^N \leq \widetilde{\theta}_{n-1}^N + \frac{x}{N} \mid \widetilde{\theta}_{n-1}^N\right) \\ &= 1 - \left(1 - \frac{\left(G\left(\widetilde{\theta}_{n-1}^N + \frac{x}{N} \mid \omega\right) - G\left(\widetilde{\theta}_{n-1}^N \mid \omega\right)\right)}{1 - G\left(\widetilde{\theta}_{n-1}^N \mid \omega\right)}\right)^{N-n}.\end{aligned}$$

Noting that as $N \to \infty$, we have $\widetilde{\theta}_{n-1}^N \xrightarrow{\mathcal{P}} \underline{\theta}$ and $\frac{x}{N} \to 0$, and therefore we have:

$$N \cdot \frac{\left(G\left(\widetilde{\theta}_{n-1}^N + \frac{x}{N} \mid \omega\right) - G\left(\widetilde{\theta}_{n-1}^N \mid \omega\right)\right)}{1 - G\left(\widetilde{\theta}_{n-1}^N \mid \omega\right)} \xrightarrow{\mathcal{P}} g\left(\underline{\theta} \mid \omega\right) \cdot x.$$

Noting also that

$$\lim_{N\to\infty}\left(1 - \frac{g\left(\underline{\theta} \mid \omega\right) \cdot x}{N}\right)^{N-n} = e^{-g(\underline{\theta}|\omega)\cdot x},$$

we have:

$$\lim_{N\to\infty} \Pr\left(\Delta Y_n^N \leq x \mid Y_{n-1}^N\right) = 1 - e^{-g(\underline{\theta}|\omega)\cdot x}.$$

This means that ΔY_n^N converges in distribution to an exponentially distributed random variable with parameter $g\left(\underline{\theta} \mid \omega\right)$ that is independent of all lower order statistics. ■

Note that the limit distribution of ΔY_n^N does not depend on n. Therefore, $Y_n^N = \Sigma_{i=1}^n \Delta Y_n^N$ converges to a sum of independent exponentially distributed random variables, which means that the limiting distribution of Y_n^N is Gamma distribution:

Corollary 13 *For all* n,

$$Y_n^N = \sum_{i=1}^n \Delta Y_i^N \xrightarrow{\mathcal{D}} \sum_{i=1}^n \Delta Y_i^\infty \equiv Y_n^\infty,$$

where $Y_n^\infty \sim \Gamma\left(n, g\left(\underline{\theta} \mid \omega\right)\right)$.

Proposition 12 means that when N is large, observing the n lowest order statistics is observationally equivalent to observing n independent exponentially distributed random variables. This has an important implication for the Bayesian updating based on order statistics: observing only the n:th order statistic $\widetilde{\theta}_n^N$ is informationally equivalent to observing $\left\{\widetilde{\theta}_i^N\right\}_{i=1}^n$ that contains all order statistics up to n. This is due to the "memoryless" nature of exponential random variables. To see this formally, write the posterior belief of an observer who updates her belief on the state of the world based on the realization $\left\{\widetilde{\theta}_i^N\right\}_{i=1}^n$ (approximating the joint distribution of $\left(\widetilde{\theta}_i^N - \widetilde{\theta}_{i-1}^N\right) \cdot N$ by exponential distribution based on Proposition 12). As can be seen, this posterior depends only on the realization of $\widetilde{\theta}_n^N$:

$$\pi\left(\omega \mid \left\{\widetilde{\theta}_i^N\right\}_{i=1}^n\right) \approx \frac{\pi^0(\omega) \cdot \prod_{i=1}^n g(\underline{\theta} \mid \omega) e^{-g(\underline{\theta}|\omega)\left(\widetilde{\theta}_i^N - \widetilde{\theta}_{i-1}^N\right) \cdot N}}{\sum_{\omega \in \Omega} \pi^0(\omega) \cdot \prod_{i=1}^n g(\underline{\theta} \mid \omega) e^{-g(\underline{\theta}|\omega)\left(\widetilde{\theta}_i^N - \widetilde{\theta}_{i-1}^N\right) \cdot N}}$$

$$= \frac{\pi^0(\omega) \cdot (g(\underline{\theta} \mid \omega))^n e^{-g(\underline{\theta}|\omega)\left(\widetilde{\theta}_n^N - \underline{\theta}\right) \cdot N}}{\sum_{\omega \in \Omega} \pi^0(\omega) \cdot (g(\underline{\theta} \mid \omega))^n e^{-g(\underline{\theta}|\omega)\left(\widetilde{\theta}_n^N - \underline{\theta}\right) \cdot N}}.$$

So far, we have discussed the properties of the order statistics of the signals without linking them to the equilibrium behavior. Now we turn to the properties of the informative equilibrium, and show that in the large game limit the equilibrium path can be approximated by a simple algorithm that samples sequentially the order statistics. To make this statement precise, we now fix N and define two different sequences of random variables, both obtained as mappings from realized signal values to real numbers.

First, for each N, denote by $T_n^*(N)$ the real time at which the number of players that stop exceeds n in the unique informative equilibrium:

$$T_n^*(N) \equiv \min\left\{T^k \mid Q^k \geq n\right\}.$$

The increasing sequence $\{T_n^*(N)\}_{n=1}^N$ contains the real stopping moments of all N players in the game.

Second, we define a sequence of stopping times $\widehat{T}_n(N)$ calculated directly on the basis of the order statistics. As an intermediate step, denote by $\widetilde{T}_n(N)$ the optimal stopping moment given the information contained in the n lowest order statistics:

$$\widetilde{T}_n(N) \equiv \inf\left\{t \geq 0 \middle| \mathbb{E}\left[v(t,\omega) \middle| \left\{\widetilde{\theta}_i^N\right\}_{i=1}^n\right] \geq \mathbb{E}\left[v(t',\omega) \middle| \left\{\widetilde{\theta}_i^N\right\}_{i=1}^n\right] \text{ for all } t' \geq t\right\}.$$

Next, define random variable $\widehat{T}_n(N)$ as:

$$\widehat{T}_n(N) \equiv \max_{i=1,\dots,n} \widetilde{T}_n(N). \tag{7}$$

Hence, $\left\{\widehat{T}_n(N)\right\}_{n=1}^N$ is the sequence of optimal stopping times based on sequential sampling of order statistics under an additional constraint that one is never allowed to "go back in time", i.e. choose a stopping time lower than some previously chosen stopping time. Note that both $\{T_n^*(N)\}_{n=1}^N$ and $\left\{\widehat{T}_n(N)\right\}_{n=1}^N$ are weakly increasing sequences of random variables.

The next proposition says that for any fixed n, the difference between $\widehat{T}_n(N)$ and $T_n^*(N)$ vanishes as N goes to infinity (in the sense of convergence in probability). The key for this result is the finding that inference on order statistics becomes informationally equivalent to inference based on independent exponentially distributed random variables. This means that a player that conditions on having the lowest signal among the remaining players does not learn anything more by conditioning on exact realizations of the signals lower than hers. Thus, inference based on the exact realizations of lowest order statistics becomes the same as the inference of the marginal player in equilibrium, who knows the lowest signal realizations only to the extent that they lie within some fixed intervals.

Proposition 14 *Fix $n \in \mathbb{N}_+$ and consider random variables $T_n^*(N)$ and $\widehat{T}_n(N)$. As $N \to \infty$, we have:*

$$\widehat{T}_n(N) - T_n^*(N) \xrightarrow{\mathcal{P}} 0.$$

Proof. Fix n. As $N \to \infty$, the updating based on the realizations of the n lowest signals is informationally equivalent to observing n exponentially distributed random variables with parameter $g(\underline{\theta} \mid \omega)$. Consider the player that has the n:th lowest signal $\widetilde{\theta}_n^N$. As N is increased, this signal is of course arbitrarily close to $\underline{\theta}$ at a probability arbitrarily close to one. In equilibrium, this player is the n:th to stop (possibly together with some other players). By (1), her real stopping time $T_n^*(N)$ is optimal conditional on information that some $n' < n$ players have signals within $[\underline{\theta}, \theta']$ for some $\theta' \leq \widetilde{\theta}_n^N$, no player as signals within $\left(\theta', \widetilde{\theta}_n^N\right)$, and she herself has signal $\widetilde{\theta}_n^N$. In contrast, $\widetilde{T}_n(N)$ is optimal conditional on n players having signals within $\left[\underline{\theta}, \widetilde{\theta}_n^N\right]$, which by MLRP and super-modularity means that for any $\varepsilon > 0$,

$$\lim_{N \to \infty} \Pr\left(\widetilde{T}_n(N) - T_n^*(N) > \varepsilon\right) = 0.$$

Since for all N, we have

$$\begin{aligned}
\widehat{T}_n(N) &\equiv \max_{i=1,\dots,n} \widetilde{T}_n(N) \text{ and} \\
T_n^*(N) &\geq \max_{i=1,\dots,n} T_i^*(N),
\end{aligned}$$

we have also

$$\lim_{N \to \infty} \Pr\left(\widehat{T}_n(N) - T_n^*(N) > \varepsilon\right) = 0.$$

To show that $\Pr\left(T_n^*(N) - \widehat{T}_n(N) > \varepsilon\right) \to 0$ is conceptually similar. ■

5 Simulating the informative equilibrium path

In this section we illustrate the main properties of the game by Monte-Carlo simulations. Proposition 14 gives a simple way to simulate the informative equilibrium directly in the limit $N \to \infty$. A sample path of the equilibrium is generated as follows. i) First, fix prior $\pi^0(\omega)$ and the true state of world ω'. ii) Draw a sequence $\{y_i\}_{i=1}^M$ of independent exponentially distributed random variables with parameter $g(\underline{\theta} \mid \omega')$. For this sequence, the corresponding sequence of posteriors is:

$$\pi^i(\omega) = \frac{\pi^{i-1}(\omega)\, g(\underline{\theta} \mid \omega)\, e^{-g(\underline{\theta}|\omega)\cdot y_i}}{\sum\limits_{\omega\in\Omega} \pi^{i-1}(\omega)\, g(\underline{\theta} \mid \omega)\, e^{-g(\underline{\theta}|\omega)\cdot y_i}},\ \ i = 1, ..., M.$$

For each $i = 1, ..., M$, calculate the stopping time $\widehat{T}_i$ as:

$$\widehat{T}_i = \max\left\{\widehat{T}_{i-1}, \arg\max_t \mathbb{E}\left[\sum_{\omega\in\Omega} \pi^i(\omega)\cdot v(t,\omega)\right]\right\}.$$

The generated sequence $\left\{\widehat{T}_i\right\}_{i=1}^M$ is the simulated realization of the sequence (7) in the limit $N \to \infty$. By Proposition 14, it corresponds to the real time moments at which the first M players stop the game in the large game limit. By choosing M sufficiently large, one can ensure that the belief $\pi^M(\omega)$ has converged to the true state, i.e. $\pi^M(\omega') \approx 1$ and $\pi^M(\omega) \approx 0$ for all $\omega \neq \omega'$. This means that all the remaining players will stop in equilibrium at the same real time as the M:th player (with high probability). Thus, $\widehat{T}_M$ gives the real time at which the game collapses for this particular sample.

We illustrate next the model by Monte-Carlo simulations, where we generate a large number of equilibrium paths and use those to compute the probability distributions for the players' stopping times and payoffs. We specify the model as follows:

$$\begin{aligned}
\Omega &= \left\{0, \frac{1}{S-1}, \frac{2}{S-1}, ..., \frac{S-2}{S-1}, 1\right\}, \\
\left[\underline{\theta}, \overline{\theta}\right] &= [0,1], \\
g(\theta\,|\omega) &= 1 + \gamma\left(\omega - \frac{1}{2}\right)\left(\theta - \frac{1}{2}\right), \\
v(t,\omega) &= -(\omega - t)^2.
\end{aligned}$$

Here S is the number of states and γ is a parameter measuring the precision of individual signals. In this illustration we have $S = 10$, and for the signal precision we compare two cases: $\gamma = 2$ (precise signals) and $\gamma = 0.2$ (imprecise signals).

5.1 Distribution of stopping times

We generated 10000 sample paths for each 10 state values. For each sample path, we use $M = 300000$ random variables to make sure that the posteriors have fully converged to the true state. Figure 1 shows the simulated cumulative distribution functions of the moment of collapse, conditional on state. Top panel uses precision parameter $\gamma = 2$ while the bottom panel uses $\gamma = 0.2$. This Figure demonstrates clearly the Theorem 11: the time of collapse is random and delayed as compared to the first best for all but the highest state. The delay is more sever for the lowest state values. The signal precision has an expected effect: with less precise signals there is on average more delay.

Figure 1

5.2 Payoffs

Using the distributions of stopping times generated by the Monte-Carlo simulation, we can easily compute the ex-ante value of a player in equilibrium according to (3).

The following table shows the ex-ante values defined in section 4.2 and computed with the two precision parameters used in the simulations:

	V^0	V^1	V^*	V^P	V^S
$\gamma = 2$	-0.1019	-0.0984	-0.0690	0.0035	0.0294
$\gamma = 0.2$	-0.1019	-0.1018	-0.0989	0.000035	0.0029

The obvious result in this table is that the more precise the private signals, the more valuable private learning: V^P is higher for the precise signals. What is less obvious is that the social value behaves similarly: the more precise the private signals, the more valuable is the additional value of the social learning on top of the private value of the signals. In fact, it is easy to show formally that in the limit where private signals are made uninformative in the sense that V^P goes to zero (in our model specification this would mean $\gamma \to 0$), also V^S must go to zero.

Figures 2 and 3 show the values conditional on signal and state, respectively. The value of an isolated player conditional on signal is U-shaped: extreme signal realizations are ex-ante good news in the quadratic payoff case, since they make large mistakes unlikely. In equilibrium, high signals are good news: they indicate that the optimal timing is more likely to be late, and social learning is particularly valuable if that is the case. Learning from others causes delay, which is valuable if late action is ex-post optimal, but it is costly if the early action would have been optimal. This can be seen more clearly in Figure 3 that shows the value functions conditional on state. Social learning makes payoffs more sensitive on true state: actions are delayed which is good if state is high but bad if state is low.

Figure 2

Figure 3

6 Discussion

Our results are quite different from related models in Chamley & Gale (1994) and Chamley (2004). To understand why this is the case, it is useful to note that we can embed the main features of those models as a special case of our model. For this purpose, assume that $\omega \in \{0, \infty\}$, and

$$v(t, 0) = e^{-rt}, v(t, \infty) = -ce^{-rt}.$$

If it is optimal to invest at all in this version of the model, then the investment time is insensitive to the information of the players. In other words, investment is good either immediately or never. Private signals only affect the relative likelihood of these two cases. This leads to the conclusion that it is never optimal to invest at $t > 0$ conditional on no other investments within $(t - \varepsilon, t)$, since then it would have been optimal to invest immediately. As a result, a given stage k ends either immediately if at least one player stops at time $t = 0$ and the play moves to stage $k + 1$, or the stage continues forever and the game never moves to stage $k + 1$. This means that all investment must take place at the beginning of the game, and with a positive probability investment stops forever even when $\omega = 0$. The models in Chamley & Gale (1994) and Chamley (2004) are formulated in discrete time, but their limiting properties as the period length is reduced corresponds exactly to this description.

We get an intermediate case by setting $\Omega = \{\omega_1, ..., \omega_S, \infty\}$ with $P(\infty) > 0$. In this case, the game has some revelation of information throughout the game. Nevertheless, it is possible that all investment ends even though $\omega < \infty$, and as a result, the game allows for a similar possibility of incorrect actions as Chamley & Gale (1994).

There are a number of directions where the analysis in this paper should be extended. Exogenous uncertainty on the payoff of investment plays an important role in the literature on real options. Our paper can be easily extended to cover the case where the profitability of the investment depends on an exogenous (and stochastic) state variable p and on private information about common market state ω. In this formulation, the stage game is one where the players pick a Markovian strategy for optimal stopping. With our monotonicity assumptions this is equivalent to selecting a threshold value $p_i(\theta_i)$ at which to stop conditional on their signal. The stage ends at the first moment when the threshold value of some player is hit.

The analytical simplicity of the model also makes it worthwhile to consider some alternative formulations. First, it could be that the optimal time to stop for an individual player i depends on the common parameter ω as well as her own signal θ_i. The reason for considering this extension would be to demonstrate that the form of information aggregation demonstrated in this paper is not sensitive to the assumption of pure common values. Second, by including the possibility of payoff externalities in the game we can bring the current paper closer to the auction literature. We plan to investigate these questions in future work.

