PIER Working Paper 17-008

Safe Assets

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http://ssrn.com/abstract=2968481

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May 10, 2017

Abstract

A safe asset's real value is insulated from shocks, including declines in GDP from rare macroeconomic disasters. However, in a Lucas-tree world, the aggregate risk is given by the process for GDP and cannot be altered by the creation of safe assets. Therefore, in the equilibrium of a representative-agent version of this economy, the quantity of safe assets will be nil. With heterogeneity in coefficients of relative risk aversion, safe assets can take the form of private bond issues from low-risk-aversion to high-risk-aversion agents. The model assumes Epstein-Zin/Weil preferences with common values of the intertemporal elasticity of substitution and the rate of time preference. The model achieves stationarity by allowing for random shifts in coefficients of relative risk aversion. We derive the equilibrium values of the ratio of safe to total assets, the shares of each agent in equity ownership and wealth, and the rates of return on safe and risky assets. In a baseline case, the steady-state risk-free rate is 1.0% per year, the unlevered equity premium is 4.2%, and the quantity of safe assets ranges up to 15% of economy-wide assets (comprising the capitalized value of GDP). A disaster shock leads to an extended period in which the share of wealth held by the low-risk-averse agent and the risk-free rate are low but rising, and the ratio of safe to total assets is high but falling. In the baseline model, Ricardian Equivalence holds in that added government bonds have no effect on rates of return and the net quantity of safe assets. Surprisingly, the crowding-out coefficient for private bonds with respect to public bonds is not 0 or -1 but around -0.5, a value found in some existing empirical studies.

1 Introduction

In a Lucas-tree world (Lucas, 1978), the aggregate risk reflects the uncertainty in the process for GDP, which corresponds to the fruit that drops from the tree. This process may include rare macroeconomic disasters, which correspond to sharp and possibly permanent drops in the productivity or number of the trees. A safe asset in this world can be viewed as one whose real value is insulated from shocks, including the declines in GDP due to the rare disasters. However, if the GDP process is given, safe assets cannot mitigate overall risk but can only redistribute this risk across agents. In a representative-agent setting, the redistribution of aggregate risk cannot occur, and the economy's equilibrium quantity of safe assets will be nil.

To put this observation another way, it is possible to construct safe assets by issuing risk-free private bonds, by creating a financial structure with risk-free tranches, by entering into a variety of insurance contracts, and so on. However, the creation of any of these safe assets always goes along with a corresponding expansion in the riskiness of (levered) claims on the underlying asset, which is the Lucas tree. In equilibrium, the representative agent ends up holding the representative share of the overall risk, and this overall magnitude is unaffected by the various financial arrangements. The bottom line is that a meaningful analysis of safe assets requires heterogeneity across agents.

Differences in the degree of risk aversion are a natural form of heterogeneity for a study of safe assets. The present analysis relies on these differences in risk aversion and uses the simplest possible setup, where there are two types of agents. Type 1 has comparatively low risk aversion and type 2 has comparatively high risk aversion. Specifically, an agent of type i has a constant coefficient of relative risk aversion γ_i, where we assume $0 < \gamma_1 \leq \gamma_2$, so that agent type 1 is at least as willing as agent type 2 to absorb risk.

We focus on a model in which the desire to redistribute risk across agents is the source of safe private assets. In equilibrium, the representative agent with relatively low risk aversion, type 1, issues safe bonds (or equivalent claims) that are held by the representative agent with relatively high risk aversion, type 2. Correspondingly, agent 1 owns a disproportionate share of risky assets, which are equity claims on the Lucas tree. The quantity of safe assets in this economy equals the magnitude of the bonds issued by agent 1 and held by agent 2. The equilibrium amount of these assets depends on the differences in risk aversion across the agents, the levels of risk-aversion coefficients, the characteristics of the stochastic process (including rare disasters) that generate aggregate GDP, and some other parameters.

The equilibrium requires an enforcement mechanism for repayments of safe claims; that is, agent 1 has to make payments of principal and interest to agent 2 even in bad states of the world, such as realizations of macroeconomic disasters. Repayment mechanisms may involve

collateral, liquidity, and contractual features related to the legal system. However, these mechanisms are not the subject of the present analysis, which focuses on the underlying supply of and demand for safe private assets. Potentially complementary research that emphasizes liquidity, collateral, and asymmetric information includes Holmström and Tirole (1998) and Gorton and Ordoñez (2013).

A pure claim on the Lucas tree corresponds to unlevered equity. A match with the empirically observed high equity premium requires the expected rate of return on this equity to be substantially higher than the risk-free rate, which equals the rate of return on non-contingent, private bonds. Previous analyses with rare-disaster models, summarized in Barro and Ursúa (2012), found that the replication of this high equity premium requires first, a coefficient of relative risk aversion, γ, around 3-4 (for a representative agent) and, second, the presence of fat-tailed uncertainty, such as a non-negligible potential for drops in GDP in the short run by more than 10%. The present analysis incorporates these features.

With the familiar specification where utility is time separable with a power form, a coefficient of relative risk aversion, γ, of 3-4 implies an intertemporal elasticity of substitution (IES) of $1/3 - 1/4$, which seems unrealistically low.[1] Specifically, the high γ needed to generate a realistic equity premium precludes the case of log utility in the sense of IES=1. More generally, in the standard utility formulation, it is impossible for all agents to have the same IES, such as IES=1, along with coefficients of relative risk aversion that differ across agents.

As is well known, the Epstein-Zin/Weil (henceforth EZW) form of recursive utility, based on Epstein and Zin (1989) and Weil (1990), allows for a separation between the coefficient of relative risk aversion and the IES. Typically, this benefit from EZW comes at the cost of analytical complexity, when compared with time-separable power utility. However, with heterogeneity in risk-aversion coefficients, the EZW specification allows for a simpler analysis. The key property of EZW is that it allows for high values of the γ_i coefficients that can differ across agents i, while maintaining values of the IES that are of reasonable magnitude and the same for each agent. The rate of time preference, ρ, is also assumed to be the same for each agent.

Previous models of asset pricing with two types of agents distinguished by their coefficients of relative risk aversion include Dumas (1989); Wang (1996); Chan and Kogan (2002); Garleanu and Pedersen (2011); Longstaff and Wang (2012); Gennaioli, Shleifer, and Vishny (2012); Caballero and Farhi (2014); and Gârleanu and Panageas (2015). Except for the last reference, these analyses assume time-separable power utility, augmented in Chan and Kogan (2002) to include an external habit in household utility.

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[1]For example, the well-identified estimation in Gruber (2013) estimates an IES around 2. Bansal and Yaron (2004) and Barro (2009) argue that an IES below 1 produces puzzling patterns in the relation of growth rates and uncertainty to ratios of stock prices to earnings.

In Wang (1996), one agent has log utility and the other has square-root utility—coefficients of relative risk aversion of 1 and 0.5, respectively. In Garleanu and Pedersen (2011), one agent has log utility and the other has a coefficient of relative risk aversion greater than one. In the main analysis of Longstaff and Wang (2012), one agent has log utility and the other has squared utility—coefficients of relative risk aversion of one and two, respectively. Gennaioli, Shleifer, and Vishny (2012) assume that one agent is risk neutral and the other has infinite risk aversion, and Caballero and Farhi (2014) use an analogous setup. Gârleanu and Panageas (2015) allow for two agents with Epstein-Zin utility, so that coefficients of relative risk aversion are not constrained to equal corresponding reciprocals of intertemporal elasticities of substitution. However, in practice, they focus on cases in which each agent's IES is close to the reciprocal of its coefficient of relative risk aversion. This specification means that risk-aversion coefficients well above one are constrained to associate with IES values well below one. Hall (2016) uses the heterogeneous risk-aversion framework developed in a working-paper version of our study, Barro and Mollerus (2014), to analyze the evolution of safe real interest rates.

