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A Walrasian Theory of Sovereign Debt Auctions with Asymmetric Information

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Abstract

Sovereign bonds are highly divisible, usually of uncertain quality, and auctioned in large lots to a large number of investors. This leads us to assume that no individual bidder can affect the bond price, and to develop a tractable Walrasian theory of Treasury auctions in which investors are asymmetrically informed about the quality of the bond. We characterize the price of the bond for different degrees of asymmetric information, both under discriminatory-price (DP) and uniform-price (UP) protocols. We endogenize information acquisition and show that DP protocols are likely to induce multiple equilibria, one of which features asymmetric information, while UP protocols are unlikely to sustain equilibria with asymmetric information. This result has welfare implications: asymmetric information negatively affects the level, dispersion and volatility of sovereign bond prices, particularly in DP protocols.

1 Introduction

Sovereign debt auctions have a key set of characteristics: a large quantity of the debt is typically sold at one time to a large set of investors, and investors are free to try and

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buy as many units as they can afford. These auctions are almost invariably conducted using one of two formats - uniform-price auctions and discriminating-price auctions - with discriminating-price auctions being slightly more prevalent and uniform-price auctions being the standard method in the United States.¹ Spreads on sovereign bonds vary substantially both across countries and across time, suggesting that expectations and information about the likelihood of default or renegotiation seems to play an important role.²

In this paper we propose a novel model of auctions for goods with three properties which characterize sovereign bonds: (i) the good being auctioned is perfectly divisible, (ii) the number of bidders is large, and (iii) there is both common uncertainty about quality of the good about the mass of investors who participate in the auction. Given these three characteristics, the price-quantity strategic aspects of standard auction theory disappears, and a price-taking, or Walrasian, analysis naturally emerges. This insight makes these auctions particularly tractable and allows for an analysis of the role of information on equilibrium prices.

Our paper fills an important gap in the sovereign debt literature which has typically focused on the strategic decision of the government; something we completely neglect.³ While the literature has considered multiple equilibria stemming from self-fulfilling beliefs on the part of investors, it has neglected modeling how information acquisition and inference play a role in determining bond spreads.⁴ In fact, it has typically taken investors to be risk neutral and required that the return, adjusted for the probability of default, equals the risk-free rate. This means that the literature has neglected information acqui-

¹The heterogeneity of treasury auction formats is well-documented. Bartolini and Cottarelli (2001) show that 39 out of 42 countries in their sample use discriminatory price auctions. Brenner, Galai, and Sade (2009) show that market-oriented economies tend to use uniform-price auctions while economies who are less market oriented tend to use discriminatory-price auctions.

²See Aguiar et al. (2016b).

³See for example Eaton and Gersovitz (1981), the review articles by Aguiar and Amador (2013) and Aguiar et al. (2016b), and the recent quantitative literature by Aguiar and Gopinath (2006), Arellano (2008), Chatterjee and Eyigungor (2012), Bocola and Dovis (2016). This focus on the strategic decisions of the government has not just been with respect to default but also into how government's reach settlements and regain access to credit markets (see Cole, Dow, and English (1995), Benjamin and Wright (2009) and Yue (2010).

⁴There are two main approaches in this literature, following either Calvo (1988) (with successor papers Lorenzoni and Werning (2013) or Ayres et al. (2016)) or Cole and Kehoe (2000) (with successor papers Aguiar et al. (2015)) and Aguiar et al. (2017).

sition and its aggregation in prices. Moreover, while there has been some attention to the timing of decisions in bond markets and the impact of debt maturity (see Aguiar et al. (2016a)), the actual mechanics of sovereign bond auctions and their impact on prices has been ignored. Our paper focuses on the neglected role of investors and information acquisition when auction protocols are explicitly modeled. There has been a recent effort to empirically document the implications of two various auction protocols for the revenue of governments (for an excellent survey of this empirical literature see Hortaçsu (2011)). As our setting is very different, we discuss our paper's relationship with this literature once we have presented our model. There we also discuss the relationship to the rational expectations literature along the lines of Grossman and Stiglitz (1980).

Our model includes both a shock to the quality of the bond being auctioned, and a demand shock to the number of investors that show up at the auction. No investor is informed about the demand shock. Some investors may be informed about the quality shock while others are not. The quality shock thus is the source of asymmetric information in our model.

Because all bidders are price-takers and the bond is perfectly divisible, there is a marginal price in each state of the world that determines the lowest bids that the government must accept in order to raise the required funds. The set of state-contingent marginal prices can be computed ex-ante by all investors. Because all investors are uninformed about the demand shock and only some investors are informed about the quality shock, there is uncertainty about the *realized* marginal price, however. As a result, investors typically submit menus of bids at various marginal prices. The advantage of being an informed investor is that knowing the quality shock provides a better idea of the set of marginal prices that may be realized, and allows investors to tailor their bids more closely to the quality of the bond.

In each state of the world, the government sells its debt at the highest possible average price by executing all bids submitted at or above the state-specific marginal price. The price at which these bids are executed depends on the auction protocol. We study two protocols that are widely used in practice: a discriminatory-price (DP) protocol in which bids above the marginal price are executed at the bid price (also known as "pay-as-you-

bid”), and a uniform-price (UP) protocol in which bids are executed at the marginal price (the price of the lowest accepted bid).

Because the government accepts the highest-price bids first, investors know that bids submitted at a high marginal price (designed to be accepted when the quality of the bond is high) will also be accepted in states where the bond quality, and thus the marginal price, turns out to be low. As a result, there is uncertainty over the quality of bonds that investors will obtain when submitting a bid at a particular price. This uncertainty gives rise to a complicated pattern of complementarities between bids in different states that is determined by the rank order of the marginal prices.

We assume that all of our bidders are ex-ante identical. This leads to there being a *representative informed bidder* (w.r.t. the quality shock) and a *representative uninformed bidder*. We characterize the equilibrium bid choices of our representative bidders and the marginal prices at which they bid. We then develop a simple quantitative illustration to contrast how equilibrium prices change with the degree of information about the bond quality (as measured by the share of informed bidders), and how it differs between the two auction protocols.

We show that our two types of auction protocols generate similar bond prices and quantities and deliver similar utility levels to the government and investors in two special cases: when there is symmetric ignorance (no investor is informed about the quality shock), or when there is symmetric information (all investors are informed about the quality shock). When information is asymmetric information (some investors are informed about the quality shock), however, the two protocols deliver sharply different outcomes.

In the uniform-price auction, the gains from being informed are limited by the fact that all bids are executed at the same price. Uninformed investors may thus not always buy the right quantity of bonds, but they never pay the wrong price. Under certain conditions, in fact, uninformed investors may be able to perfectly replicate the portfolio of an informed bidder. We show that this is the case when there are enough informed investors and the default probabilities of the high and low quality bond are sufficiently different. In contrast, when the number of informed investors is sufficiently low, bonds of different quality tend to share the same price under different demand shocks, leading to an infer-

ence problem similar to that in Grossman and Stiglitz (1980) that renders the uninformed unable to perfectly replicate the informed portfolio.

The price-discriminating auction protocol works very differently. The uninformed must now worry about buying the wrong quantity *and* paying the wrong price. As a result, the fundamental inference problem for each bid is with respect to the overall probability of default for all states in which this bid is in-the-money, fixing the bid price. This leads to a strong adverse selection problem in which the uninformed may avoid buying bonds at high prices (associated with high quality bonds) because they worry that these bids will be executed even when bond quality is low. This generates higher information rents for the informed, because infra-marginal per-capita rents are strictly larger when the share of investors who participate in an auction is lower. Nevertheless, we can show that the bond prices on the high and the low quality schedules must converge to a common price schedule almost everywhere as the share of the informed investors goes to zero. This is the case for both auction protocols.

Finally, we endogenize the share of informed bidders by assuming that all investors are ex-ante uninformed but can pay a fixed utility cost to become informed. With endogenous information acquisition, the level of information about the bond quality in equilibrium is very different for our two auction protocols. In the uniform auction, the gains from becoming informed are strictly decreasing in the in the number of informed investors. The discriminatory auction instead features both complementarity and substitutability in information acquisition, and the gains from information are initially increasing but gradually decreasing. As a result, the discriminatory auction naturally generates equilibrium multiplicity. When this is the case, there are two stable equilibria: an informed equilibrium in which a strictly positive share of investors is informed, and an uninformed equilibrium in which no investor acquires information.

The economic mechanism that delivers this multiplicity is as follows. Starting in the uninformed equilibrium, prices are invariant to the quality shock because no investor is informed. An increase in the number of informed investors gradually generates a spread between the price of a high-quality and low-quality bonds as information is impounded into prices. Uninformed investors who bid on the high-price schedule are now subject to

adverse selection, as their bids will be executed even when the bond quality is low. As the number of informed investors increases, so do spreads, and the uninformed gradually withdraw from bidding on the high-price schedule. This allows informed investors to earn larger infra-marginal rents, pushing up the value of being informed (complementarity). Once uninformed investors fully withdraw from bidding on the high-price schedule, however, a further increase in the number of informed investors only leads to cannibalization of information rents and a decrease in the value of being informed (substitutability).

In the uniform-price auction instead, information rents can be sustained only at relatively low share of informed bidders because all bids are executed at the same price. Hence the uniform-price protocol naturally leads to fewer investors becoming informed, if any, and thus is much less likely to induce asymmetric information in equilibrium than a discriminatory-price protocol. For this reason, uniform-price auctions are typically characterized by higher prices on average, less volatility and lower debt burdens for the government. However, at the extremes of symmetric ignorance and symmetric information, both types of auction protocols work very similarly in terms of average prices and debt-burdens.

2 Model with Exogenous Information Asymmetry

2.1 Environment

This is a two-period model featuring a measure one of identical potential investors and a government. The government is modeled mechanically: it needs to raise D units of the numeraire good in period one by auctioning a bond that promises repayment in period two. This bond is risky because it constitutes a claim to one real unit in period two only if the government does not default. If the government defaults, then investors cannot recover any of the investment. The probability of default, κ , is random and takes on two values $\kappa_g < \kappa_b$. The ex-ante probability of each value is given by $f(g)$ and $f(b)$ respectively. Since the default probability determines the expected repayment of the bond, we refer to

the realization of κ as the quality shock. The government sells these bonds in an auction in period 1. If the amount of money raised at auction falls short of D , then we assume that the government simply defaults on any bonds that it sold in period one (we can take this to also mean that they defaulted on the bonds coming due in period 1).

The objective of investors is to maximize their expected, strictly concave, flow utility functions U over their second period consumption. Each investor has wealth W in period one and can either invest in a risk-free bond (storage) or the risky bond being auctioned by the government. In addition to the quality shock that determines the probability of default, there is a demand shock, which we model as a random share of investors who show up to the government's auction. We denote the random fraction of the potential investors who make it to the auction by $1 - \eta$. Those that do not make it to the auction have no choice but to invest all of their wealth in the risk-free bond and eat the proceeds in the second period. The investors who do make it to the auction have the option to bid and invest a fraction of their wealth in the risky government bond, with the remainder invested in storage.⁵

We assume that η is continuously distributed on the interval $[0, \eta_M]$ according to a continuous density function $g(\eta)$ that is nonzero everywhere on the interior of the interval, with $\eta_M < 1$. We will denote the set of possible values of η by \mathcal{H} , and refer to (θ, η) as the state of the world. Here $\theta \in \{g, b\}$ and $\kappa = \kappa_g$ if $\theta = g$, and $\kappa = \kappa_b$ if $\theta = b$. The set of states is denoted by $\mathcal{S} = \{g, b\} \times \mathcal{H}$.

At the auction, investors can submit multiple bids. Each bid is a price and quantity pair $\{P, B\}$ representing a commitment to purchase B units of the bond either at price P (discriminatory auction) or at the marginal price (uniform auction) should the government decide to execute the bid. The government treats each bid independently, sorts all bids from the highest to the lowest bid price, and accepts all bids in descending order to the highest bid price at which the amount D is raised. We refer to this highest possible "lowest" accepted price as the marginal price \bar{P} , and to bids above the marginal price as *bids in the money*.

⁵This shock to the demand for the bond can be also interpreted as a shock to its supply, or the amount of funds that the government needs to raise at the auction in period one, $D/(1 - \eta)$.

The price that an investor has to pay when a bid is accepted depends on the auction protocol. We consider two protocols that are widely used in large volume auctions of a common good. In the first, we will assume that the government sells bonds using a *discriminatory-price* (DP) auction (bonds are sold at the bid price, or “pay as you bid”). In the second, we assume that the government sells bonds using a *uniform-price* (UP) auction (all accepted bids are executed at the lowest accepted, or marginal, price). We assume that the government and the investors take the auction protocol as given. If the marginal price does not exactly clear the market (i.e. it generates revenue greater than D), then only a fraction of the bonds at the marginal price are accepted and bonds are rationed pro-rata among investors.

There will be two types of investors at the auction: those who are informed about θ and those who are not. We denote by $i \in \{I, U\}$ the type of investor and use $n \in [0, 1]$ to denote the share of investors who are *informed* (I), with $1 - n$ denoting the share who are *uninformed* (U). Because informed (uninformed) investors are otherwise identical, they behave the same and we can refer to a representative informed (uninformed) investor. No investor is informed about η , which means that all investors face uncertainty about the minimum price at which they can buy the bond conditional on their information (or lack thereof) about θ . Consistent with our mechanical modeling of the government we assume that it observes neither θ nor η before the auction.