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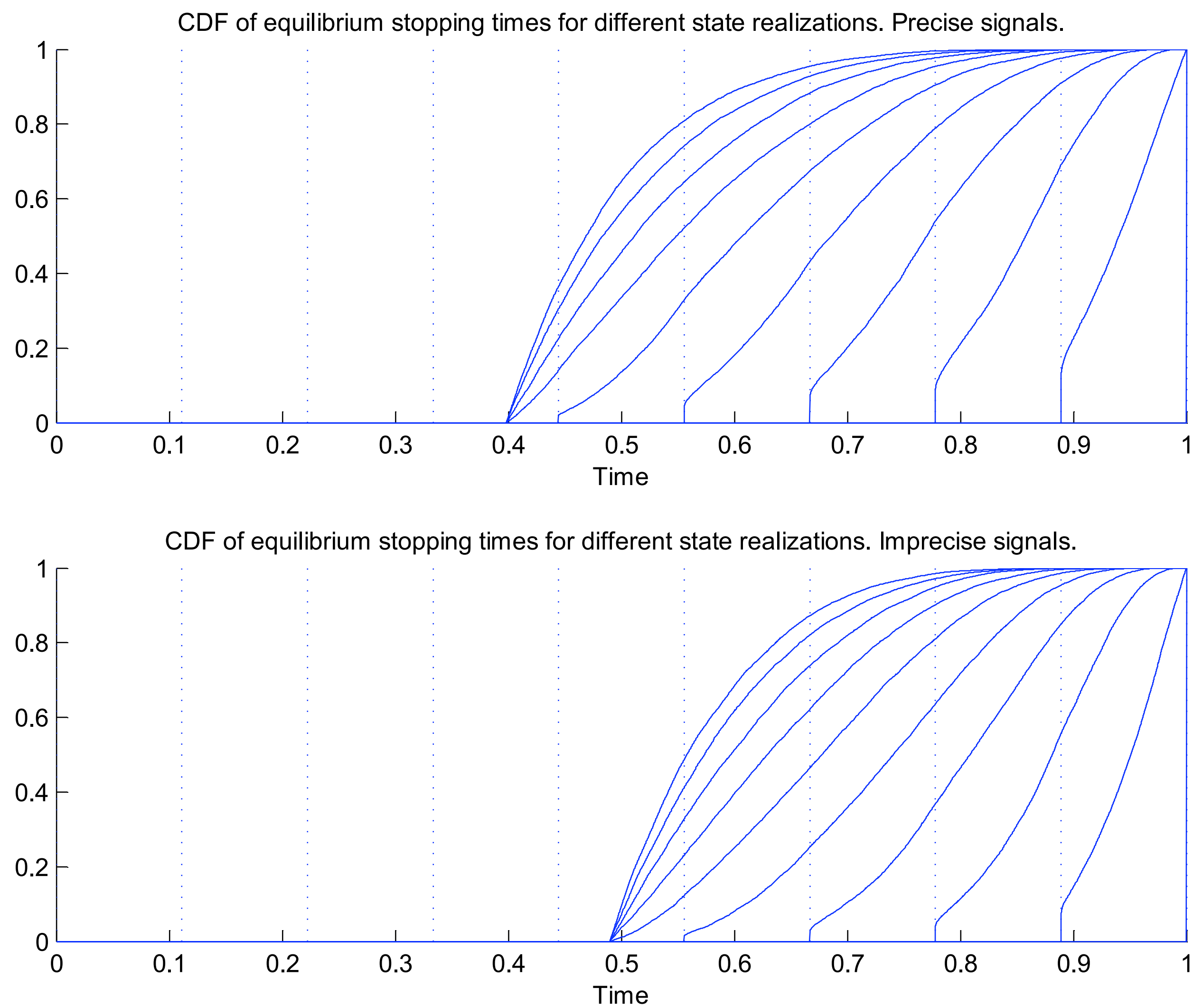
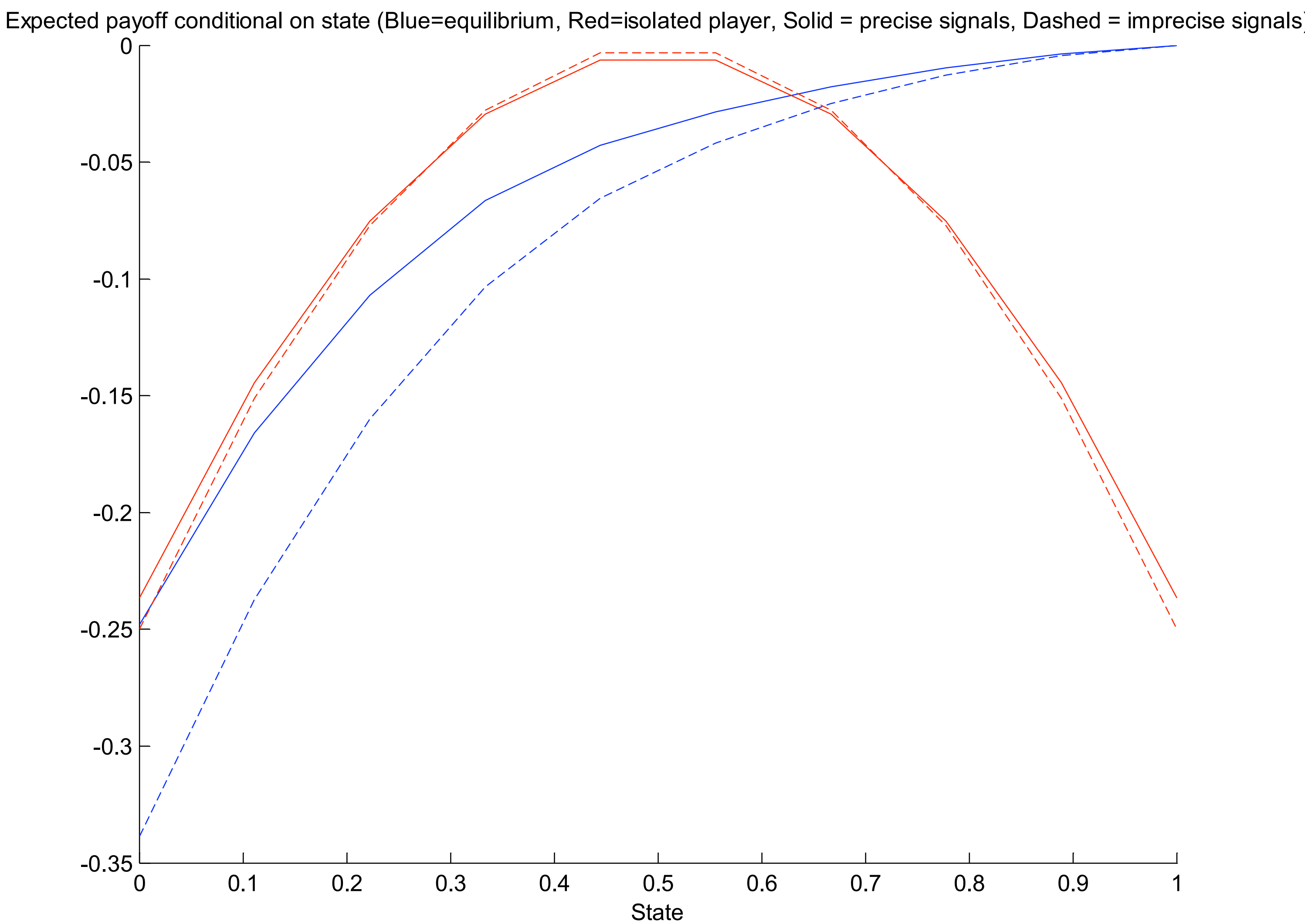
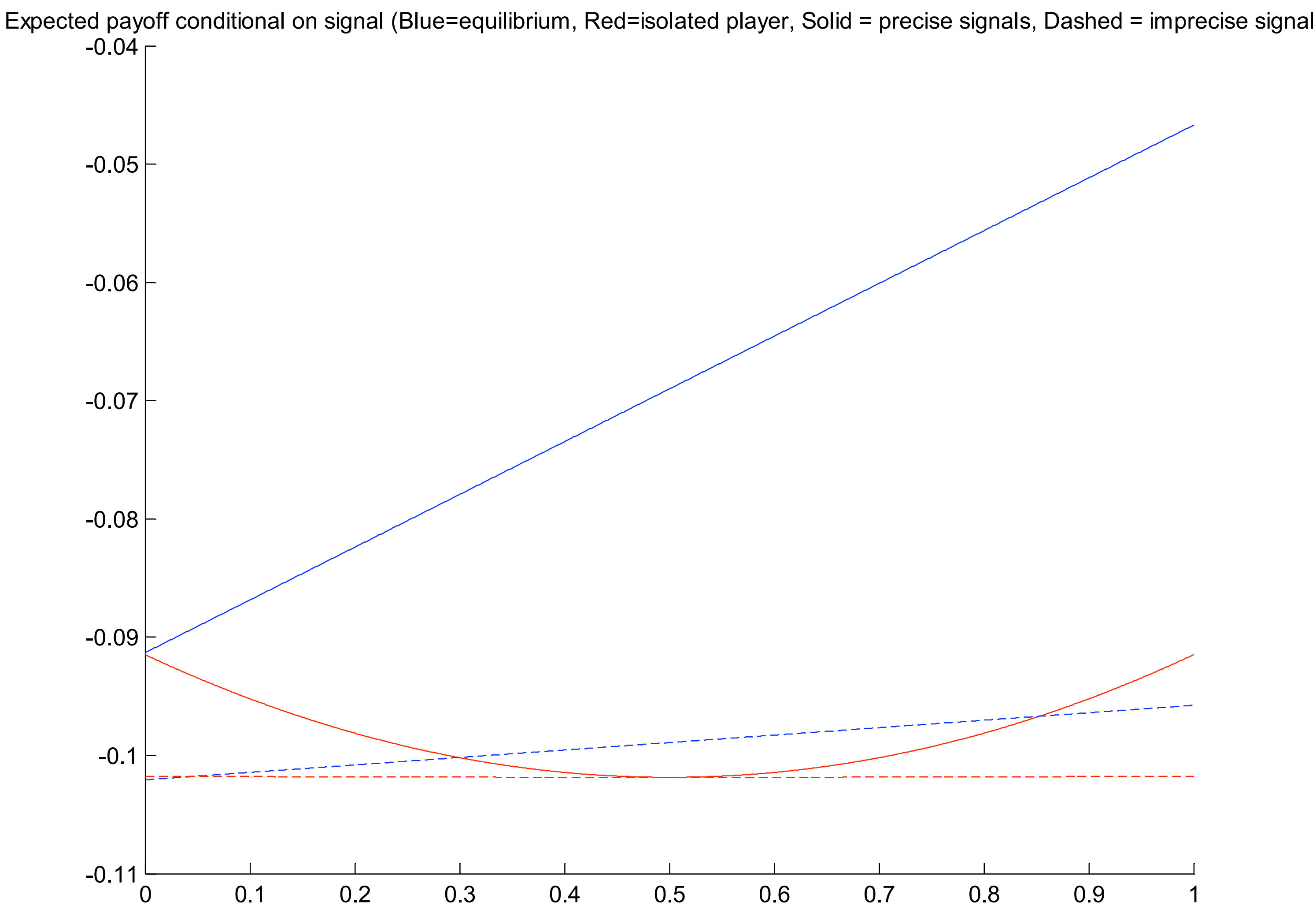

CDF of equilibrium stopping times for different state realizations. Precise signals.
1
0.8
0.6
0.4
0.2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Time
CDF of equilibrium stopping times for different state realizations. Imprecise signals.
1
0.8
0.6
0.4
0.2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Time


Expected payoff conditional on state (Blue=equilibrium, Red=isolated player, Solid = precise signals, Dashed = imprecise signals)
0
-0.05
-0.1
-0.15
-0.2
-0.25
-0.3
-0.35
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
State


Expected payoff conditional on signal (Blue=equilibrium, Red=isolated player, Solid = precise signals, Dashed = imprecise signals
-0.04
-0.05
-0.06
-0.07
-0.08
-0.09
-0.1
-0.11
0
0.1
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1

Learning and Information Aggregation in an Exit Game*

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Abstract

We analyze information aggregation in a stopping game with uncertain common payoffs. Players learn from their own private experiences as well as by observing the actions of other players. We give a full characterization of the symmetric mixed strategy equilibrium, and show that information aggregates in randomly occurring exit waves. Observational learning induces the players to stay in the game longer. The equilibria display aggregate randomness even for large numbers of players.

Keywords: Learning, optimal stopping, dynamic games.

JEL Classification: C73,D81,D82,D83

1 Introduction

Learning in dynamic decision problems comes in two different forms. Players learn from their own individual, and often private, observations about the fundamentals of their economic environment. At the same time, they may learn by observing the behavior of other players in analogous situations. In this paper, we analyze the interplay of these two modes of learning in a timing game with pure informational externalities. We show

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that even though private information accumulates steadily over time, it is revealed in occasional bursts.

There are a number of examples where both forms of learning are important. Learning about the quality of a service, the profitability of a new technology, or the size of a new market are examples of this type. In all these instances, it is reasonable to assume that part of the uncertainty is common to all agents and part is idiosyncratic. Demand may be high or low. For a population of monopolistically competing firms, the market is profitable to a larger fraction of firms if demand is high. Learning from others is useful to the extent that it can be used to determine the overall demand level. It is not sufficient, however, as it may be that the product of an individual firm does not appeal to the consumers even when demand is high.

We use a standard discounted single-player experimentation model in discrete time to represent private learning. Players do not know their type at the beginning of the game. Over time, they learn by observing signals that are correlated with their true payoff type. We assume binary types. Good types gain in expected terms by staying in the game while bad types gain by exiting the game. We assume that information accumulates according to a particularly simple form. Good types observe a perfectly informative signal with a constant probability in each period that they stay in the game while bad types never see any signals.[1] Uninformed players become more pessimistic as time passes and the optimal strategy is to exit the game once a threshold level of pessimism is reached.

Observational learning matters if a number of players face the same decision problem and if their types are correlated. We model this correlation by assuming that there is a binary state of the world that determines the probability distribution of individual types. In the high state, a higher fraction of the players are of the good type. Conditional on the state, the players' types are identically and independently distributed. Whenever the exit decisions of a given player are sensitive to her information, her actions reveal information about her information and hence also about the state of the world (since more players are informed in the high state). Uninformed players gain from additional information on the state, which creates an incentive to wait as in Chamley & Gale (1994). But in contrast to Chamley & Gale (1994), private learning prevents the players from waiting indefinitely. Our model strikes a balance between the benefits from delaying in order to learn more from others and the costs from increased pessimism as a result of private learning.

We show that the game has a unique symmetric equilibrium in mixed strategies. In

[1] The actual form of information revelation is not very important for the logic of our model. The important assumption is that it takes time for even the most pessimistic individual player to exit the game.

order to highlight the effects of learning and waiting, we eliminate the observation lags by reducing the time interval between consecutive decision moments. We show that the symmetric equilibrium can be characterized by two modes of behavior: In the *flow mode*, bad news from no informative signals is balanced by the good news from the observation that no other player exits. Exits are infrequent and prior to any exit, the beliefs of the uninformed players evolve smoothly.

When a player exits, the beliefs of the other players become more pessimistic. Immediate exit by all uninformed players would release so much information that an individual player might find it optimal to wait (since the cost of delay is small for frequent periods). As a result, the equilibrium must be in mixed strategies that balance the incentives to exit and wait. If there are further exits from the market as a result of the randomization, pessimism persists, and another round of randomizations is called for. We call this phase of consecutive exits an *exit wave*. As soon as there is a period with no exits, a sufficient level of optimism is restored in the market and the exit wave ends. An exit wave thus ends either in a collapse of the game where the last uninformed player exits, or in a reversion to the flow mode following a period with no exits. In the symmetric equilibrium, play fluctuates randomly between these two modes until a collapse ends the game.[2]

When the number of players is increased towards infinity, the pooled information on the aggregate state becomes accurate. One might conjecture that conditional on the state aggregate randomness would vanish by the law of large numbers. We show that this is not the case. Even in the case with a large number of players, transitions between the phases remain random. The size of an individual exit wave as measured by the total number of exits during the wave also remains random. Information is thus aggregated during quick random bursts. We compute the exit probabilities during exit waves and the hazard rate for their occurrence when the number of players is large.

We show that information is aggregated efficiently in the high state if there is a large number of players. By this we mean that almost all uninformed players exit in the high state as if they knew the true state. But if the state is bad, information aggregation fails: players learn the state too late, and as a result, they delay exit. In terms of the payoffs, the message of our paper is that observational learning helps the good types while it hurts the bad types.

Related Literature

This paper is related to the literature on herding and observational learning where

[2]Examples of models that display waves of action that resemble our exit waves include Bulow & Klemperer (1994) and Toxvaerd (2008). However, these models depend on the direct payoff externalities arising from scarcity.

players have private information about a common state variable at the beginning of the game. Early papers in this literature assumed an exogenously given order of moves for the players, e.g. Banerjee (1992), Bikhchandani, Hirshleifer & Welch (1992), and Smith & Sorensen (2000). A number of later papers have endogenized the timing of action choices. Among those, the most closely related to ours is Chamley & Gale (1994).[3] In that paper a number of privately informed players consider investing in a market of uncertain aggregate profitability. The model exhibits herding with positive probability: the players' beliefs may get trapped in a region with no investment even if the market is profitable. In our model, private learning during the game prevents the beliefs from getting trapped. The difference between the models is best seen by eliminating observation lags, i.e., letting period length go to zero. In Chamley and Gale, information aggregates incompletely in a single burst at the start of the game. In our model, information is revealed eventually, but at a slow rate.

Caplin & Leahy (1994) and Rosenberg, Solan & Vieille (2007) consider models with private learning about common values. While these papers are close to ours in their motivation, each makes a crucial modeling assumption that leads to qualitatively different information aggregation properties to ours. Caplin and Leahy assume a continuum of players from the beginning. This assumption leads to some problems with the existence of an equilibrium and also rules out what is a key feature of our model. In our model, the actions of a large number of players result in a moderate rate of information revelation. Rosenberg, Solan & Vieille (2007) assume a finite number of players like we do, but they assume signals that may make a player so pessimistic after one period that exiting is the dominant strategy right away. As a result, when the number of players is increased, the exit behavior after the first period reveals the state by the law of large numbers. Due to these modeling assumptions, the aggregate behavior in the large game limit is essentially deterministic conditional on state both in Caplin & Leahy (1994) and Rosenberg, Solan & Vieille (2007). Our model adds to these papers by showing that information may also aggregate slowly through randomly occurring exit waves, even when the pooled information is precise.

Another difference to the literature mentioned above is that by combining common and idiosyncratic uncertainty, our paper relaxes the assumption of perfect payoff correlation across players made in Chamley & Gale (1994), Caplin & Leahy (1994), and Rosenberg, Solan & Vieille (2007). The pure common values case is obtained in our model as a special case.

[3]See also a more general model Chamley (2004). An early contribution along these lines is Mariotti (1992).

Our paper is also related to the literature on strategic experimentation. That literature focuses on the private provision of public information rather than aggregation of privately held information. Examples of such models are Bolton & Harris (1999) and Keller, Rady & Cripps (2005). The key difference is that in those models the signals of all players are publicly observable, whereas in our model the players see only each other's actions.

The paper is organized as follows. Section 2 sets up the discrete time model and Subsection 2.1 presents an alternative interpretation for the model as a model of irreversible investment. Section 3 describes the flow of information in the game, and Section 4 provides the analysis of the symmetric equilibrium. In Section 5, we discuss information aggregation in large games. In Section 6, we characterize the symmetric equilibrium in the continuous time limit. Section 7 concludes. The proofs are in the Appendix.