Section 2 works out a baseline model that derives equilibrium holdings of equity claims and private bonds by the two types of agents, distinguished by their degrees of relative risk aversion. We assume Epstein-Zin utility, so that intertemporal elasticities of substitution for consumption need not correspond to reciprocals of the coefficients of relative risk aversion. For heuristic purposes, we begin with an approximate solution to a tractable case where utility is logarithmic so that the IES for both agents equals one. We show subsequently with more powerful but less transparent numerical techniques that the results generalize to settings without log utility.

The initial model with permanent differences in coefficients of relative risk aversion is non-stationary because, in the long run, the wealth share of the group with comparatively low risk aversion tends to approach one. Our main approach for achieving stationarity is to assume that agents are continually replaced by new agents (possibly children) who are randomly assigned one of the two possible coefficients of relative risk aversion. We focus on a metaphor in which infinite-lived agents randomly experience changes in coefficients of relative risk aversion but where these changes do not have "wealth effects" (so that the potential for these future changes does not influence earlier plans). Because of this churning of types, the economy has a steady state in which the mean wealth shares of each agent are interior in the sense of being between 0 and 1. We also note that analogous steady-state results can be attained by introducing a system of public finance with lump-sum transfers financed by a graduated-rate income tax. With this graduation (but not otherwise), the economy can again have an interior steady state.

Section 3 carries out quantitative analyses based on specifications of the underlying parameters, including coefficients of relative risk aversion and the characteristics of the macro-disaster process. We focus on parameters that generate "reasonable" steady-state values of the risk-free interest rate (around 1.0% per year) and the unlevered equity premium (around 4.2%). When the gross replacement rate is 2% per year, the steady-state ratio of safe to total assets ranges up to 15%. We carry out dynamic analyses for two cases: the realization of a macroeconomic disaster and the experience of tranquility (no disasters) for 40 years.

Section 4 distinguishes further the gross quantity of private bonds from the net quantity corresponding to loans from group 2 to group 1. With infinitesimal transaction costs for paying interest and principal payments on bonds, agents in our model will not be simultaneously holding and issuing bonds. This condition pins down the equilibrium gross amount of safe assets. We then introduce public debt. Added government bonds create more safe assets while simultaneously creating corresponding "safe liabilities" in the form of the present value of taxes. In the baseline setting, where the government and private sector are equally good at creating safe assets, Ricardian Equivalence holds, in the sense that changes in the quantity of government bonds do not affect rates of return and the net quantity of safe assets. More surprisingly, the model predicts that an increase in government bonds by 1 unit crowds out private bonds by around 0.5 units. This prediction accords with some existing empirical evidence.

Section 5 relates the model's predictions on the quantity of safe assets to empirical estimates of this quantity. We argue that the model accords with the observed stability of ratios of safe to total assets. And we argue further that reasonable modifications of the model make it consistent with estimates of safe assets at 30-35% of total assets.

Section 6 concludes with suggestions for future research.

2 Baseline Model

2.1 Structure and First-Order Conditions

The model is set up for convenience in discrete time, where we think of the period as short. Agent i, for $i = 1, 2$, has an Epstein and Zin (1989) / Weil (1990) utility function, given by:

$$U_{i,t} = \left\{ \left(\frac{\rho}{1+\rho} \right) C_{i,t}^{1-\theta} + \left(\frac{1}{1+\rho} \right) \left[\mathbb{E}_t(U_{i,t+1}^{1-\gamma_i}) \right]^{(1-\theta)/(1-\gamma_i)} \right\}^{1/(1-\theta)} \tag{1}$$

The coefficients of relative risk aversion satisfy $0 < \gamma_1 \leq \gamma_2$; that is, agent 1 is the comparatively low-risk-aversion agent. The IES, $1/\theta > 0$, and the rate of time preference, $\rho > 0$, are the same for the two agents. We simplify initially by assuming $\theta = 1$ (log utility).

In the representative-agent case, this specification implies that the consumption of each agent, C_1 and C_2, equals ρ multiplied by a measure of each agent's assets. It also follows here that the price of equity is independent of parameters that govern expected growth and uncertainty.[2]

Parts of the structure parallel Longstaff and Wang (2012). The single Lucas tree generates real GDP of Y^t in period t. This GDP is consumed by the two agents:

$$C_{1,t} + C_{2,t} = Y_t \tag{2}$$

Ownership of the tree is given by $K_{1,t}$ and $K_{2,t}$, which add to full ownership, normalized to one:

$$K_{1,t} + K_{2,t} = 1 \tag{3}$$

We use a convention whereby $K_{i,t}$ applies at the end of period t, after the payment of the dividend, $K_{i,t-1} \cdot Y_t$, to agent i in period t. This timing convention is unimportant when the length of the period is short. The price of the tree in period t in units of consumables is P_t.

The stochastic process that generates Y_t corresponds to previous rare-disaster models, except for the omission of a normally-distributed business-cycle shock, which is quantitatively unimportant. The probability of a disaster is the constant p per period. With probability $1-p$, real GDP grows over one period by the factor $1+g$, where $g \geq 0$ is constant. With probability p, a disaster occurs and real GDP grows over one period by the factor $(1+g) \cdot (1-b)$, where $b > 0$ is the size of a disaster. In the present simplified setting, disasters last for only one "period" and have a single size. The expected growth rate per year of GDP, denoted g^*, is given when the length of the period is short by

$$g^* \approx g - pb, \tag{4}$$

where g and p are measured per year.

The analysis can be extended to allow for a time-invariant size distribution of disasters, as in Barro and Ursúa (2012). With more complexity, the analysis can be modified to allow disasters to have stochastic duration and cumulative size and to be followed by a tendency for recovery in the sense of above-normal growth rates (Nakamura, Steinsson, Barro, and Ursúa, 2013 and Barro and Jin, 2016).[3] Other feasible extensions include time variation in the

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[2]This result means that the expected rate of return on equity, r^e, is independent of uncertainty parameters. Therefore, with $\theta = 1$, all of the effects of uncertainty parameters on the equity premium work through the risk-free rate, r^f, rather than r^e. We know from previous analyses of this i.i.d. setting with a representative agent, such as Barro (2009), that the equity premium is independent of the parameter θ. Therefore, in this context, the setting of $\theta = 1$ would not affect the model's implications for the equity premium.

[3]The recovery tendency lowers the effective size, b, of a disaster. Therefore, for some purposes, we could allow for recoveries within the present framework by adjusting b.

disaster probability, p, as in Gabaix (2012), and the growth-rate parameter, g (as in Bansal and Yaron, 2004). The baseline calibration specifies $p = 0.04$ per year. This probability corresponds to the empirical frequency of disasters—defined as short-term declines in real per capita GDP by at least 10%—in a long-term panel of countries. The effective disaster size—in the sense of the single value in the representative-agent economy that generates an equity premium corresponding roughly to the full size distribution of disasters—is set at $b = 0.32$. The growth-rate parameter, intended to correspond to the non-disaster mean growth rate of real per capita GDP or consumption, is set at $g = 0.025$ per year.

The analysis assumes that agents can deal in two types of assets. The first type is an unlevered equity claim, $K_{i,t}$, on the tree. Individual agents are allowed to go short on this claim. We also consider a non-contingent, one-period private bond, $B_{i,t}$. The quantity $B_{i,t}$ is negative for a borrower (issuer of a bond) and positive for a lender (holder of a bond). Since the analysis assumes a closed economy, the total quantity of these private bonds, when added up across the two types of agents, is always zero:

$$B_{1,t} + B_{2,t} = 0. \tag{5}$$

The analysis ignores possibilities of default and neglects transaction costs associated with interest and principal payments; that is, with collecting on loans.[4] In this case, bonds pay the risk-free interest rate for period t, denoted r_t^f. The amount of principal and interest received or paid on bonds by agent i in period t is $(1 + r_t^f)B_{i,t-1}$.

In the baseline case with a single disaster size and no normally-distributed shock, the two forms of assets are sufficient to generate a complete-markets solution. More generally, a wider array of assets would be needed to span the possible outcomes.