Investors lack commitment in two important dimensions. First, they cannot commit to honor any intertemporal contracts. We will take this to mean that they cannot borrow at the risk-free rate, nor can they make negative bids at the auction. Investors must therefore bid nonnegative quantities ($B \geq 0$) and can spend no more than their wealth W on bonds. Second, they cannot commit to credibly share their information about θ . We will take this to mean that there is no market for information about θ .

A unit of the bond is a claim to a real unit of the numeraire good in period two. As this claim either pays 1 or 0, the range of possible prices is $P \in [0, 1]$. Since investors will typically find it optimal to submit multiple bids, we start by taking the investors’ strategy to be a bid function $B^I(P|\theta)$ for the informed and $B^U(P)$ for the uninformed where $B : [0, 1] \rightarrow [0, W]$.

If $\bar{P}(\theta, \eta)$ is the marginal price in state (θ, η) , then the amount that the government raises in this state in a UP auction is

$$\bar{P}(\theta, \eta)(1 - \eta) \int_{\bar{P}(\theta, \eta)}^1 [nB^I(P|\theta) + (1 - n)B^U(P)] dP.$$

The left-hand expression is simply the marginal price $\bar{P}(\theta, \eta)$ multiplied by the accepted number of bids given this marginal price. The number of accepted bids is decreasing in the marginal price $\bar{P}(\theta, \eta)$. Since the price paid is increasing there can be multiple price points at which the government raises the necessary amount D . The auction protocol we use is to always use the highest such price.

The amount raised in a DP auction in state (θ, η) given marginal price $\bar{P}(\theta, \eta)$ is

$$(1 - \eta) \int_{\bar{P}(\theta, \eta)}^1 [nB^I(P|\theta) + (1 - n)B^U(P)] PdP,$$

which is declining in the marginal price $\bar{P}(\theta, \eta)$ since the amount raised per accepted bid is fixed and this reduces the number of accepted bids.

We assume that investors have rational expectations: the set of marginal prices, their probabilities and the states associated with them are all common knowledge before submitting the bids. After the auction has been performed and the realization of the marginal price has been revealed, informed and uninformed investors can make inferences with respect to the state. For the informed investor this is straightforward since they know θ and can infer η by inverting the price schedule. For the uninformed this is somewhat more complicated. If the price $\bar{P}(\theta, \eta)$ is uniquely generated by this state, then they too can infer the state. If there exist two states (g, η_g) and (b, η_b) that have a common price, then they will still be able to update their beliefs about the set of possible states and their probabilities from observing the price. *However*, this ex-post information is of limited use since all of the investors must choose their bids prior to observing the price.

At the time they make their bids, the informed investors know θ and the probability distribution over η , while the uninformed investors know the probabilities distributions over θ and η . They can also compute $\bar{P}(\theta, \eta)$ so they know how the realized state will de-

termine the marginal price at the auction. With this information they can make inferences about the set of states and their probabilities in the event they are able (or not able) to buy at a bid price of P .

In the DP auction it is a strictly dominating strategy to bid only at the possible prices $\bar{P}(\theta, \eta)$ for any price whose bid will be accepted with positive probability. Otherwise, if the bid is made at a price slightly above $\bar{P}(\theta, \eta)$ the same bid is accepted but the investor pays a higher price. In the UP auction it is a weakly dominating strategy to do so. In light of this, we restrict our agents to only bid at marginal prices. With this restriction, we no longer need to think of our agents having a bidding strategy. We can instead think of them as choosing how many bonds to bid for at the marginal price for each state. Since bids only happen at marginal prices, we drop the notation \bar{P} and just refer to P . Also, for this reason, we switch to a simpler and starker specification of prices and bids.

Definition 1. For each state $(\theta, \eta) \in \mathcal{S}$, the marginal price is denoted $P(\theta, \eta)$ and the set of marginal prices by \mathcal{P} . An action for the uninformed investors is a function $B^U(\theta, \eta)$ which denotes the number of units bid at the marginal price $P(\theta, \eta)$. An action for the informed investors is a function $B^I(\theta, \eta | \hat{\theta})$ which denotes their bids at the various possible states when the realized θ is $\hat{\theta}$.

Remark 2. This stark specification is particularly helpful when the set of η 's is finite. In that case the set of possible marginal prices is finite as well. In our original specification of actions as bids on the set of all potential prices $P \in [0, 1]$ this would mean that the bid function would be positive only at a finite set of points corresponding to those marginal prices. But even when η is continuous, the set of marginal prices is a strict subset of the set of potential prices.

Next, we can define expenditures. The total value of the bids that will be accepted in a UP auction for an investor $i \in \{U, I\}$ whose bid schedule is $B^i(\hat{\theta}, \hat{\eta})$ is given by

$$X^i(\theta, \eta) = \sum_{\hat{\theta} \in \{g, b\}} \int_{\hat{\eta}} I \{ P(\hat{\theta}, \hat{\eta}) > P(\theta, \eta) \} B^i(\hat{\theta}, \hat{\eta}) P(\theta, \eta) d\hat{\eta}, \quad (\text{X UP})$$

where $I \{ \cdot \}$ is an indicator function. Note that in this case an investor is buying all of his units of bonds at the marginal price for the state, $P(\theta, \eta)$.

The total value of the bids that will be accepted by an investor i with this bid schedule

in a *DP auction* is given instead by

$$X^i(\theta, \eta) = \sum_{\hat{\theta} \in \{g, b\}} \int_{\hat{\eta}} I \{P(\hat{\theta}, \hat{\eta}) > P(\theta, \eta)\} B^i(\hat{\theta}, \hat{\eta}) P(\hat{\theta}, \hat{\eta}) d\hat{\eta} \quad (\text{X PD})$$

Note that an investor who submits multiple bids will buy the bond at multiple prices.

The gross return on risky bonds is 0 if the government defaults, and

$$R^i(\theta, \eta) = \sum_{\hat{\theta} \in \{g, b\}} \int_{\hat{\eta}} I \{P(\hat{\theta}, \hat{\eta}) > P(\theta, \eta)\} B^i(\hat{\theta}, \hat{\eta}) d\hat{\eta} - X^i(\theta, \eta). \quad (\text{R})$$

if the government repays. The investor's consumption thus is $W - X^i$ if the government defaults and $W + R^i$ if it repays.

Note that the informed investor's bid schedule will be conditioned upon the true θ . When it is useful to make this explicit we shall write $B^I(\hat{\theta}, \hat{\eta}|\theta)$, $X^I(\hat{\theta}, \hat{\eta}|\theta)$ and $R^I(\hat{\theta}, \hat{\eta}|\theta)$. Note also that both the expenditure function X and the gross return function R are functions of the bid function of the investor. To economize on notation we suppress this dependence.

Remark 3. For any pair of states (g, η_g) and (b, η_b) s.t. $P(g, \eta_g) = P(b, \eta_b)$, bids at these states are perfect substitutes since they will be accepted and rejected in identical circumstances across realized states. Thus, the investor is bidding the quantity $B(g, \eta_g) + B(b, \eta_b)$ at the price $P = P(g, \eta_g) = P(b, \eta_b)$. In this case, the uninformed investors will not be able to infer the state ex-post for this price P .

The payoff to an informed investor can be written as

$$E_{\eta} \{U(W - X^I(\theta, \eta)) \kappa_{\theta} + U(W + R^I(\theta, \eta)) (1 - \kappa_{\theta}) | \theta\},$$

with the expectation taken η conditional on θ .

The payoff for an uninformed investor is

$$E_{\theta, \eta} \{U(W - X^U(\theta, \eta)) \kappa_{\theta} + U(W + R^U(\theta, \eta)) (1 - \kappa_{\theta})\}.$$

with the expectation taken over both η and θ . Because of the short-sale constraint and limited wealth, it follows that

$$0 \leq X^i(\theta, \eta) \leq W. \quad (\text{T EXP})$$

In what follows we assume sufficient risk aversion to ignore the upper bound constraint. We will examine the investor problems in detail after defining an equilibrium.

2.2 Equilibrium

Market clearing. Market clearing defined in terms of expenditures is

$$(1 - \eta) [nX^I(\theta, \eta|\theta) + (1 - n)X^U(\theta, \eta)] = D \quad \forall (\theta, \eta) \in \mathcal{S}. \quad (\text{MK CL})$$

In defining this market-clearing condition we impose that there is no rationing of investor bids. As we will see, this is true in equilibrium: if an investor were rationed over a range of η 's, then they would have a preferred realized value over that range. By bidding at a slightly higher price, they could obtain their preferred outcome. Hence being rationed over a range of positive measure is not consistent with optimization. In addition, if an investor were rationed at a price P in a DP auction, then they would buy the portion that they were unable to buy at the price P for all prices below P .⁶

Bid overhang constraint. Because the marginal price is defined to be *highest* price such that demand is enough to cover the government's supply of debt, bids and prices must also satisfy an additional constraint that we call the *bid-overhang constraint*. This constraint requires that there *cannot* exist a state (θ', η') such that, at the marginal price $P(\theta', \eta')$, there is enough demand to cover the supply in a state (θ, η) with a lower marginal price. This is, it cannot be the case that

$$P(\theta', \eta') > P(\theta, \eta) \quad \text{and} \quad (1 - \eta) \{nX^I(\theta', \eta'|\theta) + (1 - n)X^U(\theta', \eta')\} \geq D. \quad (\text{BD OV})$$

⁶While there can be rationing at extremes like η_M , there is an equivalent equilibrium in which everyone simply bids the additional amount that they purchase at each marginal price. For this reason we focus on an equilibrium arrangement in which the market just clears without rationing.

The constraint arises because aggregate bids are jointly determined by investor's per-capita bids and the realization of the demand shock η . Holding per-capita bids constant, bids submitted in a high η state may therefore be sufficient to clear the market in a low η state, because a larger share of investors shows up at the auction. The bid overhang constraint ensures that the marginal price must be equal in both states in this case. By construction, the bid overhang constraint never binds in a DP auction because all bids are executed at the bid price. It will typically bind in UP auctions, however.

Definition 4. *An equilibrium is defined as a function $P : \mathcal{S} \rightarrow [0, 1]$, and bidding functions $B^U : \mathcal{S} \rightarrow [0, W]$ and $B^I : \mathcal{S} \times \{g, b\} \rightarrow [0, W]$, such that*

1. *each type of investor optimizes bidding at each state $(\theta, \eta) \in \mathcal{S}$, given prices $P \in \mathcal{P}$ and subject to constraint (T EXP).*
2. *the market clearing condition (MK CL) is satisfied for all $(\theta, \eta) \in \mathcal{S}$, and*
3. *the bid-overhand constraint (BD OV) is also satisfied at each $(\theta, \eta) \in \mathcal{S}$.*

Remark 5. *Formulating an equilibrium in this stark fashion where bids are defined as functions of the state is isomorphic to a more standard formulation where bids are defined as functions of prices as well. The price functions are the same, as $B^U(P(\theta, \eta)) = B^U(\theta, \eta)$ and 0 elsewhere, while $B^I(P(\theta, \eta)|\hat{\theta}) = B^I(\theta, \eta|\hat{\theta})$ and 0 elsewhere. The main difference is that the standard formulation defines the bid function over all potential prices rather than marginal prices only. This standard formulation is poorly behaved in the sense that the bid function is discontinuous around marginal prices because no investor bids on non-marginal prices. This is particularly problematic when η is discrete.*

Proposition 6. *For both the UP and the DP auctions, the equilibrium marginal price, $P(\theta, \eta)$, is declining in η for each θ . A bid $B^i(\theta, \eta)$ made at a price $P(\theta, \eta)$ is thus the money for all $\hat{\eta} \geq \eta$ given θ , and it is in the money for all $\hat{\eta} \geq \bar{\eta}$ given $\bar{\theta} \neq \theta$, where $\bar{\eta}$ is the value of η for which*

$$\bar{\eta}_{\bar{\theta}}(\theta, \eta) : P(\bar{\theta}, \bar{\eta}) = P(\theta, \eta),$$

Notice that $\bar{\eta}_{\theta}(\theta, \eta) = \eta$.

Because the price schedule conditional on θ is bounded and monotonic, it follows that it is both continuous and differentiable almost everywhere.

Proof. Ordering prices in decreasing order for each θ , an increase in η implies that fewer investors are available to purchase bonds, so that the government needs to raise more funds per investor. This is possible only at a lower marginal price. For continuity see Rudin (1964, p. 96) and for differentiability see Royden (1968, p. 96). \square

2.3 Investor Problems

We now explicitly examine the decision problems of the informed and the uninformed investors for each auction protocol.

Inference Problem. An important input to the investor problem is an inference problem determining the quality of the bond an investor expects receive conditional on a given bid being executed. Given that bids are executed depending on the realization of the marginal price and that the quality of a bond is fully pinned down by its default probability, this inference problem is equivalent to computing the expected default probability of a bond given the realization of a marginal price. We denote this conditional expected default probability by $\tilde{\kappa}$. For the informed, $\tilde{\kappa}(P(\theta, \eta)|\theta) = \kappa_\theta$ because they know the true θ . For the uninformed there are two cases:

1. For any (θ, η) such that $\nexists (\theta', \eta')$ with $P(\theta, \eta) = P(\theta', \eta')$, then $\tilde{\kappa}(P(\theta, \eta)) = \kappa_\theta$.
2. If there are two states (θ, η) and (θ', η') such that $P(\theta', \eta') = P(\theta, \eta)$ and $\theta' \neq \theta$ the solution to the uninformed investor's inference problem is as follows.