2 Model

The model is in discrete time with periods $t = 0, 1, ..., \infty$. The discount factor per period is $\delta = e^{-r\Delta}$, where Δ is the period length, and $r > 0$ is the pure rate of time preference. The set of players is $\{1, ..., N\}$.

Before the game starts, nature chooses the (aggregate) state randomly from two alternatives: $\theta \in \{H, L\}$. Let q^0 denote the common prior $q^0 = \Pr\{\theta = H\}$. After choosing the state, nature chooses randomly and independently the individual type for each player. Each player is either good or bad. If $\theta = H$, the probability of being good is ρ^H, while if $\theta = L$, the probability of being good is ρ^L, where $0 \leq \rho^L < \rho^H \leq 1$. In the special case, where $\rho^H = 1$ and $\rho^L = 0$, the players' types are perfectly correlated and the game is one of pure common values. Conditional on the state, the player types are drawn independently for all players. All types are initially unobservable to all players, but the parameters q^0, ρ^H, and ρ^L are common knowledge.

The information about nature's choices arrives gradually during the game as follows. In each period, each player receives a random signal $\zeta \in \{0, 1\}$. Signals have two functions: they generate payoffs and transmit information. For a bad-type player, $\zeta = 0$ with probability 1. For a good player, $\Pr\{\zeta = 1\} = \lambda\Delta$, where λ is a commonly known parameter. Notice that informative signals arrive at a rate that depends linearly on the period length, and as a result, the real-time rate of information arrival is independent of the period length The signal realizations across periods and players (conditional on the state and the type) are assumed to be independent. We call the signal $\zeta = 1$ a *positive* signal, since it entails a positive payoff (see next paragraph) and reveals to a player that her type is good. Each player observes only her own signals. We use the terms *informed* and *uninformed* to refer to the players' knowledge of their own type: players who have had a positive signal are informed, other players are uninformed.

At the beginning of each period t, all active players i make a binary decision a_i^t. They either exit, $a_i^t = 0$, or continue, $a_i^t = 1$. Exiting is costless, but irreversible: once a player exits, she becomes inactive and receives the outside option payoff 0. Hence we require that whenever $a_i^t = 0$, then $a_i^s = 0$ for all $s > t$. We call player i active in period t if she has stayed in the game up to that point in time. We denote by $\mathcal{N}$ the set of active players and we let n denote their number.

If the player continues in the game, she pays the (opportunity) cost $c \cdot \Delta$, observes a signal $\zeta \in \{0, 1\}$ that yields payoff $\zeta \cdot v$, and then moves to the next period. The cost c and the benefit v are parameters for which we assume $c < \lambda v$. We also assume risk neutrality (i.e. we measure the costs and benefits in utils). The expected payoff per period is $(\lambda v - c)\,\Delta > 0$ for a good player and $-c\Delta < 0$ for a bad player. This means that if the players knew their types, bad types would exit immediately, and good types would never exit.[4]

Within each period the players act simultaneously, but they know each others' previous actions. However, they do not observe each others' signals, and therefore they do not know whether the others are informed or uninformed.

The history of player i consists of the private history of her own signals, and the public history consisting of the actions of all the players. Since a positive signal reveals fully the player's type, the uninformed have never observed the good signal. Conditional on a good signal, it is a dominant strategy to stay in the game forever. Strategies are therefore fully described by the exit behavior of the uninformed players. For the uninformed players, all relevant information is contained in the public history of past actions, and therefore we call this public information simply the *history*. Formally, a history h^t in period t is a sequence of actions:

$$h^t = \left\{a^0, a^1, ..., a^{t-1}\right\},$$

where $a^t = (a_1^t, ..., a_N^t)$. Denote by H^t the set of all such histories up to t and let $H = \bigcup_{t=0}^{\infty} H^t$. A history $h^\infty = \{a^t\}_{t=0}^{\infty}$ gives a sequence of action profiles for the entire game.

A (behavior) strategy for an uninformed player i is a mapping

$$\sigma_i : H \to [0, 1]$$

that maps all histories where i is active to an exit probability. A strategy profile in the

[4]In Section 7, we discuss the possibility that the value of the signal is different in the two states.

game is a vector $\boldsymbol{\sigma} = (\sigma_1, ..., \sigma_N)$.

A player maximizes her expected discounted sum of future cash flows as estimated on the basis of her own signal history, observations of other players' behavior, and initial prior probability q^0. By equilibrium, we mean a Perfect Bayesian Equilibrium of the above game. In an equilibrium, all actions in the support of $\sigma_i(h^t)$ are best responses to $\boldsymbol{\sigma}_{-i}$ for all i and for all h^t.

2.1 Interpretation as an Investment Game

We can interpret the game as an investment model where a number of firms have the option of making an irreversible investment. The project qualities are correlated across firms. A good project yields $c\Delta > 0$ per period whereas a bad project yields $(c - \lambda v)\Delta < 0$ per period. The fixed investment cost is normalized to zero. Before undertaking the project, each firm learns about the quality of her individual potential project as follows. In each period, a "failure" signal occurs with probability $\lambda\Delta$ if and only if the project is bad (see Décamps & Mariotti (2004) for another investment model with this kind of learning).

To see that this is equivalent to our exit game, consider the capitalized value of undertaking the action "exit" in our original model. If the player type is bad, then by exiting the player avoids the fixed cost $c\Delta$ from today to eternity. Therefore the capitalized value of exit is equal to the value of investing in a "good" project in the investment model. A "good" type in the original model avoids the cost $c\Delta$ by exiting, but at the same time she forgoes the expected payoff $\lambda v\Delta$ per period. The net capitalized value of exit is then equal to the value of investing in a "bad" project. This shows that the two models are isomorphic (with "good" types interpreted as "bad" projects and "bad" types as "good" projects).

3 Beliefs

In this section we fix a strategy profile $\boldsymbol{\sigma}$ and describe the two different forms of learning in our model. First, as long as a player stays in the game, she receives in every period a direct signal on her type. The strength of this signal is exogenously given, and Bayesian updating resulting from such signals has been studied extensively in the literature. Subsection 3.1 below describes this private learning.

The second form of learning depends on the publicly observed actions and is endogenous in our model. Since all past exit decisions are observable to all players remaining in the game, these decisions may convey information on the aggregate state. Since types are correlated with states, this information is also relevant for learning each player's own type. We let $\xi_i^\theta(h^t)$ denote the probability with which player i exits after history h^t in state θ. If we denote by $\pi_i^\theta(h^t)$ the posterior probability with which player i is uninformed given history h^t, we have

$$\xi_i^\theta\left(h^t\right) := \sigma_i\left(h^t\right)\pi_i^\theta\left(h^t\right). \tag{1}$$

As long as $\xi_i^H(h^t) \neq \xi_i^L(h^t)$, other players learn about the true aggregate state by observing the exit decisions of player i. In contrast to the individual learning, the information content in observational learning from others is endogenous. First of all, if $\sigma_i(h^t) = 0$, then $\xi_i^H(h^t) = \xi_i^L(h^t) = 0$ and exit decisions are uninformative. Second, the amount of information depends on $\pi_i^\theta(h^t)$. It is important to note that $\pi_i^\theta(h^t)$ depends on $\sigma_i(h^s)$ for $s < t$ and hence is also endogenously determined. Subsection 3.2 describes how beliefs are updated based on observational learning from others. Subsection 3.3 combines the two forms of learning to derive the beliefs of an uninformed player in our model.[5]

3.1 Private Learning

We start with the analysis of an isolated player that can only learn from her own signals. Denote by p_t the current belief of an uninformed player about her type, i.e.

$$p_t := \Pr\{\text{"type is good"} \mid \text{"player } i \text{ is uninformed in period } t\text{"}\}.$$

If the player continues for another period, and does not receive a positive signal, the new posterior p_{t+1} is obtained by Bayes' rule:

$$p_{t+1} = \frac{p_t(1-\lambda\Delta)}{p_t(1-\lambda\Delta)+1-p_t}. \tag{2}$$

The updating formula is essentially the same if the player knows the true aggregate state. We let p_t^θ denote the player's belief on her own type conditional on state θ. Using $p_0^\theta = \rho^\theta$, equation (2) gives us the formula for p_t^θ:

$$p_t^\theta = \frac{\rho^\theta(1-\lambda\Delta)^t}{\rho^\theta(1-\lambda\Delta)^t+(1-\rho^\theta)}. \tag{3}$$

Notice that p_t^θ is a monotonic function of t and since it conditions on the state of the world, it will not be affected by learning from others.

[5]The reader more interested in the equilibrium analysis may jump to Section 4 and return to the present section as needed.

To make the connection to the case with learning from others, we also describe the isolated player's beliefs on the aggregate state. We let q_t denote the probability that the individual player assigns on the state being H. By the law of iterated expectation,

$$p_t = q_t p_t^H + (1 - q_t)\, p_t^L. \tag{4}$$

Therefore

$$q_t = \frac{p_t - p_t^L}{p_t^H - p_t^L}, \tag{5}$$

and as a consequence, we see that the beliefs p_t and q_t move tightly together.

3.2 Observational Learning

To describe observational learning in our model, we consider for the moment how player i learns from the behavior of players $j \neq i$. We denote by $\widehat{q}_i(h^t)$ the belief of i on the aggregate state, when learning is based only on the behavior of the other players. Alternatively, we may think of $\widehat{q}_i(h^t)$ as the belief of player i as an *outside observer* to the game.

Recall that $\xi_j^\theta(h^t)$ denotes the probability with which an active player $j \in \mathcal{N}(h^t)$ exits at history h^t. If $\xi_j^L(h^t) > \xi_j^H(h^t)$ and j does not exit, then Bayes' rule implies that i believes that j is more likely to be informed. As a result, i also believes that state H is relatively more likely. To describe the belief updating, we denote by $A_{-i}(h^t)$ the random vector containing the actions of all active players, excluding i, at history h^t. The probability of a given exit vector a_{-i}^t is then:

$$P_\theta\left(A_{-i}\left(h^t\right) = a_{-i}^t\right) = \prod_{j \neq i} \left((1 - a_j^t)\xi_j^\theta\left(h^t\right) + a_j^t\left(1 - \xi_j^\theta\left(h^t\right)\right)\right), \tag{6}$$

where we use shorthand notation P_θ to denote probability conditional on state:

$$P_H(\cdot) := \Pr(\cdot\,|\theta = H),\; P_L(\cdot) := \Pr(\cdot\,|\theta = L).$$

After observing the exit vector a_{-i}^t, player i updates her belief $\widehat{q}_i(h^t)$ according to Bayes' rule as follows:

$$\widehat{q}_i\left(h^{t+1}\right) = \frac{\widehat{q}_i(h^t)\, P_H\left(A_{-i}(h^t) = a_{-i}^t\right)}{\widehat{q}_i(h^t)\, P_H\left(A_{-i}(h^t) = a_{-i}^t\right) + (1 - \widehat{q}_i(h^t)) P_L\left(A_{-i}(h^t) = a_{-i}^t\right)}. \tag{7}$$

Note that $\widehat{q}_i(h^{t+1})$ depends on $\xi_j^\theta(h^t)$ through (6), which in turn depends on $\pi_j^\theta(h^t)$ through (1). Therefore, in order to complete the description of observational learning in our model, we must also specify the evolution of $\pi_i^\theta(h^t)$, $i = 1, ..., N$, for the fixed strategy profile $\boldsymbol{\sigma}$.

At the beginning of each period exit decisions are realized. If player i continues, then the other players calculate their updated beliefs using Bayes' rule as follows:

$$\pi_i'^{\theta}\left(h^t\right)=\frac{\pi_i^{\theta}\left(h^t\right)\left(1-\sigma_i\left(h^t\right)\right)}{1-\sigma_i\left(h^t\right)\pi_i^{\theta}\left(h^t\right)},\ \theta\in\{H,L\}. \tag{8}$$

In the second step, uninformed players become informed within the current period with probability $1-p_t^{\theta}\lambda\Delta$ (conditional on not exiting). Combining these two steps, the updated belief after history h^{t+1} is:

$$\pi_i^{\theta}\left(h^{t+1}\right)=\frac{\pi_i^{\theta}\left(h^t\right)\left(1-\sigma_i\left(h^t\right)\right)}{1-\sigma_i\left(h^t\right)\pi_i^{\theta}\left(h^t\right)}\cdot\left(1-p_t^{\theta}\lambda\Delta\right),\ \theta\in\{H,L\}. \tag{9}$$

3.3 Combined Learning

The remaining task is to combine the two forms of learning to derive the beliefs of the uninformed players in the game. The easiest way of doing this connects the information contained in $\widehat{q}_i\left(h^{t+1}\right)$ with the information that i is uninformed.

Let π_t^{θ} denote the ex-ante probability with which a player (that stays in the game with probability 1) is uninformed in period t, conditional on state θ. The player is a bad type with probability $\left(1-\rho^{\theta}\right)$, and all bad types remain uninformed. The player is a good type with probability ρ^{θ}, and good types remain uninformed with probability $(1-\lambda\Delta)$ in each period. Hence we have:

$$\pi_t^{\theta}=\left(1-\rho^{\theta}\right)+\rho^{\theta}\left(1-\lambda\Delta\right)^t. \tag{10}$$

We denote by $q_i\left(h^t\right)$ the belief of i on the aggregate state (conditional on being uninformed). This belief differs from $\widehat{q}_i\left(h^t\right)$ only to the extent that the private history of i affects her belief, and therefore the relationship between the two is given by Bayes' rule as follows:

$$q_i\left(h^t\right)=\frac{\widehat{q}_i\left(h^t\right)\pi_t^H}{\widehat{q}_i\left(h^t\right)\pi_t^H+\left(1-\widehat{q}_i\left(h^t\right)\right)\pi_t^L}. \tag{11}$$

The belief of i on her own type follows now from the law of iterated expectation:

$$\begin{aligned}p_i\left(h^t\right) &= q_i\left(h^t\right)p_t^H+\left(1-q_i\left(h^t\right)\right)p_t^L\\ &= \frac{\widehat{q}_i\left(h^t\right)\pi_t^H p_t^H+\left(1-\widehat{q}_i\left(h^t\right)\right)\pi_t^L p_t^L}{\widehat{q}_i\left(h^t\right)\pi_t^H+\left(1-\widehat{q}_i\left(h^t\right)\right)\pi_t^L}.\end{aligned} \tag{12}$$

where p_t^L and p_t^H are given by (3).

We end this section with two propositions that characterize learning in our model. First, we show that the likelihood of exit in state L is strictly larger than the likelihood of exit in H. Furthermore, the likelihood ratio of exit across the states is increasing over time. This guarantees that an exit is always informative about the aggregate state.

Equation (10) implies that the ex-ante likelihood ratio across states of being uninformed changes monotonically over time:

$$\frac{\pi_t^L}{\pi_t^H} > \frac{\pi_{t-1}^L}{\pi_{t-1}^H} > \cdots > \frac{\pi_1^L}{\pi_1^H} = \frac{1-\rho^L\lambda\Delta}{1-\rho^H\lambda\Delta} > 1. \tag{13}$$

With this observation at hand, we can prove our first result on the informativeness of exits.

Proposition 1 *For any strategy profile $\boldsymbol{\sigma}$, we have*

$$\frac{\xi_i^L\left(h^t\right)}{\xi_i^H\left(h^t\right)} \geq \frac{\pi_t^L}{\pi_t^H} > 1$$

for all $t > 0$ and h^t such that $\sigma_i\left(h^t\right) > 0$.

We give next a proposition that ranks strategy profiles according to their informativeness. For a profile $\boldsymbol{\sigma}$ and a history h^t, we let the random variable $P_i^{t+1}\left(h^t, \boldsymbol{\sigma}\left(h^t\right)\right)$ denote the posterior of player i on her own type at the beginning of period $t+1$, assuming that she is uninformed at the beginning of period t. The randomness in the posterior arises from i's private signal realization and the realized exit decisions of the players other than i. The following Proposition shows that higher exit probabilities by other players induce a mean preserving spread (in the sense of Rothschild and Stiglitz, 1970) on the posterior.

Proposition 2 *Take an arbitrary history h^t with $t > 0$ and two strategy profiles $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ with $\boldsymbol{\sigma}\left(h^{t'}\right) = \boldsymbol{\sigma}'\left(h^{t'}\right)$ for $t' = 0, ..., t-1$. Then $P_i^{t+1}\left(h^t, \boldsymbol{\sigma}\left(h^t\right)\right)$ dominates $P_i^{t+1}\left(h^t, \boldsymbol{\sigma}'\left(h^t\right)\right)$ in the sense of second order stochastic dominance if $\sigma_j'\left(h^t\right) \geq \sigma_j\left(h^t\right)$ for all $j \neq i$ and $\sigma_j'\left(h^t\right) > \sigma_j\left(h^t\right)$ for some $j \neq i$.*

The economic content of this Proposition is rather immediate. Since all players' types are correlated with the state of the world θ, having maximal information on the state is also maximal information on an individual type. The total amount of information available to the players is captured by the vector of information types for the players, i.e. an enumeration of all players that are informed. A pure strategy profile $\boldsymbol{\sigma}\left(h^t\right) = 1$ transmits all this information, since under this strategy players exit if and only if they are uninformed. The profile with $\boldsymbol{\sigma}\left(h^t\right) = 0$ conveys no information. Any intermediate exit probability can be seen as a convex combination of these two signal structures, and it is to be expected that the combination with a higher weight on the informative signal is more informative with respect to the true informational state of a player.

