Juggling Tasks*

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Abstract

This paper studies the work practices of a worker who can split his effort among several projects. The worker opens projects at a given rate and devotes the same effort to all open projects. Working on too many projects at once is referred to as task-juggling. We find that when the worker opens projects at too high a rate, task juggling leads to decreased output rate and projects take longer to complete. As time goes by, the duration of projects increases linearly with time. We obtain a precise formula linking the input rate, the complexity of the projects, and the worker's effort, to the output rate. We provide a model in which task juggling result as the Nash equilibrium of a lobbying game among the worker's clients. We also show that if there is value in the projects clearing intermediate goals, and not only in completion, then large input rates are not necessarily inefficient; in this case task-juggling can be optimal despite durations which increases linearly over time. We compare two ways of incentivizing the worker when the worker can multitask across projects of different complexity: rewarding output leads the worker to focus effort on on low-complexity projects, whereas penalizing duration leads the worker to focus on relatively more complex projects. We conclude by analyzing certain extensions of the model including one where the worker is forgetful, and so his productivity per task decreases as a function of the time it takes him to complete two consecutive tasks of the same project. The presence of this friction gives rise to the possibility of multiple growth paths.

*Thanks to Debraj Ray.

1 Introduction

Most of us seem to be working on too many projects at once. With too much on one's plate, one's effort gets dispersed and so only incremental work gets done on any one project. As a result projects never get done, deadlines are pushed forward, etc. We refer to this work practice as *task juggling.* Sounds familiar? This because this problem is ubiquitous. As academic researchers, perhaps under the pressure of different co-authors, we sometimes find ourselves jumping from project to project, and only being able to devote fractions of their time to each paper. But task juggling is not only limited to researchers: many other types of workers also engage in task juggling. In related empirical work (Coviello *et al.* 2010), we show that judges in Italy also engage in task juggling, and we measure the effect of this practice on the duration of trials. Of course, it can be efficient to work on a number of projects simultaneously, and our theory will accommodate this possibility. To be precise, therefore, we reserve the term "task juggling" for the case in which more than the efficient number of projects are worked on simultaneously, resulting in a slowdown of project completion. This paper develops a theory of task juggling.

To understand what we do, let's start with a simple static model of a worker who can split one hour between two projects, the first requiring one hour to get done, the second requiring two hours. If the worker commits the same amount of time to both projects, then each project receives 1/2 hour and neither project gets done. If instead the worker focused on the first project, he would at least gets one done. We say that this worker is juggling tasks, in the sense that by working on too many projects simultaneously, he slows down his output. The lesson here is that if the goal is completing projects, then it pays to focus on the one that is closest to being done. Of course, in this case the technology that produces the output (a complete case) was very stark, and so the effect of overcommitment on productivity was easy to describe. Imagine now a more realistic world in which the worker receives a constant stream of projects and can work on many projects at a time. Here the projects that get done in one instant are exactly those projects which, the instant before, were almost done. In other words, the evolution of productivity over time depends on the composition of the case load (how many projects are almost done at any point in time), as well as on the speed at which projects are moved along. Now this gets complicated: the composition of the case load, like an inventory, contains projects in different stages of completion and evolves over time. How the case load composition evolves depends on how much the worker works in each instant, how much work it takes to complete each project, how many projects he can complete in each instant given how many he is working on currently, and how many new projects he opens. In other words, the technology that determines output is now a high-dimensional Markov problem whose dimensionality must be sufficient to fully describe the evolution of case load composition.

The paper proceeds as follows. In Section 2 we lay out a continuous-time model which describes *how much* the worker works and *how* he works—how many projects he keeps open at the same time. This second element matters because, by assumption, the worker splits his effort equally among all open projects. The main variables of the model are: the rate at which the worker starts working on projects; the stock in each instant in time of projects being worked on distinguished according to their level of "done-ness" of each project; and the output rate, the rate at which projects are completed. Formally, the model is described by the system of partial differential equations (1) through (4).

In Sections 3 to 5 we provide closed-form solutions for this system of partial differential equations, under the assumption that the rate at which projects are started and the worker's effort are stationary. This solution fully characterizes the dynamic production function that turns opened projects and effort into completed projects. Our analysis is most interesting and challenging when the rate at which projects are started exceeds that at which projects are completed. In this case, over time the worker juggles an increasing number of tasks, the duration of his projects will increase (linearly with time, as we will show), and his output rate *decrease.* That the output rate should decrease as the input rate increases follows because, by assumption, if a new project is opened the worker will devote just as much effort to it as to any other open project. Thus when the case load is above its efficient level, opening a new project reduces the effort devoted to projects that are closer to completion. This externality creates the slowdown associated with task juggling.

What is the production function of output when task juggling is a concern? It is expression (5) on page 12. This expression links the effort rate η, the complexity X of the project, the rate ν at which projects are started, and the output rate ω. This relationship, therefore, represents the technology that yields output ω. We show that when ν exceeds the threshold η/X then ω becomes decreasing in ν.[1] That is, the higher the input rate, the worse the task juggling and the lower the output rate. Moreover, studying expression (5) reveals that ν and η are strategic substitutes in the production of output.[2] This suggests that anything that makes workers juggle more tasks will also, indirectly, induce lower effort on the part of the worker. We will use this information later.

If task juggling is bad, then why is it so prevalent? Why don't workers keep the input rate down to η/X and maximize their output rate? Or, from a different perspective, why does the worker split his effort equally among too many open projects? While the answer will depend on the specific circumstances—whether the worker is a judge, a university researcher, or whatever else—Section 6 offers an explanation that could fit several scenarios. The idea

[1] Figure 3 portrays the relationship implied by equation (5) between the input rate ν and the output rate ω.

[2] This means that the cross partial of output with respect to ν and η is negative.

is that effort is allocated "under duress." We provide a framework in which projects being worked on belong to clients, each of whom wants their project to be done quickly. These clients have the ability to, in each instant, lobby the worker (at a cost) and have their project skip the line and get worked on immediately. For an academic researcher, the competing clients might be co-authors; for a judge, they might be the plaintiffs in the trials, particularly in countries where trials can last decades. In our model, no matter how low the cost of lobbying, the input rate that results from the equilibrium of this lobbying game is shown to be inefficiently large. In other words, lobbying causes task juggling. In addition, we fully characterize the equilibrium of the lobbying game and show that, if the cost of lobbying is constant, then the equilibrium input rate is time invariant. This result provides a justification for focusing on time-invariant input rates, as we do throughout the paper.

A different hypothesis could be that task juggling is strategically engineered by a principal, to induce the worker to work harder. However, this hypothesis is not consistent with the theory. We mentioned before that ν and η are strategic substitutes in the production of output. Therefore, if the worker maximizes (an increasing function of) output, then causing the worker to juggle more tasks results in lower effort. Our analysis, therefore, suggests that even if a principal faces limitations on the monetary rewards he can pay for performance (say, in the public sector because money is scarce, or in the private sector because of union rules), still he should not induce his workers to open too many new cases in hopes of improving their effort or their output. This finding has practical implications for the organization of trials in Italy. Italian judges are required by law to open cases as soon as they have been assigned to them, thereby forcing ν to be very high. In light of our analysis, this is detrimental for both output rates and effort. In a separate empirical paper, we apply some of the ideas developed here to the study of the productivity of Italian judges. The same logic suggests that workers should be protected from lobbying by clients.

A still different view of task juggling is that in fact it could be *efficient*. In Section 7 we show that if value is created when a project completes early intermediate goals, then it may be optimal to push the input rate ν above η/X. This is because if the primary goal is not (only) to complete the project, but there is also value in *starting* it, then the negative effect of large input rates on output rates is less important. In this case, the progressive slowdown of projects and the associated increase in duration are in fact efficient. As an application, consider certain types of trials in which preliminary injunctions may need to be issued early in the trial, before the case is decided.

In Section 8 we explore variable speed strategies, that is, strategies that calibrate the amount of effort devoted to a case depending on the degree of completion of the case. A special case are the "equal treatment" strategies considered in the rest of the paper, which devote the same proportion of effort to each open case at any point in time. We give conditions under which the optimal variable speed strategy is an equal treatment one.

In Section 9 we analyze the different options available to a principal who needs to incentivize the worker. The principal may chose to reward the aggregate output rate, or penalize the average duration of projects. Although these two performance measures are closely related, if the worker can strategically elect to work on projects of different complexity, then it matters which measure is used. In Section 9 we show that if the worker is compensated based on aggregate output irrespective of the complexity of the project, then that worker will totally ignore complex projects and focus on easy ones. Under this strategy, complex problems never get done. If instead the worker is penalized based on the average duration of assigned projects, then we show that a worker will work on cases of all complexities, and in fact will devote relatively more effort to complex cases.

Section 10 analyzes several extension to the main model. We deal first with the case in which projects have different degrees of complexity, but they cannot be treated disparately (if they can, then the analysis of Section 9 applies). We caracterize the ratio of input to output rates as a function of the complexity of the project and show that more complex projects have a worse input to output ratio. Next we deal with the case in which the worker is forgetful, that is, as the completion time grows and any open project is worked on less and less per unit of time, the worker starts to forget about each individual project and thus before he can make progress he needs additional effort on each case to "remind himself" of where he left off. The additional effort required slows down production relative to a constant growth path. More remarkably, in our model introducing forgetfulness may generate a multiplicity of growth paths. Finally, we discuss how the model would change in the presence of a "time to build" constraint whereby the worker cannot complete a project under a certain time threshold, no matter how large the effort.

1.1 Related Literature

What we call task juggling is an inefficiency that is related to the concept of "bottleneck" in the literatures on project management and project planning (see Model et al, 1983). Our model is also related to the literature on network queuing, originating with Jackson (1963). The focus of these literatures is on how to identify and eliminate bottlenecks in the production or service processes. In our model this is easily done (set the input rate ν equal η/X) and is not the main point of the paper. In contrast, our analysis focuses on the observable features of an inefficient dynamic production function, culminating in equation (5) which, to our knowledge, is a new mathematical result. This result is our starting point for a study of incentives: how private incentives to lobby the worker might result in task juggling, and how to incentivize the worker. Incentives considerations such as the ones we study are largely absent from these literatures.

The model of lobbying by clients presented in Section 6 is a version of common-pool problem, of a kind that has been studied in economics under various guises (see e.g. Ostrom 1990).

Section 9 analyzes the effect of different incentive schemes when the worker can arbitrage across projects of different complexity. Such problems have been studied by Holmstrom and Milgrom (1991).

In Section 10.2 we study the case in which coming back to a project too seldom can slow down the productivity on each individual task of the project. This idea is explored in a recent plychological literature on multitasking and task interruptions (see e.g. Mark et al. 2008).

2 The Model

The model lives in continuous time, starting from $t = 0$. There is a continuum of projects. Each project takes X steps to complete and is characterized, at any point in time, by its degree of completion $x \in [0, X]$. Before a worker starts working on a project, that project's degree of completion is $x = X$. We call a project **completed** when $x = 0$. Note that, because x is a continuous variable, we are assuming (with some abuse of the English language) that there is a continuum of steps for each project. X can properly be interpreted as a measure of the complexity of the project.

As soon as the worker starts working on a project, that project becomes **active**. The project stops being active when it is completed. At any time t, the worker has A_t active projects, in various degrees of completion. The distribution $\varphi_t(x)$ represents the mass of projects which are exactly x steps away from being done.[3] By definition, the number of active projects at time t is

$$A_t = \int_0^X \varphi_t(x)\,dx \tag{1}$$

We assume that all active projects become more complete at a rate η/A_t, where a large η denotes a hard-working or fast worker. Informally, this means that in the time interval between t and $t + \Delta$, the worker's work shaves off approximately $(\eta/A_t)\,\Delta$ steps from *each* active project.[4] This formulation captures the idea that the worker divides a fixed amount of working hours equally among all projects active at time t. We refer to this procedure as working "in parallel." Parallel work means that all projects proceed at the same speed; this

[3] Technical note: φ is a measure but not a probability measure: it does not sum to 1.

[4] Note that this formulation requires $A_t > 0$.

means that after Δ has elapsed, the distribution $\varphi_t(x)$ is translated horizontally to the left (refer to Figure 1), and so for Δ "small enough" we can write intuitively

$$\varphi_{t+\Delta}\left(x - \frac{\eta}{A_t}\Delta\right) = \varphi_t(x).$$

To express this condition rigorously, bring $\varphi_t(x)$ to the right-hand side, divide by Δ and let $\Delta \to 0$ to get

$$\frac{\partial \varphi_t(x)}{\partial t} - \frac{\partial \varphi_t(x)}{\partial x}\frac{\eta}{A_t} = 0. \tag{2}$$

This partial differential equation embodies the assumption of parallel work.

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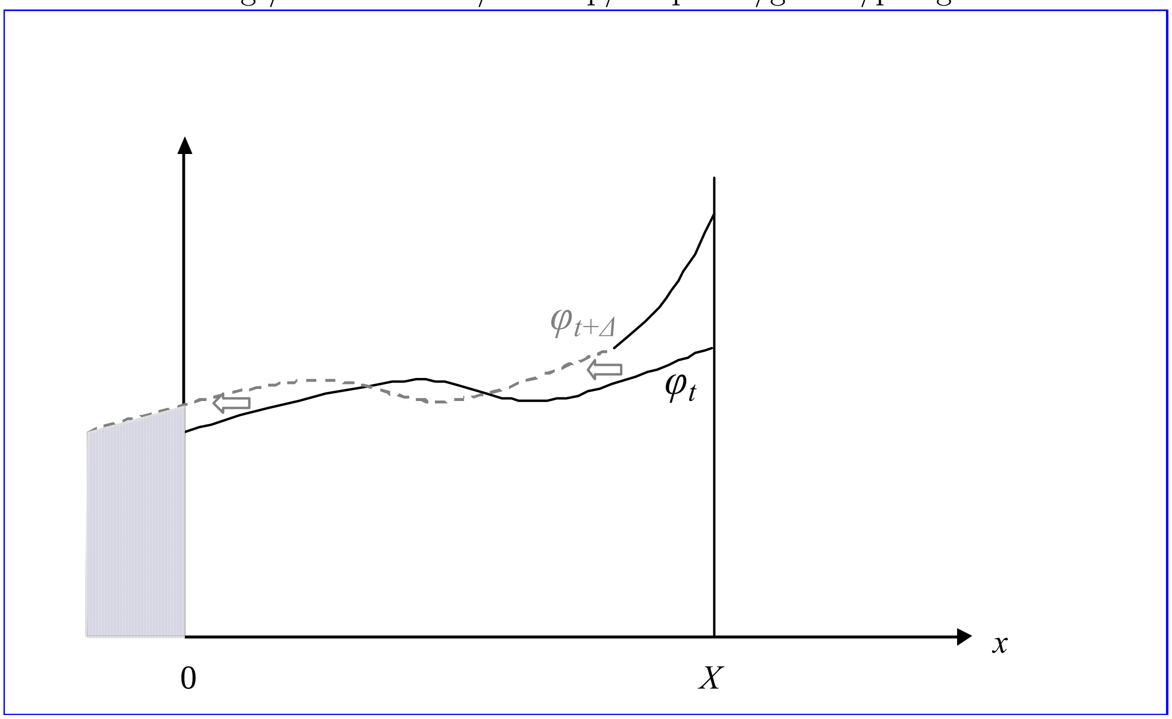

Figure 1: The function φ_t is traslated horizontally to the left as time passes. Newly opened cases are added to the right. The grey mass of cases to the left of zero are completed.