Given $P(\theta, \eta)$, define $\eta = \phi(P|\theta)$, where ϕ is the inverse function of the price w.r.t. η .⁷ Define the probability of an interval of prices $\mathbf{P} \subset \mathcal{P}$ conditional on θ as

$$h(\mathbf{P}|\theta) = \int_{\{\eta: P(\theta, \eta) \in \mathbf{P}\}} g(\eta) d\eta = \int_{\tilde{P} \in \mathbf{P}} g(\phi(\tilde{P}|\theta)) \frac{\partial \phi(\tilde{P}|\theta)}{\partial \tilde{P}} d\tilde{P}.$$

⁷Note from proposition 6 that $P(\theta, \eta)$ is continuous almost everywhere and since rationing does not occur in equilibrium, strictly monotonic, then it is invertible almost everywhere. See Rudin (1964, p. 90).

Note that the slope of the inverse function with respect to the price determines the size of the set of η 's that are associated with the prices in \mathbf{P} (given θ). The unconditional probability of the set of prices is then given by

$$h(\mathbf{P}) = \sum_{\theta} f(\theta)h(\mathbf{P}|\theta),$$

and the probability of θ conditional on a price in \mathbf{P} is simply $f(\theta)h(\mathbf{P}|\theta)/h(\mathbf{P})$. We can define the probability density function of a particular price $P \in \mathbf{P}$, given θ , by shrinking the set $\mathbf{P} \rightarrow P$, and observing h in the limit, or

$$Pr(P|\theta) = \lim_{\mathbf{P} \rightarrow P} \frac{h(\mathbf{P}|\theta)}{\Delta(\mathbf{P})},$$

where $\Delta(\mathbf{P})$ is the length of the price interval. This then leads to the inferred default probability

$$\tilde{\kappa}(P) = \frac{\sum_{\theta} f(\theta)Pr(P|\theta)\kappa_{\theta}}{\sum_{\theta} f(\theta)Pr(P|\theta)}.$$

Payoffs. Given the conditional probability of default, we can define the *conditional payoff* for investor i given state (θ, η) as $V(B^i(\cdot)|\theta, \eta)$, where $B(\cdot)$ is the set of bids submitted for all states $(\hat{\theta}, \hat{\eta}) \in \mathcal{S}$. Then

$$V(B^i(\cdot)|\theta, \eta) = U(W - X^i(\theta, \eta))\tilde{\kappa}(P(\theta, \eta)) + U(W + R^i(\theta, \eta)) [1 - \tilde{\kappa}(P(\theta, \eta))].$$

The overall payoff of an *uninformed agent* is then given by

$$V^U = \max_{B^U(\cdot)} \sum_{\theta} f(\theta) \int_{\eta} V(B^U(\cdot)|\theta, \eta)g(\eta)d\eta \quad \text{s.t. } B^U(\hat{\theta}, \hat{\eta}) \geq 0 \quad \forall (\hat{\theta}, \hat{\eta}) \in \mathcal{S} \quad (1)$$

The overall payoff for an *informed agent* is defined analogously as

$$V^I = \max_{B^I(\cdot|\theta)} \sum_{\theta} f(\theta) \int_{\eta} V(B^I(\cdot|\theta)|\theta, \eta)g(\eta)d\eta \quad \text{s.t. } B^I(\hat{\theta}, \hat{\eta}|\theta) \geq 0 \quad \forall (\hat{\theta}, \hat{\eta}, \theta) \in \mathcal{S} \times \{g, b\}. \quad (2)$$

The key distinction is that the informed can condition their bids on the realization of θ .

Impact of auction protocols. The auction protocol determines how bids $B(\cdot)$ affect each agent's conditional payoff. To make this explicit, we will write $V^{UP}(B(\cdot)|\theta, \eta)$ for the UP auction and $V^{DP}(B(\cdot)|\theta, \eta)$ for the DP auction.

In a UP auction, the derivative of the conditional payoff in state (θ, η) with respect $B(\hat{\theta}, \hat{\eta})$, the bid in state $(\hat{\theta}, \hat{\eta})$, is

$$\frac{\partial V^{UP}(B(\cdot)|\theta, \eta)}{\partial B(\hat{\theta}, \hat{\eta})} = I \left\{ P(\hat{\theta}, \hat{\eta}) \geq P(\theta, \eta) \right\} \times \left\{ \begin{array}{l} U'(W - X(\theta, \eta))\tilde{\kappa}(P(\theta, \eta))P(\theta, \eta) \\ + U'(W + R(\theta, \eta)) [1 - \tilde{\kappa}(P(\theta, \eta))] [1 - P(\theta, \eta)] \end{array} \right\}.$$

Importantly, this derivative is the same for any bid $B(\hat{\theta}, \hat{\eta})$ such that $P(\hat{\theta}, \hat{\eta}) \geq P(\theta, \eta)$ because any such bid is executed at the state-specific marginal price $P(\theta, \eta)$. The derivative is also identically equal to zero for all bids $B(\hat{\theta}, \hat{\eta})$ such that $P(\hat{\theta}, \hat{\eta}) < P(\theta, \eta)$ because no such bid is executed. Total expenditures $X(\theta, \eta)$ and revenues $R(\theta, \eta)$ in state (θ, η) are thus determined by the sum of all in-the-money bids in that state, but not by their *distribution* over in-the-money states. As we will see below, this feature of the UP auction allows us to solve for the optimal bids in a recursive fashion.

The same is not true for the for the DP auction. Here, the derivative is given by

$$\frac{\partial V^{DP}(B(\cdot)|\theta, \eta)}{\partial B(\hat{\theta}, \hat{\eta})} = I \left\{ P(\hat{\theta}, \hat{\eta}) \geq P(\theta, \eta) \right\} \times \left\{ \begin{array}{l} U'(W - X(\theta, \eta))\tilde{\kappa}(P(\theta, \eta))P(\hat{\theta}, \hat{\eta}) \\ + U'(W + R(\theta, \eta)) [1 - \tilde{\kappa}(P(\theta, \eta))] [1 - P(\hat{\theta}, \hat{\eta})] \end{array} \right\}.$$

The key difference is that the derivative is now evaluated at the bid price $P(\hat{\theta}, \hat{\eta})$ rather than at the state-specific marginal price $P(\theta, \eta)$. As a result, the derivative is longer the same for all in-the-money bids, and the distribution of bids over in-the-money states now crucially determines payoffs. Optimal bids can thus no longer be solved for recursively. Note that it is still true that the derivative is identically equal to zero for any bids $B(\hat{\theta}, \hat{\eta})$ such that $P(\hat{\theta}, \hat{\eta}) < P(\theta, \eta)$ because these bids are not executed.

The next section studies the implication of these observations for equilibrium bids and prices in more detail.

3 Uniform-Price Auctions

We start with the uniform-price auction. Recall that the fraction of informed investors was denoted by n , and that this fraction determines the degree of asymmetric information in our model. Accordingly, we will use $P^{UP}(\theta, \eta; n)$ to denote the price function for each state and level of asymmetric information in the UP auction. When there is no risk of confusion, we sometimes also simply write $P(\theta, \eta)$. In a similar fashion, we will use $B^{UP,I}(\theta, \eta; n)$ for the bids of the informed and $B^{UP,U}(\theta, \eta; n)$ for the bids of the uninformed, but may sometimes use the simpler notation $B^I(\theta, \eta)$ and $B^U(\theta, \eta)$ when there is no risk of confusion.

3.1 Symmetric Information Benchmarks

We begin by considering the two symmetric information benchmarks: the symmetric ignorance equilibrium in which no investor is informed ($n = 0$), and the symmetric information equilibrium in which all investors are informed ($n = 1$).

3.1.1 Symmetric Ignorance

Since there are no informed investors in the symmetric ignorance equilibrium, marginal prices cannot depend upon θ , and we must have $P(g, \eta) = P(b, \eta)$ for all $\eta \in \mathcal{H}$. Hence, we can simplify our notation and write $P(\eta)$, $B(\eta)$, $X(\eta)$ and $R(\eta)$, and state the market-clearing condition (MK CL) as

$$D = (1 - \eta)X(\eta) = (1 - \eta)\mathcal{B}(\eta)P(\eta),$$

where $\mathcal{B}(\eta) \equiv [\int_0^\eta B(\hat{\eta})d\hat{\eta}]$ denotes the total bids of uninformed investors when the demand shock is η and the marginal price is $P(\eta)$. $\mathcal{B}(\eta)$ is strictly increasing in η since $P(\eta)$ is strictly decreasing. This implies that the short-sale constraint cannot bind for any η .

The problem for our uninformed investor then is

$$\max_{\mathcal{B}(\eta)} \int_0^{\eta_M} \left\{ \begin{array}{l} U(W - \mathcal{B}(\eta)P(\eta)) \kappa^U \\ U(W + \mathcal{B}(\eta)[1-P(\eta)]) (1 - \kappa^U) \end{array} \right\} g(\eta) d\eta,$$

where $\kappa^U = f(g)\kappa_g + f(b)\kappa_b$ is the ex-ante probability of default. Note that investors are optimizing with respect to the *total* number of bids purchased at each η . This is without loss of generality since all bids all in-the-money bids are executed at the same price. The individual state-specific bids can be recovered from the identity $B(\eta) = d\mathcal{B}(\eta)$. Note also that we have ignored the short-sale constraint because it cannot bind in equilibrium.

The first-order condition for this problem at η^* is

$$\int_{\eta^*}^{\eta_M} \left\{ \begin{array}{l} -U'(W - \mathcal{B}(\eta)P(\eta)) \kappa^U P(\eta) \\ +U'(W + \mathcal{B}(\eta)[1-P(\eta)]) (1 - \kappa^U)[1-P(\eta)] \end{array} \right\} g(\eta) d\eta = 0. \quad (3)$$

Note that the version of this condition at $\eta^* + \varepsilon$ implies that the integral from $[\eta^* + \varepsilon, \eta_M]$ is equal to 0. Since this is true for all $\eta > \eta^*$, it follows that

$$\left\{ \begin{array}{l} -U'(W - \mathcal{B}(\eta)P(\eta)) \kappa^U P(\eta) \\ +U'(W + \mathcal{B}(\eta)[1-P(\eta)]) (1 - \kappa^U)[1-P(\eta)] \end{array} \right\} = 0 \text{ for all } \eta \in \mathcal{H}. \quad (4)$$

We can then solve for the equilibrium by imposing market clearing to give

$$\left\{ \begin{array}{l} -U' \left(W - \frac{D}{1-\eta} \right) \kappa^U P(\eta) \\ +U' \left(W + \frac{D}{1-\eta} \frac{1-P(\eta)}{P(\eta)} \right) (1 - \kappa^U) [1 - P(\eta)] \end{array} \right\} = 0 \text{ for all } \eta \in \mathcal{H}. \quad (5)$$

We denote the equilibrium price function that solves this equation by $P^{UP}(\theta, \eta; n = 0)$. Note that prices are invariant to θ . We use notation including θ only to facilitate comparison with other equilibria. The total bond purchases can be then derived using this price function and market clearing. We denote this bid function by $B_U^{UP}(\theta, \eta; n = 0)$, where subscript U indicates that these bonds are purchased by uninformed investors.

Example 7. *If we assume log preferences, then our equilibrium can be solved in closed form.*

First-order condition (4) becomes

$$\frac{(1 - \kappa^U)[1 - P(\eta)]}{W + \mathcal{B}(\eta)[1 - P(\eta)]} - \frac{\kappa^U P(\eta)}{W - \mathcal{B}(\eta)P(\eta)} = 0.$$

which implies

$$\mathcal{B}(\eta) = \frac{W [1 - \kappa^U - P(\eta)]}{P(\eta)[1 - P(\eta)]}.$$

The equilibrium price then is the solution to the market-clearing condition $P(\eta)\mathcal{B}(\eta) = D/(1 - \eta)$,

which leads to

$$P(\eta) = \frac{(1 - \kappa^U)W - \frac{D}{1 - \eta}}{W - \frac{D}{1 - \eta}} = 1 - \kappa^U - \frac{(1 - \kappa^U)\frac{D}{1 - \eta}}{W - \frac{D}{1 - \eta}}.$$

By inspection we can see that $P(\eta)$ is equal to the risk-free price $1 - \kappa^U$ if $D = 0$, and that it is declining in $D/(1 - \eta)$.

Informed investors in an uninformed world. Looking ahead to our results on information acquisition in Section 6, we now briefly discuss how an informed investor chooses his portfolio when all other investors are uninformed. Since a single investor's bids have no price impact, the price schedule is fixed at $P(\eta)$. Since the informed investor knows θ , his bids satisfy the first-order condition (3) evaluated at either κ_g or κ_b instead of κ^U . The advantage of being informed in an uninformed world thus is the ability to adjust quantities at fixed prices. Generically, this implies the informed will thus bid more when $\theta = g$ and less when $\theta = b$. Note that in the case that short-sale constraints do not bind the state-specific bids satisfy the analog of condition (4).