4 Equilibrium Analysis

4.1 Isolated Player

It is again useful to start with the case of an isolated player. The decision problem of the isolated player is to choose whether to continue or exit at period t. Standard arguments show that the problem is a Markovian optimal stopping problem with the posterior probability $p = p_t$ as the state variable. We let $V_m(p)$ denote the value function of the isolated player. Stopping at posterior p yields a payoff of 0. If the player continues, she pays the cost $c\Delta$, and gets a positive signal with probability $p\lambda\Delta$. In this case, the player learns that her expected payoff per period is $(\lambda v - c) \cdot \Delta$, and thus the value function jumps to

$$V^+ := V_m(1) = \frac{(\lambda v - c) \cdot \Delta}{1 - \delta}.$$

Without a positive signal p falls to p_{t+1} according to (2). The Bellman equation for the optimal stopping problem can thus be written as:

$$V_m(p) = \max\left\{0, -c\Delta + p\lambda\Delta\left(v + \delta V_m(1)\right) + (1 - p\lambda\Delta)\,\delta V_m\left(\frac{p(1-\lambda\Delta)}{p(1-\lambda\Delta) + (1-p)}\right)\right\}. \tag{14}$$

The optimal policy is to stop as soon as p falls below a threshold level, denoted $p^*(\Delta)$. Standard arguments establish that value function $V_m(p)$ is increasing, convex and continuous in p. The threshold $p^*(\Delta)$ is obtained from (14) by setting $V_m(p^*(\Delta)) = 0$:

$$p^*(\Delta) = \frac{c}{\lambda\left(v + \delta V_m(1)\right)}. \tag{15}$$

We shall see that $p^*(\Delta)$ plays a crucial role also in the model with many players. Denote by $t^*(\Delta)$ the period in which p falls below $p^*(\Delta)$ if there is no positive signal:

$$t^*(\Delta) := \min\left\{t \in \mathbb{N} \,|p_t \leq p^*(\Delta)\right\}.$$

We denote the optimal strategy of the isolated player by

$$a_m(p_t) = \begin{cases} 1 \text{ if } p_t > p^*(\Delta), \\ 0 \text{ if } p_t \leq p^*(\Delta). \end{cases}$$

4.2 Symmetric Equilibrium

In this subsection, we show that the exit game with observational learning has a unique symmetric equilibrium.[6] Furthermore, the equilibrium value functions of the individual

[6] The game has also asymmetric equilibria, where the players act in a predetermined order conditioning their actions on the outcomes of the previous moves by the other players. Since the properties of such

players can be written as functions of their belief on their own type only. With symmetric strategies all uninformed players have identical beliefs, and therefore we drop the subscripts i from the beliefs and the strategies of the uninformed players. In particular, we let $\sigma(h^t)$ denote the probability with which each uninformed player exits at history h^t in symmetric equilibrium.

We start by showing that if a symmetric equilibrium exists for the stopping game, then the equilibrium value function $V(h^t)$ is closely related to the value function of the isolated player. The key observation for this result is that as long as $\sigma(h^t) = 0$, there is no observational learning and thus the information available in the game is identical to the information available to the isolated player. On the other hand, when $\sigma(h^t) > 0$, the players can learn from each other, but their the value must be zero since they exit with a positive probability.

Lemma 1 *For any symmetric equilibrium of the exit game,*

$$V\left(h^t\right) = V_m\left(p\left(h^t\right)\right).$$

Lemma 1 allows us to derive recursively the symmetric equilibrium strategy profile. To see this, note that if the symmetric equilibrium is given for periods $0, ..., t-1$, we can calculate the beliefs of uninformed players at history h^t as explained in Section 3. Consider then exit probabilities at history h^t. By Lemma 1, the payoff for the next period is given by $V(h^{t+1}) = V_m(p(h^{t+1}))$, and therefore, all we have to do is to find an exit probability $\sigma(h^t)$ that induces a probability distribution for $p(h^{t+1})$ that makes the players indifferent between exiting and staying. This indifference condition must equate the discounted expected value for the next period with the cost of staying for one period, so we can write it as:

$$\delta \mathbb{E} V_m\left(P^{t+1}\left(h^t; \sigma\left(h^t\right)\right)\right) = c\left(h^t\right) \cdot \Delta,$$

where we use notation $c(h^t)$ to denote cost of staying net of expected payoff per time unit:

$$c\left(h^t\right) := c - p\left(h^t\right)\lambda v.$$

The next Lemma shows that increasing the exit probabilities for the current period increases the players' incentive to stay. This result follows from two observations. First, by Proposition 2, increasing the exit probability for the current period induces a mean preserving spread for the next period belief $P^{t+1}(h^t, \sigma(h^t))$. Second, we know from Subsection 4.1 that the isolated player's value function V_m is convex.

equilibria are essentially the same as in the herding models with exogenous order of moves, we do not discuss this issue further (details about asymmetric equilibria are available from authrors upon request).

Lemma 2 *The expected continuation payoff* $\mathbb{E}V_m\left(P^{t+1}\left(h^t;\sigma\left(h^t\right)\right)\right)$ *is weakly increasing in* $\sigma\left(h^t\right)$. *Furthermore, for each* h^t *there is at most one exit probability* $\sigma\left(h^t\right)$ *satisfying*

$$\delta\mathbb{E}V_m\left(P^{t+1}\left(h^t;\sigma\left(h^t\right)\right)\right)=c\left(h^t\right)\cdot\Delta.$$

Lemma 2 guarantees that for each h^t, at most one exit probability makes the players indifferent between exiting and staying. However, it may not be possible to induce the players to stay in the game. The pure strategy profile $\boldsymbol{\sigma}\left(h^t\right)=[1,...,1]$ releases all information available to the players. Nevertheless, it is possible that the uninformed players are so pessimistic that even the release of all this information is not sufficient to compensate for the one-step loss of $c\left(h^t\right)\Delta$ from waiting. When this is the case, all uninformed players exit with probability 1, and we say that the game collapses.

With these preliminaries, we are ready to prove the main result of this section.

Theorem 1 *The stopping game has a unique symmetric equilibrium where the exit probability at history* h^t *is given by:*

$$\sigma\left(h^t\right)=\begin{cases}0 & \text{if } \delta\mathbb{E}V_m\left(P^{t+1}\left(h^t;0\right)\right)>c\left(h^t\right)\Delta,\\ \sigma^*\left(h^t\right)\in[0,1] & \text{if } \delta\mathbb{E}V_m\left(P^{t+1}\left(h^t;0\right)\right)\leq c\left(h^t\right)\Delta\leq\delta\mathbb{E}V_m\left(P^{t+1}\left(h^t;1\right)\right),\\ 1 & \text{if } \delta\mathbb{E}V_m\left(P^{t+1}\left(h^t;1\right)\right)<c\left(h^t\right)\Delta,\end{cases}$$

where $\sigma^*\left(h^t\right)$ *solves for all* h^t

$$\delta\mathbb{E}V_m\left(P^{t+1}\left(h^t;\sigma^*\left(h^t\right)\right)\right)=c\left(h^t\right)\Delta.$$

The symmetric equilibrium has a simple structure. Whenever the players' beliefs on their own type are above the threshold of the isolated player, i.e. $p\left(h^t\right)>p^*\left(\Delta\right)$, then $\delta\mathbb{E}V_m\left(P^{t+1}\left(h^t;0\right)\right)>c\left(h^t\right)\Delta$ and thus the equilibrium actions coincide with those prescribed by the optimal decision rule of the isolated player (i.e. stay). On the other hand, if the players are very pessimistic, then $\delta\mathbb{E}V_m\left(P^{t+1}\left(h^t;1\right)\right)<c\left(h^t\right)\Delta$, and again equilibrium actions coincide with the isolated player (i.e. exit with probability one). With intermediate beliefs equilibrium behavior differs from isolated player: in equilibrium the exits take place with a probability that exactly balances the players' incentives to exit and wait, whereas an isolated player would exit with probability one.

We end this section with a corollary that gives a lower bound for $p\left(h^t\right)$ along histories where no player has exited. This will be useful in obtaining a sharper characterization of the symmetric equilibrium behavior in the limit where the time interval between successive periods is short. To see how the corollary follows from Theorem 1, note that in order to have $\delta\mathbb{E}V_m\left(P^{t+1}\left(h^t;1\right)\right)>0$, there must be a positive probability that $p\left(h^{t+1}\right)>p^*\left(\Delta\right)$.

Since the highest possible value for $p(h^{t+1})$ results when no player exits at history h^t, it follows that whenever $p(h^t)$ is below p^*, the next period belief $p(h^{t+1})$ is again above p^* if no player exits.

Corollary 1 *Consider the history $h^t = (\mathbf{1}, \mathbf{1}, ..., \mathbf{1})$, i.e. the history without any exits. In the symmetric equilibrium,*

$$p\left(h^t\right) \geq \frac{p^*(\Delta) \cdot (1-\lambda\Delta)}{p^*(\Delta) \cdot (1-\lambda\Delta) + 1 - p^*(\Delta)}.$$

5 Information Aggregation in Large Games

In this section, we consider information aggregation in the symmetric equilibrium of the game as the number of players grows without bound. As a point of comparison, we use the case where the players share all information with each other. If the number of players is large and all information is pooled, then the (weak) law of large numbers implies that the players can determine the true aggregate state with arbitrarily high accuracy. Nevertheless, idiosyncratic uncertainty about player types remains: conditional on the state, each player is still uncertain about her own type. As a result, the efficient benchmark in terms of information aggregation for large games is the one where all the uninformed players know the aggregate state θ. In state θ , an uninformed player believes that she is a good type with probability p_t^θ. Therefore it is optimal for her to exit as soon as p_t^θ falls below $p^*(\Delta)$. We denote the efficient exit period in state θ by $t_\theta^*(\Delta)$:

$$t_\theta^*(\Delta) := \min\left\{t : p_t^\theta \leq p^*(\Delta)\right\}, \ \theta = H, L.$$

The main result of this section is Theorem 2, which says that by decreasing the period length, we eliminate the possibility that a large number of players exit too early relative to this efficient benchmark. This means that for large games, information is aggregated efficiently in state $\theta = H$ because $t_H^*(\Delta)$ is an upper bound for all exit times of the uninformed players. However, if $\theta = L$, information aggregation fails: all players exit too late in expectation.

Since we vary the number of players N and period length Δ while keeping all the other parameters of the model fixed, we denote by $\Gamma(\Delta, N)$ the game parametrized by Δ and N. We denote by $X(h^t)$ the number of players that exit the game at history h^t in the unique symmetric equilibrium of the game:

$$X\left(h^t\right) := n\left(h^t\right) - n\left(h^{t+1}\right).$$

As a first step towards Theorem 2, we consider the effect of a large number of exits on the beliefs. In Proposition 1, we showed that individual exit probabilities are different across the two states, which allows the players to make inferences based on observed exits. It is therefore natural to expect that if a large number of players exit, then all the remaining players should have accurate beliefs on the aggregate state. Proposition 3 shows that this must indeed be the case with a high probability:

Proposition 3 *For all $\varepsilon > 0$, there is some $K \in \mathbb{N}$ such that*

$$P_H\{h^\infty \;:\; n\left(h^t\right) \leq N - K \text{ and } q\left(h^t\right) < 1 - \varepsilon \text{ for some } h^t \in h^\infty\} < \varepsilon, \tag{16}$$

$$P_L\{h^\infty \;:\; n\left(h^t\right) \leq N - K \text{ and } q\left(h^t\right) > \varepsilon \text{ for some } h^t \in h^\infty\} < \varepsilon, \tag{17}$$

for any game $\Gamma(\Delta, N)$.

A couple of remarks are in order. First, Proposition 3 holds for all strategy profiles $\boldsymbol{\sigma}$ (equilibrium or not) as long as some private information has been accumulated before first exits. Second, the bound K for the number of exits in the Proposition is independent of Δ and N. By increasing N, we can make sure that the state is revealed if an arbitrarily small fraction of players exit.

Proposition 3 implies that once K players have exited and $\theta = H$, then with a high probability, no further exits take place before $t^*_\theta(\Delta)$. This would suggest that the total number of suboptimally early exits must be bounded. However, we must also consider the possibility that an unbounded number of players exit within *a single period* before they learn the true state. Our second step towards Theorem 2 is to show that by reducing period length Δ, we can eliminate this possibility. This is established in Proposition 4 below.

We need some notation to keep track of the passage of real time as we vary Δ.[7] Let τ_θ denote the efficient exit time corresponding to state θ in the limit $\Delta \to 0$:

$$\tau_\theta := \lim_{\Delta \to 0} \left[t^*_\theta(\Delta) \cdot \Delta\right], \theta = H, L.$$

To link real time to the corresponding period of a discrete time model, we define $t(\tau, \Delta)$ as the last period before an arbitrary real time τ:

$$t(\tau, \Delta) := \max\{t : t \cdot \Delta \leq \tau\}. \tag{18}$$

[7] This notation will also be useful in the following Section where the continuous time limit of the model is considered.

Proposition 4 *For all $\tau < \tau_H$ and $\varepsilon > 0$, there are constants $\overline{\Delta} \in \mathbb{R}^+$ and $K \in \mathbb{N}$ such that*

$$P_H\{h^\infty : X\left(h^t\right) > K \text{ for some } t \leq t\left(\tau, \Delta\right)\} < \varepsilon,$$

for any game $\Gamma\left(\Delta, N\right)$ with $\Delta < \overline{\Delta}$.

The proof of Proposition 4 is lengthy, but the intuition is straight-forward. If the players were to adopt a strategy that induces a large number of exits with non-negligible probability within a single period, then this would generate a very informative signal about the state. For all $\tau < \tau_H$, the value of such a signal is positive. If the waiting cost is low enough, then all the players would have a strict preference to observe the signal rather than exit contradicting the hypothesized positive probability of exits.

Combining Propositions 3 and 4 gives us Theorem 2, which bounds the total number of suboptimally early exits in the game. The result means that in the double limit where we increase the number of players and reduce the period length, the fraction of players that exit suboptimally early shrinks to zero, and thus, information is aggregated efficiently if $\theta = H$. Nevertheless information aggregation fails if $\theta = L$ since in that state, the players exit *too late.*

Theorem 2 *For all $\tau < \tau_H$ and $\varepsilon > 0$, there are constants $\overline{\Delta} \in \mathbb{R}^+$ and $K \in \mathbb{N}$ such that*

$$P_H\left\{h^\infty : \sum_{t=0}^{t(\tau,\Delta)} X\left(h^t\right) > K\right\} < \varepsilon,$$

for any game $\Gamma\left(\Delta, N\right)$ with $\Delta < \overline{\Delta}$.

We end this section with a remark on the restriction to the symmetric equilibrium. Although we have assumed symmetry throughout this section, there is very little that depends on this restriction. The proof of Proposition 3 is valid for any asymmetric equilibrium strategy profile as well. The proof of Proposition 4 uses symmetry in two lemmas (Lemma 6 and Lemma 8 in the Appendix). However, even there symmetry is used for notational convenience (the number of exits within a period is binomially distributed, which leads more easily to the desired results).

6 Exit Waves

In this section, we characterize the symmetric equilibrium in the limit as $\Delta \downarrow 0$. We have several reasons for this. The first reason is substantive. In a model with endogenous timing decisions, it is important to know if the results depend on an exogenously imposed reaction lag Δ. Second, it turns out that the inherent dynamics of the model are best displayed in the limit: information aggregation happens in randomly occurring bursts of sudden activity. We call these bursts of activity *exit waves.* Third, when we also let $N \to \infty$, we can compute the statistical properties of the equilibrium path in an explicit form.

We may view the public history h^∞ generated by the symmetric equilibrium $\boldsymbol{\sigma}(\Delta, N)$ in the game $\Gamma(\Delta, N)$ with period length Δ and N players from a slightly different angle. Suppose that the players are to be treated anonymously. Then the vector $\boldsymbol{t}(\Delta, N) = (t_1(\Delta, N), ..., t_N(\Delta, N))$ where $t_k(\Delta, N)$ indicates the period in which k^{th} exit took place gives a full account h^∞. The profile $\boldsymbol{\sigma}(\Delta, N)$ induces a probability distribution on $\mathbb{R}^N$ on instants of exit measured in continuous time τ (that is, k^{th} exit takes place at time $\tau_k = t_k(\Delta, N) \cdot \Delta$). We denote this distribution, conditional on state, by $F^\theta_{\Delta,N}(\boldsymbol{\tau})$. We investigate the limiting distribution

$$F^\theta_N(\boldsymbol{\tau}) = \lim_{\Delta \downarrow 0} F^\theta_{\Delta,N}(\boldsymbol{\tau}),$$

where the convergence is taken to be in the sense of weak convergence. Observational learning then results from the differences between $F^H_N(\boldsymbol{\tau})$ and $F^L_N(\boldsymbol{\tau})$.

In Subsection 6.1 we keep the number of players fixed at N. We show that when there was no exit in the previous period, the proability of exit within the current period is of the order Δ. This means that exits arrive according to a well defined hazard rate, and we say that the game is in the *flow mode.* On the other hand, if there was an exit in the previous period, then the probability of exit in the current period is bounded away from zero, and we say that the game is in an *exit wave.* We also show that each exit wave ends in collapse with a strictly positive probability.

In Subsection 6.2, we consider the limiting distributions

$$F^\theta(\boldsymbol{\tau}) = \lim_{N \to \infty} F^\theta_N(\boldsymbol{\tau})$$

defined on the set of sequences of exit times $\{\tau_k\}_{k=1}^\infty$. In particular, we compute the distributions for the first K exit instants and we also calculate the probability of the event that the market collapses by time instant τ, i.e. the probability of the event $\{\tau_k \leq \tau\}$ for all k. We make use of Poisson approximations and Theorem 2 when computing the size of exit events and the probability of collapse given an exit event.

6.1 The Structure of the Symmetric Equilibrium

In this subsection, we keep the number of players N fixed. Since we are interested in the limit $\Delta \to 0$, we parametrize the game and its histories with the period length Δ. We say that the game is in the *flow mode* at history h^t if no players exited at history h^{t-1}, i.e. if $X(h^{t-1}) = 0$. The game is in an *exit wave* at history h^t if $X(h^{t-1}) > 0$. Finally, we say that the game *collapses* at history h^t if $\sigma(h^t) = 1$. Collapse is an absorbing state: since all uninformed players exit, the game is effectively over, and $\pi\left(h^{t'}\right) = 0$ for all $t' > t$. This means that for a game with a given Δ, we have three mutually exclusive sets of histories, corresponding to flow mode, exit wave, and collapse, respectively:

$$\begin{aligned} H^f(\Delta) &:= \left\{h^t : X\left(h^{t-1}\right) = 0 \text{ and } \pi\left(h^t\right) > 0\right\} \\ H^e(\Delta) &:= \left\{h^t : X\left(h^{t-1}\right) > 0 \text{ and } \pi\left(h^t\right) > 0\right\} \\ H^c(\Delta) &:= \left\{h^t : \pi\left(h^t\right) = 0\right\} \end{aligned}$$

In order to relate the discrete decision periods to real time instants, we define

$$p^* := \lim_{\Delta \downarrow 0} p^*(\Delta),$$
$$\tau^* := \lim_{\Delta \downarrow 0} t^*(\Delta) \cdot \Delta,$$

where $p^*(\Delta)$ and $t^*(\Delta)$ are understood as the belief threshold and the exit time as defined in Section 4.1.