Each agent's budget constraint for period t is:

$$C_{i,t} + P_t K_{i,t} + B_{i,t} = (Y_t + P_t)K_{i,t-1} + (1 + r_t^f)B_{i,t-1}. \tag{6}$$

The choice for period t of $C_{i,t}$ and the portfolio allocation, $(K_{i,t}, B_{i,t})$, occur when Y_t, P_t, and r_{t+1}^f are known but Y_{t+1} and P_{t+1} are unknown.

Let R_{t+1} represent the gross return on an asset between periods t and $t + 1$. This return equals $(Y_{t+1} + P_{t+1})/P_t$ for equity and $(1 + r_{t+1}^f)$ for bonds. Each agent seeks to maximize expected utility, given in equation (1), subject to the budget constraint in equation (6) and the levels of initial assets. The first-order optimization conditions for each agent can be expressed by means of a perturbation argument for periods t and $t + 1$ as:

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[4]In the cases considered, debtors always have sufficient assets to make the prescribed principal and interest payments on bonds.

$$\left[\mathbb{E}_t\left(U_{i,t+1}^{1-\gamma_i}\right)\right]^{\left(\frac{\theta-\gamma_i}{1-\gamma_i}\right)} = (\frac{1}{1+\rho})\mathbb{E}_t\left[U_{i,t+1}^{\theta-\gamma_i} * \left(\frac{C_{i,t+1}}{C_{i,t}}\right)^{-\theta} * R_{t+1}\right]. \tag{7}$$

This expression simplifies in straightforward ways under log utility, $\theta = 1$.

Previous analyses (Giovannini and Weil, 1989, Obstfeld, 1994, and Barro, 2009) showed that, in a representative-agent model with Epstein-Zin/Weil preferences and i.i.d. shocks, the measure of utility, $U_{t+1}^{1-\gamma}$, can be expressed as a positive constant multiplying $(C_{t+1})^{1-\gamma}$. This result suggests looking for an approximate solution to the present two-agent model in which $U_{i,t+1}^{1-\gamma}$ is a positive constant (different for each agent) multiplying the analogous object for agent i, $(C_{i,t+1})^{1-\gamma_i}$. This condition implies that equation (7) can be rewritten as

$$\left[\mathbb{E}_t\left(\frac{C_{i,t+1}}{C_{it}}\right)^{1-\gamma_i}\right]^{\left(\frac{\theta-\gamma_i}{1-\gamma_i}\right)} \approx \left(\frac{1}{1+\rho}\right)\mathbb{E}_t\left(\left(\frac{C_{i,t+1}}{C_{it}}\right)^{-\gamma_i} R_{t+1}\right). \tag{8}$$

When R_{t+1} equals the risk-free return, $1+r_{t+1}^f$, and $\theta = 1$, equation (8) implies

$$1 + r_{t+1}^f \approx (1+\rho)\frac{\mathbb{E}_t(\frac{C_{i,t+1}}{C_{it}})^{(1-\gamma_i)}}{\mathbb{E}_t(\frac{C_{i,t+1}}{C_{it}})^{-\gamma_i}}. \tag{9}$$

Thus, a key implication of the first-order conditions is that, in equilibrium, the right-hand side of equation (9) has to be the same for each agent; that is, for γ_1 and γ_2, respectively. This correspondence implies that the prospective paths of uncertain consumption levels for the two agents have to accord with the differences in the coefficients of relative risk aversion.

Under log utility, $\theta = 1$, each agent's consumption in period t is approximately the multiple ρ of that agent's resources for period t:[5]

$$C_{it} \approx \rho * [(Y_t + P_t) * K_{i,t-1} + \left(1 + r_t^f\right) B_{i,t-1}]. \tag{10}$$

Equations (10) and (9) jointly determine agent i's choices of consumption, $C_{i,t}$, and portfolio allocation, $(K_{i,t}, B_{i,t})$.

Adding up equation (10) for the two agents and using the conditions from equations (2), (3), and (5)—total consumption equals GDP, equity holdings add to one, and bond holdings add to zero—leads to

$$P_t \approx Y_t \cdot (1 - \rho)/\rho \approx Y_t/\rho, \tag{11}$$

where the second approximation assumes that the length of the period is negligible. Thus,

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[5]This condition holds exactly in a representative-agent economy, as discussed by Giovannini and Weil (1989, Appendix B).

under log utility, the equity price and, hence, the value of total assets is independent of parameters related to expected growth and uncertainty and the degree of risk aversion. This result implies that the expected rate of return on equity, r^e, is the dividend yield, ρ, plus the expected rate of capital gain, which equals g^*, the expected growth rate of GDP and consumption:

$$r^e \approx \rho + g^* \approx \rho + g - pb, \qquad (12)$$

where g^* is given in equation (4).

2.2 Market Equilibrium

Agent i's wealth at the end of period $t-1$ is

$$W_{i,t-1} = P_{t-1}K_{i,t-1} + B_{i,t-1},$$

so that agent 1's wealth share at the end of period $t-1$ is

$$\frac{W_{1,t-1}}{W_{t-1}} = K_{1,t-1} + \frac{\rho B_{1,t-1}}{(1-\rho)Y_{t-1}}.$$

Note that total wealth, W_{t-1}, equals the equity price, P_{t-1}.

The analysis requires an initial value for agent 1's wealth share. For example, with an equal number of agents of each type, this share might start at 0.5 in period 0. Heuristically, if $\gamma_1 < \gamma_2$, there is an incentive in this initial position for agent 1 to issue risk-free bonds, so that $B_{11} < 0$ in period 1, and these bonds will be held by agent 2, so that $B_{21} > 0$. That is, agent 1 borrows from agent 2 on a safe basis. Correspondingly, agent 1 uses its bond issue to increase its share of equity, so that $K_{11} > 0.5$ and $K_{21} < 0.5$. In a richer model, this process of safe credit creation would affect the equilibrium amount and composition of investment.

The pattern of bond and equity positions shifts risk from the high-risk-aversion agent 2 to the low-risk-aversion agent 1. However, the process does not entail complete risk shifting; rather, enough bond issue occurs so that the resulting stochastic paths of future consumption for each agent make the right-hand side of equation (9) the same for each agent i. This equation also determines r_1^f.

We can use equations (9) and (10), along with the agents' budget constraints, to find numerically the equilibrium values for period 1 of r_1^f, each agent's consumption, and each agent's allocation of assets between equity and bonds.[6] The realization for Y_1 (disaster or no disaster in the present case) then determines each agent's wealth at the end of period 1 and,

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[6]We carried out this analysis numerically using periods of quarterly length.

hence, agent 1's wealth share at the end of period 1. This share determines the equilibrium values for period 2, and so on.

Using the budget constraint from equation (6) and the condition for consumption in equation (10), we can show that agent 1's wealth share at the end of period t relates to asset holdings from the end of period $t-1$ in accordance with

$$\frac{W_{1,t}}{W_t} = K_{1,t-1} + \frac{\rho(1 + r_t^f)B_{1,t-1}}{Y_t}. \tag{13}$$

We can also show that the change in agent 1's wealth share from $t-1$ to t is

$$\frac{W_{1,t}}{W_t} - \frac{W_{1,t-1}}{W_{t-1}} \approx \frac{\rho B_{1,t-1}}{Y_t}(r_t^f - \rho - g_t), \tag{14}$$

where $g_t \equiv (Y_t/Y_{t-1} - 1)$ is the stochastic growth rate of GDP.