The projects that fall below 0 (grey mass in Figure 1) are the ones that get completed within the interval Δ. These are the projects whose x at t is smaller than $\frac{\eta}{A_t}\Delta$. Therefore, the mass of output between t and $t+\Delta$ is approximately

$$\int_0^{\frac{\eta}{A_t}\Delta} \varphi_t(x)\,dx.$$

To get the **output rate** ω_t, divide this expression by Δ and let $\Delta \to 0$ to get

$$\omega_t = \lim_{\Delta \to 0} \frac{1}{\Delta} \int_0^{\frac{\eta}{A_t}\Delta} \varphi_t(x)\, dx = \frac{\eta}{A_t} \varphi_t(0)\,. \tag{3}$$

Cases are assigned to the worker at rate α_t, but the worker is not required to open projects as soon as they are assigned. Rather, we allow the worker to open new projects at a rate ν_t. A larger ν_t will, ceteris paribus, mean more task juggling—more projects being worked on simultaneously. This ν_t is seen either as a choice on the part of the worker, or as determined by lobbying, or else imposed by some regulation. For Δ small, the change in the mass of projects active at t is approximately

$$A_{t+\Delta} - A_t = \nu_t \cdot \Delta - \omega_t \cdot \Delta.$$

Divide both sides by Δ and let $\Delta \to 0$ to get the formally correct expression

$$\frac{\partial A_t}{\partial t} = \nu_t - \omega_t. \tag{4}$$

Graphically, the mass of newly opened projects is squeezed in at the back of the queue in Figure 1, just to the left of X, in whatever space is vacated on the horizontal axis by the progress made in Δ on the pre-existing open projects.

We close this section by defining two measures of productivity.

Definition 1 *For a project assigned at t we define the **duration** D_t as the time which elapses between t and the completion of the project. For a project opened at t (and thus assigned at a time before t), we define **completion time** C_t the time which elapses between t and the completion of the project.*

This completes the description of our model. In this model, two variables are interpreted (for now) as exogenously given: η and ν_t. The first describes *how much* the worker works, the second *how* he works—how many projects he keeps open at the same time. These two variables will determine, through the process described mathematically by equations (1) through (4), the key variable of interest, the output rate ω_t. This variable, in turn, will determine the duration of a project and its completion time. The two (endogenous) functions A_t and $\varphi_t(x)$ are, perhaps, of merely instrumental interest: they describe the state of the worker's docket at any point in time—how many projects he has open, and the degree of completeness of each. Our first major task is to uncover the law through which η and ν_t affect ω_t. We turn to this next.

3 Growth Paths and Fundamental Theorem

Theorem 1 in this section identifies the law through which ν_t determines ω_t. In terms of language, however, rather than talking about ν_t *determining* ω_t, it is more convenient to talk about ν_t and ω_t (and also $\varphi_t(x), A_t$) *belonging to a growth path.*[5] This is the language we use in this section.

Definition 2 *A* ***growth path*** *is a quadruple of positive real functions* $[\nu_t, \varphi_t(x), A_t, \omega_t]_{\substack{t\in(0,\infty)\\ x\in[0,X]}}$ *that satisfies (1), (2), (3) and (4). A* ***constant growth path*** *is a growth path where input and output rates are constant,* $\nu_t = \nu$ *and* $\omega_t = \omega$.

A growth path is a set of functions which are linked together in that they jointly satisfy a system of functional equations. While this definition is somewhat abstract, the way to think about a growth path is as a time-varying production function. The input rate ν_t is given exogenously, and the output rate ω_t is the output of a complicated, time-evolving "production function," which is characterized by the time-varying quantities $\varphi_t(x)$ and A_t.

In general, finding a quadruple that constitutes a growth path is a daunting mathematical problem. However, when ν_t is constant equal to ν, then a growth path can be fully characterized. We start by guessing a functional form for $\varphi_t(x)$ and A_t. Let

$$\varphi_t^*(x) = \frac{(\nu-\omega)}{\eta}\omega\ t\ e^{\frac{\nu-\omega}{\eta}x},$$

and

$$A_t^* = (\nu-\omega)\ t.$$

The theorem states that the pair $\varphi_t^*(x), A_t^*$, paired with constant input and output rates ν and ω, satisfies conditions (1) - (4), and thus are part of a growth path, if and only if a certain equation holds relating ν and ω.

Theorem 1 *The quadruple* $[\nu, \varphi_t^*(x), A_t^*, \omega]$ *is a constant growth path if and only if the pair* ν, ω *solves the equation* $X\frac{\nu-\omega}{\eta} = \log(\nu) - \log(\omega)$.

Proof. One can verify directly that for any K, λ, the pair $\varphi_t(x) = Kte^{\frac{\lambda}{\eta}x},\ A_t = \lambda t$ solves (2) above. Moreover, for any λ the triple $\varphi_t(x) = Kte^{\frac{\lambda}{\eta}x},\ A_t = \lambda t,\ \omega_t$ satisfies (3) if and only if $K = \frac{\lambda}{\eta}\omega_t$, which implies $\omega_t = \omega$. Finally, the triple ν_t, A_t, ω satisfies (4) if and only if $\lambda = \nu_t - \omega$, which implies $\nu_t = \nu$.

[5] In what follows we will omit the subscripts $t \in (0,\infty)$ and $x \in [0, X]$ when no confusion can arise.

This shows that, for any ν, ω, the quadruple $[\nu, \varphi_t^*(x), A_t^*, \omega]$ satisfies all but one of the equalities which define a constant growth path. However, we do not yet know which values of ν and ω are compatible with each other along a growth path. We now show that the pair $\varphi_t^*(x) = Kte^{\frac{\lambda}{\eta}x}, A_t^* = \lambda t$ solves (1) if and only if $X\frac{\nu-\omega}{\eta} = \log(\nu) - \log(\omega)$. Condition (1) reads

$$A_t^* = \int_0^X \varphi_t^*(x)\ dx.$$

Substituting $\varphi_t^*(x)$ and A_t^* yields

$$\begin{aligned}
\lambda t &= \int_0^X Kte^{\frac{\lambda}{\eta}x}\ dx \\
&= Kt\ \frac{\eta}{\lambda}e^{\frac{\lambda}{\eta}x}\Big|_{x=0}^{X} \\
&= \frac{\eta}{\lambda}Kt\ \left[e^{\frac{\lambda}{\eta}X} - 1\right].
\end{aligned}$$

Now substitute for $K = \frac{\lambda}{\eta}\omega$ and $\lambda = \nu - \omega$ to get

$$\begin{aligned}
\lambda t &= \frac{\eta\lambda\omega t}{\eta\lambda}\left[e^{\frac{\lambda}{\eta}X} - 1\right] \\
\lambda &= \omega\left[e^{\frac{\lambda}{\eta}X} - 1\right] \\
\nu - \omega &= \omega\left[e^{\frac{(\nu-\omega)}{\eta}X} - 1\right] \\
\frac{\nu}{\omega} &= e^{\frac{(\nu-\omega)}{\eta}X}.
\end{aligned}$$

Taking logs yields

$$\log(\nu) - \log(\omega) = X\frac{(\nu-\omega)}{\eta}.$$

Therefore, Theorem 1 is proved. ■

Along a constant growth path the function $\varphi_t^*(x)$ is exponential in x and multiplicative in t, as depicted in Figure 2. Thus as $t \to 0$ the function $\varphi_t^* : [0, X] \to \mathbb{R}$ converges to zero uniformly. As t grows, the function φ_t^* grows multiplicatively in t.

The next result shows that completion time and duration both grow linearly with t.

Proposition 1 *In a constant growth path we have* $C_t = \frac{(\nu-\omega)}{\omega}t$ *and* $D_t = \frac{(\alpha-\omega)}{\omega}t$.

Proof. The completion time C_t of a project started at t is the time that it takes all the projects in front of it to clear. These projects are A_t, and given an output rate ω that and Settings/Nicola Persico/Desktop/tempwork/giudici/figurephiequil2.wmf

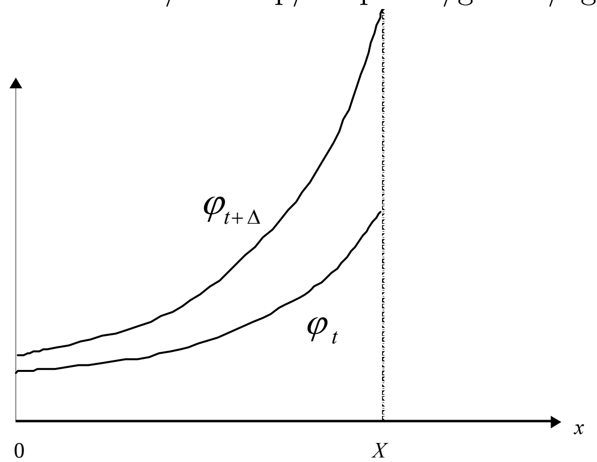

Figure 2: Distribution of active cases, by number of steps away from being done. On the growth path it is exponential.

duration is given by the solution to the following equation

$$\int_t^{t+C_t} \omega \; ds = A_t,$$

which equals

$$\omega C_t = (\nu - \omega)\, t.$$

Solving for C_t yields the desired expression. Let us now turn to duration. Given an arrival rate α, a project assigned at t finds

$$\alpha t - \omega t$$

projects in front of it. Given an output rate of ω, these projects will take

$$D_t = \frac{(\alpha - \omega)}{\omega} t$$

to complete. This is the duration of a project assigned at t. ■

The results presented until now show how the system evolves if $\nu > \omega$. In this case, as time goes by the worker juggles an increasing number of projects (formally, A_t grows linearly with t), which makes projects proceed at a progressively slower pace (the pace is η/A_t). As projects grind along more and more slowly, the constant input ν of new cases results in more and more projects being accumulated at all degrees of completion x, especially at high x (we know this because $\varphi_t(x)$ grows multiplicatively with t). Despite all these complex dynamics, the output rate is constant through time. This remarkable property of the output rate results from two opposite effects offsetting each other: As time goes by, cases move through at progressively slower rates, which tends to progressively reduce the output rate. On the other hand, the mass of cases that are almost done increases with time (this is because $\varphi_t(0)$ grows multiplicatively with t), which tends to progressively increase the output rate. These two effects exactly offset each other along a constant growth path, and thus the output rate is time-invariant.

4 Characterization of the Output Rate

Theorem 1 goes a long way towards characterizing a constant growth path, but there is still some work to do. We need to characterize the relationship that links ν and ω along a growth path or, said differently, we need to understand what level of output is possible given an input rate.

According to Theorem 1, the relationship between ν and ω along a growth path is

$$(\nu - \omega)\frac{X}{\eta} = \log(\nu) - \log(\omega). \tag{5}$$

Define

$$h(y) = \frac{X}{\eta}y - \log(y).$$

Then equation (5) reads

$$h(\nu) = h(\omega).$$

The next lemma characterize the function $h(\cdot)$.

Lemma 1 *The function $h(y)$ is strictly convex on $(0, \infty)$, converges to infinity at $y = 0$ and $y = +\infty$, and it has its minimum at $y = \eta/X$.*

Proof. One can easily verify that $h(0) = +\infty = h(\infty), h'(y) = \frac{X}{\eta} - \frac{1}{y}$, and finally $h''(y) = \frac{1}{y^2}$. ■

Figure 3 depicts $h(y)$. Not all solutions to equation (5) can be part of a growth path. Which solutions are consistent with a growth path is described in the next proposition.

Proposition 2 ***(conditions for a growth path)*** *ν is compatible with a constant growth path if and only if $\nu > \frac{\eta}{X}$. In that case, the quadruple $[\nu, \varphi_t^*(x), A_t^*, \omega]$ is a constant growth path if and only ω is the unique solution to equation (5) that is smaller than $\frac{\eta}{X}$.*

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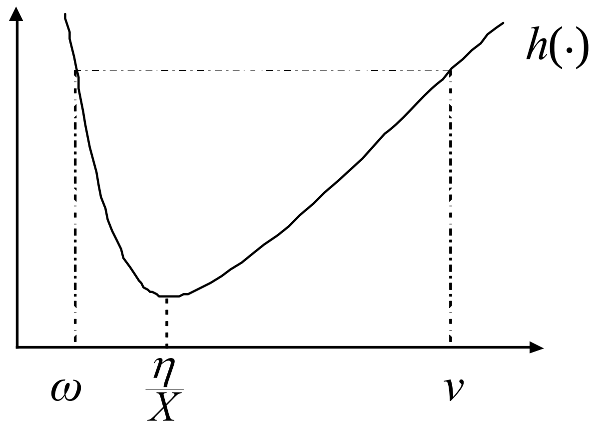


Figure 3: Relationship between input and output rates on a growth path.

Proof. The solution $\omega = \nu$ to equation (5) is not acceptable because then $A_t^* \equiv 0$ and (3) is not well-defined. Nor can we accept solutions where $\omega > \nu$, for then $\varphi_t^*(x)$ and A_t^* would be negative and thus the quadruple identified in Theorem 1 would not meet the definition of a growth path. So we need to find solutions with $\omega < \nu$. This implies $\nu > \frac{\eta}{X}$. The rest of the Proposition follows immediately from the previous analysis. ■

This proposition shows how to construct the entire growth path associated to any constant input rate ν. Given a constant input rate $\nu > \frac{\eta}{X}$, one can uniquely identify the corresponding output rate $\omega < \frac{\eta}{X}$ which solves equation (5). The (ν, ω) pair thus obtained is substituted into $\varphi_t^*(x), A_t^*$ to obtain a full characterization of the growth path.

Proposition 2 shows that a constant growth path only exists if the input rate is sufficiently large. If not, then the worker can solve projects faster than he opens them, and in that case our model predicts $A_t \equiv 0$. In this case we do not have a model of task juggling, but rather one of "undercommitment." We conclude this section by analyzing this case. The next proposition shows that if $\nu < \frac{\eta}{X}$ then A_t shrinks over time, and if $\nu = \frac{\eta}{X}$ then A_t is constant. Some of these results will be useful in future sections.

Proposition 3 ***(steady-state and shrink paths)*** *If $\nu = \frac{\eta}{X}$ then there are a continuum of steady-state paths, indexed by the mass of projects active at time zero, A_0. In each of these steady states $A_t \equiv A_0$, the output rate is equal to η/X, and the duration of projects is increasing in A_0.*

If $\nu < \frac{\eta}{X}$ then whatever the value of A_0, after a transition period it will be $A_t \equiv 0$ and, from then on, the duration of projects will be zero and the output rate will be increasing in ν.

Proof. Let's start with the case $\nu < \frac{\eta}{X}$. In this case the setup of the model described in Section 2 is no longer applicable, since that setup implicitly required that $A_t > 0$, which now cannot be guaranteed. In fact, if we start at time 0 with $A_0 > 0$ and open projects at rate $\nu < \frac{\eta}{X}$, we expect $\omega_t > \nu$, and so we are on a temporary "shrink path" where over time A_t will shrink down to zero. After A_t hits zero, the worker completes projects instantaneously as soon as they are opened, and the system settles into a long-run path with $\omega_t = \nu < \frac{\eta}{X}$, and $A_t = C_t = D_t = 0$. In this long-run steady state, increasing ν increases ω contrary to Proposition 4.

In the case $\nu = \frac{\eta}{X}$, let us conjecture $\nu = \omega$ and so by (4) we have $A_t = A$. Fix any $A_0 > 0$. Note that this requires assuming an initial load of projects at time zero. Then (3) reads

$$\omega = \frac{\eta}{A_0}\varphi_t(0),$$

whence for all $t > 0$

$$\varphi_t(0) = \frac{A_0}{\eta}\omega. \tag{6}$$

Now, by definition we have that for all $x > 0$ we have $\varphi_t(0) = \varphi_\tau(x)$ for some $\tau < t$. This observation, together with (6), implies

$$\varphi_\tau(x) = \frac{A_0}{\eta}\omega \text{ for all } x, \tau.$$

Then (1) reads

$$A_0 = \int_0^X \varphi_t(x)\,dx = \frac{X}{\eta}A_0\omega.$$

Note that this equality reduces to the identity $\omega = \eta/X$, which yields no new information. This means that any A_0 is compatible with the steady state path when $\nu = \eta/X$. Whatever is the initial condition of open projects A_0, choosing $\nu = \eta/X$ will exactly perpetuate that mass of open projects.