3.1.2 Symmetric Information

We now consider the symmetric information equilibrium in which all investors are informed ($n = 1$). Since all investors bid contingent on θ , the equilibrium can be determined conditional on the realized quality shock θ . The market clearing condition is

$$D = (1 - \eta)X(\theta, \eta) = (1 - \eta)\mathcal{B}(\theta, \eta)P(\theta, \eta) \text{ where } \mathcal{B}(\theta, \eta) \equiv \left[\int_0^\eta B(\theta, \hat{\eta}) d\hat{\eta} \right],$$

where all variables are now conditioned on the realized θ . As before, $\mathcal{B}(\theta, \eta)$ must be increasing in η for each θ . The conditional payoff to the informed investors is

$$\max_{\mathcal{B}(\theta, \eta)} \int_0^{\eta_M} \left\{ \begin{array}{l} U(W - \mathcal{B}(\theta, \eta)P(\theta, \eta)) \kappa_\theta \\ U(W + \mathcal{B}(\theta, \eta)[1 - P(\theta, \eta)]) (1 - \kappa_\theta) \end{array} \right\} g(\eta) d\eta.$$

This leads to a similar first-order condition to (3) which we can then be solved state-by-state as in (4). The only difference is that we have the conditional default probability κ_θ instead of κ^U . If we again impose market clearing we obtain the analog to (5):

$$\left\{ \begin{array}{l} U' \left(W - \frac{D}{1-\eta} \right) \kappa_\theta P(\theta, \eta) \\ U' \left(W + \frac{D}{1-\eta} \frac{1-P(\theta, \eta)}{P(\theta, \eta)} \right) (1 - \kappa_\theta) [1 - P(\theta, \eta)] \end{array} \right\} = 0 \text{ for all } \theta \in \{g, b\} \text{ and } \eta \in \mathcal{H}.$$

Denote the equilibrium values implied by this condition by $P^{UP}(\theta, \eta; 1)$ and the associated bond function by $B^{UP,I}(\theta, \eta; 1)$.

Example 8. *If we again assume log preferences the closed-form conditional price schedules is*

$$P(\theta, \eta) = 1 - \kappa_\theta - \frac{(1 - \kappa_\theta) \frac{D}{1-\eta}}{W - \frac{D}{1-\eta}}.$$

Observe that the $\theta = g$ schedule will lie above the $\theta = b$ conditional on η . Going forward, an important question will be whether the $\theta = g$ schedule also lies above the $\theta = b$ schedule for all η . It is easy to verify that this is the case if

$$1 - \kappa_g - \frac{(1 - \kappa_g) \frac{D}{1-\eta_M}}{W - \frac{D}{1-\eta_M}} > 1 - \kappa_b - \frac{(1 - \kappa_b) D}{W - D}. \quad (6)$$

If this condition is satisfied, then each marginal price is associated with a unique θ , and there is perfect ex-post inference. If not, then the price schedules overlap and some marginal prices will be associated with both values of θ . From inspection one can see that overlap is more likely the closer are the bankruptcy costs and the bigger is the span of the demand shocks.

Uninformed in an informed world. Again looking ahead to our results on information acquisition, we now study the problem of an uninformed investor if all other investors

are informed. This will provide us with intuition as to the value of information under asymmetric information more generally. As before, a single investor has no price impact. The price schedules are thus fixed from the perspective of the uninformed investor. The uninformed investor chooses his bidding strategy without knowing θ . When deciding on a particular bid, the investor must therefore form expectations regarding the states of the world in which this bid will be executed. For each marginal price $P(\theta, \eta)$, we can define $\bar{\eta}$ to be value of η such that the marginal price given $\bar{\theta} \neq \theta$ is equal to $P(\theta, \eta)$. Such a price exists if the price schedules overlap. Hence

$$\bar{\eta}_{\bar{\theta}}(\theta, \eta) : P(\bar{\theta}, \bar{\eta}_{\bar{\theta}}(\theta, \eta)) = P(\theta, \eta)$$

If the price schedules do not overlap, we say that $\bar{\eta}_{\bar{\theta}}(\theta, \eta) = 0$. Note that $\bar{\eta}_{\theta}(\theta, \eta) = \eta$ if $\bar{\theta} = \theta$. Hence $\bar{\eta}_{\bar{\theta}}(\theta, \eta)$ determines the set of states for which a bid at $P(\theta, \eta)$ is in the money. This will allow the uninformed to form expectations as to the default probability of the bond when a bid at $P(\theta, \eta)$ is executed. The uninformed investor's problem now is

$$\mathcal{L} = \max_{B^U(\theta, \eta)} \min_{\lambda(\theta, \eta)} \sum_{\theta} f(\theta) \int_0^{\eta_M} \left\{ \begin{array}{l} U \left(W - \left[\sum_{\bar{\theta}} \int_0^{\bar{\eta}_{\bar{\theta}}(\theta, \eta)} B^U(\bar{\theta}, \hat{\eta}) d\hat{\eta} \right] P(\theta, \eta) \right) \tilde{\kappa}(P(\theta, \eta)) \\ U \left(W + \left[\sum_{\bar{\theta}} \int_0^{\bar{\eta}_{\bar{\theta}}(\theta, \eta)} B^U(\bar{\theta}, \hat{\eta}) d\hat{\eta} \right] [1 - P(\theta, \eta)] \right) (1 - \tilde{\kappa}(P(\theta, \eta))) \\ + \lambda(\theta, \eta) [B^U(\theta, \eta) - 0] \end{array} \right\} g(\eta) d\eta.$$

where we have included the short-sale constraint because it is no longer clear whether or not it will bind. The first-order condition for $B^U(\cdot)$ in a given state (θ^*, η^*) for this problem is given by

$$\frac{\partial \mathcal{L}}{\partial B^U(\theta^*, \eta^*)} = \sum_{\theta} \int_{\bar{\eta}_{\theta}(\theta^*, \eta^*)}^{\eta_M} \left\{ \begin{array}{l} U \left(W - \left[\sum_{\bar{\theta}} \int_0^{\bar{\eta}_{\bar{\theta}}(\theta, \eta)} B^U(\bar{\theta}, \hat{\eta}) d\hat{\eta} \right] P(\theta, \eta) \right) \tilde{\kappa}(P(\theta, \eta)) P(\theta, \eta) \\ U \left(W + \left[\sum_{\bar{\theta}} \int_0^{\bar{\eta}_{\bar{\theta}}(\theta, \eta)} B^U(\bar{\theta}, \hat{\eta}) d\hat{\eta} \right] [1 - P(\theta, \eta)] \right) (1 - \tilde{\kappa}(P(\theta, \eta))) [1 - P(\theta, \eta)] \\ + \lambda(\theta, \eta) [B^U(\theta, \eta) - 0] \end{array} \right\} g(\eta) d\eta = 0.$$

Importantly, the first-order condition for a bid at marginal price $P(\theta^*, \eta^*)$ integrates over the set of η 's for which the bid is in the money, $[\bar{\eta}_\theta(\theta^*, \eta^*), \eta_M]$.

Without loss of generality, we can order the states in terms of their marginal prices. If the short-sale constraint does not bind at the lowest marginal price, then the f.o.c. for the state(s) at that price will be zero (with the multiplier also equal to zero). If the short-sale constraint does not bind for any marginal price, then we can recursively argue that the condition holds in terms of total bids for each marginal price, or

$$\sum_{\theta} \left\{ \begin{array}{l} U' \left(W - \left[\sum_{\bar{\theta}} \int_0^{\bar{\eta}_{\bar{\theta}}(\theta, \eta)} B^U(\bar{\theta}, \hat{\eta}) d\hat{\eta} \right] P(\theta, \eta) \right) \tilde{\kappa}(P(\theta, \eta)) P(\theta, \eta) \\ U' \left(W + \left[\sum_{\bar{\theta}} \int_0^{\bar{\eta}_{\bar{\theta}}(\theta, \eta)} B^U(\bar{\theta}, \hat{\eta}) d\hat{\eta} \right] [1 - P(\theta, \eta)] \right) (1 - \tilde{\kappa}(P(\theta, \eta))) [1 - P(\theta, \eta)] \\ + \lambda(\theta, \eta) [B^U(\theta, \eta) - 0] \end{array} \right\} \Bigg|_{\bar{\eta}_\theta(\theta^*, \eta^*)} = 0.$$

By inspection, solution to this equation is equal to the portfolio of the informed if there is a unique state that generates the marginal price $P(\theta^*, \eta^*)$. This leads to the following proposition.

Proposition 9. *In a UIP auction the uninformed will be able to replicate the total bids of the informed state-by-state, and hence their ex-ante payoff, if:*

1. *Each marginal price is associated with a unique state in \mathcal{S} .*
2. *When we order the marginal prices and the associated total bids of the informed from the highest to the lowest marginal price, the total bids of the informed are also weakly ranked from lowest to highest.*

Corollary 10. *For any equilibrium with $n > 0$, $P(g, \eta_M, n) > P(b, 0, n)$ and $\mathcal{B}^I(g, \eta_M, n) < \mathcal{B}^I(b, 0, n)$ are sufficient conditions for the uninformed to be able to completely replicate the portfolio of the informed state-by-state, and hence their payoffs.*

Note that the uninformed will not make the same *marginal bids* as the informed even when there is perfect portfolio replication. The reason is all of uninformed's bids on the $\theta = g$ schedule are accepted when $\theta = b$. This is not the case for the informed, whose bids are conditioned on θ . Thus, the uninformed make the same marginal bids on the

high-price schedule, but make lower marginal bids on the low-price schedule in order to arrive at the same *total* bids.

Example 11. *Returning to our log example, assume that condition (6) is satisfied so that the price schedules do not overlap. To verify that we can have replication when $n = 1$ we only need to check that $\mathcal{B}^I(g, \eta_M, 1) < \mathcal{B}^I(b, 0, 1)$. The equilibrium total bond purchases of the informed in the fully informed equilibrium are*

$$\mathcal{B}^I(\theta, \eta, 1) = \frac{W \frac{(1-\kappa_\theta) \frac{D}{1-\eta}}{W - \frac{D}{1-\eta}}}{\left(1 - \kappa_\theta - \frac{(1-\kappa_\theta) \frac{D}{1-\eta}}{W - \frac{D}{1-\eta}}\right) \left(\kappa_\theta + \frac{(1-\kappa_\theta) \frac{D}{1-\eta}}{W - \frac{D}{1-\eta}}\right)}.$$

We can use this result to evaluate whether or not perfect replication will be possible in the informed equilibrium for different parameter values. Indeed it is easy to generate such outcome when the default probabilities are sufficiently different and the span of the demand shocks is not too large.

Remark 12. *Assume a discrete η grid. When short-sale constraints bind, it is necessary to solve a simultaneous system of equations to determine the optimal choice of B in each state. While challenging, this system of equations has a convenient nice linear structure. Ordering marginal prices on the state space, we can index this grid by j , such that $\{(\theta_0, \eta_0), \dots, (\theta_j, \eta_j), \dots, (\theta_J, \eta_J)\}$ with $P(\theta_j, \eta_j) > P(\theta_{j+1}, \eta_{j+1})$. Next, define the following set of vectors*

$$\vec{P} = \begin{bmatrix} P(\theta_0, \eta_0) \\ \vdots \\ P(\theta_J, \eta_J) \end{bmatrix}, \quad \vec{B}^U = \begin{bmatrix} B^U(\theta_0, \eta_0) \\ \vdots \\ B^U(\theta_J, \eta_J) \end{bmatrix}, \quad (1 - \vec{P}) = \begin{bmatrix} 1 - P(\theta_0, \eta_0) \\ \vdots \\ 1 - P(\theta_J, \eta_J) \end{bmatrix},$$

$$\vec{\kappa} = \begin{bmatrix} \kappa_{\theta_0} \\ \vdots \\ \kappa_{\theta_J} \end{bmatrix}, \quad \vec{\lambda} = \begin{bmatrix} \lambda(\theta_0, \eta_0) \\ \vdots \\ \lambda(\theta_J, \eta_J) \end{bmatrix}, \quad \vec{f} = \begin{bmatrix} f(\theta_0)g(\eta_0) \\ \vdots \\ f(\theta_J)g(\eta_J) \end{bmatrix}$$

and the following set of triangular matrices

$$\mathbf{P} = \begin{cases} P_{ij} = P(\theta_i, \eta_i) & \text{if } i \leq j \\ P_{ij} = 0 & \text{o.w.} \end{cases}, \quad \mathbf{1} - \mathbf{P} = \begin{cases} 1 - P_{ij} = 1 - P(\theta_i, \eta_i) & \text{if } i \leq j \\ 1 - P_{ij} = 0 & \text{o.w.} \end{cases}$$

Then the system of equations defining the optimal bids of the uninformed can be expressed in vector form simply as

$$-U' \left(W - \mathbf{P} \times \vec{B}^U \right) \cdot \vec{P} \cdot \vec{\kappa} \cdot \vec{f} + U' \left(W + [\mathbf{1} - \mathbf{P}] \times \vec{B}^U \right) \cdot \vec{P} \cdot [1 - \vec{\kappa}] \cdot \vec{f} + \vec{\lambda} = 0$$

$$\vec{\lambda} \cdot \vec{B}^U = 0.$$

3.2 Perfect Replication and the Bid-Overhang Constraint

The previous section showed that, under certain parametric restrictions, uninformed investors could perfectly replicate the portfolio of informed investors in the $n = 1$ equilibrium. But if both types of investors choose the same portfolio, then it stands to reason that the same equilibrium should obtain for values of n less than one. In particular, setting $P(\theta, \eta, n) = P(\theta, \eta, 1)$ satisfies market-clearing, $\mathcal{B}^I(\theta, \eta, 1)$ maximizes the payoff of the informed, and given that this choice is also feasible for the uninformed, the $n = 1$ allocation should also satisfy our equilibrium conditions (1-2) even when $n < 1$. This raises the question of whether the fully informed equilibrium allocation is an equilibrium allocation for all n , and for $n \rightarrow 0$ in particular.