Since the risk-free rate, r_t^f, will be less than $\rho + g^*$, which equals the expected rate of return on equity, r^e, the expectation of the right-hand side of equation (14) is positive if $B_{1,t-1} < 0$. In other words, the expected change in agent 1's wealth share is positive whenever agent 1 (the low risk-aversion agent) is borrowing in a risk-free manner from agent 2. The reason that agent 1's wealth share tends to rise over time is that this agent's wealth is relatively concentrated in risky equity, which has a higher expected rate of return than risk-free bonds (even after factoring in the occasional macroeconomic disasters, which tend to reduce agent 1's wealth share). Consequently, we find numerically that agent 1's wealth share asymptotically approaches one, and the ratio of risk-free bonds, $B_{1,t}$, to total assets or GDP asymptotically approaches zero. In effect, there is a selection or survival effect, whereby wealth is concentrated asymptotically in the agent with relatively low risk aversion. Hence, the model behaves in the long run like a representative-agent economy with a coefficient of relative risk aversion equal to γ_1.

The non-stationarity of the initial form of the model makes it unsatisfactory for studying the determination of wealth shares and quantities of safe assets.[7] To get a satisfactory analysis, the model has to be modified to achieve stationarity; in particular, to have the expected wealth share of agent 1 asymptotically approach a value less than one.

2.3 Replacement and Stationarity

A natural way to achieve stationarity is to have agents die off randomly, with replacement by new agents who have a random (50-50) chance of being type 1 or type 2; that is, having a

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[7]An example of this problem in earlier research is Longstaff and Wang (2012, p. 3208).

coefficient of relative risk aversion of γ_1 or γ_2.[8] Since type-1 agents tend to have above-average wealth, this process tends to redistribute wealth back to type-2 agents.

The replacement agents might inherit the assets of their altruistic predecessors, who might be parents. Alternatively, as in Blanchard (1985), agents may leave no bequests and hold all of their bond-like assets as annuities, on which the returns factor in the probability of dying. In either case, a full analysis requires the optimizing choices of consumption and asset holdings to take account of the possibility of dying with replacement by children whom one may or may not care about.

When agents are linked to their descendants via operative intergenerational transfers, the main effect in the model from death and replacement is the random change in the coefficient of relative risk aversion. Therefore, we can use a simpler metaphor in which infinite-lived persons randomly experience moments in which shifts occur in their coefficients of relative risk aversion to either γ_1 or γ_2. We assume that each destination has a 50-50 chance of being picked. We denote by $\nu \geq 0$ the gross replacement rate, in the sense of the random rate at which each agent moves to a state that has a fifty-fifty chance of switching from the incumbent γ_i to the other γ_i. In this scenario, the net replacement rate for the overall population is $\nu/2$.

The structure of randomly shifting γ_i's avoids having to deal with intergenerational transfers and the degree of altruism. In general, however, each agent's current optimizing decisions on consumption and asset holdings would depend on the potential for future shifts in one's own γ_i. With EZW preferences, as in equation (1), the channel for these effects has to work through expected future utility, $\mathbb{E}U_{i,t+1}$. We choose as our replacement metaphor a "wealth-neutral" case, whereby any direct effect on $C_{i,t+1}$ from a shift in γ_i at time $t+1$ is offset by a contemporaneous shift in the utility function that keeps $C_{i,t+1}$ fixed overall. In this setting, the first-order condition in equation (9) remains valid.[9]

The replacement process is stochastic at the individual level but roughly deterministic in the aggregate. We retained the first-order conditions derived from the initial model but modified the equilibrium analysis to factor in the shifting of wealth composition across the two types of agents. Specifically, if ν is the gross replacement rate, the expression for the change in wealth share of agents of type 1 is modified from equation (14) to:

$$\frac{W_{1,t}}{W_t} - \frac{W_{1,t-1}}{W_{t-1}} \approx \frac{\rho B_{1,t-1}}{Y_t}\left(r_t^f - \rho - g_t - \nu\right) - \nu * (K_{1,t-1} - 0.5). \tag{15}$$

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[8]Chan and Kogan (2002) generate stationarity effectively by having each agent's coefficient of relative risk aversion, γ_i, be an increasing function of that agent's wealth share. The assumed sign of this effect is not obvious; that is, it is unclear that richer agents would have higher coefficients of relative risk aversion. In any event, this type of model functions in the steady state as a representative-agent model with a single coefficient of relative risk aversion. Gârleanu and Panageas (2015) have agents dying off stochastically and being replaced by unloved children in the manner of Blanchard (1985).

[9]This idea was suggested to us by Emmanuel Farhi.

Equation (15) takes account of shifts in wealth between agents of type-1 and type-2 when agents change their type. However, this shifting also means that individuals within the two groups have to be heterogeneous in wealth. Typically, someone who just moved from type 1 to type 2 will have wealth above the mean of the existing type-2 agents, and vice versa for someone who just changed from type 2 to type 1. As this process evolves, the agents within each group will have a range of wealth levels, depending on their history of past transitions.

The wealth distribution within groups might be interesting to analyze but is unimportant for the present analysis. For our purpose, what matters is the total wealth held by agents of each type, not the distribution of wealth within types. In particular, equations (9) and (10) imply that, for a given γ_i, higher wealth scales up proportionately the chosen values of consumption, without changing the proportionate amounts held of risky and risk-free claims. Thus, we can use equation (15) to gauge the changing wealth shares of groups 1 and 2, while neglecting effects from the differing levels of wealth within each group.

If $\nu = 0$, as before, the expectation of the right-hand side of equation (15) is positive if $B_{1,t-1} < 0$. As agent 1's wealth share approaches one, $B_{1,t-1}/Y_t$ asymptotically approaches zero and, therefore, equation (15) implies that the expectation of the change in agent 1's wealth share asymptotically approaches zero. Another property of the equilibrium with $\nu = 0$ is that $K_{1,t-1}$ asymptotically approaches 1.

If $\nu > 0$, when $K_{1,t-1}$ is close to 1 and $B_{1,t-1}/Y_t$ is negligible, the term on the far right of equation (15) is negative and dominates in magnitude the first term on the right. It follows that the expected change in agent 1's wealth share reaches zero before $K_{1,t-1}$ gets close to one and $B_{1,t-1}/Y_t$ becomes negligible. For this reason, the economy tends to approach a stochastic steady state in which mean wealth shares for each agent are between zero and one. We compute these steady-state mean wealth shares as well as steady-state means of safe assets (expressed relative to total assets or GDP) and risk-free rates, r_t^f.

2.4   Numerical Solution Method

In making the numerical calculations, we work directly with the first-order conditions shown in equation (7). That is, we do not rely on the approximations that allow for the form of the first-order conditions shown in equation (8). We also extend beyond the case of log utility, $\theta = 1$, to consider alternative values of the intertemporal elasticity of substitution (still assumed to be the same for the two types of agents).

We solve the model using the Taylor projection algorithm proposed by Levintal (2016). This method has been shown to work well in models with rare disasters by Fernández-Villaverde and Levintal (2016). See these two papers and the appendix for further details.

Imagine we want to find an agent's decision rule for some endogenous variable, y, as a

function of the model's n state variables, x; that is, $y = g(x)$. To identify this decision rule, Taylor projection combines features of projections and Taylor-based perturbation algorithms. In a first, "projection" step, Taylor projection postulates that $g(x)$ can be approximated by a k-order polynomial $\hat{g}(x, \Theta)$, where θ is a vector of size m of polynomial coefficients to be determined. If we plug $\hat{g}(x, \Theta)$ into the equilibrium conditions of the model (first-order conditions, resource constraints, etc.), we obtain a residual function, $\mathcal{R}(x, \Theta)$, that depends on x and the unknown coefficients θ. The term "residual" comes from the observation that if we were plugging in $g(x)$, the equilibrium conditions would hold exactly; since instead we are pluggin in $\hat{g}(x, \Theta)$, we are left with a "residual." This first step of building a residual function is the same as in any standard projection.