We start by showing that the beliefs of the uninformed players are qualitatively different in the two active modes. When the game is in the flow mode and Δ is small, beliefs are close to p^* while in an exit wave, they are bounded away from p^*.

Lemma 3 *i) For all $\varepsilon > 0$, there is a $\overline{\Delta} > 0$ such that*

$$p\left(h^t\right) \in (p^* - \varepsilon, p^* + \varepsilon)$$

for all $h^t \in H^f(\Delta)$, $t \geq t^*(\Delta)$, $\Delta < \overline{\Delta}$.
ii) There is a $\zeta > 0$ and $\overline{\Delta} > 0$ such that

$$p\left(h^t\right) < p^* - \zeta$$

for all $h^t \in H^e(\Delta)$, $\Delta < \overline{\Delta}$.

The following Proposition shows that active players also behave differently in the two modes.

Proposition 5 *i) There is a* $\kappa > 0$ *and a* $\overline{\Delta} > 0$ *such that*

$$P_H\left(X\left(h^t\right) > 0\right) < P_L\left(X\left(h^t\right) > 0\right) < \kappa\Delta \text{ for all } h^t \in H^f\left(\Delta\right), \Delta < \overline{\Delta}.$$

ii) There is a $\underline{p} > 0$ *and a* $\overline{\Delta} > 0$ *such that*

$$P_L\left(X\left(h^t\right) > 0\right) > P_H\left(X\left(h^t\right) > 0\right) > \underline{p} \text{ for all } h^t \in H^e\left(\Delta\right), \Delta < \overline{\Delta}.$$

The first claim in the above Proposition justifies our use of the term *flow mode.* The flow mode comes to an end with a well-defined hazard rate. The actual computation of the equilibrium hazard rate is not hard in principle. Nevertheless, the formula will depend on the evolution of $\pi^\theta\left(h^t\right)$ and it is not possible to give a closed-form solution for the continuous-time limit of the updating formula (9).[8] In the following subsection, we compute the explicit hazard rate in the limit as $N \to \infty$.

The second point to note is that for all finite N, any exit wave ends in a finite number of periods. Hence the real time that an exit wave lasts is bounded from above by ΔN and vanishes as $\Delta \to 0$. As a result, we may view the limit exit waves as (rather complicated) randomization events between the flow mode and collapse.

Finally, every exit wave results in a collapse with a strictly positive probability. Since an exit wave takes only a vanishing amount of real time, learning from own experience during the wave can be ignored. Recall that $p\left(h^t\right)$ is a martingale, and its value in exit wave mode is bounded away from p^* by part ii) of Lemma 3. Therefore, we conclude that a return to the flow mode cannot happen with probability 1. If the number of players is small, then the first exit starting a wave may in fact lead to an immediate collapse. Then the exit wave lasts only one period and ends up in collapse with probability one. If this is not the case, then a martingale argument establishes that the game must return to the flow mode with a strictly positive probability.

6.2 Exit Events in Large Games

The large game limit $N \to \infty$ simplifies the computations for a number of reasons. First, we can use Poisson approximations of the Binomial distribution for the number of exits within each period of an exit wave. Second, conditional on the game not having collapsed, we know that $\pi_i^\theta\left(h^t\right) \approx \pi_t^\theta$ for all t, which allows us to use the continuous time limit of equation (10) to compute the conditional probabilities for the players to be uninformed. This follows from the fact that with a large number of players, each individual player exits

[8] The complication arises because this updating depends on the equilibrium randomization probabilities $\sigma\left(h^t\right)$.

with a negligible probability (until collapse). Third, we can apply Theorem 2, which says that the probability of a collapse in state $\theta = H$ before τ_H vanishes as $N \to \infty$.

Let $p^\theta(\tau)$ and $\pi^\theta(\tau)$ denote the continuous time limits of (3) and (10):

$$\begin{aligned} p^\theta(\tau) &:= \frac{\rho^\theta e^{-\lambda\tau}}{(1-\rho^\theta)+\rho^\theta e^{-\lambda\tau}}, \\ \pi^\theta(\tau) &:= \left(1-\rho^\theta\right)+\rho^\theta e^{-\lambda\tau}, \end{aligned}$$

and let $q^*(\tau)$ denote the belief on aggregate state that corresponds to the critical private belief p^*:

$$q^*(\tau) := \frac{p^* - p^L(\tau)}{p^H(\tau) - p^L(\tau)},\ \tau_L \le \tau \le \tau_H.$$

Note that $q^*(\tau)$ is strictly increasing within $[\tau_L, \tau_H]$ and $q^*(\tau_L) = 0$ and $q^*(\tau_H) = 1$.

We compute first the hazard rate of exits in the flow mode. In particular, assume that k players have exited the game at real times $\tau_1, ..., \tau_k$, and the game is in flow mode at real time τ.Using the fact that the likelihood ratio of exit across states is given by $\pi^L(\tau)/\pi^H(\tau)$, and the fact that the belief of an uninformed player must stay close to p^* as long as no player exits (as required by Lemma 3), we can determine the hazard rate with which an additional player exits:

Proposition 6 *In the limit $N \to \infty$, the instantaneous hazard rate of $k+1^{st}$ exit at some $\tau \in (\tau_k, \tau_H)$, conditional on the first k exit times $\tau_1, ..., \tau_k$, is given by*

$$\frac{f^\theta_{k+1}(\tau\,|\tau_1, ..., \tau_k)}{1 - F^\theta_{k+1}(\tau\,|\tau_1, ..., \tau_k)} = \pi^\theta(\tau)\,\lambda \frac{p^*(1-p^*)\left(p^H(\tau) - p^L(\tau)\right)}{(p^* - p^L(\tau))\,(p^H(\tau) - p^*)\,(\pi^L(\tau) - \pi^H(\tau))}\ . \tag{19}$$

Since the expression (19) does not depend on $\tau_1, ..., \tau_k$, we may simply denote by $\psi^\theta(\tau)$ the hazard rate with which an exit wave starts at time τ:

$$\psi^\theta(\tau) = \pi^\theta(\tau)\,\lambda \frac{p^*(1-p^*)\left(p^H(\tau) - p^L(\tau)\right)}{(p^* - p^L(\tau))\,(p^H(\tau) - p^*)\,(\pi^L(\tau) - \pi^H(\tau))}.$$

Every exit wave leads either to collapse or a return to flow mode. With a large number of players, it is easy to compute the probabilities with which either possibility occurs. To see this, note that conditional on the flow mode ending at τ, the posterior after the first exit is given by:

$$q^-(\tau) = \frac{\pi^H(\tau)\,q^*(\tau)}{\pi^H(\tau)\,q^*(\tau) + \pi^L(\tau)\,(1 - q^*(\tau))}.$$

By Theorem 2, the game returns to the flow mode with a probability that converges to 1 as $N \to \infty$ in state $\theta = H$. Therefore, if the game collapses, $q(\tau+) = 0$. On the other hand, we know from Lemma 3 that if the game returns to the flow mode, we have

$q(\tau+) = q^*(\tau)$. Let $\phi^\theta(\tau)$ denote the probability of collapse given an exit event at τ, conditional on state θ. The probability of collapse estimated by a player with belief $q^-(\tau)$ is $(1 - q^-(\tau))\,\phi^L(\tau)$. Therefore, by the martingale property of belief for this player we have:[9]

$$q^-(\tau) = \left(1 - \left(1 - q^-(\tau)\right)\phi^L(\tau)\right) q^*(\tau),$$

which gives

$$\phi^L(\tau) = \frac{q^*(\tau) - q^-(\tau)}{q^*(\tau)\left(1 - q^-(\tau)\right)} = \frac{\pi^L(\tau) - \pi^H(\tau)}{\pi^L(\tau)}. \tag{20}$$

Since (19) gives the hazard rate with which an exit wave starts, and (20) gives the probability with which a given exit wave leads to collapse, we get the hazard rate of collapse by multiplying them:

Corollary 2 *In the limit $N \to \infty$, the instantaneous hazard rate of collapse at time $\tau \in (\tau_L, \tau_H)$, conditional on state, and conditional on being in flow mode at τ, is:*

$$\begin{aligned} \chi^H(\tau) &= 0 \\ \chi^L(\tau) &= \lambda \frac{p^*(1-p^*)\left(p^H(\tau) - p^L(\tau)\right)}{\left(p^* - p^L(\tau)\right)\left(p^H(\tau) - p^*\right)} \end{aligned}$$

We describe next the sequence of events within a given exit wave that takes place in real time τ. We use index $s = 1, 2, ...$ to refer to the consequtive periods within the exit wave and S to denote the total number of periods within the wave. Let q_s denote the belief in the s^{th} period of the wave, and let X_s denote the number of exits at that period. Note that since we are considering the limit $\Delta \to 0$, the duration of the exit wave in real time is zero.

Fix a period s and the corresponding belief q_s. Lemma 3 implies that we must have $q_s < q^*(\tau)$. On the other hand, the same lemma implies that if s is the last period of the exit wave (that is, no player exits), then we must have $q_{s+1} = q^*(\tau)$.

Proposition 7 *Consider period s of an exit wave taking place at time τ. As $N \to \infty$, X_s^θ converges in distribution to a Poisson random variable with parameter:*

$$\frac{\pi^\theta(\tau)}{\left(\pi^L(\tau) - \pi^H(\tau)\right)} \log\left(\frac{q^*(\tau)}{\left(1 - q^*(\tau)\right)} \cdot \frac{(1 - q_s)}{q_s}\right) \text{ for } \theta \in \{H, L\}. \tag{21}$$

If the realized number of exits is $X_s = k$, the next period belief is:

$$q_{s+1}(k) := \frac{\left(\pi^H(\tau)\right)^k q^*(\tau)}{\left(\pi^H(\tau)\right)^k q^*(\tau) + \left(\pi^L(\tau)\right)^k \left(1 - q^*(\tau)\right)}$$

[9]If the game has $N < \infty$ players, then collapse will take place at a posterior $q^C > 0$ and as a consequence, the probability of a collapse is higher than in this equation.

Notice that the number of exits in the previous stage is a sufficient statistic for the individual randomization probabilities in the current stage. Hence the limiting process of beliefs on the state of the world can be though of as a non-homogenous Poisson random walk on the positive integers.

The exit event taking place at real time instant τ reverses to flow mode if there is some s such that $X_s = 0$. The Poisson approximation formula above in equation (21), gives the probability that the exit event ends after s periods. The total number of exits at real time instant τ is given by

$$X(\tau) = 1 + \sum_{s=1}^{S} X_s.$$

7 Conclusion

We have analyzed a stopping game, where the players base their decisions on their private information and on the behavior of other players in a similar situation. Other things equal, the ability to observe the actions of others makes the players more willing to postpone their actions. But this, in turn, reduces the informativeness of their actions, thus reducing the incentives to wait. The equilibrium balances these effects and leads to aggregate delays and randomly arriving exit waves. We showed that even when the number of players gets large, aggregate uncertainty persists in equilibrium, and information aggregates gradually until a sudden collapse leads to full revelation of the aggregate state.

We have kept the model as simple as possible in order to highlight the interplay between individual and social learning. A number of generalizations are possible. We could allow the value of a signal in aggregate state θ be v^{θ}. In this case, the value of being informed, V^{θ}, would also depend on the state. As long as the signals accumulate at the same rate across the two states, the analysis remains similar. The main difference compared to the current model would be in the characterization of the indifference condition of the uninformed players. In the current paper, indifference requires (in the continuous time limit) that $p(\tau) = p^*$. In the extended model the analogous requirement would be that

$$\left[q(\tau) p^H(\tau) V^H + (1 - q(\tau)) p^L(\tau) V^L\right]$$

remain constant.

Alternatively, we could write a model where payoff signals arrive at different positive rates across the two states. While the details of the model would change a bit, the main message of the current paper would remain true in this more complicated model. Information would still be released in randomly occurring bursts of activity. In the limiting game with a large number of players, the analysis would, in fact, be almost identical to the current model. The fraction of those players that have not seen a single signal is sufficient for determining the aggregate state by the law of large numbers. Therefore we can analyze information aggregation amongst those players in the same manner as in the previous section.

Finally, a more challenging extension would be to incorporate payoff externalities in the model. The payoff could for example depend on the number of players present in the market. It seems to us that beyond the two-player case, quite different analytical techniques would be needed to cover this case. In our view, this is an interesting and challenging direction for further research.

Appendix: Shorter Proofs

Proof. [Proof of Proposition 1] Since $\xi_i^\theta(h^t) = \sigma_i(h^t)\pi_i^\theta(h^t)$, all we have to do is to show that

$$\frac{\pi_i^L(h^t)}{\pi_i^H(h^t)} \geq \frac{\pi_t^L}{\pi_t^H} \tag{22}$$

for all $t > 0$ (note that $\pi_t^L/\pi_t^H > 1$ follows from (13)).

We use induction. As an induction hypothesis, assume that (22) holds for some $t \geq 0$. Using (9) and (22), we then have

$$\begin{aligned} \frac{\pi_i^L(h^{t+1})}{\pi_i^H(h^{t+1})} &= \left(\frac{\pi_i^L(h^t)}{\pi_i^H(h^t)}\right)\left(\frac{1-\sigma_i(h^t)\pi_i^H(h^t)}{1-\sigma_i(h^t)\pi_i^L(h^t)}\right)\left(\frac{1-p_t^L\lambda\Delta}{1-p_t^H\lambda\Delta}\right) \\ &\geq \frac{\left(1-p_t^L\lambda\Delta\right)}{\left(1-p_t^H\lambda\Delta\right)}\frac{\pi_t^L}{\pi_t^H}. \end{aligned} \tag{23}$$

On the other hand, using (3) and (10), we have:

$$\pi_{t+1}^\theta = \left(1-p_t^\theta\lambda\Delta\right)\pi_t^\theta,\ \theta = H, L.$$

Combining this with (23) gives us the induction step:

$$\frac{\pi_i^L(h^{t+1})}{\pi_i^H(h^{t+1})} \geq \frac{\pi_{t+1}^L}{\pi_{t+1}^H}.$$

Noting that $\pi_i^L(h^0) = \pi_i^H(h^0) = \pi_0^L = \pi_0^H = 1$ gives us:

$$\frac{\pi_i^L(h^0)}{\pi_i^H(h^0)} \geq \frac{\pi_0^L}{\pi_0^H},$$

and therefore, the proof by induction is complete. ■

Proof. [Proof of Proposition 2] Construct an experiment X_i on $\Theta = \{H, L\}$ with outcomes in $S^{X_i} = \{0, 1\}$. The joint probabilities on the states and outcomes are given by the following stochastic matrix P^{X_i}

P^{X_i}	$\theta = H$	$\theta = L$
$s^{X_i} = 1$	$1 - \sigma_i(h^t)\, \pi_i^H(h^t)$	$1 - \sigma_i(h^t)\, \pi_i^L(h^t)$
$s^{X_i} = 0$	$\sigma_i(h^t)\, \pi_i^H(h^t)$	$\sigma_i(h^t)\, \pi_i^L(h^t)$

If we interpret the event $\{\theta = H\}$ as the event that the state is good and the event $\{s^{X_i} = 1\}$ as the decision of player i to stay in the game. The joint probability over (θ, s^{X_i}) simply reflects the conditional exit probabilities given strategy σ.

Consider next another experiment Y_i on Θ with outcomes in $S^{Y_i} = \{0, 1\}$ and the associated stochastic matrix P^{Y_i}

P^{Y_i}	$\theta = H$	$\theta = L$
$s^{Y_i} = 1$	$1 - \sigma_i'(h^t)\, \pi_i^H(h^t)$	$1 - \sigma_i'(h^t)\, \pi_i^L(h^t)$
$s^{Y_i} = 0$	$\sigma_i'(h^t)\, \pi_i^H(h^t)$	$\sigma_i'(h^t)\, \pi_i^L(h^t)$

with $\sigma_i'(h^t) > \sigma_i(h^t)$ .Then we can write

$$P^{X_i} = \Phi P^{Y_i},$$

where the stochastic matrix Φ is given by:

Φ	$s^{Y_i} = 1$	$s^{Y_i} = 0$
$s^{X_i} = 1$	1	$\frac{\sigma_i'(h^t) - \sigma_i(h^t)}{\sigma_i'(h^t)}$
$s^{X_i} = 0$	0	$\frac{\sigma_i(h^t)}{\sigma_i'(h^t)}$

Since Φ is a stochastic matrix that is independent of θ, X_i is a garbling of Y_i, and therefore Y_i is sufficient for X_i.

Since the individual exit decisions X_i are independent (conditional on the informational status of the players), the same argument as above applies for the joint experiments $X := \times_{i=1}^{n(h^t)} X_i$ and $Y = \times_{i=1}^{n(h^t)} Y_i$.