The answer is no: the equilibrium with full replication can be sustained only if n is above a threshold, but breaks down when n is below the threshold. The reason is the bid overhang constraint (BD OV). To see this, note that if the total value of the bids for the informed and the uninformed are the same (as they are in the $n = 1$ equilibrium), then they must be equal to the per capita supply of debt, or

$$P(\theta, \eta) \mathcal{B}^U(\theta, \eta, 1) = P(\theta, \eta) \mathcal{B}^I(\theta, \eta, 1) = \frac{D}{1 - \eta}.$$

The total demand coming from the uninformed is thus equal to $(1 - n)D/(1 - \eta)$. We can use this to solve for the n at which the bid-overhang constraint first binds. This threshold is given by

$$n : \frac{(1 - n)D}{1 - \eta_M} = D, \text{ or } n = \eta_M.$$

For n smaller than this threshold level, the per-capita demand for debt coming from the

uninformed in state (g, η_M) alone is sufficient to clear the market in state $(b, 0)$. Since $P(g, \eta_M, 1) > P(b, 0, 1)$, the government strictly prefers to execute the uninformed high-price bids. This implies that the marginal price in state $(b, 0)$ is $P(g, \eta_M, 1)$ rather than $P(b, 0, 1)$, which is at odds with the proposed allocation. We summarize this discussion with the following proposition.

Proposition 13. *If the UP auction exhibits perfect replication with $n = 1$, then*

1. *for $n > \eta_M$ the equilibrium price level does not change with n and nor do the total bonds purchases; i.e. the uninformed continue to be able to perfectly replicate the outcomes of the informed.*
2. *However, for $n \leq \eta_M$ the bid overhang constraint will bind and force the prices at the bottom of the g schedule to overlap with prices at the top of the b schedule.*

When the bid overhang constraint binds, points on the $\theta = b$ schedule will be forced to have a common price with points on the $\theta = g$ schedule, and the two price schedules will overlap. When this happens, the uninformed cannot perfectly infer the ex-post state but instead relies on the inferred $\tilde{\kappa}$ discussed above. The implied lack of perfect replication generates information rents for the informed. We discuss this case further in the numerical example below.

3.3 Equilibria as $n \rightarrow 0$

A natural question in light of the last section is to what extent price schedules overlap as $n \rightarrow 0$. This turns out to have a clear analytic answer spanning both UP and DP auction protocols. When $n \rightarrow 0$, then almost all expenditures must be made by the uninformed type. This implies that

$$X^U(\theta, \eta, n) \rightarrow D/(1 - \eta) \text{ as } n \downarrow 0,$$

under both auction protocols. When n is sufficiently close to 0, it must therefore be the case that

$$X^U(\theta, \eta, n) > X^U(\theta', \eta', n) \text{ if } \eta > \eta'.$$

But this in turn implies that

$$P(\theta, \eta) < P(\theta', \eta').$$

This is because the cumulated expenditures of the uninformed more than cover demand at (θ', η') at price $P(\theta, \eta)$ hence the price $P(\theta', \eta')$ cannot be lower $P(\theta, \eta)$. Note that this is true under *either* auction protocol. From this we get the following result.

Proposition 14. *The price schedules converge (for interior n) since for n sufficiently close to 0, η must partially order the price schedules $P(\theta, \eta)$ under both of our auction protocols: i.e. if $\eta > \eta'$ then $P(\theta, \eta) < P(\theta', \eta')$.*

This proposition implies that prices must be sorted by η when n is small. Thus, prices must lie between the low price at the small η and the high price at the higher η . When η is a continuous interval, this implies that the price schedules must converge at every interior point in which the price schedules are continuous. Even when η is not an interval, it follows that the uninformed investor faces the same set of in-the-money states when he buys along the high price schedule $P(g, \eta)$ as he did when he bought at $P(\eta)$ in the uninformed equilibrium. In the DP auction protocol, the price that will cause the uninformed to spend $D/(1 - \eta)$ converges to $P(\eta)$ everywhere but at the bottom when $\eta = \eta_M$.

4 Discriminating-Price Auctions

We now replicate our analysis of the symmetric ignorance and symmetric information benchmarks for the discriminatory price auction. In doing so, we highlight differences and similarities with the uniform price auction.

4.1 Symmetric Information Benchmarks

We start analyzing the DP auction with the two symmetric information benchmarks, the uninformed equilibrium ($n = 0$) and the informed equilibrium ($n = 1$).

With symmetric ignorance ($n = 0$), we can again simplify our notation to have $P(\eta)$,

$B(\eta)$, $X(\eta)$ and $R(\eta)$,. With this change our market clearing condition (MK CL) is

$$D = (1 - \eta)X(\eta) = \int_0^\eta B(\hat{\eta})P(\hat{\eta})d\hat{\eta}.$$

Note the first important difference between UP and DP auctions: expenditures and hence the clearing condition in the DP auction do not just depend upon the total number of bids, but also on the *prices* at which the individual bids are executed. Because of this, $X(\eta)$ *must* be monotonically increasing in η . Market clearing implies that the bids must always be positive, i.e. $B(\eta) > 0$ for all $\eta \in \mathcal{H}$. This in turn implies that the short-sale constraint cannot bind for any η . The problem for our uninformed investor is thus simply given by

$$\max_{B(\eta)} \int_0^{\eta_M} \left\{ \begin{array}{l} U(W - \int_0^\eta B(\hat{\eta})P(\hat{\eta})d\hat{\eta}) \kappa^U \\ U(W + \int_0^\eta B(\hat{\eta})[1 - P(\hat{\eta})]d\hat{\eta}) (1 - \kappa^U) \end{array} \right\} g(\eta)d\eta,$$

The first-order condition for this problem at η^* is

$$\int_{\eta^*}^{\eta_M} \left\{ \begin{array}{l} -U'(W - \int_0^\eta B(\hat{\eta})P(\hat{\eta})d\hat{\eta}) \kappa^U P(\hat{\eta}) \\ +U'(W + \int_0^\eta B(\hat{\eta})[1 - P(\hat{\eta})]d\hat{\eta}) (1 - \kappa^U)[1 - P(\hat{\eta})] \end{array} \right\} g(\eta)d\eta = 0. \quad (7)$$

Notice a second key difference to the UP auction: the cumulation of the marginal utilities are being multiplied by the bid price $P(\hat{\eta})$ and the bid return $(1 - P(\hat{\eta}))$ rather than by the marginal prices. This means that the system is not block-recursive as in the UP auction. Instead, it must be solved simultaneously. As in the UP auction, however, the only benefit from being informed when all other investors are uninformed is the ability to adjust bid quantities knowing θ , taking prices as given.

Remark 15. *To solve this problem, we can use the same linear algebra structure that we used in UP auction, as the short-sale constraints do not bind. Again, assume that we have a fine grid on the space of η 's $\{\eta_0, \dots, \eta_J\}$ which is indexed by j and where $\eta_0 = 0$, $\eta_J = \eta_M$ and the η 's are*

increasing in j . Next, we denote the following set of vectors:

$$\vec{P} = \begin{bmatrix} P(\eta_0) \\ \vdots \\ P(\eta_J) \end{bmatrix}, \quad \vec{B}^U = \begin{bmatrix} B^U(\eta_0) \\ \vdots \\ B^U(\eta_J) \end{bmatrix}, \quad (1 - \vec{P}) = \begin{bmatrix} 1 - P(\eta_0) \\ \vdots \\ 1 - P(\eta_J) \end{bmatrix},$$

And we denote the following set of triangular matrices

$$\mathbf{P} = \begin{cases} P_{ij} = P(\eta_i) \text{ if } i \leq j \\ P_{ij} = 0 \text{ o.w.} \end{cases}, \quad \mathbf{1} - \mathbf{P} = \begin{cases} 1 - P_{ij} = 1 - P(\eta_i) \text{ if } i \leq j \\ 1 - P_{ij} = 0 \text{ o.w.} \end{cases}$$

Then with this notation, the system of equations can be expressed in vector form as

$$-U' \left(W - \mathbf{P} \times \vec{B}^U \right) \cdot \vec{P} \cdot \kappa^U + U' \left(W + [\mathbf{1} - \mathbf{P}] \times \vec{B}^U \right) \cdot [1 - \vec{P}] * [1 - \kappa^U] = 0.$$

With symmetric information ($n = 1$), we can solve for the equilibrium separately for each θ . This requires replacing κ^U with the appropriate conditional default probability κ_θ , but proceeds in the analog manner thereafter.

Uninformed in an informed world. Just as in the UP case we can gain a lot of insight from the problem of an individual uninformed investor in the completely informed equilibrium. Given the price function $P(\theta, \eta)$, we can define the $\bar{\eta}$ values at which the price for a particular $\bar{\theta}$ matches this marginal price. This value is

$$\bar{\eta}_{\bar{\theta}}(\theta, \eta) : P(\bar{\theta}, \bar{\eta}_{\bar{\theta}}(\theta, \eta)) = P(\theta, \eta)$$

if such a price exists, and 0 otherwise. Again, if $\bar{\theta} = \theta$, then of course $\bar{\eta}_\theta(\theta, \eta) = \eta$.

The Lagrangian for this problem is given by

$$\mathcal{L} = \max_{B^U(\theta, \eta)} \min_{\lambda(\theta, \eta)} \sum_{\theta} f(\theta) \int_0^{\eta_M}$$

$$\left\{ \begin{array}{l} U \left(W - \left[\sum_{\bar{\theta}} \int_0^{\bar{\eta}_{\bar{\theta}}(\theta, \eta)} B^U(\bar{\theta}, \hat{\eta}) P(\bar{\theta}, \hat{\eta}) d\hat{\eta} \right] \right) \tilde{\kappa}(P(\theta, \eta)) \\ U \left(W + \left[\sum_{\bar{\theta}} \int_0^{\bar{\eta}_{\bar{\theta}}(\theta, \eta)} B^U(\bar{\theta}, \hat{\eta}) [1 - P(\bar{\theta}, \hat{\eta})] d\hat{\eta} \right] \right) (1 - \tilde{\kappa}(P(\theta, \eta))) \\ + \lambda(\theta, \eta) [B^U(\theta, \eta) - 0] \end{array} \right\} g(\eta) d\eta.$$

The first-order condition with respect to $B^U(\cdot)$ in state (θ^*, η^*) is

$$\frac{\partial \mathcal{L}}{\partial B^U(\theta^*, \eta^*)} = \sum_{\theta} \int_{\bar{\eta}_{\theta}(\theta^*, \eta^*)}^{\eta_M}$$

$$\left\{ \begin{array}{l} U' \left(W - \left[\sum_{\bar{\theta}} \int_0^{\bar{\eta}_{\bar{\theta}}(\theta, \eta)} B^U(\bar{\theta}, \hat{\eta}) P(\bar{\theta}, \hat{\eta}) d\hat{\eta} \right] \right) \tilde{\kappa}(P(\theta, \eta)) P(\theta^*, \eta^*) \\ U' \left(W + \left[\sum_{\bar{\theta}} \int_0^{\bar{\eta}_{\bar{\theta}}(\theta, \eta)} B^U(\bar{\theta}, \hat{\eta}) [1 - P(\bar{\theta}, \hat{\eta})] d\hat{\eta} \right] [1 - P(\theta, \eta)] \right) (1 - \tilde{\kappa}(P(\theta, \eta))) [1 - P(\theta^*, \eta^*)] \\ + \lambda(\theta, \eta) [B^U(\theta, \eta) - 0] \end{array} \right\} g(\eta) d\eta = 0.$$

Again we are integrating over the set of η 's for which this bid is in the money, $[\bar{\eta}_{\theta}(\theta^*, \eta^*), \eta_M]$, given that the bid is made at price $P(\theta^*, \eta^*)$. The crucial difference to the UP auction is that we are multiplying the marginal utilities by the bid price $P(\theta^*, \eta^*)$ and the bid return $[1 - P(\theta^*, \eta^*)]$ rather than the state-specific marginal prices. This means that we cannot solve this problem recursively even if the short-sale does not bind.

An even more important implication is that the uninformed can never replicate the portfolio of the informed, no matter whether the short-sale constraints binds or not. To see why, consider the case where $P(g, 0) > P(b, 0)$ and an informed investor buys a positive quantity of bonds both at $P(g, 0)$ when $\theta = g$ and at $P(b, 0)$ when $\theta = b$. If the uninformed investor bids a positive amount at $P(g, 0)$, then when the state is $(b, 0)$ he is going to spend $P(g, 0)B^U(g, 0) + P(b, 0)B^U(b, 0)$, while the informed investor will have spent $P(b, 0)B^I(b, 0)$. Thus even if $B^U(g, 0) + B^U(b, 0) = B^I(b, 0)$, and they are both buying the same quantity of bonds, the uninformed is paying more. Then, note that even if $P(g, 0) = P(b, 0)$ the informed investor will want to alter the quantity he bids in response to the quality shock, which the uninformed cannot do in this case since the price is the same. This leads to the following proposition.

Proposition 16. *In a DP auction, the uninformed will never be able to replicate the payoffs and bids of the informed so long as:*

1. $\kappa_g \neq \kappa_b$ and $f(g)$ and $f(b)$ are both positive
2. the informed investor is buying positive amounts for both $\theta = g$ and $\theta = b$ for some values of η .

A second important difference between the auction protocols is the nature of the inference problem solved by an uninformed investor. This can be illustrated by considering a risk-neutral investor.

Remark 17. Consider the special case of a risk-neutral investor and assume that the short-sale constraint does not bind. The UP investor is concerned with the default probability at the marginal price of his bid $(\bar{B}, \bar{P}); \tilde{\kappa}(\bar{P})$. The DP investor is concerned with the default probability of the entire set of states at which his bid (\bar{B}, \bar{P}) is in the money; i.e. $E \{ \tilde{\kappa}(P) | P \leq \bar{P} \}$.

The bid overhang constraint in discriminatory auctions. The bid-overhang constraint is critical for preventing perfect replication in UP auctions. We have already seen that the uninformed can never replicate the informed portfolio in a DP auction. In line with this result, the bid overhang constraint never binds in a DP auction. The reason is that total expenditures are cumulated at the bid price, but not at the marginal price. Total expenditures must therefore be strictly increasing in η no matter the slope of the price schedules, so long as marginal bids are positive. Moreover, marginal bids must be positive by market-clearing.