In a second, "Taylor" step, Taylor projection expands $\mathcal{R}(x, \Theta)$ around a point x_0 with a k-order Taylor series. Then it determines θ as those coefficients that zero the terms of this series: if all the Taylor coefficients up to the kth-order are zero, then $\mathcal{R}(x, \Theta) \approx 0$ around x_0, as desired. This procedure requires finding values for θ that satisfy:

$$\mathcal{R}(x_0, \Theta) = 0,$$

$$\left. \frac{\partial \mathcal{R}(x, \Theta)}{\partial x_i} \right|_{x_0} = 0, \ \forall i = 1, \ldots, n,$$

$$\vdots$$

$$\left. \frac{\partial^k \mathcal{R}(x, \Theta)}{\partial x_{i_1} \ldots \partial x_{i_k}} \right|_{x_0} = 0, \ \forall i_1, \ldots, i_k = 1, \ldots, n.$$

This system is solved using the Newton method with the analytic Jacobian. This second step is similar but not identical to the series expansion undertaken by a perturbation. The main difference is that we can apply Taylor projection at any point x_0, whereas perturbation is applicable only at the steady state of a deterministic version of the model. In our application, this difference is important, because the model economy travels to states that are far away from the deterministic steady state.[10]

In contrast to our procedure, a standard projection finds the values of θ by making $\mathcal{R}(x, \Theta)$ as close as possible to zero over the whole domain of x, instead of zeroing the terms of the Taylor series. This projection has the advantage of high global accuracy, but it comes at the cost of wasting much effort getting $\mathcal{R}(x, \Theta) \approx 0$ for infrequently traveled areas of the

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[10]Note also that a traditional perturbation finds a Taylor series of the agent's decision rules by perturbing the volatility of the shocks of the model around zero. Taylor projection, instead, considers the true volatility of the shocks when building $\mathcal{R}(x, \Theta) \approx 0$ and evaluating its derivatives. Considering the true volatility of shocks is a crucial advantage in models with rare disasters, which are characterized by large shocks.

domain of x. This projection therefore suffers from an acute curse of dimensionality, while the same problem is much milder with Taylor projection. Furthermore, Taylor projection exploits the information embedded in the derivatives of the residual function, information that is ignored in projection methods. Finally, the Jacobian resulting from solving the system above is much smaller and sparser than the one resulting from a standard projection and, thus, faster to solve. Levintal (2016) shows how the main cost of Taylor projection, the computation of all the required derivatives, can be accomplished efficiently by a chain-rule method that exploits symmetry, permutations, repeated partial derivatives, and sparsity.

Fernández-Villaverde and Levintal (2016) document that the simulated moments and impulse-response functions of a model with rare disasters solved with Taylor projection are nearly indistinguishable from those when the model is solved with a much costlier and less scalable standard projection. They also show that Taylor projection generates very small errors throughout different simulations. Finally, much of the economics of rare disasters is not in what happens after a disaster, but on how the positive probability of a future disaster changes consumption, saving, and asset pricing in non-disaster times. Therefore, obtaining accuracy in normal periods, as Taylor projection does by taking an expansion around x_0, is important.[11]

3   Quantitative Analysis of Stationary Model

Aside from the coefficients of relative risk aversion, γ_1 and γ_2, the baseline parameter values, listed in the notes to Table 1, are $\rho = 0.04$ per year (rate of time preference), $g = 0.025$ per year (growth-rate parameter), $p = 0.04$ per year (disaster probability), and $b = 0.32$ (effective disaster size). These values accord with the prior empirical analysis summarized in Barro and Ursúa (2012). These parameter values imply from equation (4) that the expected growth rate is

$$g^* = g - p \cdot b = 0.0122 \text{ per year.} \tag{16}$$

The baseline analysis assumes log utility, $\theta = 1$.

3.1   A Representative Agent

Table 1 considers a representative agent, where $\gamma_1 = \gamma_2 = \gamma$. In these cases, if we start with agent 1's wealth share at 0.5, $B_{i,t}$ and $K_{i,t}$ stay constant over time at 0 and 0.5, respectively, irrespective of the realizations of Y_t. Because of log utility, the expected rate of return on

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equity, r^e, is fixed at $\rho + g^*$, where $\rho = 0.04$ per year and $g^* = 0.0122$ (equation 16), so that $r^e = 0.052$ per year. A higher γ lowers the risk-free rate, r^f, and, thereby, raises the equity premium. Specifically, Table 1 shows that r^f ranges from 0.046 at $\gamma = 1$ to -0.055 at $\gamma = 6$.[12] An unlevered equity premium between 0.03 and 0.06 (corresponding to historical data) requires γ to be between 3 and 4.5. For a given γ, r^f is fixed over time, regardless of the realizations of Y_t. This risk-free rate is a shadow rate in the sense that no risk-free borrowing and lending occur in equilibrium. That is, no net safe assets are created in this representative-agent environment.

3.2   Heterogeneity in Risk Aversion

Table 2 allows for differences between γ_1 and γ_2. We begin with a gross replacement rate of $\nu = 0.02$ per year, which corresponds roughly to adult mortality rates. The implied net replacement rate for the γ_i coefficients is $\nu/2 = 0.01$ per year.

The table shows combinations of γ_1 and γ_2 that generate a mean steady-state risk-free rate of $r^f = 0.010$ and a mean steady-state unlevered equity premium of $r^e - r^f = 0.042$. That is, these combinations of γ_1 and γ_2 accord roughly with empirically observed averages of the risk-free rate and the equity premium. The table shows the corresponding steady-state means of a set of variables: agent 1's share of risky assets, K_1, and wealth, W_1/W, and the ratio of the amount of safe assets, $|B_1|$, to wealth and GDP. Because economy-wide assets equal annual GDP times 25 ($1/\rho$) in this model, the amount of safe assets expressed relative to annual GDP is 25 times the ratio to total assets. Note that total assets correspond to the capitalization of the entire flow of GDP, effectively including human capital as well as physical capital.

The first row of Table 2 shows that, if $\gamma_1 = \gamma_2$, the value of γ_1 and γ_2 needed to generate a mean steady-state r^f of 0.010 is 3.86 (see Table 1). Columns 1 and 2 of Table 2 show that values of γ_1 below 3.86 require higher values of γ_2. For example, $\gamma_1 = 3.6$ matches up with $\gamma_2 = 4.25$, $\gamma_1 = 3.4$ with $\gamma_2 = 4.9$, $\gamma_1 = 3.2$ with $\gamma_2 = 7.0$, and $\gamma_1 = 3.1$ with $\gamma_2 = 8.7$. For still lower values of γ_1, the required value of γ_2 explodes. However, our numerical procedure does not work well in this extreme range.

In column 5, the steady-state mean of the share of risky assets held by agent 1, K_1, equals 0.50 when $\gamma_1 = \gamma_2$, then rises toward 1.0 as γ_1 falls and γ_2 rises. When $\gamma_1 = 3.1$ and $\gamma_2 = 8.7$, the steady-state mean of K_1 is 0.92.

In column 6, the steady-state mean of the wealth share, W_1/W, starts at 0.50 when $\gamma_1 = \gamma_2$, then rises as γ_1 falls and γ_2 rises. This wealth share equals 0.77 when $\gamma_1 = 3.1$ and $\gamma_2 = 8.7$.

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[12]In the present model (which lacks risk-free and costless storage of final product), there is nothing special about a risk-free rate of zero.

Note that equity ownership is much more unequally distributed than overall wealth.

Column 7 shows that $|B_1|/W$, the steady-state mean of the ratio of the magnitude of safe to total assets, rises from 0 when $\gamma_1 = \gamma_2$ to 3.4% when $\gamma_1 = 3.6$ ($\gamma_2 = 4.25$), 7.1% when $\gamma_1 = 3.4$ ($\gamma_2 = 4.9$), 12.8% when $\gamma_1 = 3.2$ ($\gamma_2 = 7.0$), and 14.7% when $\gamma_1 = 3.1$ ($\gamma_2 = 8.7$). For subsequent purposes, we are particularly interested in the model's predictions about the size of safe assets. From this perspective, an important result is that the predicted quantity of safe assets remains below 15% of economy-wide assets as long as γ_2 is less than 8.7, which is a high degree of relative risk aversion. In column 8, the corresponding ratio to GDP is 3.7.