The completion time of a newly opened project is the interval of time it takes the worker to process the A_0 projects that have precedence over it. We are looking for the time interval C_t it takes for a worker to complete A_0 projects. At a completion rate ω, C_t solves

$$\begin{aligned} A_0 &= \int_t^{t+C_t} \omega\ ds \\ &= \omega C_t = \frac{\eta}{X}C_t, \end{aligned}$$

whence the completion time of a newly activated project is $C_t = \frac{A_0}{\eta}X$, which is increasing in A_0. Given an arrival rate α, a project assigned at t finds

$$A_0 + \alpha t - \omega t$$

projects in front of it. The duration of a project assigned at t is the time it takes to complete these projects given an output rate ω. Thus the duration of a project assigned at t is also increasing in A_0. ■

5 Comparative Statics For the Output Rate

In this section we derive a number of comparative static results for the output rate in a constant growth path. To this end, it is convenient to think of the output rate as *generated* by the other parameters of the model. The next definition introduces the notation Ω for the production function of ω. Throughout this section we implicitly assume that $\nu \geq \frac{\eta}{X}$.

Definition 3 *For each pair $(\nu, \eta/X)$ denote by $\Omega(\nu; \eta/X)$ the unique $\omega < \nu$ that solves (5).*

The next proposition presents the comparative statics results for Ω.

Proposition 4 *a) $\Omega(\nu; \eta/X)$ is decreasing in ν.*

b) $\Omega(\nu; \eta/X)$ is increasing in η/X.

c) $\frac{\partial \Omega(\nu;\eta/X)}{\partial \nu \partial \eta} < 0$, which means that ν and η are strategic substitutes in $\Omega(\nu; \eta/X)$.

d) The function $\Omega(\cdot; \cdot)$ is homogeneous of degree 1.

e) $\Omega(\eta/X; \eta/X) = \eta/X$.

Proof. a): Immediate from inspection of Figure 3.

b), c): See the Appendix.

d) Suppose the triple $\left(\nu, \omega, \frac{\eta}{X}\right)$ solves (5). We need to show that for any scalar $r > 0$, the triple $\left(r\nu, r\omega, r\frac{\eta}{X}\right)$ also solves (5). Write

$$\begin{aligned} r\frac{\eta}{X}\left[\log(r\nu) - \log(r\omega)\right] &= r\frac{\eta}{X}\left[\log(\nu) - \log(\omega)\right] \\ &= r(\nu - \omega) = (r\nu - r\omega). \end{aligned}$$

where the second equality follows because the triple $\left(\nu, \omega, \frac{\eta}{X}\right)$ solves (5). The equality between the first and the last element in this chain of equalities shows that the triple $\left(r\nu, r\omega, r\frac{\eta}{X}\right)$ solves (5).

e) Immediate from inspection of Figure 3. ■

Part a) is the effect of task juggling: increasing ν increases the mass of active projects and reduces output. The proposition shows that η/X is the maximum feasible output rate. When $\nu = \omega = \eta/X$ we have a steady state where $A_t \equiv 0$; this is a case treated in Proposition 3. Part b) simply says that if a worker works more then the output rate is larger.

Part c) deals with the complementarity of inputs in the production of the output rate. It says that the returns to effort decrease when ν increases. Intuitively, this is because A_t is larger and so an increase in effort needs to be spread over a greater number of projects.

Part d) admits two economic intepretations. First, the lemma expresses constant returns to scale with respect to "inputs" ν and η. One implication is the following. Imagine that instead of treating all projects in the same pool, the worker splits his projects into two pools: projects only dealt with in the morning, and projects only dealt with in the afternoon. Suppose the worker allocates a fraction $r < 1$ of his projects to the morning, and the remaining $(1 - r)$ to the afternoon. Suppose also that the worker splits his total effort η in the same proportion. Then his total output is the sum of the outputs of morning and afternoon projects, which by the lemma are, respectively, $r\omega$ and $(1 - r)\,\omega$. The total sum is ω, which shows that the worker does not gain or loses from splitting projects into smaller pools. The second interpretation obtains when $r > 1$. Setting $r > 1$ means that the entire system is working at a faster pace: per unit of time, we have more input, more effort, and more output, all in the same proportion. Another way to think about it is as if time was speeding up. One could, for example, make r a time-varying function of t, and in that case $r > 1$ would correspond to the case where the system operates at a faster speed than calendar time.

The results reported in Proposition 4 can be used to prove that increasing the input rate, and thus the degree of task juggling, increases the inefficiency.

Corollary 2 *In a constant growth path, completion time C_t and duration D_t are increasing in ν.*

Proof. From Proposition 1 we have

$$C_t = \left(\frac{\nu}{\Omega\left(\nu; \eta/X\right)} - 1\right) t = \left(\frac{1}{\Omega\left(1; \eta/\nu X\right)} - 1\right) t,$$

where the second equality follows from Proposition 4 d). From Proposition 4 (b) we have that Ω is increasing in its second argument, whence increasing ν decreases $\Omega\left(1; \eta/\nu X\right)$ and increases C_t.

As for duration, from Proposition 1 we have

$$D_t = \left(\frac{\alpha}{\Omega\left(\nu; \eta/X\right)} - 1 \right) t.$$

From Proposition 4 (a) we have that Ω is decreasing in its first argument, whence increasing ν decreases $\Omega\left(\nu; \eta/X\right)$ and increases D_t. ■

6 Input Rate Determined Through Lobbying Equilibrium

In the previous sections we have assumed that ν_t, the exogenous input rate, is constant through time and, furthermore, that it exceeds the efficient input rate η/X. We have not discussed what process might gives rise to a ν_t with these features. In this section we introduce a game in which each project has a client who in each instant can lobby the worker to devote effort to his project. In this model the input rate ν_t is determined endogenously by the clients' lobbying. In the equilibrium of this game, the input rate turns out to be constant through time and inefficiently high. Therefore, the results in this section provides a foundation for the model analyzed in the previous sections. At the end of this section we also allow effort to be chosen endogenously by the worker.

The basic setup is that clients can, at a cost, force the worker to work on their project, regardless of its order of assignment, in addition to the projects he would be working on otherwise. The private benefit of lobbying is that a client avoids its project waiting unopened and gets the worker working on it immediately. But, by forcing the worker to work on their projects, lobbyists force a worker to distribute effort among more projects. This will increase the number of active projects, which slows down all projects. This externality, which is not internalized by the lobbyists, gives rise to an inefficiency.

The model is as follows. Judicial effort η is fixed and constant through time (we will relax the first assumption later). Lobbying is modeled as a technology whereby, at any instant t, a client can pay $\kappa \cdot \Delta$ and force activity on its project during the interval $(t, t + \Delta)$. Activity on the project means that the project moves forward by $(\eta/A_t) \cdot \Delta$. The rate κ is interpreted as the per-unit of time cost of lobbying. If κ is not paid then the project sits idle at some x until either lobbying is restarted or the never-lobbied projects of its vintage (those assigned at the same time) *catch up* to x, at which time the project becomes active again and stays active without any need of, or benefit from, further lobbying. In every instant, $\underline{\nu}$ "never lobbied" projects are opened, in the order they were assigned. Once a never-lobbied project is opened, it forever remains active whether or not it is lobbied. The rate $\underline{\nu}$ represents the input rate that would prevail in the absence of any lobbying by the clients.[6] Here A_t denotes the mass of all projects active in instant t and it is composed of the two type of projects: all those that are lobbied in that instant, and some that are not.[7]

We assume that clients minimize B times the duration of their project, from assignment to completion, plus κ times the time spent lobbying. B represents the rate of loss experienced by a client whose project is not completed. We assume no discounting for simplicity. The problem of a client who has lobbied in the past and is not caught up is given by

$$V_t(x) = \max\left\{V_{t+\Delta}\left(x - \frac{\eta}{A_t}\Delta\right) - \kappa\Delta,\ V_{t+\Delta}(x)\right\}.$$

The value function $V_t(x)$ represents the client's value of being x steps away from completion at time t. This problem captures the choice between lobbying in interval Δ at cost $\kappa\Delta$ which advances the project by $\frac{\Delta}{\theta A_t}$, or not lobbying and finding oneself at x in time $t + \Delta$. Since no loss is eperienced once the project is complete, we have $V_t(0) = 0$ for all t. The value function of a client whose project was assigned at time τ and is caught up at time t is given by $B \cdot (\tau + D_\tau - t)$.

Since our goal is to explain why lobbying makes the input rate ν inefficiently large, let's tie our hands by stipulating that the input rate of never-lobbied projects $\underline{\nu}$ is "low," that is, it belongs to the interval $\left[0, \frac{\eta}{X}\right]$. This choice of baseline ensures that any slowdown in the output rate cannot be attributed to an excessively large $\underline{\nu}$.

Projects are indexed by the time τ they are assigned and by an index a that runs across the set of the α projects assigned at time τ. We now introduce the notion of lobbying strategy and lobbying equilibrium.

Definition 4 *A **lobbying strategy** for project (a,τ) is a measurable indicator function $S_{a\tau}(t)$ defined on the interval $[\tau,\infty)$ which takes value 1 if project a is lobbied in instant*

[6] One could be concerned that in equilibrium there might not be enough never-lobbied cases to open, and that therefore it would be more precise to state that in every instant the judge opens the minimum of $\underline{\nu}$ never-lobbied cases and the balance of the never lobbied cases. However, we will see that in equilibrium the balance of never-lobbied cases never falls below $\underline{\nu}$.

[7] Under these rules, for a case that has been lobbied in the past, two scenarios are possible in instant t. First, the case may have been "caught up" by the never-lobbied cases of its own assignement vintage; in other words, the case was lobbied in the past, but then the lobbying lapsed and the case is now at the same stage of advancement (same x) as its never-lobbied assignment vintage. Such a case is worked on without the need for further lobbying and proceeds at speed η/A_t. The second scenario is that the case has not been caught up at time t. In this scenario the case is worked on in the interval Δ and makes $\eta\Delta/A_t$ progress if $\kappa\Delta$ is spent; otherwise, the case does not proceed.

t, and is zero otherwise. A ***lobbying equilibrium*** *is a set of strategies such that, for each project (a,τ), the strategy $S_{a\tau}(t)$ minimizes κ times the time spent lobbying plus B times the project's duration.*

Although strategies are defined for the infinite future, project (a,τ) will be completed at a certain time; the shape of the strategy after that time is payoff-irrelevant. Equilibrium strategies could potentially be quite unwieldy, featuring complex patterns of activity interspersed with periods of no lobbying. The next lemma affords some simplification. It suggests that we should look for equilibria in which clients play just two simple strategies.

Lemma 2 *In any lobbying equilibrium in which the number of active projects grows, two strategies payoff-dominate all others: strategy $\mathbf{1}(\cdot)$ which denotes immediate and perpetual lobbying starting from time of assignment, and strategy $\mathbf{0}(\cdot)$ which denotes never lobbying.*

Proof. We prove that any strategy $S_\tau(\cdot)$ (typically displaying "intermittent" lobbying) is dominated either by strategy $\mathbf{0}(\cdot)$ or by strategy $\mathbf{1}(\cdot)$. Let us show this next. First, if $S_\tau(\cdot)$ is caught up, then it is dominated by the strategy $\mathbf{0}(\cdot)$ which achieves the same completion date at a lower lobbying cost. This is because after a strategy is caught up, it cannot go any faster than its assignment vintage. Suppose then that $S_\tau(\cdot)$ is not caught up.

Denote

$$\chi(t) = \int_\tau^t S_\tau(s)\ ds$$

where by construction $\chi(\cdot)$ is non-decreasing, $\chi(\tau) = 0$ and $\chi(t) \le t - \tau$. The function $\chi(t)$ can be interpreted as a measure representing how much activity has occurred on the project between τ and t or, equivalently, the state of advancement of the project. When strategy S_τ is employed, the project's advancement at time t is given by

$$\begin{aligned} x_S(t) &= X - \int_\tau^t \dot{x}_S(r)\ dr \\ &= X - \int_\tau^t \frac{\eta}{A_r}\ d\chi(r). \end{aligned}$$

Denote by T the time at which the project is done, that is, T is the smallest value that solves $x_S(T) = 0$. Create a new strategy $\widetilde{S}(t)$ which equals 1 for $t \in [\tau, \tau + \chi(T)]$ and 0 for

$t > \tau + \chi(T)$. Then we have

$$\begin{aligned} 0 &= x_S(T) \\ &= X - \int_\tau^T \frac{\eta}{A_r}\, d\chi(r) \\ &= X - \int_\tau^{\tau+\chi(T)} \frac{\eta}{A_{\chi^{-1}(y-\tau)}}\, dy \\ &\geq X - \int_\tau^{\tau+\chi(T)} \frac{\eta}{A_y}\, dy \\ &= X - \int_\tau^{\tau+\chi(T)} \frac{\eta}{A_y} \widetilde{S}_\tau(y)\ dy = x_{\widetilde{S}}(\tau + \chi(T)) \end{aligned}$$

where the third equality reflects a change of variable $y = \tau + \chi(r)$, and the inequality follows because $\chi(y) \leq y - \tau$, hence $\chi^{-1}(y-\tau) \geq y$ and $A_{\chi^{-1}(y-\tau)} \geq A_y$. The inequality shows that strategy S is just done at time T, whereas strategy $\widetilde{S}$ is more than done already by time $\tau + \chi(T) \leq T$. This means that the duration under strategy $\widetilde{S}$ is smaller than that under strategy S. . Denote by $\widetilde{T} \leq \tau + \chi(T)$ the time strategy $\widetilde{S}$ is done. Let us now turn to lobbying expenditures. Strategy S's lobbying expenditure is given by $\kappa\chi(T)$. Strategy $\widetilde{S}$'s lobbying expenditure is given by $\kappa\left(\widetilde{T} - \tau\right)$. Since $\widetilde{T} \leq \tau + \chi(T)$, strategy $\widetilde{S}$'s lobbying expenditure is smaller than strategy S's.

Summing up, we have shown that duration and lobbying expenditure are smaller under strategy $\widetilde{S}$ than under strategy S. Thus strategy $\widetilde{S}$ dominates S. Notice that, since under $\widetilde{S}$ a project ends at $\widetilde{T} \leq \tau + \chi(T)$, strategy $\widetilde{S}$ is payoff-equivalent to strategy $\mathbf{1}(\cdot)$. Thus strategy S is dominated by strategy $\mathbf{1}(\cdot)$. ■

The intuition behind Lemma 2 is the following. Lobbying "buys advancement" at the speed of η/A_t. If it is profitable to lobby at the assignment of the project, then it makes no sense to have interludes of no lobbying. During those interludes the project does not advance, but the mass of active projects A_t keeps growing, making lobbying (once it is restarted) less productive.

Even taking Lemma 2 into account, lobbying equilibria could potentially be very complex because of the possibility of non-steady growth equilibria in which the input rate is not constant through time. Inspired by Lemma 2, we look for a simple class of equilibria in which a time-invariant fraction z of the α newly assigned projects is never lobbied, and the remaining fraction $(1-z)\alpha$ is lobbied immediately upon assignment and then continuously during the entire duration of their trial. We will call these equilibria **constant-growth lobbying equilibria**. Note that the definition of constant-growth lobbying equilibrium does not restrict the strategy space.

If players adopt the strategies of a constant-growth lobbying equilibrium, the input rate $\nu(z)$ is determined by z via the identity

$$\nu(z) = \underline{\nu} + (1 - z)\,\alpha.$$

The percentage of lobbyists $(1 - z^{*})$, and hence the input rate $\nu(z^{*})$, are determined in equilibrium. In what follows we show that a constant-growth lobbying equilibrium always exists, and we characterize it.