5 Numerical Example

We now use a numerical example to better understand how our two different types of auctions. In order to sharply illustrate the mechanics of the two auction protocols, and the differences between them, we focus on a stark example in which there is perfect replication in the UP auction when n is sufficiently close to 1. The parameters are as follows.

1. Preferences are log.
2. The default probabilities are determined by $\kappa_g = 0.15$, $\kappa_b = 0.35$ and $f(g) = 0.4$.

3. The demand shock η is uniformly distributed on $[0, 0.3]$
4. The per-capita wealth of lenders is $W = 250$, and the debt to be rolled over is $D = 60$.

5.1 Symmetric Ignorance

We start by computing the uninformed equilibrium for both the UP and the DP auctions. Both the price schedules and the bid schedules are shown in the first panel of figure 1. For both auction protocols, the price declines modestly with the demand shock, because a reduction in the mass of investors forces each investor to hold a higher per-capita share of exposure to the government's debt. This increases the required risk premium. Reflecting this increased exposure, the total quantity of bonds purchased by investors increases in η . More surprising, perhaps, is that equilibrium prices and bids are very similar across the two auction protocols.

There are some slight differences. For low demand shocks, marginal prices are higher in the UP auction. This reflects the fact that investors do not have to worry about overpaying relative to the marginal price. For high values of the demand shock, marginal prices in the DP auction. This reflects the fact that bids submitted at lower demand shocks are executed at higher prices than in the UP auction, so that a larger share of the debt D has already been sold. Consistent with this logic, the total bid schedules cross in a similar manner.

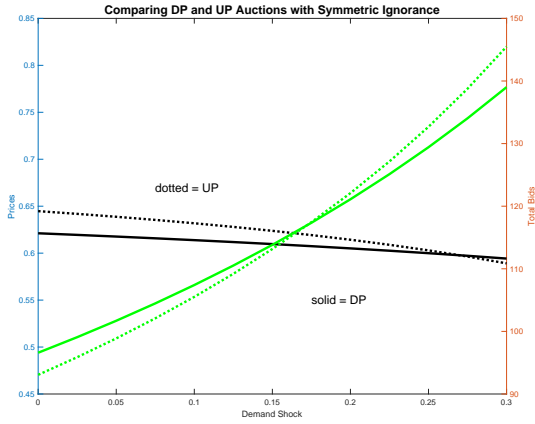
5.2 Symmetric Information

Given that prices are so similar under symmetric ignorance, it is perhaps not surprising that they are also very similar when all investors are informed about the quality shock. The price schedules in the second panel of figure 1 are similar to those in the first panel, except that they are shifted up (and down) in response to a lower (higher) bankruptcy probability.

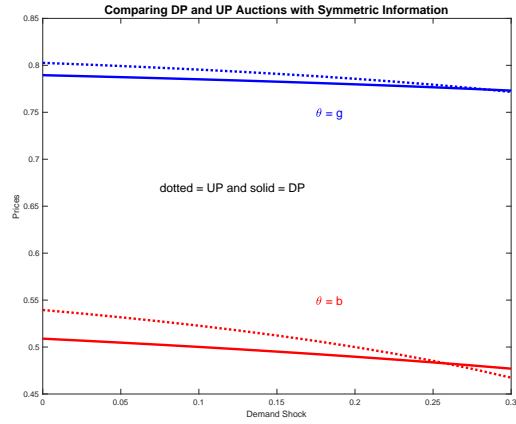
Table 1 reports the average debt burden of the government (computed as the expected total amount that the government has to repay in exchange of raising $D = 60$ to investors

Figure 1: Symmetric Ignorance and Information for UP and DP Auctions

(a) Symmetric Ignorance



(b) Symmetric Information



in case of not defaulting) at the two auctions under our two informational extremes. Here too the UP and DP auctions are very close to each other for each one of the benchmarks.

Table 1: Avg. Debt Burdens by n

# of Informed	UP	DP
$n = 0$	96.7	96.8
$n = 1$	101.3	101.4

5.3 UP Auctions with Asymmetric Information

We now turn to examining what happens in our numerical example when we shrink the number of informed from $n = 1$ to $n = 0$ in a UP auction. The results are shown in figure 2. At first, lowering n has no impact because the debt overhang constraint does not bind and we have perfect replication (see Proposition 13). However, when n falls below η_M , the binding debt overhang constraint forces the price schedules to overlap. Since the constraint first binds when $n = \eta_M$, the initial overlap is such that states (g, η_M) and $(b, 0)$ share a common price. As n falls further, the bid-overhang constraint binds at progressively lower values of the demand shock on the high quality schedule, and

there is a larger overlap. The reason is that the *per-capita demand* that is required for the uninformed's bid to cover the total demand when the demand shock is at its smallest value is lower when the the fraction of uninformed investors is high.

To gain further intuition, we now explain in detail how prices and quantities are determined when the bid-overhang constraint binds. Consider two demand shocks η_g and $\eta_b < \eta_g$ on the high and low price schedule, respectively, and must share a common price P . For P to be an equilibrium, this price needs to clear the market in state (g, η_g) and (b, η_b) . The respective market-clearing conditions are

$$n \left(\frac{1 - \kappa_g - P}{1 - P} \right) + (1 - n) \left(\frac{1 - \tilde{\kappa} - P}{1 - P} \right) = \frac{D}{W} \frac{1}{1 - \eta_g},$$

and

$$n \max \left[\left(\frac{1 - \kappa_b - P}{1 - P} \right), 0 \right] + (1 - n) \left(\frac{1 - \tilde{\kappa} - P}{1 - P} \right) = \frac{D}{W} \frac{1}{1 - \eta_b}.$$

The first term on the left-hand side of these two expressions represents the total expenditures of the informed. Note that we have taken into account the fact that the short sale constraint can bind for the informed investor when $\theta = b$, and in fact will bind when $1 - \kappa_b - P \leq 0$. The reason is that the bond is worse deal in the bad state. The second term on the left-hand side of this expressions represents the total expenditures of the uninformed, which must be independent of the quality shock because the uninformed do not observe θ . Note, however, that the uninformed's expected default probability $\tilde{\kappa}$ is an equilibrium object. In particular, Section 2.3 argued that the inferred default probability at a particular price P is determined by the slope of the price schedule. We can therefore solve for P and $\tilde{\kappa}$ from these two constraints. If we think in terms of a hyperfine grid (that approximates the continuum distribution of η), then the level of $\tilde{\kappa}$ determines the slope of the price function as the ratio of good and bad state points according to

$$\tilde{\kappa} = \frac{(\#g)(1 - a)\kappa_g + (\#b)a\kappa_b}{(\#g)(1 - a) + (\#b)a}.$$

This in turn tells us how fast we need to change η_g and η_b for a given change in our price

schedule. The next value of η_g and η_b along our grid is determined according to

$$\eta'_b = \eta_b + \frac{(\#b)}{(\#g)}\Delta \text{ and } \eta'_g = \eta_g + \Delta.$$

Because the initial price at which our price schedules will overlap is on the high quality schedule, $\tilde{\kappa}$ will have to be close to κ_g to prevent the uninformed from reducing their demands. In fact, when the short-sale constraint binds at this price for the informed it is easy to see that it must in fact be equal to κ_g since the initial binding point occurs when the demands of the uninformed are just sufficient to cover the total supply of D when $\eta = 0$ and $\theta = b$. This means that η_g is changing much faster than η_b as we go down the prices which arise from common states. In figure 2 this can be seen by noting that the slope of the high quality schedule is much flatter than that of the low quality schedule when n is fairly high and the demands of the informed are substantial.

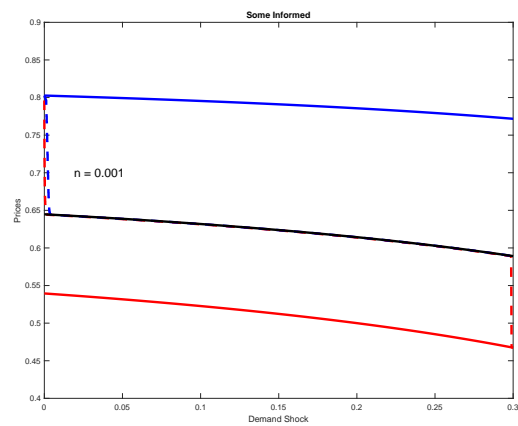
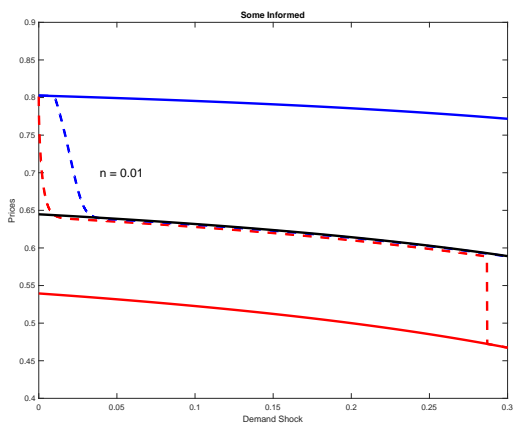
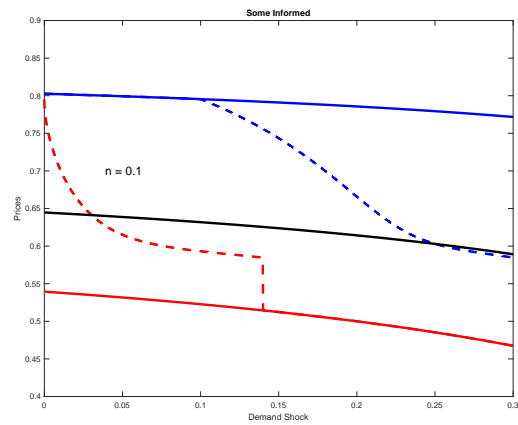
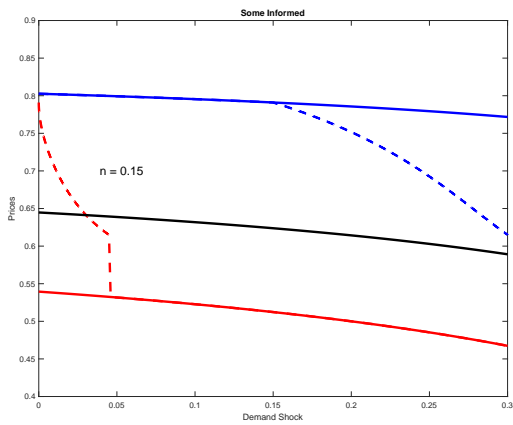
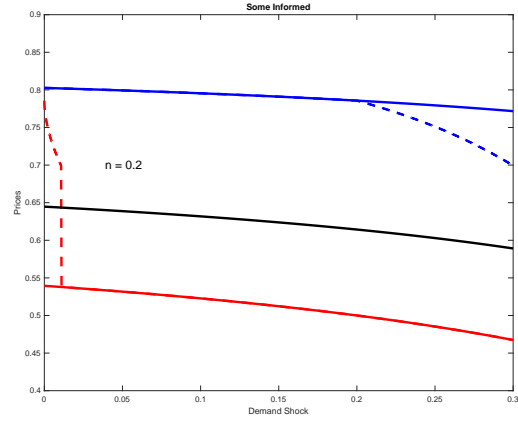
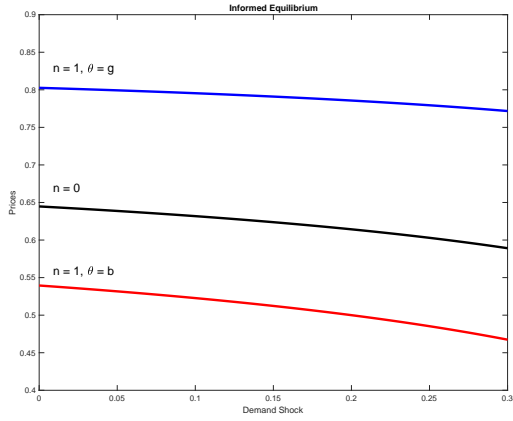
The fact that the prices are falling as the gap between η_g and η_b widens comes from the need to have a greater spread in the total bids of the informed at the two different values of θ . As we can see from the figure, this can even lead to prices that are below those on the uninformed price schedule (i.e. the $n = 0$ equilibrium).

The figure also shows there is more overlap as n shrinks, and that the high and low quality price schedules are converging to the uninformed price schedule all interior η . This is because as n gets small, the gap in the η 's must also get small since there can be less and less variation coming from the total bids of the informed. This in turn leads the η 's varying one-to-one. This implies $\tilde{\kappa} = \kappa^U$ and that the prices lies on the $n = 0$ schedule.

Figure 2: UP Auctions as Information Shrinks

(a) $n > 0.3$. Debt overhang not binding.

(b) Debt Overhang now binding



5.4 DP Auctions with Asymmetric Information

We now turn DP auctions and examine how the equilibrium prices change as we shrink n from 1 to 0. Some examples are plotted in Figure 3. It is immediately clear that equilibrium prices are very different compared to the UP auction. The first important observation is that the low quality-price schedule associated with $\theta = b$ is independent of n if n is close enough to 1, while the high quality-price schedule associated with $\theta = g$ is sensitive to n .