Table 3 redoes the analysis for alternative settings of four of the parameters: the reciprocal of the IES, θ, is allowed to be 0.5 or 2.0, rather than 1.0; the gross replacement rate, ν, is 0.05 per year, rather than 0.02; the disaster probability, p, is 0.02 per year, rather than 0.04; and the population share of type 1 agents is 0.25, rather than 0.5. In each case, the table shows steady-state values of r^e and r^f and the other variables for three of the combinations of (γ_1, γ_2) considered in Table 2. Aside from the parameter value that changed, the other parameters are held fixed at the values assumed in Table 2.

Shifts in the reciprocal of the IES, θ, have only moderate effects on the steady-state equilibrium. Consider, as an example, the case where $\gamma_1 = 3.4$ and $\gamma_2 = 4.9$. With $\theta = 1$ (in Table 2), the rates of return are $r^e = 0.052$ and $r^f = 0.010$. Table 3 shows that these rates of return change to 0.053 and 0.013, respectively, when $\theta = 0.5$ and to 0.049 and 0.006, respectively, when $\theta = 2$. Correspondingly, the steady-state means of K_1 and W_1/W were 0.695 and 0.625, respectively, in Table 2. These values change in Table 3 to 0.750 and 0.687, respectively, when $\theta = 0.5$ and to 0.661 and 0.584, respectively, when $\theta = 2$. The steady-state mean of $|B_1|/W$ was 0.071 in Table 2 and changes in Table 3 to 0.063 when $\theta = 0.5$ and 0.077 when $\theta = 2$. Correspondingly, the steady-state mean of $|B_1|/Y$ was 1.76 in Table 2 and changes in Table 3 to 1.52 when $\theta = 0.5$ and 2.08 when $\theta = 2$. A key point is that the results for the magnitude of safe assets show little sensitivity to the assumed IES. Or, to put it another way, the results in the simplified setting of log utility, $\theta = 1$, are likely to be reasonably accurate.

An increase in the replacement rate, ν, means that the higher risk-aversion type, group 2, counts more for the steady-state equilibrium. Therefore, the rise in ν to 0.05 in the middle of Table 3 lowers the steady-state shares of agent 1 in equity and wealth. For the case where $\gamma_1 = 3.4$ and $\gamma_2 = 4.9$, the values of K_1 and W_1/W go from 0.695 and 0.625, respectively, in Table 2 to 0.631 and 0.556, respectively, in Table 3. Correspondingly, the rise in ν lowers the risk-free rate, r^f, which falls from 0.0100 in Table 2 to 0.0080 in Table 3. However, the change in ν has only a minor effect on the size of safe assets. $|B_1|/W$ goes from 0.071 in Table 2 to 0.076 in Table 3, and $|B_1|/Y$ goes from 1.76 in Table 2 to 1.88 in Table 3. Therefore,

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an important finding is that the results—particularly with regard to the quantity of safe assets—do not change greatly when ν is 0.05, rather than 0.02.

The next part of Table 3 sets the disaster probability, p, at 0.02 per year, rather than the value 0.04 assumed in Table 2. The decrease in p makes the steady-state risk-free rate, r^f, sharply higher, around 0.036, rather than 0.010. Correspondingly, the equity premium becomes too low in Table 3, compared with empirically observed averages. Thus, as in previous research, the model does not accord with regularities on mean rates of return unless the disaster risk is sufficiently high. A similar conclusion arises if the disaster size, b, is lowered substantially below its initially assumed value of 0.32.

The last part of Table 3 shows the effects from setting the population share, N_1, of the low risk-aversion agents to 0.25, rather than 0.50 (see the appendix for the corresponding equilibrium conditions). This shift effectively lowers the supply of safe assets (from agents of type 1) compared to the demand (from agents of type 2) and results, thereby, in a drop in the risk-free rate, r^f, and a corresponding rise in the equity premium.

3.3 Tax/Transfer Systems

Some forms of tax/transfer systems provide an alternative to our type-replacement setup as a way to achieve a stationary equilibrium in the steady state. The results depend on the details of the structure of taxes and transfers, and this kind of public-finance analysis is not the focus of our present analysis. Therefore, we limit the present discussion to a description of a simple income-tax system that generates results analogous to those from our type-replacement mechanism.

One possibility is that the basis of the tax system is income from dividends and interest, but not capital gains. (We would interpret dividends in the model to encompass labor income.) In this setup, interest expenses are deductible for income-tax purposes. For type i, taxable income is then

$$I_{i,t} = Y_t * K_{i,t-1} + r_t^f * B_{i,t-1}. \tag{17}$$

The average tax rate levied on this income is assumed to be linear in type i's relative income:

$$\tau_{i,t} = a_0 + a_1 * \left(\frac{I_{i,t}}{I_t}\right), \tag{18}$$

where I_t is aggregate income. If $a_1 = 0$, the income-tax system is proportional at rate a_0, whereas if $a_1 > 0$, the system is graduated with marginal income-tax rate equal to $a_0 + 2a_1 * (\frac{I_{i,t}}{I_t})$. Any aggregate taxes collected are assumed to be remitted as lump-sum transfers, equally to the two agents.

We find that a purely proportional income tax $(a_1 = 0)$ does not generate an interior

equilibrium for our model in the steady state. That is, as in the initial setup with no replacement and no taxes, agent 1 ends up asymptotically with all of wealth, equity ownership, and consumption. In contrast, a graduated-rate system ($a_1 > 0$) can support an interior equilibrium, analogous to those studied in the regime with replacement. The results under alternative systems of public finance will be studied in future research.

3.4 Dynamics

The dynamics of the economy reflects the evolution of the share of agent 1 in total wealth, W_1/W. Disaster shocks and long periods free of disasters affect this wealth share and, thereby, have persisting influences on the risk-free interest rate, r^f, the ratio of safe to total assets, and other variables. We consider first the dynamic effects from a disaster and then examine the consequences from a long period free of disasters.

3.4.1 Aftermath of a Disaster.

Figure 1 shows the dynamics of the economy starting from a steady state (that is, with all variables at their mean steady-state values) and assuming the realization of a disaster of size $b = 0.32$ in period 1. The results correspond to the parameter combination $\gamma_1 = 3.3$ and $\gamma_2 = 5.6$ in Table 2. The paths of variables in Figure 1 assume no further disasters and are, therefore, deterministic in our specification. The variables considered over ten years are agent 1's wealth share, W_1/W, the risk-free interest rate, r^f, agent 1's share of total equity, K_1, and the ratio of the magnitude of safe to total assets, $|B_1|/W$.

Because of agent 1's relatively high concentration in risky assets, this agent's wealth share, W_1/W, falls with the disaster from 0.670 to 0.626. The share rises thereafter (in the absence of further disasters) but remains below the steady-state value even after 10 years, when the share reaches 0.654. Another way to look at this pattern is that relatively low inequality of wealth and consumption persist for a long time after a disaster shock. However, the recovery toward the steady state is accompanied by rising inequality. These patterns also appear in agent 1's share of equity, K_1. This share falls on impact from its steady-state value of 0.767 to 0.731, then rises to 0.755 after 10 years.

For the risk-free rate, r^f, we can view the disaster shock and consequent shift in relative wealth toward agent 2 as raising the demand for safe bonds (from agent 2) compared to the supply (from agent 1). In response to the shift in excess demand, r^f falls on impact from its steady-state mean value of 0.0100 to 0.0083. That is, the disaster leads to a low risk-free interest rate. In the recovery period, r^f rises but remains below its steady-state value. After 10 years, r^f reaches 0.0094.

The enhanced wealth share of agent 2 is accompanied on impact by a rise in the ratio of the magnitude of safe to total assets, $|B_1|/W$. This ratio increases initially from its steady-state mean value of 0.097 to 0.106. Thus, safe assets are comparatively large immediately after a disaster. The ratio then falls gradually and reaches 0.101 after 10 years.