Finally, consider two experiments $X^\omega = (X, Z)$ and $Y^\omega = (Y, Z)$ on $\Omega = \{G, B\}$ where X and Y are as above and Z is an experiment with outcomes in $S^Z = \{0, 1\}$. Since θ is correlated with ω, the information contained in X and Y is also information on Ω. We interpret Z as the individual learning experiment on own type and hence the matrix of conditional probabilities for that experiment is given by P^Z :

P^Z	$\omega = G$	$\omega = B$
$s^Z = 1$	$\lambda\Delta$	0
$s^Z = 0$	$1 - \lambda\Delta$	1

Since (X, Z) is a garbling of (Y, Z) by the argument above, we know that (Y, Z) is sufficient for (X, Z) with respect to Ω. The assertion that $P_i^{t+1}(h^t, \boldsymbol{\sigma}(h^t))$ second order stochastically dominates $P_i^{t+1}(h^t, \boldsymbol{\sigma'}(h^t))$ follows from Blackwell's theorem. ■

Proof. [Proof of Lemma 1] i) Let $p(h^t)$ be the belief of an uninformed player at h^t under symmetric equilibrium $\boldsymbol{\sigma}$. Compute the sequence of beliefs p_s for $s > t$ starting at $p_t = p(h^t)$ and using the private updating formula (2). Denote the optimal strategy of the isolated player starting from prior $p(h^t) = p_t$ by $a_m(p_s)$ for all $s \geq t$. Since $a_m(p_s)$ is a feasible strategy for each player and since equilibrium strategy $\sigma(h^t)$ is optimal by the definition of an equilibrium, we have

$$V\left(h^t\right) \geq V_m\left(p\left(h^t\right)\right).$$

This implies that $\sigma(h^t) = 0$ if $p(h^t) > p^*$.

ii) Consider next any history h^t such that $p(h^t) = p \leq p^*$. Let

$$\tau = \{\min s \geq t \,|\sigma(h^s) > 0\}.$$

If $\tau = \infty$, then $p(h^s) = p_s$ for all $s > t$. But this contradicts the optimal strategy calculated for the isolated player, so we must have $\tau < \infty$. Since exiting is in the support of the equilibrium strategy at h^τ, we have

$$V(h^\tau) = 0.$$

Since $p(h^s) = p_s$, for all $t \leq s < \tau$, $V(h^s) > 0$ for some $s \geq t$ implies a contradiction with the optimal policy of the isolated player. Therefore

$$V\left(h^t\right) = 0 \text{ if } p\left(h^t\right) \leq p^*.$$

iii) Since $V(h^t) = 0$ whenever $p(h^t) \leq p^*$, and since $\sigma(h^t) = 0$ for $p(h^t) > p^*$, the pure strategy $a_m(p(h^t))$ is a best response for each player after each history h^t, given the strategy profile $\boldsymbol{\sigma}$. Therefore, $V(h^t) = V_m(p(h^t))$ for each h^t. ■

Proof. [Proof of Lemma 2] By Lemma (1),

$$\mathbb{E}V\left(h^{t+1}\right) = \mathbb{E}V_m\left(P^{t+1}\left(h^t; \sigma\left(h^t\right)\right)\right).$$

Furthermore, $V_m(p)$ is convex in p and $P^{t+1}(h^t; \sigma(h^t))$ is second order stochastically decreasing in $\sigma(h^t)$ by Proposition 2 and hence the first claim follows.

To prove the second claim, suppose that there exists a σ such that

$$\delta \mathbb{E}V_m\left(P^{t+1}\left(h^t; \sigma\left(h^t\right)\right)\right) = c\left(h^t\right)\Delta.$$

We claim that for all $\sigma'(h^t) > \sigma(h^t)$,

$$\mathbb{E}V_m\left(P^{t+1}\left(h^t;\sigma'\left(h^t\right)\right)\right) > c\left(h^t\right)\Delta. \tag{24}$$

To see this, consider the exit decision of player i when all players use the symmetric strategy σ. There must be an exit decision vector $\widehat{a}^t_{-i}$ such that at $h_0^{t+1} := h^t \cup \left(\widehat{a}^t_{-i}, 0\right)$,

$$p\left(h_0^{t+1}, \boldsymbol{\sigma}\right) < p^*$$

and at $h_1^{t+1} := h^t \cup \left(\widehat{a}^t_{-i}, 1\right)$

$$p\left(h_1^{t+1}, \boldsymbol{\sigma}\right) > p^*.$$

Furthermore,

$$\Pr\{a^t_{-i} = \widehat{a}^t_{-i}\} > 0.$$

Suppose next that player i exits with probability $\sigma_i' > \sigma(h^t)$ and all other players exit with probability $\sigma(h^t)$ after history h^t. We consider the beliefs of an arbitrary player $j \neq i$ following this change in the strategy profile at history h^t.

Denote the profile where all players but i exit with probability $\sigma(h^t)$ and i exits with probability σ_i' by (σ_{-i}, σ_i'). By Lemma 2, and by the convexity of $V_m(p)$, we know that for every a^t_{-i}

$$\delta \mathbb{E}_{a_i} V\left(h^t \cup \left(a^t_{-i}, a_i\right); (\sigma_{-i}, \sigma_i')\right) \geq \delta \mathbb{E}_{a_i} V\left(h^t \cup \left(a^t_{-i}, a_i\right); \sigma\left(h^t\right)\right).$$

Therefore, the payoff of players other than i is strictly increasing in σ_i', if for $\widehat{a}^t_{-i}$ we have

$$\delta \mathbb{E}_{a_i} V\left(h^t \cup \left(\widehat{a}^t_{-i}, a_i\right); (\sigma_{-i}, \sigma_i')\right) > \delta \mathbb{E}_{a_i} V\left(h^t \cup \left(\widehat{a}^t_{-i}, a_i\right); \sigma\left(h^t\right)\right).$$

But this follows immediately from the facts that

$$p\left(h_0^{t+1}; (\sigma_{-i}, \sigma_i')\right) = p\left(h_0^{t+1}; \sigma\left(h^t\right)\right),$$
$$p\left(h_1^{t+1}; (\sigma_{-i}, \sigma_i')\right) > p\left(h_1^{t+1}; \sigma\left(h^t\right)\right),$$

and

$$0 = \frac{\partial^- V_m\left(p\left(h^t \cup \left(\widehat{a}^t_{-i}, 0\right)\right)\right)}{\partial p} < \frac{\partial^- V_m\left(p\left(h^t \cup \left(\widehat{a}^t_{-i}, 1\right)\right)\right)}{\partial p},$$

where $\frac{\partial^- V_m(p)}{\partial p}$ denotes the derivative from the left (which exists by the convexity of $V_m(p)$) of V_m at p.

Starting with the strategy profile (σ_{-i}, σ_i'), change the exit probability of all players $j \neq i$ to $\sigma_j' = \sigma_i'$, and denote the resulting symmetric profile by $\sigma'(h^t)$. By Proposition 2, the payoff to all players is weakly increased. Therefore for all j,

$$\begin{aligned} \mathbb{E}V_j\left(h^{t+1}; \sigma'\left(h^t\right)\right) &= \mathbb{E}V_m\left(P^{t+1}\left(h^t; (\sigma_{-i}', \sigma_i')\right)\right) \geq \mathbb{E}V_m\left(P^{t+1}\left(h^t; (\sigma_{-i}, \sigma_i')\right)\right) \\ &> \mathbb{E}V_m(P^{t+1}\left(h^t; \sigma\left(h^t\right)\right)) = \mathbb{E}V_j\left(h^{t+1}; \sigma\left(h^t\right)\right). \end{aligned}$$

■

Proof. [Proof of Theorem 1] All we have to do is to check that $\sigma(h^t)$ is optimal for all players under all three cases given in the Theorem, and that this is the only symmetric exit probability with this property.

Lemma 1 implies that it is optimal to stay (exit) at h^t iff $\sigma(h^t)$ satisfies

$$\delta \mathbb{E} V_m \left(P^{t+1}\left(h^t; \sigma\left(h^t\right)\right)\right) \geq (\leq)\, c\left(h^t\right) \Delta. \tag{25}$$

Consider now cases i) - iii) below. These cases cover all possibilities and are mutually exclusive, because $\mathbb{E}V_m(P^{t+1}(h^t; \sigma(h^t)))$ is increasing in $\sigma(h^t)$ by Lemma 2.

i) Assume

$$\delta \mathbb{E} V_m \left(P^{t+1}\left(h^t; 0\right)\right) > c\left(h^t\right) \Delta.$$

Then it is strictly optimal for all the players to stay under $\sigma(h^t) = 0$. Moreover, by Lemma 2, $\delta\mathbb{E}V_m(P^{t+1}(h^t; x)) > c(h^t)\Delta$ for all $x \geq 0$, so $\sigma(h^t) = 0$ is the unique symmetric equilibrium action in that case.

ii) Assume

$$\delta \mathbb{E} V_m \left(P^{t+1}\left(h^t; 0\right)\right) \leq c\left(h^t\right) \Delta \leq \delta \mathbb{E} V_m \left(P^{t+1}\left(h^t; 1\right)\right).$$

First note that $\mathbb{E}V_m(P^{t+1}(h^t; \sigma(h^t)))$ is continuous in $\sigma(h^t)$ as a result of the continuity of the Bayes' rule in $\sigma(h^t)$. Lemma 2 then implies that there is a unique value $\sigma^*(h^t)$ for which

$$\delta \mathbb{E} V_m \left(P^{t+1}\left(h^t; \sigma^*\left(h^t\right)\right)\right) = c\left(h^t\right) \Delta.$$

Moreover, for all $\sigma(h^t) < \sigma^*(h^t)$ the strictly optimal action is to exit, and for all $\sigma(h^t) > \sigma^*(h^t)$ the strictly optimal action is to stay. Thus, $\sigma^*(h^t)$ is the unique symmetric equilibrium action in this case.

iii) Assume

$$\delta \mathbb{E} V_m \left(P^{t+1}\left(h^t; 1\right)\right) < c\left(h^t\right) \Delta.$$

Then it is strictly optimal for all the players to exit under $\sigma(h^t) = 1$. Moreover, by Lemma 2, $\delta\mathbb{E}V_m(P^{t+1}(h^t; x)) < c(h^t)\Delta$ for all $x \leq 1$, so $\sigma(h^t) = 1$ is the unique symmetric equilibrium action in that case. ■

Proof. [Proof of Lemma 3] i) We claim that for each $\varepsilon > 0$ there exists a $\overline{\Delta} > 0$ such that for all $\Delta < \overline{\Delta}$,

$$p\left(h^t\right) < p^* + \varepsilon \text{ for all } h^t \text{ with } t \text{ such that } t\Delta \geq \tau^*.$$

To prove the claim, suppose to the contrary. Clearly the losses from continuing in the game are bounded from above by $c\Delta$.

$$c\Delta \geq \left(p^* - p\left(h^t\right)\right) \lambda(v + e^{-r\Delta} V^+(\Delta)). \tag{26}$$

Indifference requires that

$$
\begin{aligned}
& \left(p^* - p\left(h^t\right)\right)\lambda\Delta(v + e^{-r\Delta}V^+(\Delta)) && (27)\\
= \;& \Sigma_{a^t}\Pr\{\mathbf{A}^t = a^t\}e^{-r\Delta}V_m\left(p_{t+1}\left(h_t, a^t\right)\right) && (28)\\
\geq \;& \Pr\{\mathbf{A}^t = \mathbf{1}\}e^{-r\Delta}V_m\left(p_{t+1}\left(h_t, \mathbf{1}\right)\right) && (29)\\
\geq \;& \Pr\{\mathbf{A}^t = \mathbf{1}\}e^{-r\Delta}V_m\left(p^* + \varepsilon\right), && (30)
\end{aligned}
$$

where $\mathbf{A}^t$ is the random vector of exits in period t and $\{\mathbf{A}^t = \mathbf{1}\}$ is the event that nobody exits. Since $V_m(p)$ is strictly increasing for $p > p^*$, there is an $\eta > 0$ such that

$$e^{-r\Delta}V_m\left(p^* + \varepsilon\right) > \eta.$$

Hence we have:

$$\Pr\{\mathbf{A}^t = \mathbf{1}\} \leq \frac{c\Delta}{e^{-r\Delta}V_m\left(p^* + \varepsilon\right)} = \frac{c}{\eta}\Delta.$$

Since the individual randomization probabilities are independent, symmetric equilibrium implies that

$$\left(1 - \xi_N\left(h^t\right)\right)^{N-1} \leq \frac{c}{\eta}\Delta,$$

where $\xi_N\left(h^t\right)$ is the individual exit probability in the game with N players after history h^t. Therefore

$$\xi_N\left(h^t\right) \geq 1 - \left(\frac{c}{\eta}\Delta\right)^{\frac{1}{N-1}}. \tag{31}$$

Since only uninformed player exit, we have

$$\xi_N\left(h^t\right) \leq 1 - \pi\left(h^t\right) \leq 1 - \pi_t < 0,$$

and equation (31) leads to a contradiction for small enough Δ and the claim is established. Corollary 1 and the above claim establish part i) of the Lemma.

ii) Let

$$h^t(k) = h^{t-1} \cup a^t(k),$$

where $a^t(k)$ is a vector of exit decisions where exactly k active players exit at history h^{t-1}. By Bayes' rule, we know that

$$\frac{1 - q\left(h^t(k)\right)}{q\left(h^t(k)\right)} = \frac{1 - q\left(h^t(0)\right)}{q\left(h^t(0)\right)}\left(\frac{\pi^L}{\pi^H}\right)^k.$$

By Proposition 1, there is an $\eta > 0$ such that $\frac{\pi^L}{\pi^H} > 1 + \eta$. Therefore, for all k, there is an η' such that

$$q\left(h^t(k)\right) < q\left(h^t(0)\right) - \eta'.$$

By equation (12), there exists a $\zeta > 0$ such that

$$p\left(h^t(k)\right) < p\left(h^t(0)\right) - \zeta.$$

By part i), for all $\varepsilon > 0$, there exists a $\overline{\Delta} > 0$ such that for all $\Delta < \overline{\Delta}$,

$$p\left(h^t(0)\right) < p^* + \varepsilon.$$

Since $\varepsilon > 0$ can be chosen arbitrarily small, the claim follows. ■

Proof. [Proof of Proposition 5] i) If $X\left(h^{t-1}\right) = 0$, and $p\left(h^{t-1}\right) < p^*$, then $p\left(h^t, \mathbf{1}\right) > p^*$. If $p\left(h^{t-1}\right) \geq p^*$, then there are no exits and the only learning comes from private observations. Hence there exists an $\eta > 0$ and a $\overline{\Delta}$ such that for all $\Delta < \overline{\Delta}$,

$$p\left(h^t\right) > p^* - \eta\Delta.$$

The loss from staying in the game for an additional period to an uninformed player is

$$\begin{aligned} & c\lambda\Delta - \lambda\Delta p\left(h^t\right)\left(v + e^{-r\Delta}V^+\right) \\ = \; & \lambda\Delta(c - p\left(h^t\right))\left(v + e^{-r\Delta}V^+\right)) \\ \leq \; & \lambda\Delta(c - p^* + \eta\Delta)\left(v + e^{-r\Delta}V^+\right)) \\ \leq \; & \lambda\Delta\eta\Delta\left(v + V^+\right). \end{aligned}$$

Suppose to the contrary of the claim that no such $\kappa, \overline{\Delta}$ exist. Then there exist a sequence $\Delta_n \to 0$ and a sequence $\kappa_n \to \infty$

$$\frac{P_L\left(X\left(h^t\right) > 0\right)}{\Delta_n} > \kappa_n \text{ for some } h^t \in H^f\left(\Delta_n\right).$$

But then Bayes' rule implies that there exists an $\alpha > 0$ such that conditional on no exit,

$$p\left(h^{t+1}\right) \geq p^* + \alpha\kappa_n\Delta_n.$$

Since

$$V\left(p\left(h^t\right)\right) = V_m\left(p\left(h^t\right)\right),$$

we can compute a lower bound for the expected value from stying in the game as:

$$\Pr\{\text{no exit}\}[V_m\left(p^*\right) + V_m'\left(p^*\right)\left(p\left(h^t\right) - p^*\right) + \frac{1}{2}V_m''\left(p^*\right)\left(p\left(h^t\right) - p^*\right)^2 + \text{higher order terms.}$$

Notice that $V_m'\left(p^*\right) = 0$ by smooth pasting. Since the losses are of order Δ_n^2, and since $V_m''\left(p^* > 0\right)$, we see that the gains exceed the losses contradicting the optimality of exits.

ii) Follows immediately from part ii) of the previous Lemma and Bayes' rule. ■

Proof. [Proof of Proposition 6] We define

$$p(\tau) := q(\tau)\, p^H(\tau) + (1 - q(\tau))\, p^L(\tau), \tag{32}$$

we have

$$\dot{p}(\tau) = \dot{p}^L(\tau) + \dot{q}(\tau)\left(p^H(\tau) - p^L(\tau)\right) + q(\tau)\left(\dot{p}^H(\tau) - \dot{p}^L(\tau)\right) = 0. \tag{33}$$

Using 2, we have along the history with no exits:

$$\dot{p}^\theta(\tau) = -\lambda p^\theta(\tau)\left(1 - p^\theta(\tau)\right) \text{ for } \theta \in \{H, L\}. \tag{34}$$

Substituting from 34 and 32 into 33 gives:

$$\dot{q}(\tau) = -\lambda q(\tau)\left(1 - q(\tau)\right)\left(p^H(\tau) - p^L(\tau)\right) + \frac{p^*(1 - p^*)}{p^H(\tau) - p^L(\tau)}. \tag{35}$$

On the other hand, we have for short intervals $d\tau$ of real time

$$q + dq =$$

$$\frac{q\left(1 - p^H + p^H(1 - \lambda d\tau)\right)\left(1 - \xi^H d\tau\right)^{n-1}}{q\left(1 - p^H + p^H(1 - \lambda d\tau)\right)\left(1 - \xi^H d\tau\right)^{n-1} + (1 - q)\left(1 - p^L + p^L(1 - \lambda d\tau)\right)\left(1 - \xi^L d\tau\right)^{n-1}},$$

Therefore

$$\dot{q}(\tau) = q(\tau)\left(1 - q(\tau)\right)\left(-\lambda p^H(\tau) - (n-1)\,\xi^H(\tau) + \lambda p^L(\tau) + (n-1)\,\xi^L(\tau)\right). \tag{36}$$

Equating 35 and 36 and using 32 and **??**, we get

$$\xi^\theta(\tau) = \pi^\theta(\tau)\,\frac{\lambda p^*(1 - p^*)\left(p^H(\tau) - p^L(\tau)\right)}{(n-1)\left(p^* - p^L(\tau)\right)\left(p^H(\tau) - p^*\right)\left(\pi^L(\tau) - \pi^H(\tau)\right)}.$$

■

Proof. [Proof of Proposition 7] By Lemma 3, we have

$$q^*(\tau) = \lim_{n\to\infty} \frac{\left(1 - \sigma_n \pi^H(\tau)\right)^{n-1} q_s}{\left(1 - \sigma_n \pi^H(\tau)\right)^{n-1} q_s + \left(1 - \sigma_n \pi^L(\tau)\right)^{n-1}(1 - q_s)}.$$

Therefore

$$\frac{1 - q^*(\tau)}{q^*(\tau)} = \frac{1 - q_s}{q_s} \lim_{n\to\infty} \frac{\left(1 - \sigma_n \pi^L(\tau)\right)^{n-1}}{\left(1 - \sigma_n \pi^H(\tau)\right)^{n-1}}.$$

Evaluating the limits, we have:

$$e^{\sigma\left(\pi^L(\tau) - \pi^H(\tau)\right)} = \frac{1 - q_s}{q_s}\,\frac{q^*(\tau)}{1 - q^*(\tau)},$$

where

$$\sigma = \lim \frac{\sigma_n}{n}.$$

Therefore the claim follows by taking logarithms and computing the probability of no exit for each state.