Proposition 5 *Suppose $\alpha > \frac{\eta}{X}$. Then, for any $\underline{\nu}$ and any cost of lobbying κ,*

a) a constant-growth lobbying equilibrium exists;

b) in any constant-growth lobbying equilibrium $\nu(z^{*}) > \frac{\eta}{X}$, i.e., the input rate is inefficiently high;

c) the constant-growth lobbying equilibrium is unique;

d) the fraction $(1 - z^{*})$ of projects that are lobbied in equilibrium is increasing in $\frac{\alpha}{\underline{\nu}}$ and $\frac{\eta}{X}$, and decreasing in $\frac{\kappa}{B}$;

e) the equilibrium input rate $\nu(z^{*})$ is decreasing in $\frac{\kappa}{B}$ and increasing in $\frac{\alpha}{\underline{\nu}}$ and $\frac{\eta}{X}$.

Proof. a) We show that there is a time-invariant z such that the value at the time of assignment of two players who follow the two different equilibrium strategies (lobby and not) are the same. The lobbyist's value at the time of assignment for a project assigned at τ, assuming the project is lobbied from assignment through to completion, is $(-\kappa - B)\,C_{\tau}$ where C_{τ} is the completion time of a project started at τ. Substituting for C_t from Proposition 1, the value is given by

$$VL_{\tau}(z) = (-\kappa - B)\ \left[\frac{\nu(z)}{\Omega(\nu(z);\eta/X)} - 1\right]\tau.$$

The value of the non-lobbyist at the time of assignment for a project assigned at τ, assuming that he never lobbies, is computed as follows. First, the fraction of non-lobbyist projects inputed in each instant is given by $\frac{\underline{\nu}}{\nu(z)}$, and consequently the output rate is made up of a fraction $\frac{\underline{\nu}}{\nu(z)}$ of non-lobbyist projects. Thus, a project assigned at τ finds

$$z\alpha\tau - \frac{\underline{\nu}}{\nu(z)}\Omega(\nu(z);\eta/X)\,\tau$$

non-completed projects in front of it. These projects are completed at rate $\frac{\underline{\nu}}{\nu(z)}\Omega(\nu(z);\eta/X)$, so it takes

$$\left[\frac{z\alpha}{\frac{\underline{\nu}}{\nu(z)}\Omega\left(\nu\left(z\right);\eta/X\right)}-1\right]\tau$$

before all non-lobbied projects assigned before τ are completed. Therefore the value of a non-lobbyist at the time of assignment, assuming that he never lobbies in the future, is

$$VN_{\tau}\left(z\right)=-B\left[\frac{z\alpha}{\frac{\underline{\nu}}{\nu(z)}\Omega\left(\nu\left(z\right);\eta/X\right)}-1\right]\tau$$

In an equilibrium with lobbyists and non-lobbyists, z^{*} solves $VL_{\tau}\left(z^{*}\right)=VN_{\tau}\left(z^{*}\right)$, or

$$\left(-\kappa-B\right)\ \left[\frac{\nu\left(z^{*}\right)}{\Omega\left(\nu\left(z^{*}\right);\eta/X\right)}-1\right]=-B\left[\frac{\alpha}{\underline{\nu}}z^{*}\frac{\nu\left(z^{*}\right)}{\Omega\left(\nu\left(z^{*}\right);\eta/X\right)}-1\right] \tag{7}$$

It is important to note that condition is independent of τ. Thus, if a z^{*} exists that verifies equation (7), this z^{*} will be time-invariant, consistent with the definition of constant-growth lobbying equilibrium. We conclude the proof by showing that at least one z^{*} exists that verifies equation (7) and it lies between $\frac{\underline{\nu}}{\alpha}$ and $\frac{\underline{\nu}}{\alpha}+\frac{1}{\alpha}\left(\alpha-\frac{\eta}{X}\right)$.

The lowest possible value of z^{*} is $\frac{\underline{\nu}}{\alpha}$. If z falls below this level, there are not enough non-lobbyists to fill $\underline{\nu}$, and then non-lobbied projects get started immediately. Formally, in this project the expression in brackets on the RHS of (7) is no greater than the brackets on the LHS, whence $VN_{\tau}\left(z\right)>VL_{\tau}\left(z\right)$. So $z\leq\frac{\underline{\nu}}{\alpha}$ is not consistent with equilibrium. The highest possible value of z^{*} is that for which $\nu\left(z^{*}\right)=\eta/X$. At this level the LHS of (7) is zero, and so $VN_{\tau}\left(z\right)<VL_{\tau}\left(z\right)$. Intuitively, if z^{*} were any higher, then $\nu\left(z^{*}\right)<\eta/X$ and then completion times would be zero, and then lobbyists would lobby at zero cost whence Thus such z cannot be part of the equilibrium. To find an expression for this bound, write $\eta/X=\nu^{*}=\underline{\nu}+\left(1-z\right)\alpha$, and solving for z yields $z=\frac{\underline{\nu}}{\alpha}+\frac{1}{\alpha}\left(\alpha-\frac{\eta}{X}\right)$. We have shown that on the lower bound of the interval $z\in\left(\frac{\underline{\nu}}{\alpha},\frac{\underline{\nu}}{\alpha}+\frac{1}{\alpha}\left(\alpha-\frac{\eta}{X}\right)\right)$ we have $VN_{\tau}\left(z\right)>VL_{\tau}\left(z\right)$, and on the upper bound $VN_{\tau}\left(z\right)<VL_{\tau}\left(z\right)$. Since the two functions $VN_{\tau}\left(z\right)$ and $VL_{\tau}\left(z\right)$ are continuous in z over the interval, they must cross at least once. Any crossing is consistent with an equilibrium.

b) Suppose not, so that $\nu^{*}\leq\frac{\eta}{X}$. Then $\alpha>\nu^{*}$, and so a project assigned at τ finds a backlog of $\left(\alpha-\nu^{*}\right)\tau$ unopened projects in front of it. Since projects are opened at rate ν^{*}, the time it takes the last project in the backlog to be opened is

$$\frac{\left(\alpha-\nu^{*}\right)}{\nu^{*}}\tau.$$

This expression, which we will call the unopened duration, is positive and grows linearly with τ. This time can be eliminated by lobbying from assignment time all the way through completion, at a total cost that is proportional to completion time. Proposition 3 proves that when $\nu^* \leq \frac{\eta}{X}$ completion time is stationary, i.e., it is the same for projects opened at any τ. Therefore, the strict best response of all projects assigned after a certain $\widehat{\tau}$ is to lobby all the way through completion time, in order to eliminate the unopened duration which exceeds lobbying costs. But then for every $t > \widehat{\tau}$ not lobbying cannot be equally profitable as lobbying. Therefore we have shown that if $\nu^* \leq \frac{\eta}{X}$, a positive mass cannot be not lobbying after $\widehat{\tau}$. Yet the construction requires that in any instant $\alpha - \nu^*$ projects are not lobbied, and this mass is positive because by assumption $\alpha > \frac{\eta}{X} \geq \nu^*$. Contradiction.

c) The equilibrium z^* solves (7), which can be rearranged as

$$\frac{\kappa + B}{B} \left[\frac{\nu(z)}{\Omega(\nu(z); \eta/X)} - 1\right] = \left[\frac{\alpha}{\underline{\nu}} z \frac{\nu(z)}{\Omega(\nu(z); \eta/X)} - 1\right]$$

and rewritten as

$$\left[\frac{\kappa + B}{B} - \frac{\alpha}{\underline{\nu}} z\right] \frac{\nu(z)}{\Omega(\nu(z); \eta/X)} = \left[\frac{\kappa + B}{B} - 1\right]. \tag{8}$$

The LHS in (8) is the product of two positive and decreasing functions of z, and therefore it is decreasing in z. The RHS does not depend on z. Therefore equation (8) admits a unique solution z^*.

d) Rewrite slightly (8) as

$$H\left(z; \frac{\kappa}{B}, \frac{\alpha}{\underline{\nu}}\right) = \left[\frac{\kappa}{B} + 1 - \frac{\alpha}{\underline{\nu}} z\right] \frac{\nu(z)}{\Omega(\nu(z); \eta/X)} = \frac{\kappa}{B}. \tag{9}$$

The function $H\left(z; \frac{\kappa}{B}, \frac{\alpha}{\underline{\nu}}\right)$ is decreasing in $\frac{\alpha}{\underline{\nu}}$ and $\frac{\eta}{X}$, so increasing $\frac{\alpha}{\underline{\nu}}$ or $\frac{\eta}{X}$ results in a downward shift of the function. Since the function is decreasing in z, shifting the function downward results in a shift to the left of the intersection point between the function and the constant line $\frac{\kappa}{B}$. Thus z^* is decreasing in $\frac{\alpha}{\underline{\nu}}$ and $\frac{\eta}{X}$.

The function $H\left(z; \frac{\kappa}{B}, \frac{\alpha}{\underline{\nu}}\right)$ is increasing in $\frac{\kappa}{B}$, and increasing $\frac{\kappa}{B}$ by δ results in an upward shift of $\delta \frac{\nu(z)}{\Omega(\nu(z);\eta/X)} > 1$ in the function. So increasing $\frac{\kappa}{B}$ results in the function shifting upward by more than $\frac{\kappa}{B}$. So, start from a given $\frac{\kappa}{B}$ and focus on the resulting equilibrium z^*, which is the z at which the function H attains height $\frac{\kappa}{B}$. Then increase $\frac{\kappa}{B}$. At z^*, the function H moves up by more than $\frac{\kappa}{B}$. This means that z^* is to the left of the new equilibrium. Thus z^* is increasing in $\frac{\kappa}{B}$.

e) Follows directly from d) and the definition $\nu(z) = \underline{\nu} + (1 - z)\alpha$. ■

Part b) of the proposition establishes a strong presumption of inefficiently large input rates in our lobbying environment. The intuition is clear: if input rates were efficient, say $\nu < \eta/X$, then completion time would be zero. This means that the cost of lobbying would be zero and, also, that a project which is lobbied would be completed instantaneously. Therefore lobbying is a dominant strategy, which would give rise to an input rate $\nu = \alpha > \eta/X$. Thus an equilibrium input rate ν cannot be smaller than η/X.

Part e) of the proposition also points to κ as a plausible source of heterogeneity in input rates across workers. If a worker is less susceptible to lobbying, which we can model as κ being larger, then the proposition indicates that the worker will have a smaller input rate and a larger output rate. Moreover, the proposition shows that there is more lobbying when the assignment rate is larger, which is intuitive because then the time spent waiting for a project to be opened becomes larger. Finally, harder working workers and easier projects will give rise to more lobbying. Intuitively, this is because then the completion time gets shorter relative to the duration of a non-lobbied project.

A few words of comment on the causes of inefficiency. The source of a slowdown in output is that, if an additional project is lobbied, that project does not replace other projects; rather, it adds to them increasing A_t. In this respect, our model is analogous to models of common resource extraction ("common pool" models) where utilizers cannot be excluded from the pool. We think it is natural that a worker who is lobbied on one additional project may not have the power (or the inclination) to commit to exclude ("bump") one other project: for one thing, bumped projects are treated worse, at least ex post; moreover, bumped projects would often have to be projects which are lobbied themselves.[8] For both these reasons, the worker may be reluctant to bump projects. [9]

Finally we turn to the case in which η, rather than being exogenous, is chosen by the worker. Suppose η is determined as the solution to the problem

$$\max_{\eta} \Omega\left(\nu\left(z^*\right); \frac{\eta}{X}\right) - c\left(\eta\right). \tag{10}$$

According to this formulation, the worker chooses η by trading off the output rate (increasing in η) against a cost of effort $c(\eta)$. Note that since z^* is taken as given in problem (10), the worker does *not* behave as a Stackelberg leader. This assumption reflects the idea that the worker cannot commit to maintain a given level of effort regardless of lobbying. We

[8] To see this, consider the extreme case when when $\underline{\nu} = 0$; in this case, the only cases that could possibly be bumped would be another lobbied case.

[9] If the judge has the ability to bump cases in response to lobbying then, just as in common pool problems, there need not be an inefficiency. One could imagine non-bumping models in which the judge auctions off a limited number of "undivided units of attention" to the highest bidders, thus keeping A_t fixed.The rationing would take place through a "price" κ which would clear the lobbying market. In such a model there would be greater expenditure on lobbying, but ν would not necessarily be pushed to inefficiently high levels. Therefore lobbying would not necessarily generate inefficiency.

now augment the notion of a lobbying equilibrium by allowing the worker's effort η^* to be determined endogenously.

Definition 5 *A **lobbying equilibrium with endogenous effort** is a lobbying equilibrium in which effort η^* solves (10).*

To ensure that the equilibrium effort level is greater than zero and smaller than αX we assume $c'(0) = 0$, and $c'(\alpha X) = \infty$.

Proposition 6 *Consider a lobbying equilibrium with endogenous effort. If κ increases, then the input rate decreases and the worker's effort increases.*

Proof. Suppose by contradiction that, as κ increases to $\widehat{\kappa}$, we have $\widehat{z} < z^*$. Then by definition we have $\nu(\widehat{z}) > \nu(z^*)$. Since by Proposition 4 c) $\frac{\partial\Omega(\nu;\eta/X)}{\partial\nu\partial\eta} < 0$, it follows from problem (10) that $\widehat{\eta} < \eta^*$. Then $\Omega(\nu(\widehat{z}); \widehat{\eta}/X) < \Omega(\nu(z^*); \eta^*/X)$, and then the entire LHS of equation (9) becomes larger. Since the RHS stays unchanged, equation (9) can no longer be satisfied, and so we do not have an equilibrium. Therefore it must be that as κ increases to $\widehat{\kappa}$, the input rate decreases. It then follows from problem (10) that the worker's effort increases. ■

This proposition highlights another dimension of inefficiency associated with lobbying. Not only does lobbying slow down projects, but it also induces the worker to slack off.

7 Input Rate Determined By Maximization of an Objective Function

In most of the paper we assume that, insofar as the worker has an objective function, it is to maximize the output rate. We also implicitly assume that this is the unique social goal. Under this assumption the (socially and privately) optimal input rate is, by Proposition 4 a), the lowest possible, namely η/X. In this section we explore a different hypothesis: that some private or social value may be generated when projects clear *intermediate goals*. So, we consider the possibility that value may accrue when a project is merely being opened, or when it gets half-done, etc. One example is a judge who may issue preliminary injunctions early on in the trial, which might increase social welfare.

When weight is placed on clearing intermediate goals, the optimal input rate need no longer equal η/X. To see this, consider an extreme case in which value is generated *only* when a project is opened (e.g., completion does not matter); in this case, the optimal input rate is the largest possible, $\nu = \alpha$. This observation suggests that the framework developed in this section can provide a meaningful theory of the size of the input rate.

In what follows we assume that effort is fixed. We also assume "equal treatment" in the sense that all open projects proceed at the same speed according to equation (2). Equal treatment can be justified on regulatory grounds—it may not be legal to treat open projects disparately by focusing on some and leaving others behind. It may also be that projects are unobservably heterogeneous, and the worker does not know *ex ante* which projects benefit from clearing which intermediate goal. For example, a judge may not know at which point, if any, a case might settle. That said, the equal treatment assumption has bite here, and it will be discussed in the next section.

The first goal is to specify the (private or social) objective function. To this end, some preliminaries need to be introduced.

Definition 6 ***(rate of clearing intermediate goals)*** *Denote by $\omega_t(x)$ the rate at which, at each instant t, projects clear a given $x \in [0, X]$.*

For example, $\omega_t(X/2)$ denotes the rate at which projects clear the $X/2$ mark, that is, the rate at which projects become half done. In this notation, the output rate ω_t corresponds to $\omega_t(0)$. We allow for weight to be placed on any number of these "intermediate rates" by postulating the following objective function:

$$\int_0^\infty e^{-\rho t} \left[\int_0^X u(\omega_t(x)) \; P(dx) \right] dt. \tag{P}$$

Here ρ represents a (social or private) discount factor. The function u is increasing and its curvature measures the degree to which low clearing rates are penalized in the objective function. $P(x)$ is a probability measure that specifies the weight placed on intermediate step x, so for example if one third of the value is created when projects clear the $X/2$ mark then $P(X/2) = 1/3$. A slightly different interpretation of $P(X/2) = 1/3$ is that for half the projects two thirds of the value is created by clearing the $X/2$ mark, and for the other half no value is created. In this latter interpretation projects are heterogeneous with respect to when value is created.