To summarize, disasters generate low but rising wealth (and consumption) inequality, low but rising risk-free real interest rates, and high but declining ratios of safe to total assets. In particular, low inequality and risk-free interest rates and high safe-asset ratios are all symptoms of a gradual recovery from a serious adverse shock to the economy.

An important feature of the disaster shock that we examined is that it disproportionately affects the low-risk-aversion agent, group 1, and, therefore, shifts the wealth share initially toward the high-risk-aversion agent, group 2. This pattern arises because the shock affects the value of equity, which is disproportionately held by group 1. Hart and Zingales (2014) argue that this kind of pattern characterizes some macro-financial shocks, such as the bursting of the Internet boom in 2000. They argue, however, that other shocks—notably the Great Recession of 2007-2009—feature the erosion in value of assets that were previously viewed as nearly safe. In the 2007-2009 case, this pattern applied particularly to claims associated with real estate, whose safety had been greatly exaggerated.

In our model, we could analyze the Hart and Zingales case by allowing for an unexpected decline in the value of the existing "safe" assets, which are the private bonds. That is, the zero-probability event of large losses on safe assets could be viewed as a one-time happening. In this case, agent 1's wealth share would initially shift discretely above its steady-state value.[13] The subsequent dynamics corresponds to that described in our next example.

3.4.2 Forty years of tranquility.

Figure 2 assumes that, starting from the steady state, the economy has a long period with no disasters ("40 years of tranquility"). This situation accords broadly with the U.S. experience from the 1950s up to the Great Recession of 2007-2009. The parametric assumptions for Figure 2 are the same as those for Figure 1.

In Figure 2, agent 1's wealth share rises gradually above its steady-state value of 0.670. Conditional on no disasters, this ratio rises after 40 years to 0.711—and would asymptotically approach a higher value, 0.723, if no disaster ever occurred. The value 0.723 is a kind of steady-state wealth share (shown in quotes in the figure) in that it applies asymptotically conditional on the realization of no disasters. In contrast, the lower steady-state mean wealth

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[13]A counter-vailing force in 2007-2009 is that large financial institutions, including Lehman, experienced sharp losses in the value of assets linked to real estate. This aspect of the shock tends to lower group 1's wealth share and, thereby, works like the disaster realization that was already analyzed.

share of 0.670 is defined inclusive of the occasional occurrence of disasters.

The dynamic path of the wealth share in Figure 2 shows that sustained tranquility is accompanied by rising inequality, in the sense of growing wealth (and consumption) shares of group 1. The dynamics also features a rising risk-free rate, which increases above its steady-state value of 0.010 and eventually approaches 0.0121. The ratio of safe to total assets falls from its steady-state value of 0.10 and gradually approaches 0.087.

In the paths shown in Figure 2, agent 1's wealth share would never rise above the "steady-state" value of 0.723. However, a shock mentioned before—where the value of safe assets declines sharply because this safety had been exaggerated—could put agent 1's wealth share above 0.723. In that case, the post-shock dynamic paths (conditional on no further disasters) would feature a gradually declining wealth share of agent 1, with this share asymptotically approaching from above the value 0.723. Correspondingly, the risk-free rate would rise initially above its "steady-state" value and then fall gradually, whereas the ratio of safe to total assets would fall initially below its "steady-state" value and then rise gradually.

4   Gross versus Net Lending and Ricardian Equivalence

The bond holdings, B_1, shown in Table 2 correspond to net safe lending from the high-risk-aversion agent, group 2, to the low-risk-aversion agent, group 1. There is a sense, however, in which gross bond issuance is not pinned down, because the model would admit unlimited borrowing and lending within groups. That is, agent 1 could effectively issue an arbitrary amount of bonds to himself, and analogously for agent 2.

If the model were augmented to include an infinitesimal amount of transaction costs for bond issuance or collection of interest and principal, then borrowing and lending within groups would not occur in equilibrium in the present model. In this case, the quantity of bonds, B_1, shown in Table 2 would be the unique equilibrium for the gross amount outstanding.

If transaction costs associated with bonds are substantial, the quantity of net bond issuance and the risk-free rate might differ significantly from the values shown in Table 2. Correspondingly, the risk-free rate received by lenders (group 2) would deviate from that paid by borrowers (group 1). For example, if transaction costs were prohibitive, the results would correspond to autonomy for groups 1 and 2 and, therefore, to the results shown in Table 1. The quantity of net bond issuance would be 0, and the share of capital held by each group would be 0.5. As an example, if $\gamma_1 = 3.0$, the shadow risk-free rate for group 1 would be 0.025 (from Table 1) and if $\gamma_2 = 5.0$, the shadow risk-free rate for group 2 would be -0.019 (again from Table 1). That is, members of group 1 would be willing to pay a rate of 0.025 per year at the margin on risk-free borrowing, whereas members of group 2 would be willing to accept

a rate of -0.019 per year at the margin on risk-free lending. However, no issue of safe debt occurs because of the prohibitive transaction costs.

Suppose now that the government issues one-period bonds with characteristics corresponding to those of private bonds. The real interest rate on government bonds held from t to $t+1$ must then be r_{t+1}^f, the same as that on private bonds. The simplest way to introduce public debt is for the government to make a lump-sum transfer of bonds in year t in the aggregate quantity B_t^g. This distribution is assumed to go 50-50 to members of groups 1 and 2. The aggregate principal and interest, $(1 + r_{t+1}^f)B_t^g$, is paid out to government bondholders in period $t+1$. This payout is financed by lump-sum taxes, levied equally in period $t+1$ on members of groups 1 and 2.

What is the impact of this government bond issue on private bond issue, the risk-free interest rate, and so on? The government bond issue does not affect the households' first-order conditions, which appear in equations (7) and (9). There is also no effect on households' budget constraints in equation (6) (updated to apply to periods t and $t+1$), once one factors in the transfer payments in year t and the taxes levied in year $t+1$. Therefore, it is immediate that the equilibrium involves the same net borrowing and lending as before between groups 1 and 2, the same risk-free interest rate, r_{t+1}^f, the same equity price, P_t, and the same expected rate of return on equity, r^e. That is, the equilibrium features Ricardian Equivalence with respect to net quantities of safe assets and the various rates of return.

Consider now how the added government bonds end up being held by groups 1 and 2. One possibility, assumed in the upper part of Table 4, is that each group holds the 50% of the government bonds that they initially received. These quantities correspond to the present value of the (certain) tax liabilities imposed on each group. The quantity of net private borrowing and lending, corresponding to $B_{1,t}$, is then the same as before.

The problem with this proposed equilibrium is that type-1 agents are simultaneously holding government bonds and issuing private bonds. Since government and private bonds are assumed to be indistinguishable, we can think of them as trading on a single bond market with a single rate of return. Therefore, in the upper part of Table 4, type-1 agents would be operating simultaneously on both sides of this bond market. As before, if there are infinitesimal transaction costs for bond issuance or collection of interest and principal, this type of equilibrium would be ruled out. Specifically, starting from the configuration in the upper part of Table 4, type-1 agents would be motivated to sell their government bonds and use the proceeds to retire private bonds.

In the full equilibrium, shown in the lower part of Table 4, the magnitude of the reduction in private bonds equals the amount of government bonds received by group 1.[14] Since this amount

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[14]This result assumes that the gross quantity of private bonds outstanding was initially at least as large as

was assumed to be one-half of the government bond issue, it follows that the magnitude of the reduction in private bonds expressed as a ratio to government bonds issued equals one-half. That is, the *crowding-out coefficient* for private bonds with respect to government bonds is minus one-half. More generally, this coefficient equals minus the share of the government bond issue that goes to group 1—the group that is issuing the private bonds.[15]

When compared to the equilibrium prior to the government bond issue, the only difference in the lower part of Table 4 is that some of the borrowing and lending between groups 1 and 2 is purely private, while some works through the government as intermediary (collecting taxes from group 1 and using the proceeds to pay principal and interest on half of the government bonds held by group 2). When viewed this way, the finding of Ricardian Equivalence is not surprising—it corresponds to the assumption that the private sector and the government are equally good at arranging for loans between groups 1 and 2.