The mechanism underpinning this result is tightly linked to the auction protocol. When n is large, there is a large spread between the high-price schedule and the low-price schedule. Uninformed investors who bid at the high-price schedule thus face an adverse selection problem because their bids are executed even when the bond quality is low. To avoid this issue, uninformed investors do not bid at all on the high-price schedule when n is large enough. This has two effects. First, the uninformed know that their bids on the low-price schedule will only be accepted when $\theta = b$. *Conditional* on buying, they are thus perfectly informed about the quality shock, and they thus choose the same $\theta = b$ portfolio as the informed. This means that the low-quality schedule is the same as in the fully informed equilibrium as long as the informed do not participate on the high-quality schedule. Second, precisely because the uninformed do not participate in the high state, the informed are disproportionately exposed to the government's default risk in that state. As a result, marginal prices must fall in order to deliver an increase in the risk premium. Accordingly, the figure shows a fall in prices as n declines.

This process continues, forcing prices on the high quality schedule lower and lower until at around $n = 0.4$, the high-quality schedule is low enough that the uninformed begin buying on both schedules. At that point, the bids made at high prices on the high quality schedule mean that there is less extra demand that has to be squeezed out when the quality is low. Hence this shift in the bidding of the uninformed both slows the fall in the high quality schedule and raises the prices on the low quality schedule. However, the adverse selection effect is so strong, that as n falls, it forces more and more of the high-quality schedule below the *uninformed* price schedule. The presence of informed

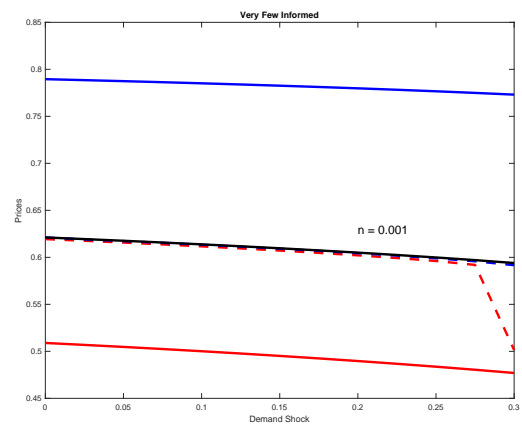
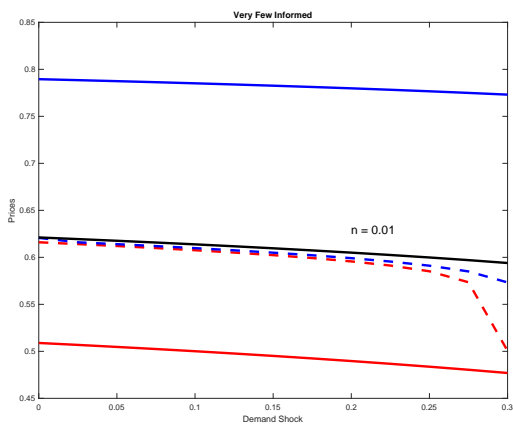
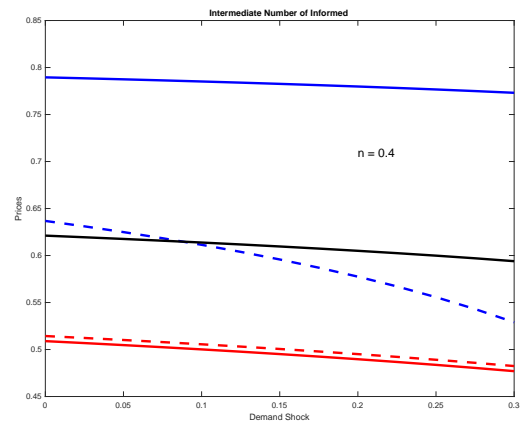
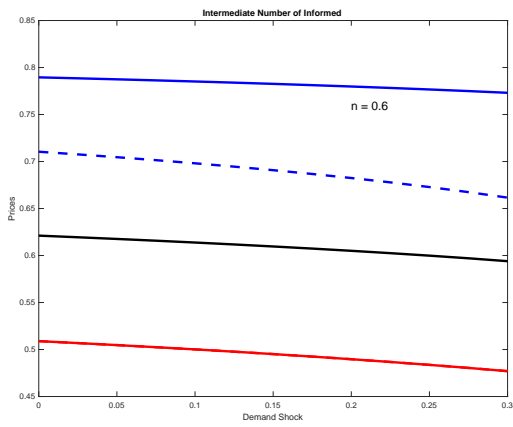
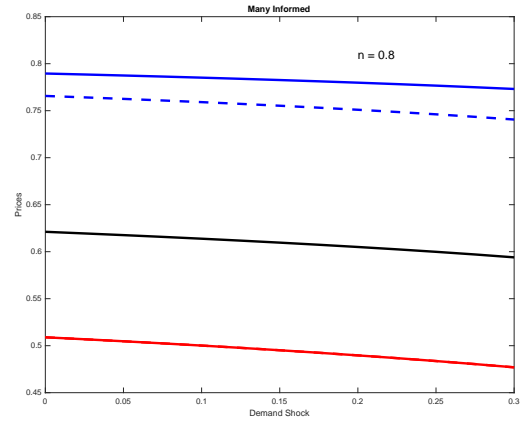
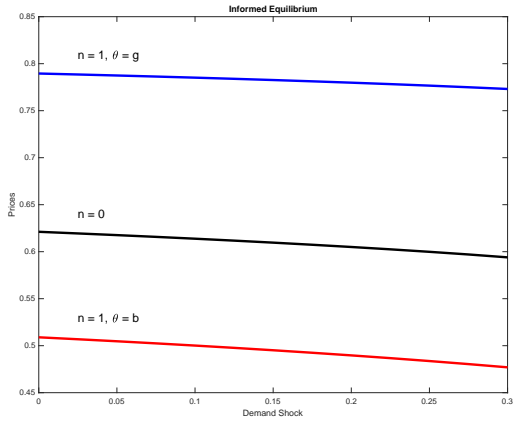
investors may thus lead *all* prices to fall below those obtained in the absence of any information.

As $n \rightarrow 0$ and the uninformed dominate the market, the likelihood of buying on the high and low quality schedules converges everywhere, except at the bottom of the low quality schedule, and hence the prices converge almost everywhere to the $n = 0$ schedule.

Figure 3: DP Auctions as Information Shrinks

(a) $n = 1$

(b) Small fall in n



It is important to realize that, almost by definition, the range of prices at which bids are executed in equilibrium is very different across the two auction protocols. With a UP auction, bids are executed at a unique marginal price given the realized state (θ, η) . Nevertheless, the range of realized prices across can vary quite a bit across states, especially when $n > \eta_M$ and the price schedules are therefore far apart. In contrast, with a DP auction both the range of realized marginal prices and the range of prices at which investors purchase bonds given the realized state may vary a lot. In general, the range of potential prices at which an uninformed investor's bids might be executed is large when (i) n is large enough to generate a substantial spread between the high-quality price schedule and the low-quality price schedule, and (ii) n is small enough to lower the prices on the high-quality schedule enough to induce the uninformed investors to bid on both schedules. In our example this is the case for values of n from 0.4 to 0.5.

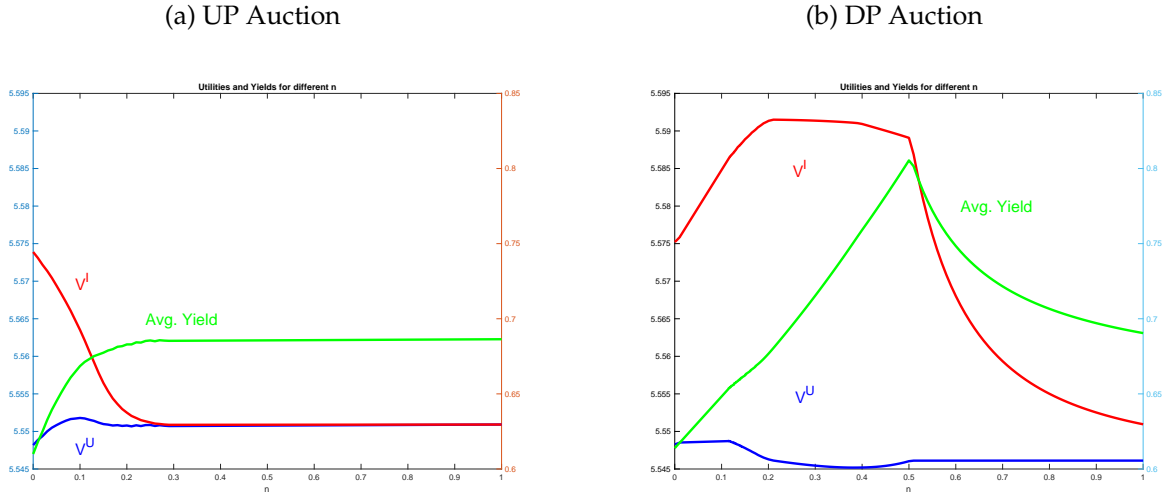
5.5 Payoffs and Yields in UP and DP Auctions

We now want to examine the implications of these two protocols for the ex-ante payoffs to the investors and the government and how these vary with the share of informed investors. For the investors this is straightforward: we simply compute their expected utility for different values of n . We have not specified a payoff function for the government. Assuming risk-neutrality, we will take their payoff to be isomorphic to the average yield on their bonds. The yield is simply the promised return on the bonds that they sell and is given by $(1 - P)/P$. We plot the results for both types of auctions in figure 4.

Not surprisingly, given that the prices and total bids are very similar across auction protocols under symmetric ignorance ($n = 0$) and symmetric information ($n = 1$), the payoffs to the informed, the uninformed and the yield for the government are similar across protocols in these benchmarks. They differ widely across protocols when there is asymmetric information, however.

In the UP auction, the payoff to being uninformed is almost invariant to the degree of information in the market, while the payoff to being informed is monotonically declining in n until it hits the point of complete replication at $n = \eta_M$. From then on, there are

Figure 4: Payoffs to Investors and Bond Yields



no information rents and expected utility is constant for all investors as n increases to 1. The yield on the government debt rises monotonically up until the point of complete replication, and is constant thereafter as the equilibrium becomes invariant w.r.t. n .

Remark 18. *If the bankruptcy probabilities are closer together, then the uninformed will not be able to replicate the total bids of the informed. This occurs for two reasons: (1) the short-sale constraint will bind as the total bids by the informed at $(b, 0)$ become less than those at (g, η_M) , and (2) because the price schedule when $\theta = g$ and b will overlap even at $n = 1$. In this case the gains to being informed will stay positive (but small) for values of $n > \eta_M$ and the price schedules will vary with n even for values close to 1. These statements depend critically on the distribution of η not being degenerate, since in that case, perfect replication is always possible.*

In sharp contrast, both the payoff to the informed and the yield on the government's debt are hump-shaped in the DP protocol. When the fraction of informed investors is low, the adverse selection effect discourages participation by the uninformed and depresses prices both in good states (because the uninformed participate less) and in bad states (because the informed bid less). Because this effect is initially stronger the larger the spread between price schedules, the overall impact on yields is hump-shaped and yields reach their maximum at intermediate levels of n .

Once the fraction of informed is large enough that the uninformed no longer partici-

pate in the high state, the informed earn rents only because they participate in both states. These high-state information rents are gradually competed away as the share of informed investors increases. This cannibalization effect raises prices in the good state and reduces the yield as n approaches 1.

Remark 19. *In contrast to the UP auctions, in DP auctions the gains from being informed are always positive for all values of n , and even when η is degenerate. This is true even when the price schedules do not overlap and completely reveal the state.*

6 Model with Information Acquisition

We now endogenize the share of informed investors by allowing for information acquisition. All investors are initially uninformed. After learning whether they will make it to the auction, investors can learn the true value of θ by paying a *utility cost* of K . An investor will choose to become informed so long as the differential benefit of doing so is weakly positive. Otherwise, everyone will choose to be uninformed. Similarly, an investor will choose to stay uninformed if the benefit from doing so is weakly better than becoming informed, otherwise everyone will be informed. Recall that Equation (1) defined V^U , the expected utility of an uninformed investor, while Equation (2) defined V^I , the expected utility of an informed investor. Given that equilibrium prices are a function of the share of informed n , we write $V^I(n)$ and $V^U(n)$ to highlight this dependence. The optimality conditions determining the equilibrium level of n thus are

$$V^I(n) - K \geq V^U(n) \text{ if } n > 0 \tag{8}$$

$$V^I(n) - K \leq V^U(n) \text{ if } n < 1. \tag{9}$$

Both of these equations must hold simultaneously in an interior equilibrium in which $n \in (0, 1)$. This requires that both conditions hold with equality. We are now ready to define an equilibrium with information acquisition.

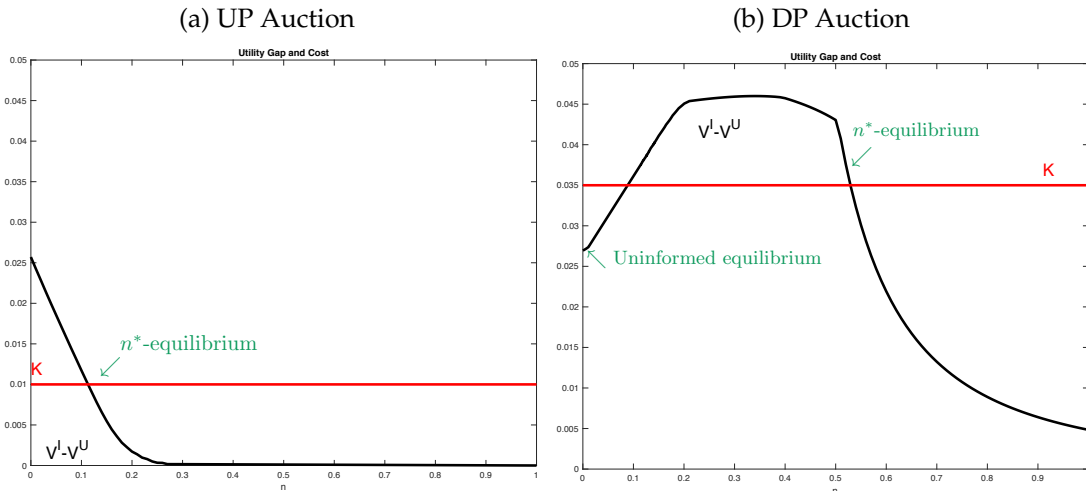
Definition 20. *For both the PD and UP economies, an equilibrium of the model with endogenous information acquisition consists of the measure of informed traders n^* , a price schedule $P(\theta, \eta)$, a*

bid schedule for the uninformed $B^U(\theta, \eta)$, and a conditional bid schedule for the informed $B^I(\theta, \eta)$. The bid schedules must be solutions to the investors' problems given P . The bids and price schedules must satisfy market clearing for all (θ, η) , and n^* must satisfy the information acquisition criterion in (8) and (9).