Considerable simplification could be achieved in (**P**) if we knew that $\omega_t(x)$ was stationary, i.e., time independent. Fortunately the next lemma ensures that is so along a constant growth path, and in addition it offers a convenient expression for $\omega_t(x)$ as a geometric combination of ν and ω with weight x/X.

Lemma 3 *Along a constant growth path,* $\omega_t(x) = \nu^{\frac{x}{X}} \cdot \omega^{1-\frac{x}{X}}$. *Thus, along a constant growth path* $\omega_t(x)$ *is stationary and is denoted by* $\overline{\omega}(x)$.

Proof. The mass of projects that clear x between t and $t+\Delta$ is approximately equal to

$$\int_x^{x+\frac{\eta\Delta}{A_t}} \varphi_t(s)\ ds \approx \varphi_t(x)\frac{\eta\Delta}{A_t},$$

and the rate $\omega_t(x)$ obtains dividing by Δ and letting $\Delta \to 0$. Formally,

$$\omega_t(x) = \frac{\eta}{A_t}\varphi_t(x).$$

We have

$$\omega_t(x) = \frac{\eta\varphi_t(x)}{A_t} = \frac{\eta\varphi_t(0)}{A_t}e^{\frac{(\nu-\omega)}{\eta}x} = \omega e^{\frac{(\nu-\omega)}{\eta}x}, \tag{11}$$

where the second equality holds along a constant growth path. Now,

$$\frac{(\nu-\omega)}{\eta}x = \frac{x}{X}\frac{(\nu-\omega)}{\eta}X = \frac{x}{X}\log\frac{\nu}{\omega},$$

where the last equality follows from equation (5) which must hold along a constant growth path. Taking exponentials on both sides and substituting into (11) yields

$$\omega_t(x) = \omega\left(\frac{\nu}{\omega}\right)^{\frac{x}{X}},$$

which is equivalent to the expression in the lemma. ■

In light of this lemma, the inner integral in problem (**P**) is stationary and so, along a constant growth path, (**P**) simplifies to

$$\frac{1}{\rho}\int_0^X u(\overline{\omega}(x))\ P(dx).$$

If, moreover, we assume $u(\cdot) = \log(\cdot)$, then (**P**) equals

$$\begin{aligned}&\frac{1}{\rho}\int_0^X \log\left(\nu^{\frac{x}{X}}\ \omega^{\left(1-\frac{x}{X}\right)}\right)\ P(dx)\\ &= \frac{1}{\rho}\left[\log(\omega) + \frac{\mathbb{E}(P)}{X}\log\left(\frac{\nu}{\omega}\right)\right]\\ &= \frac{1}{\rho}\log(\overline{\omega}(\mathbb{E}(P))).\end{aligned} \tag{P'}$$

where $\mathbb{E}(P) = \int_0^X x \, P(dx)$ denotes the expected value of the c.d.f. P. Expression (**P'**) shows that, despite the fact that the measure P may place weight on clearing a great number of intermediate goals, when $u(\cdot) = \log(\cdot)$ the worker's problem can always be reduced to caring about clearing a single "average" goal $\mathbb{E}(P)$. This is a considerable analytical advantage.

The next proposition contains the main result of this section. It shows that, the more we care about clearing early intermediate goals, the higher the optimal input rate ν. Thus we have the basis for a theory optimally chosen input rates that might account for rates exceeding η/X.

Proposition 7 *Suppose the pair (ν, ω) is part of a constant growth path given η/X. If ν is chosen to maximize (**P'**) then ν is strictly increasing in $\mathbb{E}(P)$, and the associated ω is therefore strictly decreasing in $\mathbb{E}(P)$.*

Proof. Along a constant growth path (**P'**) is proportional to

$$U\left(\nu; \widetilde{P}\right) = \log\left(\Omega\left(\nu; \eta/X\right)\right) + \frac{\mathbb{E}(P)}{X} \log\left(\frac{\nu}{\Omega\left(\nu; \eta/X\right)}\right)$$

We know from Proposition 4 (a) that $\log\left(\frac{\nu}{\Omega(\nu;\eta/X)}\right)$ is strictly increasing in ν. It follows that, if $\mathbb{E}\left(\widetilde{P}\right) > \mathbb{E}(P)$, then the expression

$$U\left(\nu; \widetilde{P}\right) - U\left(\nu; P\right)$$

is an increasing function of ν. Then for any pair $\nu < \nu'$ we have

$$U\left(\nu; \widetilde{P}\right) - U\left(\nu; P\right) < U\left(\nu'; \widetilde{P}\right) - U\left(\nu'; P\right).$$

Rearranging yields

$$U\left(\nu'; \widetilde{P}\right) - U\left(\nu; \widetilde{P}\right) > U\left(\nu'; P\right) - U\left(\nu; P\right).$$

Now set $\nu' = \nu^* = \sup\left\{\arg\max U\left(\nu; P\right)\right\}$. Then the right-hand side is no smaller than zero, which implies that $U\left(\nu^*; \widetilde{P}\right) > U\left(\nu; \widetilde{P}\right)$ for any $\nu < \nu^*$. This shows that the maximizer(s) of $U\left(\cdot; \widetilde{P}\right)$ must be at least as large as ν^*. To finish the proof we need to show that the maximizer(s) of $U\left(\cdot; \widetilde{P}\right)$ are in fact strictly larger than ν^*. The fact that the function $\Omega(\nu)$ is differentiable in ν guarantees that $\partial U\left(\nu; P\right)/\partial\nu$ is zero at ν^*. But then $\partial U\left(\nu; \widetilde{P}\right)/\partial\nu$

cannot be zero at ν^* (recall that $U\left(\nu;\widetilde{P}\right) - U(\nu;P)$ is a strictly increasing function of ν). Therefore the maximizer(s) of $U\left(\cdot;\widetilde{P}\right)$ cannot include ν^* and thus they must be strictly larger than ν^*. ■

This result makes intuitive sense if we think about polar cases. If $\mathbb{E}(P)$ assumes the largest possible value, namely X, then the worker only cares about the rate at which projects are opened. In this case, it makes sense for the worker to chose the largest ν possible because the negative consequences on the completion rate are irrelevant. Conversely, if $\mathbb{E}(P)$ assumes the smallest possible value, namely 0, then the worker only cares about the rate at which projects are completed. In this case, it makes sense for the worker to chose the smallest ν, that is, η/X.

The previous analysis is predicated on equal treatment of all open projects, and we have mentioned earlier some reasons why the equal treatment assumption makes sense in some circumstances. If, however, it is possible to treat projects disparately then it may be optimal to do so. This is the subject of the next section.

8 Variable-Speed Work Strategies

Until now we have assumed that all projects receive the same fraction of the worker's attention and all proceed at the same speed, irrespective of how close they are to being done. In this section we relax this assumption and consider work strategies that allow the worker to focus more effort on just-arrived projects, or almost-done projects, etc. Such strategies can dominate equal treatment strategies when, as in problem (**P**) in Section 7, we care about clearing intermediate goals.

To see why, consider a situation in which all projects are identical, and value is realized only when projects are opened and/or completed. In our notation, $P(0) + P(X) = 1$. If it is possible to move projects along at different rates, then the optimal strategy is the following: all assigned projects are opened immediately, thus setting the input rate equal to α; but, as soon as projects clear X, they are placed in a queue from which projects are retrieved at rate η/X. As soon as a project is retrieved,it is worked on continuously until its completion. This "stop and go" strategy achieves an average value of $\alpha P(X) + \frac{\eta}{X} P(0)$. Since $\alpha \geq \nu$ and $\frac{\eta}{X} \geq \Omega(\nu;\eta/X)$, this average value is larger than $\nu P(X) + \Omega(\nu;\eta/X) P(0)$, which is the average rate that can be achieved with an input rate of ν and equal treatment of projects. This simple example shows that if it is valuable to clear intermediate goals then a completeness-dependent work strategy can dominate an equal-treatment one.

We now define a strategy where effort can be tailored according to a project's level of completion, and moreover the worker is allowed to keep some partially-completed projects in a queue.

Definition 7 *Consider a partition of* $[0, X]$ *with generic element* $I_i = (x_{i-1}, x_i)$ *, where we posit* $x_0 = 0, x_i < x_{i+1},$ *and* $x_N = X$*. A* ***variable-speed work strategy*** *is a vector of pairs* $(\eta_i, \nu_i)_{i=1}^N$ *such that*

(a) $\sum_i \eta_i = \eta$*, and*

(b) $\nu_i \leq \Omega\left(\nu_{i+1}; \eta_{i+1}\frac{1}{x_{i+1}-x_i}\right)$.

The interpretation is as follows. I_i is an interval of completion levels, η_i the effort devoted to projects whose level of completion at any point in time belongs to I_i, and ν_i is the rate at which projects are allowed to transit into I_i. The worker is allowed to distribute his total effort η in any way he chooses across these intervals, which means that he is allowed to focus on projects with different completion levels. Moreover, the worker is allowed to tailor the input rate ν_i for I_i, that is, the worker chooses how fast to feed into I_i projects coming out of I_{i+1}. Once η_{i+1} and ν_{i+1} are set, the output rate out of interval I_{i+1} is given by $\Omega\left(\nu_{i+1}; \eta_{i+1}\frac{1}{x_{i+1}-x_i}\right)$. Obviously, the worker cannot feed projects into I_i any faster than they come out of I_{i+1}; this accounts for the inequality in part (b) of Definition 7. When strict inequality holds, the worker is slowing down projects coming out of interval I_{i+1} and putting them in a queue of projects waiting to enter interval I_i.

Definition 8 *An* ***equal treatment*** *work strategy is one in which* $\nu_i = \Omega\left(\nu_{i+1}; \eta_{i+1}\frac{1}{x_{i+1}-x_i}\right)$ *for all* i*.*

The strategies considered in the previous sections are in fact equal treatment strategies. To see this, observe that in a strategy where all projects move to the right at the same rate $\eta\frac{1}{A_t}$ we can fix any x and think of projects that cross x (moving from right to the left) as projects that have just outputed "completion level below x" and are just being inputed into "completion rate higher than x." Obviously, these two artificial output and input rates are the same.

Proposition 8 *(a) For any variable-speed work strategy, there is an equal treatment strategy that yields the same output rate and requires (weakly) less effort.*

(b) Consider the profile of intermediate output rates $\overline{\omega}(x)$ *generated by an equal treatment strategy with effort* η*. The sequence of variable-speed work strategies which in the limit yields the same profile requires the same amount of effort* η *in the limit.*

*(c) Fix η and ν, and let ω be the output rate along the associated constant growth path. Suppose we want to maximize **(P)** with $u(\cdot) = \log(\cdot)$ and intermediate outputs being valued according to $\overline{P}(dx) = \overline{\omega}(x)\ dx$. Then equal treatment strategies do just as well as variable-speed work strategies.*

Proof. (a) Suppose there is an $j > 0$ such that

$$\nu_j < \Omega\left(\nu_{j+1}; \eta_{j+1}\frac{1}{x_{j+1} - x_j}\right). \tag{12}$$

Then we can decrease η_{j+1} slightly without changing any other element of $(\eta_i, \nu_i)_{i=1}^N$ and obtain a new completion-dependent strategy which requires less effort than the original one, and which has the same output since ν_j and all the other variables with index j or lower are unaffected. Moreover, the new strategy satisfies part (b) of Definition 7 for all j if the decrease in η_{j+1} is small enough. Continue this process until (12) binds for all j, and an equal treatment strategy is obtained with the desired property.

(b) Fix η and ν in an equal treatment strategy, and let ω be the associated completion rate along the constant growth path. These ν and ω generate an entire profile $\overline{\omega}(x)$ according to Lemma 3. We turn now to the variable-speed strategy which is going to approximate the profile $\overline{\omega}(x)$. For each N consider the partition $\left(0, \frac{X}{N}, \frac{2X}{N}..., X\right)$. The cheapest way to generate a given output rate ω out of an interval of size $\frac{X}{N}$ is to have input rate ω and effort $\frac{X}{N}\omega$. So the cheapest way to generate intermediate output rates $\left(\omega, \overline{\omega}\left(\frac{X}{N}\right), \overline{\omega}\left(\frac{2X}{N}\right)..., \nu\right)$ is with a variable-speed strategy $(\eta_i, \nu_i)_{i=0}^N = \left(\frac{X}{N}\overline{\omega}\left(\frac{i}{N}X\right), \overline{\omega}\left(\frac{i}{N}X\right)\right)_{i=0}^N$. The effort required by this strategy is

$$\eta = \sum_{i=0}^{N} \eta_i = \sum_{i=0}^{N} \frac{X}{N}\overline{\omega}\left(\frac{i}{N}X\right).$$

Taking the limit as $N \to \infty$, this effort converges to

$$\begin{aligned}\int_0^X \overline{\omega}(x)\ dx &= \int_0^X \nu^{\frac{x}{X}} \cdot \omega^{1-\frac{x}{X}}\ dx \\ &= \omega \int_0^X \left(\frac{\nu}{\omega}\right)^{\frac{x}{X}}\ dx\end{aligned}$$

where the first equality follows from Lemma 3. Performing the change of variable $y = x/X$

yields

$$\begin{aligned}\eta &= \omega \int_0^1 \left(\frac{\nu}{\omega}\right)^y X\ dy \\ &= \frac{\omega X}{\log\left(\frac{\nu}{\omega}\right)} \left(\frac{\nu}{\omega}\right)^y \Big|_{y=0}^1 \\ &= \frac{\omega X}{\log\left(\frac{\nu}{\omega}\right)} \left(\frac{\nu}{\omega} - 1\right).\end{aligned}$$

Rearranging yields $(\nu - \omega)\frac{X}{\eta} = \log(\nu) - \log(\omega)$, which is exactly equation (5). So the effort required by the limit of the sequence of cheapest variable-speed strategies is the same effort that generates ω in the equal-treatment strategy.

c) For each N consider the partition $\left(0, \frac{X}{N}, \frac{2X}{N}..., X\right)$ and the associated variable-speed strategy $(\eta_i, \nu_i)_{i=0}^N$. Given this strategy space, the maximization of problem (**P**) reads

$$\begin{aligned}&\max_{(\eta_i,\nu_i)_{i=0}^N} \sum_i \int_{i\frac{X}{N}}^{(i+1)\frac{X}{N}} \log\left([\nu_i]^{\frac{x}{X}} \cdot \left[\Omega\left(\nu_i, \frac{N}{X}\eta_i\right)\right]^{1-\frac{x}{X}}\right) \overline{P}(dx) \\ &\text{s.t. } \nu_i \leq \Omega(\nu_{i+1}, N\eta_{i+1}) \text{ for all } i, \\ &\sum_i \eta_i = \eta.\end{aligned}$$

Since $\frac{N}{X}\eta_i \leq \nu_i$, the objective function is smaller than

$$\max_{(\eta_i,\nu_i)_{i=0}^N} \sum_i \int_{i\frac{X}{N}}^{(i+1)\frac{X}{N}} \log\left([\nu_i]^{\frac{x}{X}} \cdot \left[\Omega\left(\frac{N}{X}\eta_i, \frac{N}{X}\eta_i\right)\right]^{1-\frac{x}{X}}\right) \overline{P}(dx).$$

and also the constraint is more restrictive than

$$\nu_i \leq \Omega\left(\frac{N}{X}\eta_{i+1}, \frac{N}{X}\eta_{i+1}\right) \text{ for all } i.$$

We therefore define the relaxed problem as

$$\begin{aligned}&\max_{(\eta_i,\nu_i)_{i=0}^N} \sum_i \int_{i\frac{X}{N}}^{(i+1)\frac{X}{N}} \log\left([\nu_i]^{\frac{x}{X}} \cdot \left[\Omega\left(\frac{N}{X}\eta_i, \frac{N}{X}\eta_i\right)\right]^{1-\frac{x}{X}}\right) \overline{P}(dx). \\ &\text{s.t. } \nu_i \leq \Omega\left(\frac{N}{X}\eta_{i+1}, \frac{N}{X}\eta_{i+1}\right) \text{ for all } i. \\ &\sum_i \eta_i = \eta.\end{aligned}$$