The second claim is an immediate consequence of the Bayes' rule with k exits. ■

Appendix: Proofs for Section 5

This appendix contains the proofs for Proposition 3, Proposition 4, and Theorem 2. Our arguments rely on the convergence of what we call the outside observer's belief $\widehat{q}\,(h^t)$, and which we define as the posterior belief of the event $\{\theta = H\}$ at public history h^t:

$$\widehat{q}\,(h^t) = \frac{q_0 P_H\{h^t\}}{q_0 P_H\{h^t\} + (1-q_0)\, P_L\{h^t\}}.$$

Note the difference between $\widehat{q}\,(h^t)$ and $\widehat{q}_i\,(h^t)$, the latter defined in Section 3.2: $\widehat{q}\,(h^t)$ is based on actions of all players (and represents therefore a true *outside* observer), while $\widehat{q}_i\,(h^t)$ is based on actions of players other than i.

To link the convergence of $\widehat{q}\,(h^t)$ to the beliefs of actual players in the game, note that player i's belief $q_i\,(h^t)$ differs from $\widehat{q}\,(h^t)$ only to the extent that i's private information affects her belief. Lemma 4 below guarantees that i's private history can not overrule a sufficiently strong public history:[10]

Lemma 4 *Suppose that $\rho^H < 1$. Then for all $\varepsilon > 0$, there is some $\delta > 0$ such that the following implications hold for all i:*

$$\widehat{q}\left(h^t\right) \geq 1-\delta \Longrightarrow q_i\left(h^t\right) \geq 1-\varepsilon \text{ and} \tag{37}$$

$$\widehat{q}\left(h^t\right) \leq \delta \Longrightarrow q_i\left(h^t\right) \leq \varepsilon. \tag{38}$$

Proof. Recall from Section 3.2 that $\widehat{q}_i\,(h^t)$ is the belief based on public histories of all players other than i. In addition to this, $\widehat{q}\,(h^t)$ also conditions on actions of i. Consider the effect of this additional information. The most favorable piece of evidence in terms of state $\theta = H$ that could ever be obtained form i's actions is the one that fully reveals i to be informed. The likelihood ratio of being informed across the states is $\left(1-\pi_t^H\right)/\left(1-\pi_t^L\right)$, which gives us an upper bound for $\widehat{q}\,(h^t)$ as expressed in terms of $\widehat{q}_i\,(h^t)$:

$$\frac{\widehat{q}\,(h^t)}{1-\widehat{q}\,(h^t)} \leq \frac{1-\pi_t^H}{1-\pi_t^L}\frac{\widehat{q}_i\,(h^t)}{1-\widehat{q}_i\,(h^t)}. \tag{39}$$

On the other hand, we can write the relationship between $q_i\,(h^t)$ and $\widehat{q}_i\,(h^t)$ by using (11):

$$\frac{q_i\,(h^t)}{1-q_i\,(h^t)} = \frac{\pi_t^H}{\pi_t^L}\frac{\widehat{q}\,(h^t)}{1-\widehat{q}\,(h^t)}. \tag{40}$$

[10] In the pure common values case, where $\rho^H = 1$, the ratio $\frac{\pi_t^H}{\pi_t^L} \to 0$ as $t \to \infty$. In that case the statement below holds for all t up to an arbitrary, fixed $\bar{t}$. This modification is not essential for any of our results.

Combining (39) and (40) gives us:

$$\frac{q_i(h^t)}{1-q_i(h^t)} \geq \frac{\pi_t^H}{\pi_t^L}\frac{1-\pi_t^L}{1-\pi_t^H}\frac{\widehat{q}(h^t)}{1-\widehat{q}(h^t)}. \tag{41}$$

From (10), we have $0 < \pi_t^H < \pi_t^L$, and therefore the first equation of Lemma 4 follows directly from (41). The second equation follows trivially from the fact that $q_i(h^t) \leq \widehat{q}(h^t)$ (an uninformed player must be more pessimistic than the outside observer). ■

Proof of Proposition 3

Our proof strategy is to follow the evolution of outside observer's belief along a filtration that samples the players' actions sequentially one player at a time. We show that this belief must converge to truth as the number of exits increases, and furthermore this implies the convergence of actual players' beliefs in the original filtration where all actions within a period are sampled simultaneously. The key step in the argument is Lemma 5 below, which implies that this belief process drifts at a high rate towards truth when sampled at the points where players exit. With this Lemma at hand, the rest of the argument is a straightforward application of Theorem A.1. of Fudenberg & Levine (1992).

We use index $s \in \mathbb{N}$ to track the moments of observation starting from period $t^*(\Delta)$ in the following way. At $s=1$ the action of player 1 in period $t^*(\Delta)$ is observed. At $s=2$, the action of player 2 in period $t^*(\Delta)$ is observed, and so on. Once the decisions all N players in period $t^*(\Delta)$ have been sampled, the process moves to the next time period. At $s = N+1$ player 1's action in period $t^*(\Delta)+1$ is observed, and so on. This means that we map every $s \in \mathbb{N}$ to the corresponding period $t(s)$ and player $i(s)$ as follows:

$$\begin{aligned} t(s) \;&:= \left\lfloor \frac{s}{N} \right\rfloor + t^*(\Delta), \\ i(s) \;&:= s - N \cdot \left\lfloor \frac{s}{N} \right\rfloor. \end{aligned}$$

Let ξ_s^θ denote the exit probability of player $i(s)$ in period $t(s)$ with equilibrium strategy profile $\boldsymbol{\sigma}$ (nothing in the proof requires this to be symmetric):

$$\xi_s^\theta := \xi_{i(s)}^\theta\left(h^{t(s)}\right),$$

where we set $\xi_i^\theta(h^t) = 0$ if $a_{i(s)}^{t(s)-1} = 0$ (that is, probability of exit is zero for a player that has already exited). We use $x_s \in \{0,1\}$ as an indicator for player $i(s)$ exiting in period $t(s)$:

$$x_s = \begin{cases} 1 \text{ if } a_{i(s)}^{t(s)-1} = 1 \text{ and } a_{i(s)}^{t(s)} = 0 \\ 0 \text{ otherwise} \end{cases}.$$

Note that x_s fully describes the public history up to period $t(s) - 1$, and in addition contains the actions of players $1, ..., i(s)$ in period $t(s)$. We use notation h_s to refer to the event defined by $x_{s'}$, $s' = 1, ..., s$:

$$h_s = \bigcup_{s'=1}^{s} a_{i(s')}^{t(s')}.$$

We denote by $\widehat{q}_s$ the belief process of the outside observer, who observes the players sequentially:

$$\widehat{q}_s = \Pr\{\theta = H | h_s\}, \ s \in \mathbb{N}.$$

By Bayes rule, this belief evolves according to:

$$\begin{aligned} \widehat{q}_0 &= q_0 \\ \widehat{q}_s &\equiv \begin{cases} \frac{\widehat{q}_{s-1}\xi_s^H}{\widehat{q}_{s-1}\xi_s^H + (1-\widehat{q}_{s-1})\xi_s^L} \text{ if } x_s = 1 \\ \frac{\widehat{q}_{s-1}\left(1-\xi_s^H\right)}{\widehat{q}_{s-1}\left(1-\xi_s^H\right)+(1-\widehat{q}_{s-1})\left(1-\xi_s^L\right)} \text{ if } x_s = 0 \end{cases}, \ s = 1, 2, ... \end{aligned} \tag{42}$$

Note that for all $s = t \cdot N$, $t \in \mathbb{N}$, the belief $\widehat{q}_s$ coincides with the outside observer's belief after period t:

$$\widehat{q}_s = \widehat{q}\left(h^t\right), \ s = t \cdot N, \ t \in \mathbb{N}.$$

For all other values of s, $\widehat{q}_s$ is the belief of an outside observer who has observed only a subset of players in the last period.

Let X_∞ denote the total number of players that exit the game:

$$X_\infty := \sum_{t=0}^{\infty} X\left(h^t\right).$$

We define an increasing sequence of natural numbers $\{s(k)\}_{k=1}^{X_\infty}$ as follows:

$$\begin{aligned} s(0) &= 0, \\ s(k) &= \min\{s > s(k-1) | x_s = 1\}, \ k = 1, ..., X_\infty. \end{aligned}$$

Hence, $\left\{\widehat{q}_{s(k)}\right\}_{k=1}^{X_\infty}$ is a subset of $\{\widehat{q}_s\}_{s=1}^{\infty}$ sequence, that samples the beliefs immediately after realized exits.

Define:

$$L_k := \begin{cases} \frac{1-\widehat{q}_{s(k)}}{\widehat{q}_{s(k)}} \text{ for } k = 1, ..., X_\infty. \\ 0 \text{ for } k = X_\infty + 1, ... \end{cases} \tag{43}$$

In words, L_k is the likelihood ratio for the event $\{\theta = L\}$ sampled after realized exits. It is clear that under the event $\{\theta = H\}$, this process is a supermartingale. The next lemma is the key to our argument, and it states that this process is an *active supermartingale*, as defined in Fudenberg & Levine (1992).

Lemma 5 *There exists an $\eta > 0$ such that*

$$P_H\left(|L_{k+1}/L_k - 1| > \eta \,\middle|\, h_{s(k)}\right) > \eta \tag{44}$$

for all $L_k > 0$.

Proof. Note first that

$$\{|L_{k+1}/L_k - 1| \leq \eta\} \Longleftrightarrow (1-\eta) L_k \leq L_{k+1} \leq (1+\eta) L_k. \tag{45}$$

By Proposition 1 and (13), there is some $\gamma > 0$ such that

$$\frac{\xi_s^L}{\xi_s^H} > 1 + \gamma \tag{46}$$

for all $s \in \mathbb{N}$. Fix η small enough to ensure that

$$\frac{(1+\eta)}{(1-\eta)^2} < 1 + \gamma. \tag{47}$$

Write

$$\widetilde{L}_s := \frac{1 - \widehat{q}_s}{\widehat{q}_s},\ s \in \mathbb{N}.$$

Note that $L_k = \widetilde{L}_{s(k)}$ for $k = 1, ..., X_\infty$. Using (42) and (46), we have:

$$\widetilde{L}_s = \begin{cases} \frac{\xi_s^L}{\xi_s^H} \widetilde{L}_{s-1} > (1+\gamma) \widetilde{L}_{s-1} \text{ if } x_s = 1 \\ \frac{(1-\xi_s^L)}{(1-\xi_s^H)} \widetilde{L}_{s-1} < \widetilde{L}_{s-1} \text{ if } x_s = 0 \end{cases}$$

By definition of $s(k)$, we have $x_{s(k+1)} = 1$, and therefore, we have

$$\widetilde{L}_{s(k+1)} > (1+\gamma) \widetilde{L}_{s(k+1)-1}. \tag{48}$$

Noting that $L_{k+1} = \widetilde{L}_{s(k+1)}$ and $L_k = \widetilde{L}_{s(k)}$, and using (45) and (48), we have:

$$\{|L_{k+1}/L_k - 1| \leq \eta\} \Longrightarrow \left\{ \widetilde{L}_{s(k+1)-1} < \frac{1+\eta}{1+\gamma} \widetilde{L}_{s(k)} \right\}. \tag{49}$$

Let $\overline{s}$ be the first observation point after $s(k)$ at which $\widetilde{L}_{\overline{s}}$ is below $\frac{1+\eta}{1+\gamma}\widetilde{L}_{s(k)}$ in case there are no exits:

$$\overline{s} := \min \left\{ s' > s(k) : \left(\prod_{j=s(k)+1}^{s'} \frac{1-\xi_j^L}{1-\xi_j^H} \right) \widetilde{L}_{s(k)} < \frac{1+\eta}{1+\gamma} \widetilde{L}_{s(k)} \right\}.$$

Then it follows from (49) and (45) that:

$$\{|L_{k+1}/L_k - 1| \leq \eta\}$$
$$\Longrightarrow \left\{x_s = 0 \forall s = s(k)+1, ..., \overline{s} \text{ and } \widetilde{L}_{s(k+1)} > \frac{1-\eta}{1+\eta}(1+\gamma)\widetilde{L}_{\overline{s}}\right\}. \tag{50}$$

But, since $\widetilde{L}_s$ is a super-martingale under $\theta = H$, we have

$$\mathbb{E}\left(\widetilde{L}_{s(k+1)} \left| h^{\overline{s}}, \theta = H\right.\right) < \widetilde{L}_{\overline{s}},$$

which implies the following (using the fact that $\widetilde{L}_{s(k+1)}$ is bounded from below by 0):

$$P_H\left(\widetilde{L}_{s(k+1)} < \frac{1-\eta}{1+\eta}(1+\gamma)\widetilde{L}_{\overline{s}} \left| h^{\overline{s}}\right.\right) \geq 1 - \frac{(1+\eta)}{(1-\eta)(1+\gamma)} > \eta,$$

where the last inequality follows from (47). Combining this with (50), we note that

$$P_H\left(|L_{k+1}/L_k - 1| > \eta \left| h_{s(k)}\right.\right) > \eta.$$

■

Lemma 5 says that L_s, $s \in \mathbb{N}$ is an *active supermartingale* with activity η, as defined in Fudenberg & Levine (1992). We need this property to apply Theorem A.1. of Fudenberg & Levine (1992), which we restate here for convenience:

Theorem 3 (Fudenberg and Levine) *Let $l_0 > 0$, $\varepsilon > 0$, and $\eta \in (0,1)$ be given. For each $\underline{L}$, $0 < \underline{L} < l_0$, there is some $K < \infty$ such that*

$$\Pr\left(\sup_{k>K} L_k \leq \underline{L}\right) \geq 1 - \varepsilon$$

for every active supermartingale L with $L_0 = l_0$ and activity η.

With these preliminaries at hand, we are ready to finish the proof of Proposition 3:

Proof. [Proof of Proposition 3] Fix $\varepsilon > 0$. Consider the stochastic process L_k, $k \in \mathbb{N}$, defined in (43). Note from (43) that:

$$L_k \leq \underline{L} \Longleftrightarrow \left(\left\{\widehat{q}_{s(k)} \geq \frac{1}{1+\underline{L}}\right\} \text{ or } \{k > X_\infty\}\right).$$

We set $\underline{L}$ small enough to guarantee:

$$L_k \leq \underline{L} \Longrightarrow \left(\left\{\widehat{q}_{s(k)} > 1 - \varepsilon\right\} \text{ or } \{k > X_\infty\}\right). \tag{51}$$

By Lemma 5, we know that L_k is an active supermartingale with activity η. By Theorem 3, we can therefore set K high enough to guarantee that

$$P_H\left\{h^\infty : \sup_{k>K} L_k \leq \underline{L}\right\} \geq 1 - \varepsilon. \tag{52}$$

Combining this with (51), we have

$$P_H\{h^\infty : n\left(h^t\right) \leq N-K \text{ and } \widehat{q}\left(h^t\right) < 1-\varepsilon \text{ for some } h^t \in h^\infty\} < \varepsilon. \tag{53}$$

We have now proved the Proposition as regards equation (16). Knowing this, the part concerning equation (17) follows from Bayes' rule as follows. Define the following event:

$$A(K,\varepsilon) := \left\{h^\infty : n\left(h^t\right) < N-K \text{ and } \varepsilon < \widehat{q}\left(h^t\right) < 1-\varepsilon \text{ for some } h^t \in h^\infty\right\}.$$

Then, by the definition of $A(K,\varepsilon)$, the posterior of $\{\theta = H\}$ conditional on reaching $A(K,\varepsilon)$ must be between ε and $1-\varepsilon$:

$$\varepsilon < \frac{q_0 P_H\left(A(K,\varepsilon)\right)}{q_0 P_H\left(A(K,\varepsilon)\right) + (1-q)\, P_L\left(A(K,\varepsilon)\right)} < 1-\varepsilon. \tag{54}$$

Since (53) holds for any ε given large enough K, we know that $P_H\left(A(K,\varepsilon)\right)$ can be made arbitrarily small by increasing K. Therefore, for (54) to hold, also $P_L\left(A(K,\varepsilon)\right)$ must go to zero as K is increased, which implies that for any $\varepsilon > 0$, we can find K large enough to ensure that

$$P_L\{h^\infty : n\left(h^t\right) \leq N-K \text{ and } \widehat{q}\left(h^t\right) < \varepsilon \text{ for some } h^t \in h^\infty\} < \varepsilon.$$

■

Proof of Proposition 4

We work through a number of lemmas. First, we formalize an intuitive fact that whenever the probability that a large number of players exit within the current period is non-negligible, the realized actions generate a precise signal about the state of the world. In particular, if the true state is $\theta = H$, then the beliefs of all players must be very close to one after that period:

Lemma 6 *For all $\varepsilon > 0$ and $q > 0$, there is some $K \in \mathbb{N}$ such that*

$$P_L\left(X\left(h^t\right) > K\right) > \frac{1}{2} \Longrightarrow P_H\left(q\left(h^{t+1}\right) > 1-\varepsilon\right) > 1-\varepsilon,$$

whenever $q\left(h^t\right) > q$ and $t \geq t^*(\Delta)$.