Utility gap. Naturally, the incentives to acquire information are determined by the utility gap $V^I(n) - V^U(n)$. Figure 4 plots the utility gap for both auction protocols using our numerical example. There are marked differences: the utility gap is strictly decreasing in the UP auction (Panel a), but it is hump-shaped in the DP auction (Panel b).

This has important consequences for equilibrium information acquisition. In particular, there is a unique equilibrium in the UP auction in which the level of information acquisition (i.e. n^*) is decreasing in the utility cost of information K . In the DP auction, instead, it is easy to construct examples in which there are multiple equilibria for the same parameterization of our model. These multiple equilibria will include: (i) a stable equilibrium in which there is no information acquisition, (ii) an unstable equilibrium in which there is a small amount of information acquisition, and (iii) a stable equilibrium in which there is a large amount of information acquisition.

Figure 5: Equilibrium with Information Acquisition



Notice that the incentives to acquire information are larger in DP auctions for all levels of n . The differential gains from acquiring information are almost identical in the uninformed equilibrium. To see this recall that the price schedules and the bids in the

uninformed equilibrium are very similar in UP and DP auctions, and then the utility of the uninformed investors are also very similar. The incentives to become informed in this equilibrium comes from adjusting bids to the equilibrium prices more accurately once observing the realized probability of default. This is the reason the gains from becoming informed is almost the same (0.026 in the figures) initially. For a slightly positive fraction of informed investors, however, this differential gains from information increase for DP auctions and decrease in UP auctions, reaching zero in UP auctions (for $n \geq \eta_M$) and being always positive for DP auctions (as uninformed investors can never replicate the informed portfolio).

This comparison has striking implications for the existence of asymmetric information under these two auction protocols. When the cost of information acquisition K is large (above 0.046 in the numerical illustration), only symmetric ignorance is feasible in both auctions. When K is intermediate (higher than 0.026 but lower than 0.046 in the numerical illustration) symmetric ignorance is also an equilibrium under both protocols, but for the DP auction there is also an equilibrium with asymmetric information. When K is small (below 0.026) only equilibria with asymmetric information is feasible in both auctions but with the fraction of informed agents lower for the UP auction. Finally, when K is very small (below 0.005) only asymmetric information is sustainable in UP auctions and only symmetric information is sustainable in DP auctions.

7 Discussion of Relationship to Literature

We have already discussed our relationship to the sovereign debt literature. In this section we want to discuss the relationship between our work and to other several important branches of the literature. The first concerns the foundations of general equilibrium theory (GE) and the question of "where do prices and the information in them come from?" The second is auction theory when there are a large number of bidders for a perfectly divisible good with uncertain common value.

With respect to the question "where do prices come from?" the price vector in GE is an endogenous object that is not chosen by anyone, yet determined by the accumulated

actions of individuals who cannot affect prices. To get around the conundrum, Walras made up his fictional "auctioneer" that matches total supply and total demand in a market of perfect competition (perfect information and no transaction costs). But this has long been considered a thought experiment that did not adequately address the issue; see Hahn (1989).

One response to the price problem has been the market games literature, which seeks to provide a fuller description of the environment and in which all endogenous objects are selected by the agents (including prices) based upon noncooperative game theory.⁸ Examples of this market game approach include Rubinstein and Wolinsky (1985)'s sequential bargaining model in which buyers and sellers are paired up under complete information each period.⁹

This problem is more severe when the prices are simultaneously clearing the market and aggregating information as in Lucas (1972) and Grossman and Stiglitz (1980). This is because the market "needs to know" the realized demand to internalize the realized shock(s) in the individual demands. At the same time, the agents "need to know" both the price function *and* the realized price in order to make their inferences and determine their demands. Dubey, Geanakoplos, and Shubik (1987) consider the Nash equilibrium of a sequential trading game with incomplete information where traders make quantity offers to buy and sell and the price is determined by the ratio of the total buy versus sell offers. Here information revelation occurs largely in one-step through the vector of different prices for the different goods.

The question of "where does the information in prices come from?" has also sparked a large debate and is seen most starkly when the price system is invertible. As Grossman and Stiglitz (1976) pointed out, agents have no incentive to look at their private information since all of the information is already encapsulated in the price. As their quantity choice do not reflect their private information, then whose information gets aggregated

⁸See Gale (2000) for a survey of this literature. Another response is a cooperative approach which is able to rationalize competitive outcomes in the limit using a variety of solution concepts. However, as Gale (2000) points out, there is something unconvincing about a strategic underpinning that does not clearly specify agent's strategies, their potential moves, their expectations about others actions and where they came from, or even the maximization problem they solve.

⁹Gale (1987) shows that these sequential bargaining models converge to a common price equilibrium as the number of agents gets large.

into prices? This problem manifests as a nonexistence problem if acquiring the information is costly; the Grossman-Stiglitz paradox. If information is costly and prices are fully revealing, no individual wants to acquire information. However, if no agent gathers information, prices cannot be fully revealing. Fully revealing information prices are logically impossible. Grossman and Stiglitz (1980) added a second source of noise to prevent the price system being invertible.

There is a second, related, implementability problem, as it may not be possible to find a trading mechanism that induces a fully revealing equilibrium. Jackson (1991) proposes an alternative resolution when the number of agents is finite and we drop the price-taking assumption. As the agents internalize that the extent of information in prices depends upon the demand schedule they submit, there is a fully revealing equilibrium when information is costly. A very different and decentralized approach is Golosov, Lorenzoni, and Tsyvinski (2014), which features a sequence of bilateral meetings with take-it-or-leave-it offers. Here information revelation occurs in one-step for meetings between informed and uninformed agents when the informed agent is chosen to make the offer, but trading is party specific and the overall dissemination of information is gradual.¹⁰ Finally, Vives (2014) and Gaballo and Ordonez (2017) propose settings with large centralized markets in which the valuation of each trader has both common and private value components, and the costly signal bundles information about these two components, such that prices can be fully revealing and yet there are incentives to acquire information.

Our paper speaks to both of these problematic aspects of GE by using the structure of an auction to answer where prices come from and by obtaining the conditions for informational gains in two different auction protocols to answer how information gets into prices. In particular, our model features a specific order of moves. First, investors submit their bids (where each bid is a price-quantity pair). Second, a specific protocol is used to select the bids which are accepted and the prices at which they are executed. Information revelation occurs after the marginal price at the auction is revealed. This information revelation may be complete, as in REE. Under the UP protocol, when information rev-

¹⁰A related contribution is Albagli, Hellwig, and Tsyvinski (2014) who develop a dynamic REE with dispersed information in which information enters nonlinearly into prices.

elation is complete and the short-sales constraint does not bind, then there are no gains from being informed. However, when this is not true there are, which is a departure from Grossman and Stiglitz (1980) where by assumption there are no short-sale constraints. In DP auctions there are always gains from being informed with distinct quality shocks to the bond. For both types of auction protocols we can endogenize the acquisition of costly information and provide conditions under which prices can be fully revealing. Finally, we do this while retaining the price-taking assumption, as we assume that there are a continuum of investors.

A key aspect of the auction approach is that investors commit to their bid schedules before knowing the realized price. Hence, in the language of GE, “out-of-equilibrium trades take place”, in the sense that investors would choose to revise their bids if they knew the realized price. This is particularly apparent with DP auctions since bids are being accepted at multiple prices simultaneously.

A related paper which takes a similar auction-based approach to micro found REE is Milgrom (1981). He considers an auction in which n ex ante bidders bid on a k identical goods where $n > k \geq 1$. The bidders can acquire at most one unit of the good and are heterogeneously informed about its value by a signal about the value of good. They find it optimal to bid their valuation because the k th high price bidders win the object paying the $k + 1$ highest price. In this sense they are price-takers. In addition, the winning price does not generally convey all of the bidders’ private information since it is akin to an order statistic because bidders cannot express their valuations by seeking to acquire more than one unit at the same time. Also, the information in the price is not clouded by the presence of a demand (or supply) side shock such as Grossman and Stiglitz (1980). Our paper can be seen as relaxing both of these aspects.

Our paper also contributes technically to the auction literature by going to the extreme of a continuum of bidders who can to submit combinations of quantities and prices while understanding that their individual bids will not have any impact on the equilibrium prices, and that they can buy whatever amount they wish at the operating price. More practically, however, our work is related to recent empirical efforts to evaluate the implications (for efficiency, government revenues, investors profits, etc) of the two trea-

surety bond auction protocols that we consider. Hortaçsu and McAdams (2010) construct a structural discriminatory price auction model (multiunit auctions of Q indivisible units with N potential bidders) and apply it to the Turkish Treasury bonds to estimate bidders' marginal values and to compare its revenues with a counterfactual uniform-price protocol. Kastl (2011) uses Czech Treasury bond auctions, which follow a uniform-price protocol, to show the empirical relevance of discrete bidding in multiunit auctions.

Much of the discussion about the optimal protocol to auction sovereign bonds, their implications for government revenues, and its empirical implementation has however focused on two modeling choices. One constitute the backbone of the auction literature: the selling of a single object to bidders with independent private values.¹¹ These assumptions are good characterizations for consumer goods, for which it is plausible that individual bidders value the good quite differently, but not for treasury bonds. A second modeling choice, closer to the characteristics of treasury bonds is the analysis of auctioning a single indivisible good with correlated values (seminal papers are Milgrom and Weber (1982) and McAfee and McMillan (1987)). This rich literature focuses on solving a Nash equilibrium where the strategies are given by the price that each bidder submits for the single indivisible good as a function of the own valuation, information and history of the auction.

To capture goods such as treasury bonds, however, which are highly divisible, with a common uncertain quality and that have a large number of buyers, the literature extended these models to multi-unit auctions, in which many units of an identical good are auctioned to a finite set of bidders. This literature has faced large challenges as it involves bidders that face a large strategic problem when submitting a combination of quantities - prices and then a more complex solution of the corresponding equilibrium. Examples include Engelbrecht-Wiggans and Kahn (1998), Perry and Reny (1999), Kagel and Levin (2001) and McAdams (2006).

In this paper we depart from the bulk of this literature by going to the extreme on the number of bidders, assuming a continuum of investors that submit combinations of

¹¹Seminal papers are Vickrey (1962), Harris and Raviv (1981), Myerson (1979) and Maskin and Riley (1985).

quantities and prices understanding that their individual bids will not have any impact on the equilibrium prices, and that they can buy whatever amount they wish at the operating price. This rationalizes our approach of a Walrasian auction in which each bidder can compute the equilibrium marginal prices in each state and does not have incentives to bid at any other price. This approach makes the analysis specially tractable as it implies choosing bids in each possible state, allowing us to study the effects of asymmetric information for prices and the incentives to acquire information both in discriminatory price and uniform price auctions.

8 Conclusion

The model that we develop is applicable to a number of other important circumstances, including auctions of liquidity infusion by central banks, electricity, emission permits, gas, oil, and mineral rights. The key requirement is that the auction involves a "thick" enough market for a homogenous divisible good of uncertain quality so as to make the price-taking assumption a close approximation to reality. Our model also provides a potential mechanism to micro found competitive equilibria for the case of the uniform-price protocol and to break the circularity inherent in having prices and quantities determined simultaneously.

Our paper contributions to the wide discussion, dating back at least to Friedman (1960), of whether sovereign debt auctions should be conducted with a uniform-price or a price-discriminating protocol.¹² Our results strongly suggest that, if information does not have any benefit in terms of reallocation of resources or improvements in the efficiency of decisions, and just redistribute gains between informed and uninformed investors, then it would be optimal to conduct a uniform-price auction that discourages information. In contrast, if the generation of information about the quality of bonds is relevant for decision making, then it may be optimal to conduct discriminatory-price auctions that

¹²Friedman proposed (pp 64-65) that the U.S. Treasury abandons its previous price-discriminating practice and make all awards at the stopout price instead of at differing prices down through that price. The U.S. Treasury finally adopted this uniform-price protocol for all auctions of 2-year and 5-year notes on September 3, 1992. An excellent summary of this discussion is Chari and Weber (1992). Earliest discussions about Friedman's proposal include Goldstein (1962), Friedman (1963), Rieber (1964) and Friedman (1964).

encourage information acquisition, at the cost of higher levels, dispersion and volatility of bond prices.

In a follow-up paper (Cole, Neuhann, and Ordóñez (2016)) we examine the implications of discriminatory-price auctions within a two-country setting. We use the insights developed here to discuss spillovers of information across countries and the role of secondary markets. We show that the sources of complementarities inherent to discriminatory-price auctions extend from cross-states to cross-bonds and make debt crises contagious even in the absence of other linkages.

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