In the solution to the relaxed problem the constraints on each ν_i bind and so the relaxed problem reads

$$\begin{aligned}&\max_{(\eta_i)_{i=0}^N} \sum_i \int_{i\frac{X}{N}}^{(i+1)\frac{X}{N}} \log\left(\left[\Omega\left(\frac{N}{X}\eta_{i+1}, \frac{N}{X}\eta_{i+1}\right)\right]^{\frac{x}{X}} \cdot \left[\Omega\left(\frac{N}{X}\eta_i, \frac{N}{X}\eta_i\right)\right]^{1-\frac{x}{X}}\right) \overline{P}\,(dx)\\ &\text{s.t. } \sum_i \eta_i = \eta.\end{aligned}$$

Substituting for $\Omega\,(y, y) = y$, the relaxed problem reads

$$\begin{aligned}&\max_{(\eta_i)_{i=0}^N} \sum_i \int_{i\frac{X}{N}}^{(i+1)\frac{X}{N}} \left[\frac{x}{X}\log\left(\frac{N}{X}\eta_{i+1}\right) + \left(1-\frac{x}{X}\right)\log\left(\frac{N}{X}\eta_i\right)\right] \overline{P}\,(dx)\\ &= \max_{(\eta_i)_{i=0}^N} \sum_i \left[\log\left(\frac{N}{X}\eta_{i+1}\right) \int_{i\frac{X}{N}}^{(i+1)\frac{X}{N}} \frac{x}{X}\overline{P}\,(dx) + \log\left(\frac{N}{X}\eta_i\right) \int_{i\frac{X}{N}}^{(i+1)\frac{X}{N}} \left(1-\frac{x}{X}\right)\overline{P}\,(dx)\right],\end{aligned}$$

subject to the effort constraint. Define

$$\overline{P}_i = \int_{i\frac{X}{N}}^{(i+1)\frac{X}{N}} \frac{x}{X}\overline{P}\,(dx) + \int_{i\frac{X}{N}}^{(i+1)\frac{X}{N}} \left(1-\frac{x}{X}\right)\overline{P}\,(dx)$$

We can rewrite the relaxed problem as

$$\begin{aligned}&\max_{(\eta_i)_{i=0}^N} \sum_i \overline{P}_i \log\left(\frac{N}{X}\eta_i\right)\\ &\text{s.t. } \sum_i \eta_i = \eta.\end{aligned}$$

This is a concave problem so the solution is identified from the first order conditions of the associated Lagrangean. At the optimum these conditions imply that $\overline{P}_i \frac{1}{\eta_i^*}$ is the same for all i. Since η_i^* converges to zero as $N \to \infty$, it is convenient to write the first order conditions as

$$\overline{P}_i \frac{1}{\frac{N}{X}\eta_i^*} = \text{const for all } i.$$

As $N \to \infty, \overline{P}_i$ converges to $\overline{P}\,(x_i)$. Moreover, since $\frac{N}{X}\eta_i^* = \Omega\left(\frac{N}{X}\eta_i^*, \frac{N}{X}\eta_i^*\right)$, $\frac{N}{X}\eta_i^*$ is also equal to an intermediate output rate, which as $N \to \infty$ converges to the intermediate output rate $\omega^*\,(x_i)$ in the relaxed problem. Therefore, in the solution to the relaxed problem the intermediate output rate the first order conditions imply

$$\omega^*\,(x_i) = \text{const} \cdot \overline{P}\,(x_i) = \text{const} \cdot \overline{\omega}\,(x_i)\,,$$

where the last equality follows by definition of $\overline{P}(\cdot)$. This means that in the solution to the relaxed problem, the profile of intermediate output rates is proportional to that associated by an equal treatment strategy with effort η. The budget constraint then ensures that the constant is equal to 1, and thus that the solution to the relaxed problem is in fact an equal treatment strategy with effort η. Since this strategy is obviously feasible in the original problem, the proof is done. ■

Part (a) of this proposition shows that any output rate that can be achieved by a variable-speed strategy can also be achieved, more cheaply, by an equal treatment strategy. In this sense, there is no better strategy than an equal treatment strategy.[10] This result justifies the focus on equal treatment strategies *if we only care about output rates.* Because this result holds only if we care exclusively about output rates, it does not apply to the setup of Section 7. That is to say, part (a) of this proposition applies in a world in which equal treatment functions with $\nu > \omega$ are necessarily suboptimal.

Parts (b) and (c) rehabilitate equal treatment strategies when $\nu > \omega$. In part (b), for any pair $(\nu, \eta/X)$ these strategies are shown to be the most efficient way to get the profile of intermediate outputs they generate. So, to the extent that we deem these strategies "inefficient" when $\nu > \eta/X$, it is only because we are evaluating their profile of intermediate outputs according to a criterion that they do not meet (perhaps because we only value final output). But if we wish to generate the intermediate output profile they generate, there is no other strategy that attains it for cheaper. In part (c) we show that equal treatment strategies with $\nu > \omega$ can be optimal within the class of variable-speed strategies, by reverse-engineering the parameters that ensure that the variable-speed strategy that maximizes (**P**) is in fact an equal treatment strategy.

9 Incentives and Multitasking

In this section we set aside the question of the inefficiencies associated with too large an input rate, to focus on a different question: how to properly incentivize the worker in a setting where the worker can strategically direct his effort to different types of projects. We compare the incentive effects of two measures of aggregate productivity: the aggregate output rate, and the average duration of assigned projects. We find that rewarding the aggregate output rate leads to multitasking problems (in the sense of Holmstrom and Milgrom 1991) where the worker totally focuses his effort on the projects requiring the fewest steps and totally neglects to work on the other projects. In contrast, penalizing large average durations leads

[10] However, equal treatment strategies are not necessarily optimal if the objective is not only to increase the output rate, but also to clear intermediate goals. This point will be addressed at the end of Section 7.

to a more "Rawlsian" behavior whereby the worker focuses more effort on projects requiring more steps.

The model is as follows. In each instant, the worker is assigned α_1 projects that will take X_1 tasks to complete, α_2 projects that will take X_2 tasks to complete, ... , up to α_N projects that will take X_N tasks to complete. Without loss of generality we set $X_i < X_{i+1}$. In this setup the vector $\{\alpha_i, X_i\}_i$ fully describes how many projects are assigned of which length. The worker chooses the rate ν_i at which to open projects of type i, and the effort η_i to devote to each type of project.

A worker who is rewarded based on aggregate output maximizes

$$\int_0^\infty e^{-\rho t}\left[w\sum_{i=1}^N \omega_i - c\left(\sum_{i=1}^N \eta_i\right)\right]\,dt, \tag{13}$$

where ρ represents the worker's discount factor, w is a parameter that captures the magnitude of the incentives, ω_i is the output rate of projects of type $i, c(\cdot)$ is a convex cost of effort, and $\sum_{j=1}^N \eta_j$ represents the total effort exerted by the worker. A worker who is penalized linearly based on the average duration of his projects maximizes

$$\int_0^\infty e^{-\rho t}\left[-w\sum_{i=1}^N \alpha_i D_{it} - c\left(\sum_{i=1}^N \eta_i\right)\right]\,dt, \tag{14}$$

where D_{it} represents the duration of a project of type i assigned at t.

Why do we focus on the incentive schemes (13) and (14) which are based on aggregate outcomes, when in fact we are interested in disaggregate outcomes? The most natural interpretation is that the principal designing the incentive scheme is less informed than the worker, and cannot tell the number of steps X_i required to complete a project. This makes sense if we think of X_i as a measure of the complexity of a project, and of the worker as an expert vis a vis the non-expert principal.

Proposition 9 *(a) A worker who is rewarded based on aggregate output rates will immediately open all assigned projects requiring fewer than $\overline{X}$ steps to complete and immediately complete them. All projects requiring more than $\overline{X}$ steps are never worked on.*

(b) A worker who is penalized linearly based on the average duration of his projects will put positive effort on all projects, even those that take many steps to complete. Moreover, $(\alpha_j)^2 X_j > (\alpha_i)^2 X_i$ implies that either $\eta_j^* > \eta_i^*$ or else $\eta_j^* = \alpha_j X_j$. In other words, ceteris paribus the worker will work more on projects requiring more steps to complete, and on those assigned at a higher rate.

Proof. (a) The square bracket in the integral of (13) is time-invariant, so it can be factored out of the integral. Then maximizing (13) is equivalent to solving

$$\max_{\{\nu_i,\eta_i\}} \; w\sum_{i=1}^{N}\left[\Omega\left(\nu_i;\eta_i/X_i\right)\right] \; - c\left(\sum_{j=1}^{N}\eta_j\right),$$

Fix any constellation of $(\eta_1,...,\eta_N)$ (not necessarily optimal). Given that constellation, the worker will optimally chose the input rate $\nu_i = \frac{\eta_i}{X_i}$, because this choice achives the maximal feasible output rate given η_i which is $\Omega\left(\eta_i/X_i;\eta_i/X_i\right) = \frac{\eta_i}{X_i}$ We can therefore rewrite the worker's problem as

$$\max_{\eta_i} \; w\sum_{i=1}^{N}\frac{\eta_i}{X_i} - c\left(\sum_{j=1}^{N}\eta_j\right).$$

The maximand is concave and so the first order conditions identify a maximum. They read

$$\frac{w}{X_i} - c'\left(\sum_{j=1}^{N}\eta_j^*\right) \leq 0 \text{ for all } i \text{ such that } \eta_i^* = 0,$$

$$\frac{w}{X_k} - c'\left(\sum_{j=1}^{N}\eta_j^*\right) = 0 \text{ for the unique } k \text{ such that } \eta_k^* \in \left(0,\alpha_k X_k\right), \text{ and}$$

$$\frac{w}{X_i} - c'\left(\sum_{j=1}^{N}\eta_j^*\right) \geq 0 \text{ for all } i \text{ such that } \eta_i^* = \alpha_i X_i.$$

These conditions imply that the worker will set η_i at its maximum on projects with the lowest X_i. All other projects are ignored.

(b) Substitute from Proposition 1 into (14) to get

$$\int_0^\infty e^{-\rho t}\left[-w\sum_{i=1}^{N}\alpha_i\left(\frac{\alpha_i}{\Omega\left(\nu_i;\eta_i/X_i\right)}-1\right)t - c\left(\sum_{i=1}^{N}\eta_i\right)\right] dt$$
$$= -\frac{w}{\rho^2}\sum_{i=1}^{N}\alpha_i\left(\frac{\alpha_i}{\Omega\left(\nu_i;\eta_i/X_i\right)}-1\right) - \frac{1}{\rho}c\left(\sum_{i=1}^{N}\eta_i\right),$$

where we have used the identities $\int_0^\infty e^{-\rho t}t\; dt = -\frac{1}{\rho}\left(t+\frac{1}{\rho}\right)e^{-\rho t}\Big|_0^\infty = 1/\rho^2$ and $\int_0^\infty e^{-\rho t}\; dt = 1/\rho$. Like in part (a), the worker will optimally chose the input rate $\nu_i = \frac{\eta_i}{X_i}$, so the maximization problem simplifies to

$$\max_{\{\eta_i\}} -\frac{w}{\rho^2}\sum_{i=1}^{N}\alpha_i\left(\frac{\alpha_i X_i}{\eta_i}-1\right)-\frac{1}{\rho}c\left(\sum_{i=1}^{N}\eta_i\right).$$

The maximand is concave in each η_i and so the first order conditions identify an interior maximum. They read

$$\begin{aligned}
\frac{w}{\rho}(\alpha_i)^2 X_i\left[\frac{1}{\eta_i^*}\right]^2 - c'\left(\sum_{j=1}^{N}\eta_j^*\right) &\leq 0 \text{ for all } i \text{ such that } \eta_i^*=0,\\
\frac{w}{\rho}(\alpha_k)^2 X_k\left[\frac{1}{\eta_k^*}\right]^2 - c'\left(\sum_{j=1}^{N}\eta_j^*\right) &= 0 \text{ for all } k \text{ such that } \eta_k^*\in(0,\alpha_k X_k), \text{ and}\\
\frac{w}{\rho}(\alpha_i)^2 X_i\left[\frac{1}{\eta_i^*}\right]^2 - c'\left(\sum_{j=1}^{N}\eta_j^*\right) &\geq 0 \text{ for all } i \text{ such that } \eta_i^*=\alpha_i X_i.
\end{aligned}$$

From these conditions we see that the project $\eta_i^*=0$ is not possible for any i. Then, either $\eta_i^*=\alpha_i X_i$ for all i, or else there is an i such that η_i^* is interior, that is, $\eta_i^*<\alpha_i X_i$. If η_i^* is interior then for all j such that $(\alpha_j)^2 X_j>(\alpha_i)^2 X_i$ we have

$$\begin{aligned}
c'\left(\sum_{j=1}^{N}\eta_j^*\right) &= \frac{w}{\rho}(\alpha_i)^2 X_i\left[\frac{1}{\eta_i^*}\right]^2\\
&< \frac{w}{\rho}(\alpha_j)^2 X_j\left[\frac{1}{\eta_i^*}\right]^2\\
&\leq \frac{w}{\rho}(\alpha_j)^2 X_j\left[\frac{1}{\eta_j^*}\right]^2,
\end{aligned}$$

where the last inequality holds only for $\eta_j^*\leq\eta_i^*$. Thus if $\eta_j^*\leq\eta_i^*$ then derivative of the objective function with respect to η_j is positive at η_j^*. By the first order conditions this means $\eta_j^*=\alpha_j X_j$. Summing up, we have shown that if $(\alpha_j)^2 X_j>(\alpha_i)^2 X_i$ then either $\eta_j^*>\eta_i^*$ or else $\eta_j^*=\alpha_j X_j$. ■

10 Extensions

In this section we consider several important extension to the main model.

10.1 Heterogeneous, Equally Treated Cases

Until now we have assumed that all projects take the same number of tasks X to complete. Let us now consider the situation in which projects are heterogeneous in the number of tasks they take to complete. We still assume "equal treatment," however, in the sense that once opened, all projects proceed at the same speed according to equation (2). Such equal treatment will arise if, for example, the worker cannot distinguish which projects take fewer tasks to complete, as may be the case for legal cases that settle *unexpectedly* during trial. Another reason why equal treatment may prevail is that disparate treatment of projects may not be legal.

Consider then two projects which are opened at the same time: project 1 taking X_1 tasks to complete, and project 2 taking X_2. If $X_1 < X_2$, then we should expect the output rate of projects of type 1 to be larger, *relative to their input rate*, compared to projects of type 2. To get some intuition for this statement, imagine that type 1 projects take so few tasks to complete that $X_1 \approx 0$. Then $\omega_1 \approx \nu_1$, and thus the ratio of input to output rates approaches its theoretical maximum. In this section we provide exact formulas that characterize how the input/output ratio varies across projects with different X's.

The model is as follows. Fix the worker's effort level at η. In each instant, the worker opens ν_1 projects that will take X_1 tasks to complete, ν_2 projects that will take X_2 tasks to complete, ... , up to ν_N projects that will take X_N tasks to complete. We allow the possibility that $N = \infty$, in which case the set of different types of projects is countable.[11] In this setup the vector $\{\nu_i, X_i\}_i$ fully describes how many projects are opened of which length. For every open project, in the time interval between t and $t + \Delta$, the worker's work shaves off approximately

$$\frac{\eta}{A_t}\Delta,$$

where A_t represents the mass of all projects open at time t.

The next proposition provides an exact characterization of the output rates as a function of the input rates and of the characteristics of the project.