Proof. Denote

$$\mu_\theta \equiv \mathbb{E}\left[X\left(h^t\right)|\theta\right] = n\left(h^t\right)\xi^\theta\left(h^t\right).$$

Since $X\left(h^t\right)$ is a random variable that can only take positive values, the following must hold:

$$P_L\left(X\left(h^t\right) > K\right) > \frac{1}{2} \Longrightarrow \mu_L > \frac{1}{2}K. \tag{55}$$

By Proposition 1, we know that there is some $\gamma > 0$ such that

$$\frac{\xi^{L}\left(h^{t}\right)}{\xi^{H}\left(h^{t}\right)} > 1+\gamma$$

for all $t \geq t^{*}\left(\Delta\right)$. Consider the random variable

$$Z\left(h^{t}\right) := \frac{X\left(h^{t}\right)}{\mu_{L}}.$$

We have:

$$\mathbb{E}\left[Z\left(h^{t}\right)|\theta = H\right] = \frac{n\left(h^{t}\right)\xi^{H}\left(h^{t}\right)}{n\left(h^{t}\right)\xi^{L}\left(h^{t}\right)} < \frac{1}{1+\gamma}, \tag{56}$$

$$\mathbb{E}\left[Z\left(h^{t}\right)|\theta = L\right] = \frac{n\left(h^{t}\right)\xi^{L}\left(h^{t}\right)}{n\left(h^{t}\right)\xi^{L}\left(h^{t}\right)} = 1, \tag{57}$$

$$var\left[Z\left(h^{t}\right)|\theta = H\right] = \frac{n\left(h^{t}\right)\xi^{H}\left(h^{t}\right)\left(1-\xi^{H}\left(h^{t}\right)\right)}{\left(n\left(h^{t}\right)\xi^{L}\left(h^{t}\right)\right)^{2}} < \frac{1}{n\left(h^{t}\right)\xi^{L}\left(h^{t}\right)} = \frac{1}{\mu_{L}}, \tag{58}$$

$$var\left[Z\left(h^{t}\right)|\theta = L\right] = \frac{n\left(h^{t}\right)\xi^{L}\left(h^{t}\right)\left(1-\xi^{L}\left(h^{t}\right)\right)}{\left(n\left(h^{t}\right)\xi^{L}\left(h^{t}\right)\right)^{2}} < \frac{1}{n\left(h^{t}\right)\xi^{L}\left(h^{t}\right)} = \frac{1}{\mu_{L}}. \tag{59}$$

Consider the event

$$A = \left(Z\left(h^{t}\right) \leq \frac{1+\frac{1}{2}\gamma}{1+\gamma}\right).$$

The formulas (56) - (59) imply that

$$\lim_{\mu_{L}\to\infty} P_{H}\left(A\right) = 1 \text{ and } \lim_{\mu_{L}\to\infty} P_{L}\left(A\right) = 0.$$

By (55), assuming $P_{L}\left(X\left(h^{t}\right) > K\right) > \frac{1}{2}$ and increasing K will increase μ_{L} without bound. Hence, the result follows from Bayes' rule by considering the likelihood ratio across states of event A as K is increased. ∎

Next, Lemma 7 bounds the probability with which a large number of players may exit within an arbitrary period. By Lemma 6, a random experiment that would induce a large number of players to exit with a non-negligible probability would be very informative on the aggregate state. Any uninformed player would like to stay in the game until τ_{H} if she knew the state to be H. Suppose next that the probability of high state is bounded away from zero. As period length is reduced towards zero, the players would rather wait and observe the result of an informative experiment than exit immediately. Lemma 7 formalizes this argument.

Lemma 7 *For all $\tau < \tau_{H}$ and $q > 0$, there are some $K \in \mathbb{N}$ and $\overline{\Delta} \in \mathbb{R}^{+}$ such that*

$$q\left(h^{t}\right) > q \Longrightarrow P_{L}\left(X\left(h^{t}\right) > K\right) < \frac{1}{2}$$

whenever $\Delta < \overline{\Delta}$ and $t \leq t\left(\tau, \Delta\right)$.

Proof. Fix $\tau < \tau_H$ and $q > 0$. Lemma 6 implies that there is some $\phi : \mathbb{N} \to \mathbb{R}^+$ with

$$\lim_{K\to\infty} \phi(K) = 0,$$

such that the following implication holds for all h^t, $t \le t(\tau, \Delta)$, and $q(h^t) > q$:

$$P_L\left(X\left(h^t\right) > K\right) > \frac{1}{2} \Longrightarrow P_H\left(q\left(h^{t+1}\right) > 1 - \phi(K)\right) > 1 - \phi(K). \tag{60}$$

Recall the definition (3) that denotes by p_t^H the belief of a player on her own type conditional on state being H. If $\tau < \tau_H$, we can fix some $\eta > 0$ and $\Delta' > 0$ such that $p^H_{t(\tau,\Delta)+1} > p^*(\Delta) + \eta$ for all $\Delta < \Delta'$. This follows directly from the continuity of $p^*(\Delta)$ and the definition of τ_H. This means that we can choose K high enough so that

$$q\left(h^{t+1}\right) > 1 - \phi(K) \Longrightarrow p\left(h^{t+1}\right) > p^*(\Delta) + \eta. \tag{61}$$

We fix a K such that (60) and (61) hold for all h^t, $t \le t(\tau, \Delta)$, for which $q(h^t) > q$. Take any such history, and assume that $P_L(X(h^t) > K) > \frac{1}{2}$. Consider next the payoff that an uninformed player would get by staying in the game with probability one at that history. We want to find a lower bound for that payoff. Since $q(h^t) > q$, the posterior for $\theta = H$ is bounded from below by q. By (60) and (61), $1 - \phi(K)$ is a lower bound for the probability that $p(h^{t+1}) > p^*(\Delta) + \eta$, conditional on $\theta = H$. Finally, $V_m(p^*(\Delta) + \eta) > 0$ is the value of the isolated player at belief $p^*(\Delta) + \eta$. Therefore, the continuation payoff for a player that stays is bounded from below by:

$$V\left(h^t\right) \ge -c\Delta + e^{-r\Delta} \cdot q \cdot (1 - \phi(K)) \cdot V_m(p^* + \eta), \tag{62}$$

We see from (62) that we guarantee $V(h^t) > 0$ by setting Δ small enough and K large enough. Since then it is strictly optimal for any individual player to stay in the game, this contradicts the presumption that $P_L(X(h^t) > K) > \frac{1}{2}$. We can thus conclude that for high enough $K \in \mathbb{N}$ and small enough $\overline{\Delta} \in \mathbb{R}^+$ the implication

$$q\left(h^t\right) > q \Longrightarrow P_L\left(X\left(h^t\right) > K\right) < \frac{1}{2}$$

holds whenever $\Delta < \overline{\Delta}$ and $t \le t(\tau, \Delta)$. ■

Lemma 8 shows that if a large number or players exit within a period, then the belief of an uninformed player falls to a very low level.

Lemma 8 *For all $\tau < \tau_H$ and $q > 0$, there are some $K \in \mathbb{N}$ and $\overline{\Delta} \in \mathbb{R}^+$ such that the following implication holds on the equilibrium path of any game $\Gamma(\Delta, N)$ with $\Delta < \overline{\Delta}$:*

$$\left\{t \le t(\tau, \Delta) \wedge q\left(h^t\right) > q \wedge X\left(h^t\right) > K\right\} \Longrightarrow \left\{q\left(h^{t+1}\right) < q\right\}.$$

Proof. Fix $\tau < \tau_H$ and $q > 0$. By Lemma 7, fix $K' \in \mathbb{N}$ and Δ' such that

$$P_L\left(X\left(h^t\right) > K'\right) < \frac{1}{2} \tag{63}$$

whenever $\Delta < \Delta'$, $q\left(h^t\right) > q$, $t \le t(\tau, \Delta)$. Since $\tau < \tau_H$, the same logic that led to (61) allows us to fix $\Delta'' > 0$ and $q^* < 1$ such that whenever $t \le t(\tau, \Delta)$ and $\Delta < \Delta''$, the following implication holds

$$q\left(h^t\right) > q^* \Longrightarrow p\left(h^t\right) > p^*(\Delta). \tag{64}$$

Define $\overline{\Delta} = \min(\Delta', \Delta'')$. For the rest of the proof we assume that $\Delta < \overline{\Delta}$, and we take an arbitrary history h^t such that $t \le t(\tau, \Delta)$, $q(h^t) > q$, and $\xi^H(h^t) > 0$. Our goal is to find K such that that $X(h^t) > K$ would imply $q(h^{t+1}) < q$.

Consider the expression for the probability of k exits:

$$P_\theta\left(X\left(h^t\right) = k\right) = \binom{n}{k}\left(\xi^\theta\left(h^t\right)\right)^k\left(1 - \xi^\theta\left(h^t\right)\right)^{n\left(h^t\right)-k}. \tag{65}$$

Since $\xi^H(h^t) < \xi^L(h^t)$, it follows by straightforward algebra from (65) that

$$\frac{P_H(X(h^t) = k)}{P_L(X(h^t) = k)} > \frac{P_H(X(h^t) = k')}{P_L(X(h^t) = k')} \text{ for } k < k'. \tag{66}$$

It then also follows that

$$\frac{P_H(X(h^t) = K')}{P_L(X(h^t) = K')} < 2. \tag{67}$$

To see why, assume the contrary. Then, we have

$$P_H\left(X\left(h^t\right) \le K'\right) = \sum_{k=0}^{K'} P_H\left(X\left(h^t\right) = k\right) > 2 \cdot \sum_{k=0}^{K'} P_L\left(X\left(h^t\right) = k\right) > 2 \cdot \frac{1}{2} = 1,$$

where the first inequality uses (66) and the presumption that (67) does not hold, whereas the second inequality follows from (63). But a probability of an event can not be greater than one, so (67) must hold.

Consider next the following expression:

$$\begin{aligned}
\frac{P_H(X(h^t) = K' + K'')}{P_L(X(h^t) = K' + K'')} &= \frac{\begin{pmatrix} n \\ K' + K'' \end{pmatrix}\left(\xi^H(h^t)\right)^{K'+K''}\left(1 - \xi^H(h^t)\right)^{n\left(h^t\right)-K'-K''}}{\begin{pmatrix} n \\ K' + K'' \end{pmatrix}\left(\xi^L(h^t)\right)^{K'+K''}\left(1 - \xi^L(h^t)\right)^{n(h^t)-K'-K''}} \\
&= \left(\frac{\xi^H(h^t)}{\xi^L(h^t)}\right)^{K'}\left(\frac{1 - \xi^H(h^t)}{1 - \xi^L(h^t)}\right)^{n\left(h^t\right)-K'} \\
&\quad \cdot \left(\frac{\xi^H(h^t)}{\xi^L(h^t)}\right)^{K''}\left(\frac{1 - \xi^L(h^t)}{1 - \xi^H(h^t)}\right)^{K''} \\
&= \frac{P_H(X(h^t) = K')}{P_L(X(h^t) = K')} \cdot \left(\frac{\xi^H(h^t)}{\xi^L(h^t)}\right)^{K''} \cdot \left(\frac{1 - \xi^L(h^t)}{1 - \xi^H(h^t)}\right)^{K''}
\end{aligned}$$

By (67),

$$\frac{P_H\left(X\left(h^t\right)=K'\right)}{P_L\left(X\left(h^t\right)=K'\right)}<2.$$

Also, since $\xi^L\left(h^t\right)>\xi^H\left(h^t\right)$, we have

$$\left(\frac{1-\xi^L\left(h^t\right)}{1-\xi^H\left(h^t\right)}\right)^{K''}<1.$$

By Lemma 1, we have

$$\lim_{K''\to\infty}\left(\frac{\xi^H\left(h^t\right)}{\xi^L\left(h^t\right)}\right)^{K''}=0,$$

and therefore, we can set K'' high enough to ensure

$$\frac{P_H\left(X\left(h^t\right)=K'+K''\right)}{P_L\left(X\left(h^t\right)=K'+K''\right)}<\frac{1-q^*}{q^*}q. \tag{68}$$

Since $\xi^H\left(h^t\right)>0$, we know from (64) that $q\left(h^t\right)<q^*$ (otherwise no player would want to exit). Therefore, Bayesian rule and simple algebra leads to:

$$\begin{aligned}
& q\left(h^{t+1}\left|X\left(h^t\right)=K'+K''\right.\right) \\
= \; & \frac{q\left(h^t\right)P_H\left(X\left(h^t\right)=K'+K''\right)}{q\left(h^t\right)P_H\left(X\left(h^t\right)=K'+K''\right)+\left(1-q\left(h^t\right)\right)P_L\left(X\left(h^t\right)=K'+K''\right)} \\
< \; & \frac{q\left(h^t\right)}{1-q\left(h^t\right)}\frac{P_H\left(X\left(h^t\right)=K'+K''\right)}{P_L\left(X\left(h^t\right)=K'+K''\right)} \\
\leq \; & \frac{q^*}{1-q^*}\frac{P_H\left(X\left(h^t\right)=K'+K''\right)}{P_L\left(X\left(h^t\right)=K'+K''\right)}<q,
\end{aligned}$$

where the last inequality follows from (68). By (66), this means that

$$q\left(h^{t+1}\left|X\left(h^t\right)=k\right.\right)<q$$

for any $k>K$, where we have set $K\equiv K'+K''$. ■

Finally, we state a lemma that limits the probability with which an outside observer's belief $\widehat{q}\left(h^t\right)$ could ever get small values if $\theta=H$. This result is simply a formalization of the notion that a Bayesian observer is not likely to get convinced of the untrue state.

Lemma 9 *For all $\varepsilon>0$, there is a $q>0$ such that*

$$P_H\left\{h^\infty:\widehat{q}\left(h^t\right)\leq q \textit{ for some } h^t\in h^\infty\right\}<\varepsilon.$$

Proof. Consider the event

$$A=\left\{h^\infty:\widehat{q}\left(h^t\right)\leq q \text{ for some } h^t\in h^\infty\right\}.$$

The posterior probability of $\theta = H$ conditional on reaching A is

$$\frac{q_0 P_H(A)}{q_0 P_H(A) + (1-q_0) P_L(A)} \leq q$$

by the definition of the event A. Since $P_L(A) \leq 1$, we have:

$$P_H(A) \leq \frac{(1-q_0)\, q}{q_0 (1-q)},$$

which can be made arbitrarily small by decreasing q. ■

With Lemmas 8 and 9 at hand, it is now easy to finish the proof of Proposition 4:

Proof. [Proof of Proposition 4] Fix $\tau < \tau_H$ and $\varepsilon > 0$. Using Lemma 9, and noting that the divergence of uninformed player's belief from outside observer's belief is bounded by Lemma 4, we can fix $q > 0$ such that

$$P_H\left\{h^\infty : q\left(h^t\right) \leq q \text{ for some } h^t \in h^\infty\right\} < \varepsilon. \tag{69}$$

Next, by Lemma 8, fix $K \in \mathbb{N}$ and $\overline{\Delta} \in \mathbb{R}^+$ such that

$$\left\{t \leq t(\tau, \Delta),\, q\left(h^t\right) > q\,,\, X\left(h^t\right) > K\right\} \Longrightarrow \left\{q\left(h^{t+1}\right) < q\right\},$$

whenever $\Delta < \overline{\Delta}$. Thus, if there is some h^t, $t \leq t(\tau, \Delta)$, for which $X(h^t) > K$, we must have either $q(h^t) \leq q$ or $q(h^{t+1}) \leq q$. But by (69) this cannot happen with probability greater than ε, and as a result, we have

$$P_H\left\{h^\infty : X\left(h^t\right) > K \text{ for some } t \leq t(\tau, \Delta)\right\} < \varepsilon$$

if $\Delta < \overline{\Delta}$. ■

Proof of Theorem 2

Proof. [Proof of Theorem 2] Fix $\tau < \tau_H$ and $\varepsilon > 0$. Then, by Lemma 4, fix $q^* > 0$ and $\Delta' > 0$ such that whenever $\Delta < \Delta'$ and $t \leq t(\tau, \Delta)$, the following holds:

$$\widehat{q}\left(h^t\right) \geq q^* \Longrightarrow p\left(h^t\right) > p^*(\Delta). \tag{70}$$

Next, by Proposition 3, we can fix K' such that

$$P_H\{h^\infty : n\left(h^t\right) \leq N - K' \text{ and } \widehat{q}\left(h^t\right) < q^* \text{ for some } h^t \in h^\infty\} < \frac{\varepsilon}{2}. \tag{71}$$

Assume that $\Sigma_{t=0}^{t(\tau,\Delta)} X(h^t) \geq K'$ (we may safely ignore the case $\Sigma_{t=0}^{t(\tau,\Delta)} X(h^t) < K'$, because the Theorem holds for that case with any $K \geq K'$), and denote by $t_{K'}$ the first period with fewer than $N - K'$ players left:

$$t_{K'} := \min\left\{t : n\left(h^t\right) \leq N - K'\right\}. \tag{72}$$

Since $\Sigma_{t=0}^{t(\tau,\Delta)} X\left(h^t\right) \geq K'$, we must have

$$t_{K'} \leq t\left(\tau,\Delta\right)+1. \tag{73}$$

Equations (70) and (71) mean that the probability that any player exits in $[t_{K'},...,t\left(\tau,\Delta\right)]$ is less than $\frac{\varepsilon}{2}$:

$$P_H\left\{h^\infty : \sum_{t=t_{K'}}^{t(\tau,\Delta)} X\left(h^t\right) > 0\right\} < \frac{\varepsilon}{2}. \tag{74}$$

By definition of $t_{K'}$ in (72) we know that

$$\sum_{t=0}^{t_{K'}-2} X\left(h^t\right) < K'. \tag{75}$$

Finally, by (73) and Proposition 4, we can find $\Delta'' \in \mathbb{R}^+$ and K'' such that

$$P_H\left\{h^\infty : X\left(h^{t_{K'}-1}\right) > K''\right\} < \frac{\varepsilon}{2}. \tag{76}$$

Noting that (74) holds when $\Delta < \Delta'$ and (76) holds when $\Delta < \Delta''$, we may set $K := K' + K''$ and $\overline{\Delta} := \min\left(\Delta',\Delta''\right)$, and combine (74) - (76) to get:

$$P_H\left\{h^\infty : \sum_{t=0}^{t(\tau,\Delta)} X\left(h^t\right) > K\right\} < \varepsilon,$$

whenever $\Delta < \overline{\Delta}$. ■

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