Proposition 10 *Fix η and a constellation of $\{\nu_i, X_i\}_i$. If $\eta < \sum_{i=1}^N \nu_i X_i$ then completion time is positive for all projects, and there exists a constant $K < 1$ such that $\omega_i = \nu_i \cdot K^{X_i}$ for all i.*

If $\eta > \sum_{i=1}^N \nu_i X_i$ completion time is zero for all projects and $\frac{\nu_i}{\omega_i} = 1$ for all i.

[11] The analysis in this section generalizes immediately to the case in which the set of possible types of cases has the power of the continuum.

Proof. Think of the worker as grouping projects by type, and working on each group of projects separately. Accordingly, we denote by η_{it} the (still to be computed) amount of effort allocated to A_{it}, the mass of projects of type i at time t. By definition, $\sum_i \eta_{it} = \eta$ and $\sum_i A_{it} = A_t$. In order for this representation to be valid, the η_{it}'s must be such that all groups of projects move at the same speed, so for all i, j we must have

$$\frac{\eta_{it}}{A_{it}}\Delta = \frac{\eta_{jt}}{A_{jt}}\Delta. \tag{15}$$

We conjecture, and later verify, that there exists a unique set of time-invariant $\{\eta_{it}\}_i = \{\eta_i\}_i$ that solves this equation. In this case each group of projects follows a constant growth path, and so from Proposition 1 we have $A_{it} = (\nu_i - \omega_i)\, t$. Substituting into equation (15) yields

$$\frac{\eta_i}{(\nu_i - \omega_i)} = \frac{\eta_j}{(\nu_j - \omega_j)}. \tag{16}$$

For all i, ν_i and ω_i are linked by expression (5) and so we may replace ω_i with $\Omega\left(\nu_i; \eta_i/X_i\right)$. Since $\Omega\left(\nu; \eta/X\right)$ is increasing in η, the left- and right-hand sides of equation (16) are increasing in η_i and η_j respectively. This fact implies that there exists a unique set $\{\eta_i\}_{i=1}^N$ which solves equation (16) and simultaneously meets the constraint $\sum_i \eta_{it} = \eta$. This verifies that our conjecture was correct.

Now, recall that expression (5) reads

$$(\nu_i - \omega_i)\,\frac{X_i}{\eta_i} = \log\left(\frac{\nu_i}{\omega_i}\right).$$

Substituting into (16) yields

$$\frac{\log\left(\frac{\nu_i}{\omega_i}\right)}{X_i} = \frac{\log\left(\frac{\nu_j}{\omega_j}\right)}{X_j},$$

or equivalently

$$\left(\frac{\nu_i}{\omega_i}\right)^{X_j} = \left(\frac{\nu_j}{\omega_j}\right)^{X_i}. \tag{17}$$

Thus, if $X_i < X_j$ then $\frac{\nu_i}{\omega_i} < \frac{\nu_j}{\omega_j}$. Moreover, equation (17) is verified we replace ω_i with $\nu_i \cdot K^{X_i}$, and do the same for ω_j. This means that, given η and a constellation of $\{\nu_i, X_i\}_i$, there exists a constant K such that for all i

$$\omega_i = \nu_i \cdot K^{X_i}.$$

The constant K cannot exceed 1 for otherwise $\omega_i > \nu_i$ for all i. The constant equals 1 only if completion time is zero, which requires $\eta \geq \sum_{i=1}^N \nu_i X_i$. Otherwise, $K < 1$. ■

This proposition informs us about the *relative* magnitudes of $\frac{\nu_i}{\omega_i}$ and $\frac{\nu_j}{\omega_j}$, the input/output ratios. According to the proposition, $X_i < X_j$ implies $\frac{\nu_i}{\omega_i} < \frac{\nu_j}{\omega_j}$, that is, projects that take more tasks to complete have a worse input/output rate ratio. Note that the constant K in the proposition is an unspecified function of effort and of the entire vector $\{\nu_i, X_i\}$. Therefore Proposition 10 should not be construed as informing us about the *level* of any particular ω_i.

10.2 Forgetful Worker and "Sleeping On It"

In this section we deal with the case in which, as completion time grows and any open project is worked on less and less frequently per unit of time, the worker progressively forgets about the details of each individual project. Thus, upon taking up a project again, the worker needs additional effort to "remind himself" of where he left off before he can make progress. We also deal with the opposite case, that in which there is a gain from "sleeping on" projects, that is, letting time elapse between the completion of two consecutive tasks. Perhaps this second effect is most present in "creative" jobs.

We introduce both considerations by assuming that in the time interval between t and $t+\Delta$, the worker's effort shaves off approximately $\frac{\eta}{A_t+F_t}\Delta$ steps from each active project. The factor F_t may be positive or negative; if positive, it captures a "forgetfulness penalty," if negative, the gain from "sleeping on" a project. We assume that F_t becomes larger in absolute value with time; its exact form of will be specified later. The presence of forgetfulness or "sleeping on it" requires amending equations (2) and (3) from Section 2. The two amended equations read

$$\frac{\partial \varphi_t(x)}{\partial t} - \frac{\partial \varphi_t(x)}{\partial x}\frac{\eta}{A_t+F_t} = 0. \tag{18}$$

and

$$\omega_t = \frac{\eta}{A_t+F_t}\varphi_t(0). \tag{19}$$

For these conditions to be well-defined it will be necessary to ensure that $A_t + F_t > 0$. Equations (1) and (4) remain unchanged.

Definition 9 *The quadruple* $[\nu, \varphi_t(x), A_t, \omega]$ *is a* ***constant growth path with forgetful worker*** *(or* ***with "sleeping on it"****) if conditions (1), (4), (18) and (19) are verified.*

Now we specify F_t. We assume that both the "forgetfulness penalty" and the "sleeping on it" factor are proportional to the completion time C_t of a case opened at t. This specification makes sense because completion time C_t proxies for the interval that elapses between two "consecutive steps" of a project. If that interval is long, then a lot is forgotten by the time the worker gets around to making each consecutive step, and so before he can make progress the worker will have to work harder just to remember where he left off the previous time.

The converse interpretation holds for the "sleeping on it" effect. Note that, by making F_t proportional to C_t, we have made F_t endogenous.

Formally, we assume

$$\begin{aligned} F_t &= f \cdot C_t \\ &= f \ \frac{\nu - \omega}{\omega} \ t = F \cdot t. \end{aligned}$$

where by definition $F = f \ \frac{\nu-\omega}{\omega}$. The real number f is a scaling factor, and the second equality follows substituting for C_t from Proposition 1. From now on we will assume $f > -1$, which ensures that $A_t + F_t > 0$.

Define

$$\varphi_t^{**}(x) = \frac{(\nu - \omega) + F}{\eta} \omega \ t \ e^{\frac{(\nu-\omega)+F}{\eta} x},$$

and recall the previous definition

$$A_t^* = (\nu - \omega) \ t.$$

Theorem 3 *The quadruple* $[\nu, \varphi_t^{**}(x), A_t^*, \omega]$ *is a constant growth path with forgetful worker if and only if the pair* ν, ω *solves the equation* $\frac{X}{\eta}(\nu - \omega + F) = \log(\nu) - \log(\omega)$.

Proof. See the Appendix. ■

Substituting back for F we get that if the pair ν, ω are part of a constant growth path with forgetful worker then they must satisfythe equation

$$\log(\nu) - \log(\omega) = \frac{X}{\eta}\left(\nu - \omega + f \ \frac{\nu - \omega}{\omega}\right). \tag{20}$$

Proposition 11 ***(a)*** *If* $\nu \leq \frac{\eta}{X}$ *then the completion time in a constant growth path with forgetful worker or with "sleeping on it" is zero, and hence these paths both coincide with the constant growth path described in Sections 3 and 4, regardless of* f.

(b) (forgetful worker case) *If* $\nu > \frac{\eta}{X}$ *and* $f > 0$ *then completion time in a constant growth path with forgetful worker is positive, and the output rate is smaller than in the constant growth path described in Sections 3 and 4.*

(c) ("sleeping on it" case) *If* $\nu > \frac{\eta}{X}$ *and* $f \in (-1, 0)$ *then completion time in a constant growth path with "sleeping on it" is positive, and the output rate is larger than in the constant growth path described in Sections 3 and 4.*

Proof. See the Appendix. ■

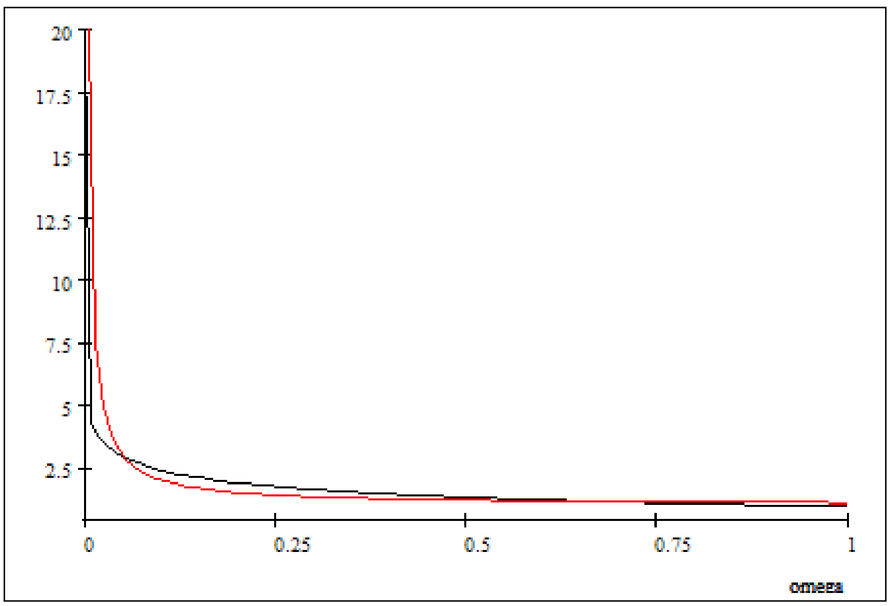
Remarkably, for a given ν there may be several values of ω that satisfy equation (20). To these are associated multiple constant growth paths with forgetful worker. We now present a numerical example displaying multiple ω that solve equation (20). Rewrite equation (20) as

$$\frac{\log\left(\frac{\nu}{\omega}\right)}{\nu-\omega}=\frac{X}{\eta}\left(1+\frac{f}{\omega}\right)$$

Now, set $\frac{X}{\eta}=1,\ \nu=1.1,\ f=0.1$ and let's plot the LHS and RHS of the equation

$$\frac{\log\left(\frac{1.1}{\omega}\right)}{(1.1-\omega)}=1+\frac{0.1}{\omega}.$$

The horizontal axis is ω, the LHS is the black line and the RHS is the red one. We see two intersection points in the interval $(0,\eta/X)$, corresponding to two solutions to the equation.



10.3 Time To Build

In certain contexts, there may be technological limitations on how fast a project can be completed. For example, a judge may need to allow the lawyers time, between two successive hearings, to produce certain evidence and to evaluate and respond to the evidence produced by the adversary. For an academic researcher, before being able to produce a final draft, time may be required to absorb one's intermediate findings and "put the puzzle together". Etc. The common thread in all these these examples is that one cannot complete a project under a certain time threshold, no matter how large the effort. Let us call this threshold $\underline{T}$. The "time to build" constraint is introduced in our model via the constraint

$$\underline{C}_t = \max\{C_t, \underline{T}\},$$

where $\underline{C}_t$ denotes the completion time for a case started at t in the model with a time-to-build constraint, and $C_t = t(\nu - \omega)/\omega$ is the completion time in a model without that constraint. One way to think about this constraint is to imagine that the system evolves exactly like in the case without constraint, except that cases that were completed in less than $\underline{T}$ are "held back" and not "released" until $\underline{T}$ has elapsed from the time they were opened. Note that, if $\nu > \omega$ then C_t grows linearly with time and so after a certain time we have $\underline{C}_t = C_t$. Thus in our formulation the time-to-build constraint stops binding after a certain time because, in a constant growth path, eventually projects take long enough to complete. In particular, the completion times for all projects started after a $\widehat{t}$ such that $C_{\widehat{t}} = \underline{T}$ are unaffected by the time-to-build constraint. One can easily verify that

$$\widehat{t} = \frac{\omega}{(\nu - \omega)}\underline{T}.$$

The output rate will, after a certain time $\widehat{\widehat{t}}$, coincide with the constant-growth-path one which is ω. The value of $\widehat{\widehat{t}}$ is given by

$$\widehat{\widehat{t}} = \widehat{t} + C_{\widehat{t}} = \frac{\nu}{(\nu - \omega)}\underline{T}.$$

For $t < \underline{T}$, the output rate is zero which is the time-to-build effect. For $t \in \left[\widehat{t}, \widehat{\widehat{t}}\right]$, the output rate will be higher than ω and decreasing.

11 Conclusion

We have developed a theory of a worker who works simultaneously on several projects, each of which takes several steps to complete. Given the complexity of the projects and the amount of effort exerted by the worker, we have identified the maximum number of projects on which the worker should work on at any given time, assuming the goal is to maximize the output rate. When this number is exceeded then the output rate decreases. The source of this inefficiency is that, for each project that is worked on but is far from complete, the completion date of the almost-done projects is pushed back.

We have focused on the case in which too many projects are worked on at the same time.

In this case, as time goes by the number of active projects grows, and each project takes longer and longer to complete after it has been started. Moreover, if projects are assigned to the worker at a constant rate, then non-started project also accumulate over time, so that it takes longer and longer between the time a project is assigned and the time it is completed. All these durations are shown to grow at a linear rate with time. We have derived an exact functional form for the "production function" that takes as inputs the rate at which the worker opens projects, the difficulty of the projects, and the amount of effort exerted by the worker, and combines all these inputs to yield an output rate. This functional form allows one, for example, to forecast the effect of additional effort on output.

Based on this "production function" we have asked what might be the cause of the inefficiently large input rate. We have posited a model in which many "clients" lobby the worker to get each step of their project done. If the worker is not able to resist this lobbying, then in equilibrium the worker necessarily works on an inefficiently large number of projects. We have also investigated a different perspective, the idea that output may not be the only goal of production. If there is value created by having projects achieve certain intermediate stages of completion, then we have shown that the optimal input rate exceeds that which is optimal when the only goal is to maximize completion rates. In this perspective, the progressive slowdown of the speed at which projects are completed is in fact optimal.

We have then investigated how different incentive schemes induce the worker to focus his efforts on different types of projects, depending on their complexity. A worker who is rewarded based on aggregate output rates will focus all his effort on non-complex projects and not ever work on complex projects. In contrast, a worker who is penalized for the average duration of projects assigned to him will work on all projects, and focus proportionally more effort on more complex projects. Which of these two schemes produces the most desirable allocation of effort will depend on the specific application.

We view the analysis in this paper and its companion (Coviello *et al.* 2010) as a first step into the theoretical and empirical analysis of work scheduling. Although the intuition for the inefficiency of task juggling is strong, measuring the quantitative effects of task juggling is far from straightforward, even at a purely theoretical level. This is because work schedules come in an almost infinite range of variations, in principle equal to all the ways in which S steps of each of N projects can be ordered, a very large cardinality indeed. This paper cuts through this complexity by providing a theory in which task juggling is parameterized by how many new projects are opened in each quarter. In separate and ongoing work (Coviello *et al.* 2010), we use this approach to estimate the productivity of Italian judges. We find that judges do juggle tasks, and that the slowdown in ouput resulting from task juggling is large. We view that paper as establishing the empirical relevance of the theory developed in this paper.

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A Proofs and Technical Results

Proof of Proposition 4 a), b).

Proof. a) There are three types of solutions to the equation $h(\nu) = h(\omega)$. The first one is $\nu = \omega$. This solution is not compatible with the analysis we have carried out because then $A_t = 0$. Then there are two kinds of solutions, one where $\nu < \frac{\eta}{X} < \omega$, which is not acceptable for then $A_t < 0$. The remaining kind of solution is $\nu > \frac{\eta}{X} > \omega$. Under this restriction, the shape of $h(\cdot)$ guarantees the required property.

b) Fix ν, and consider two values $\eta > \eta'$ with associated ω and ω'. The output rates ω and ω' solve

$$\begin{aligned} h(\omega; \eta/X) &= h(\nu; \eta/X) \\ h(\omega'; \eta'/X) &= h(\nu; \eta'/X). \end{aligned}$$

Combining these equalities yields

$$h(\omega'; \eta'/X) - h(\omega; \eta/X) = h(\nu; \eta'/X) - h(\nu; \eta/X). \tag{21}$$

Now, an easy to verify property of $h(y; \eta/X)$ that, for any $y_1 < y_2$,

$$h(y_1; \eta'/X) - h(y_1; \eta/X) < h(y_2; \eta'/X) - h(y_2; \eta/X).$$

Setting $y_1 = \omega, y_2 = \nu$, and combining with (21) gives

$$\begin{aligned} h(\omega; \eta'/X) - h(\omega; \eta/X) &< h(\omega'; \eta'/X) - h(\omega; \eta/X) \\ h(\omega; \eta'/X) &< h(\omega'; \eta'/X) \end{aligned} \tag{22}$$

Now, remember that $\omega' < \eta'/X$. Then either $\omega > \eta'/X$, in which project $\omega > \omega'$ and there is nothing to prove, or else $\omega < \eta'/X$. In this project both ω and ω' lie on the decreasing portion of the function $h(\cdot; \eta'/X)$. Then equation (22) yields $\omega > \omega'$. ■

Next we prove a technical lemma that is necessary to prove Proposition 4 c).

Lemma 4 *Take any triple* $\left(\nu, \omega, \frac{\eta}{X}\right)$ *where* $\omega = \Omega(\nu; \eta/X)$. *Then* $\left|\nu - \frac{\eta}{X}\right| > \left|\omega - \frac{\eta}{X}\right|$. *That is, along a growth path the* ***actual output*** *rate is closer to the* ***efficient output*** *rate than is the input rate.*

Proof. For any $\nu > \frac{\eta}{X} > \omega$ we can write

$$\begin{aligned} h(v) &= h\left(\frac{\eta}{X}\right) + \int_0^{\nu-\frac{\eta}{X}} h'\left(\frac{\eta}{X}+s\right) ds \\ h\left(\frac{\eta}{X}\right) &= h(\omega) + \int_0^{\frac{\eta}{X}-\omega} h'(\omega+r)\, dr. \end{aligned} \tag{23}$$

Make the change of variable $r = -\omega + \frac{\eta}{X} - s$ in the second equation, and one gets

$$\begin{aligned} h\left(\frac{\eta}{X}\right) &= h(\omega) - \int_{-\omega+\frac{\eta}{X}}^{0} h'\left(\frac{\eta}{X}-s\right) ds \\ &= h(\omega) + \int_0^{-\omega+\frac{\eta}{X}} h'\left(\frac{\eta}{X}-s\right) ds. \end{aligned}$$

Substitute into equation (23) to get

$$h(v) = h(\omega) + \int_0^{-\omega+\frac{\eta}{X}} h'\left(\frac{\eta}{X}-s\right) ds + \int_0^{\nu-\frac{\eta}{X}} h'\left(\frac{\eta}{X}+s\right) ds.$$

Since the triple $\left(\nu, \omega, \frac{\eta}{X}\right)$ solves (5), it follows that $h(v) = h(\omega)$ and so we may rewrite equation (23) once more as

$$\int_0^{\frac{\eta}{X}-\omega} -h'\left(\frac{\eta}{X}-s\right) ds = \int_0^{\nu-\frac{\eta}{X}} h'\left(\frac{\eta}{X}+s\right) ds \tag{24}$$

Now, from the proof of Lemma 1 we have $h'(y) = \frac{X}{\eta} - \frac{1}{y}$ and so

$$\begin{aligned} h'\left(\frac{\eta}{X}+s\right) &= \frac{X}{\eta} - \frac{1}{\frac{\eta}{X}+s} = \frac{X}{\eta}\left(1 - \frac{1}{1+s\frac{X}{\eta}}\right) = \frac{X}{\eta}\left(\frac{s\frac{X}{\eta}}{1+s\frac{X}{\eta}}\right) \\ h'\left(\frac{\eta}{X}-s\right) &= \frac{X}{\eta} - \frac{1}{\frac{\eta}{X}-s} = \frac{X}{\eta}\left(1 - \frac{1}{1-s\frac{X}{\eta}}\right) = -\frac{X}{\eta}\left(\frac{s\frac{X}{\eta}}{1-s\frac{X}{\eta}}\right) \end{aligned}$$

for any s such that $h'\left(\frac{\eta}{X}-s\right)$ is well defined, that is, $s < \frac{\eta}{X}$. If in addition $s > 0$ then

$$h'\left(\frac{\eta}{X}+s\right) = \frac{X}{\eta}\left(\frac{s\frac{X}{\eta}}{1+s\frac{X}{\eta}}\right) < \frac{X}{\eta}\left(\frac{s\frac{X}{\eta}}{1-s\frac{X}{\eta}}\right) = -h'\left(\frac{\eta}{X}-s\right). \tag{25}$$

Now let us turn to equation (24) and let us suppose, by contradiction, that $\nu - \frac{\eta}{X} < \frac{\eta}{X} - \omega$.

We may then rewrite that equation as

$$\int_0^{\frac{\eta}{X}-\omega} -h'\left(\frac{\eta}{X}-s\right)ds - \int_0^{\nu-\frac{\eta}{X}} h'\left(\frac{\eta}{X}+s\right)ds = 0$$

$$\int_0^{\nu-\frac{\eta}{X}} \left[-h'\left(\frac{\eta}{X}-s\right) - h'\left(\frac{\eta}{X}+s\right)\right]ds + \int_{\nu-\frac{\eta}{X}}^{\frac{\eta}{X}-\omega} -h'\left(\frac{\eta}{X}-s\right)ds = 0$$

The range of s in the above equation is at most $\left(0, \frac{\eta}{X}-\omega\right) \subset \left(0, \frac{\eta}{X}\right)$, and therefore (25) applies. This guarantees that the first integral is strictly positive. The second integral is strictly positive as well. Hence the equation cannot be verified. We therefore contradict our assumption that $\nu - \frac{\eta}{X} < \frac{\eta}{X} - \omega$. ■

Proof of Proposition 4 c)

Proof. Equation (5) reads

$$(\nu - \Omega(\nu;\eta/X)) = \frac{\eta}{X}\left[\log(\nu) - \log(\Omega(\nu;\eta/X))\right]. \tag{26}$$

Fix ν and differentiate both sides of (26) with respect to η to get

$$-\frac{\partial\Omega(\nu;\eta/X)}{\partial\eta} = \frac{1}{X}\left[\log(\nu) - \log(\Omega(\nu;\eta/X))\right] - \frac{\eta}{X}\frac{1}{\Omega(\nu;\eta/X)}\frac{\partial(\Omega(\nu;\eta/X))}{\partial\eta}.$$

Rearranging we get

$$\begin{aligned}\frac{\partial\Omega(\nu;\eta/X)}{\partial\eta}\left[\frac{\eta}{X}\frac{1}{\Omega(\nu;\eta/X)} - 1\right] &= \frac{1}{X}\left[\log(\nu) - \log(\Omega(\nu;\eta/X))\right] &(27)\\ &= \frac{1}{\eta}\left(\nu - \Omega(\nu;\eta/X)\right), &(28)\end{aligned}$$

where the second equation susbtitutes from (26). Now, fix η and differentiate (27) with respect to ν. This yields

$$\begin{aligned}\frac{\partial^2\Omega(\nu;\eta/X)}{\partial\eta\partial\nu}\left[\frac{\eta}{X}\frac{1}{\Omega(\nu;\eta/X)} - 1\right] - \frac{\partial\Omega(\nu;\eta/X)}{\partial\eta}\frac{\eta}{X}\frac{1}{(\Omega(\nu;\eta/X))^2}\frac{\partial\Omega(\nu;\eta/X)}{\partial\nu}\\ = \frac{1}{X}\left[\frac{1}{\nu} - \frac{1}{\Omega(\nu;\eta/X)}\frac{\partial\Omega(\nu;\eta/X)}{\partial\nu}\right],\end{aligned}$$

which can be rewritten as

$$\frac{\partial^2\Omega(\nu;\eta/X)}{\partial\eta\partial\nu}\left[\frac{\eta}{X}\frac{1}{\Omega(\nu;\eta/X)}-1\right]=\frac{1}{X}\left[\frac{1}{\nu}+\frac{\partial\Omega(\nu;\eta/X)}{\partial\nu}\frac{1}{\Omega(\nu;\eta/X)}\left(\frac{\partial\Omega(\nu;\eta/X)}{\partial\eta}\eta\frac{1}{\Omega(\nu;\eta/X)}-1\right)\right]. \tag{29}$$

The term in brackets on the left-hand side is positive, so $\frac{\partial^2\Omega(\nu;\eta/X)}{\partial\eta\partial\nu}$ has the same sign as the term in brackets on the right hand side of (29). We need to sign this term. To this end, substitute for $\frac{\partial\Omega(\nu;\eta/X)}{\partial\eta}$ from (28) so that the term in brackets on the right hand side of (29) reads

$$\begin{aligned}&\frac{1}{\nu}+\frac{\partial\Omega(\nu;\eta/X)}{\partial\nu}\frac{1}{\Omega(\nu;\eta/X)}\left[\frac{(\nu-\Omega(\nu;\eta/X))}{\left(\frac{\eta}{X}\frac{1}{\Omega(\nu;\eta/X)}-1\right)}\frac{1}{\Omega(\nu;\eta/X)}-1\right]\\&=\frac{1}{\nu}+\frac{\partial\Omega(\nu;\eta/X)}{\partial\nu}\frac{1}{\Omega(\nu;\eta/X)}\left[\frac{\nu-\frac{\eta}{X}}{\frac{\eta}{X}-\Omega(\nu;\eta/X)}\right].\end{aligned} \tag{30}$$

Now, to get an expression for $\frac{\partial\Omega(\nu;\eta/X)}{\partial\nu}$, fix η and differentiate both sides of (26) with respect to ν to get

$$\begin{aligned}1-\frac{\partial\Omega(\nu;\eta/X)}{\partial\nu}&=\frac{\eta}{X}\left[\frac{1}{\nu}-\frac{1}{\Omega(\nu;\eta/X)}\frac{\partial\Omega(\nu;\eta/X)}{\partial\nu}\right]\\ \frac{\partial\Omega(\nu;\eta/X)}{\partial\nu}\left[\frac{\eta}{X}\frac{1}{\Omega(\nu;\eta/X)}-1\right]&=\frac{\eta}{X\nu}-1\\ \frac{\partial\Omega(\nu;\eta/X)}{\partial\nu}&=\frac{\Omega(\nu;\eta/X)}{\nu}\frac{\frac{\eta}{X}-\nu}{\frac{\eta}{X}-\Omega(\nu;\eta/X)}.\end{aligned}$$

Substituting into (30) yields

$$\begin{aligned}&\frac{1}{\nu}-\frac{1}{\nu}\left(\frac{\nu-\frac{\eta}{X}}{\frac{\eta}{X}-\Omega(\nu;\eta/X)}\right)^2\\&=\frac{1}{\nu}\left[1-\left(\frac{\nu-\frac{\eta}{X}}{\frac{\eta}{X}-\Omega(\nu;\eta/X)}\right)^2\right].\end{aligned} \tag{31}$$

By Lemma 4,

$$\frac{\nu-\frac{\eta}{X}}{\frac{\eta}{X}-\Omega(\nu;\eta/X)}>1$$

and so equation (31) is negative. Thus the right hand side of (29) is negative, which implies $\frac{\partial\Omega(\nu;\eta/X)}{\partial\nu\partial\eta}<0$. ■

Proof of Theorem 3.

Proof. Condition (18) is verified because

$$\begin{aligned}
&\frac{\partial \varphi_t^{**}(x)}{\partial t} - \frac{\partial \varphi_t^{**}(x)}{\partial x}\frac{\eta}{A_t + F_t} \\
&= \frac{(\nu-\omega)+F}{\eta}\omega\ e^{\frac{(\nu-\omega)+F}{\eta}x} - \frac{(\nu-\omega)+F}{\eta}\omega\ t\ e^{\frac{(\nu-\omega)+F}{\eta}x}\frac{(\nu-\omega)+F}{\eta}\frac{\eta}{(\nu-\omega+F)\,t} \\
&= \frac{(\nu-\omega)+F}{\eta}\omega\ e^{\frac{(\nu-\omega)+F}{\eta}x} - \frac{(\nu-\omega)+F}{\eta}\omega\ \ e^{\frac{(\nu-\omega)+F}{\eta}x} \\
&= 0.
\end{aligned}$$

Condition (19) is verified because

$$\begin{aligned}
\omega &= \frac{\eta}{A_t + F_t}\varphi_t^{**}(0) \\
&= \frac{\eta}{(\nu-\omega+F)\,t}\frac{(\nu-\omega)+F}{\eta}\omega\ t \\
&= \omega.
\end{aligned}$$

Condition (4) can be verified immediately.

Condition (1) reads

$$A_t^* = \int_0^X \varphi_t^*(x)\ dx.$$

Substituting for $\varphi_t^{**}(x)$ and A_t^* yields

$$\begin{aligned}
(\nu-\omega)\,t &= \int_0^X \frac{(\nu-\omega)+F}{\eta}\omega\ t\ e^{\frac{(\nu-\omega)+F}{\eta}x}\ dx \\
&= \frac{(\nu-\omega)+F}{\eta}\omega\ t \int_0^X\ e^{\frac{(\nu-\omega)+F}{\eta}x}\ dx \\
&= \frac{(\nu-\omega)+F}{\eta}\omega\ t\ \ \frac{\eta}{(\nu-\omega)+F}\ e^{\frac{(\nu-\omega)+F}{\eta}x}\Big|_{x=0}^{X} \\
&= \omega\ t\quad e^{\frac{(\nu-\omega)+F}{\eta}x}\Big|_{x=0}^{X} \\
&= \omega\ t\quad \left[e^{\frac{(\nu-\omega)+F}{\eta}X} - 1\right].
\end{aligned}$$

We can rewrite this equality as

$$\frac{\nu}{\omega} - 1 = \left[e^{\frac{(\nu-\omega)+F}{\eta}X} - 1\right]$$
$$\frac{\nu}{\omega} = e^{\frac{(\nu-\omega)+F}{\eta}X}$$
$$\log(\nu) - \log(\omega) = (\nu - \omega + F)\frac{X}{\eta}$$

■

Proof of Proposition 11

Proof. (a) In this case $C_t \equiv 0$ and then we are back to the standard case of non-forgetful worker.

(b). Fix any ν and let ω^{**} be the output rate in a constant growth path with forgetful worker. Then ω^{**} solves equation (20), which can be written as

$$h(\omega) = h(\nu) + f \cdot \frac{\nu - \omega}{\omega}\frac{X}{\eta}. \tag{32}$$

Suppose $f > 0$, and by contradiction, that the output rate in a constant growth path with non-forgetful worker, call it ω^*, is smaller than ω^{**}. Since obviously, $\omega^{**} < \nu$, we have $\omega^* \leq \omega^{**} < \nu$. By definition of ω^* we have $h(\omega^*) = h(\nu)$, and since the function h is convex, it follows that $h(\omega^{**}) \leq h(\nu)$. But then since $f > 0$, the right-hand side in (32) must exceed the left-hand side, and so the equation cannot be satisfied. We have reached a contradiction.

(c) Suppose $f < 0$, and by contradiction, that the output rate in a constant growth path with non-forgetful worker, call it ω^*, is bigger than ω^{**}. We then have $\omega^{**} \leq \omega^* < \nu$. By definition of ω^* we have $h(\omega^*) = h(\nu)$, and since the function h is convex, it follows that $h(\omega^{**}) > h(\nu)$. But then since $f < 0$, the left-hand side in (32) must exceed the right-hand side, and so the equation cannot be satisfied. We have reached a contradiction. ■