The surprising part of our result is that the crowding-out coefficient for private bonds with respect to public bonds is -0.5, not 0.0 or -1.0. The one-half result came from a model with a number of simplifying assumptions; notably, there were just two groups characterized by their coefficients of relative risk aversion, γ_i, and the incidence of the present value of taxes net of transfers associated with the government bond issue was the same for each group. However, the crowding-out coefficient around 0.5 does not depend on these assumptions holding precisely. For example, the restriction to two groups is unimportant.[16] The assumption that matters most is that there is little relation across groups between γ_i and the share of taxes net of transfers applying to the group. For example, in our baseline case, the share of taxes net of transfers is one-half for each group.

The model's predicted crowding-out coefficient relates to the study by Krishnamurthy and Vissing-Jorgensen (2013, p.1), who argued "that government debt ... should *crowd out*

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the added government bonds going to group 1. We have assumed that this amount was one-half of the total government bond issue.

[15]This generalization of our one-half result was pointed out to us by Xavier Gabaix. He noted that "our basic finding was independent of the disaster theme and would come from pretty much any reason to hold debt." He then observed: "If there is a fraction f of lenders and 1-f of borrowers, then the crowding out coefficient is d(Gross private debt)/d(Govt debt) = - (1f)." The result that we stress, where f=0.5, is not general but is likely to be a good approximation because private lenders and borrowers have to be balanced in terms of dollars lent and borrowed even if not in terms of numbers of persons or wealth. Abel (2017) used the working-paper version of our analysis, Barro and Mollerus (2014), to generalize our results on crowding-out along the lines sketched by Gabaix.

[16]Suppose, for example, that there are four groups of agents, where $\gamma_1 < \gamma_2 < \gamma_3 < \gamma_4$. Suppose further that the initial equilibrium involves private bond holdings of $B_1 = -100$, B_2=-50, $B_3 = 50$, and $B_4 = 100$. Assume that the government issues 4 units of bonds, with the present value of taxes rising by 1 unit for each group. In this case, the two private borrowers go, in equilibrium, to B_1=-99 and B_2=-49, thereby preserving their positions for bond holdings net of tax liabilities (of 1 each) at -100 and -50, respectively. The two private lenders go, in equilibrium, to overall bond positions (inclusive of government bonds) of $B_3 = 51$ and $B_4 = 101$, thereby preserving their positions for bond holdings net of tax liabilities at 50 and 100, respectively. Note that the additional 4 units of government bonds crowd out the total of private bonds by 2 units.

the net supply of privately issued short-term debt." They tested this hypothesis on U.S. data for 1914-2011 and found (Table 4, Panel A) that an increase in the quantity of net U.S. government debt had a significantly negative effect on the net short-term debt created by the private financial sector. Remarkably, their estimated coefficient was close to -0.5, the value predicted by our baseline model.[17] Similarly, Gorton, Lewellen, and Metrick (2012, Table 1) estimated a crowding-out coefficient close to -0.5 for a broad concept of private financial-sector liabilities (their "high estimate") in the United States for 1952-2011.[18]

Ricardian Equivalence would not hold exactly in our model if the government is superior to the private sector in the technology of creating safe assets.[19] In particular, the government might be able to commit better than private agents to honoring payments of principal and interest on its bonds and can also use the coercive power of the tax system to ensure the financing of these payments. On the other hand, a private lending arrangement requires only that group 1 make principal and interest payments in period $t+1$ to group 2, whereas the public setup entails the government collecting taxes in period $t+1$ from group 1 and then using the proceeds to pay off group 2. Once the distorting influences from taxation are considered, it is not obvious that the public process entails lower "transaction costs" overall.[20]

An additional consideration is that the expansion of public debt and the associated taxation are poorly targeted. In our baseline case described by the lower part of Table 4, 50% of the added government bonds—held by group 2—match the added present value of tax liabilities for this group and, therefore, do not serve to shift risk toward group 1. Only the remaining 50% of government bonds corresponds to this shifting of risk. In contrast, all private bonds issued by group 1 and held by group 2 associate with risk shifting.

To highlight a case where the issue of public debt is important, suppose that the private sector's technology for creating safe assets is so poorly developed that no issue of private bonds occurs in the initial equilibrium (where no government bonds exist). In this case, analyzed at

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[17]Krishnamurthy and Vissing-Jorgensen (2013, 23) say: "These results suggest that a one-dollar increase in Treasury supply reduces the net short-term debt issued by the financial sector by 50 cents." In their theory (Section 3), they derive a crowding-out hypothesis from a model in which Ricardian Equivalence fails. However, their empirical results are actually consistent with a model in which Ricardian Equivalence holds.

[18]Gorton, Lewellen, and Metrick (2012, 1) say: "These results suggest that financial liabilities and government liabilities may be substitutes."

[19]Caballero and Farhi (2014, 3) make this assumption, although they do not clarify the elements that underlie the government's superior technology: "Public debt ... plays a central role ...as typically the government owns a disproportionate share of the capacity to create safe assets while the private sector owns too many risky assets. ... The key concept then is that of *fiscal capacity*: How much public debt can the government credibly pledge to honor should a major macroeconomic shock take place in the future?" They also do not consider that public debt issue creates additional "safe liabilities" in the form of taxes that match the added safe assets in a present-value sense.

[20]Even if the interest rate on government bonds is lower than that on private bonds, the overall transaction costs—including the distorting effects from taxation—associated with the public process might exceed that for the private process.

the beginning of this section, groups 1 and 2 are effectively autonomous, and the equilibrium for each group is the one that would apply in the corresponding representative-agent economy. The risk-free interest rate for group 1 can then diverge substantially from that for group 2.

In this environment, the government's issue of bonds can substitute for the private lending that would have occurred if the private sector had possessed the technology to create safe assets. In this setting, Ricardian Equivalence fails, and the government's debt issue moves the economy toward a more efficient outcome, where risk is shifted from type-2 to type-1 agents, and the risk-free interest rates of the two groups converge.

We can assess how much public debt is required to get the economy into the equilibrium of our baseline model with private debt. The answer—related to the crowding-out coefficient of one-half discussed before—is that the required quantity of public debt is twice the level of private bonds that arose in the initial setting. Moreover, if public debt expands beyond this quantity, it has no further effect on the equilibrium. That is, Ricardian Equivalence holds in this range at the margin even though private bonds are assumed to be absent.

5   The Quantity of Safe Assets

In the model, the quantity of safe assets corresponds to the shifting of risk from the high-risk-averse agent, group 2, to the low-risk-averse agent, group 1. Table 2 shows that, for reasonable parameter values, the steady-state mean for the ratio of safe to total assets ranges up to 15%.[21]

Using data to match the model's predictions for the quantity of safe assets is challenging because it is unclear how to measure empirically the amounts of these assets. Gorton, Lewellen, and Metrick (2012) (henceforth, GLM) define safe assets to comprise mostly liabilities of the government and the private financial sector. After making a number of adjustments—for example, to eliminate U.S. government securities held by federal trust funds and to deduct 15% of long-term debt issued by the financial sector—they focus on a "high estimate" of the amount of safe assets. Using U.S. data from 1952 to 2010, GLM report two major findings. First, the ratio of their measure of safe assets to a concept of total assets remained relatively stable over time. Second, the average size of this ratio was between 30% and 35%.

In a general sense, the observed stability of the ratio of safe to total assets accords with the model. The results in Table 2 indicate that large changes in the steady-state mean of the ratio of safe to total assets might arise from changes in the gap between the risk-aversion

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[21]We focus on the model's predictions about the ratio of safe to total assets, rather than the ratio of safe assets to annual GDP. The latter ratio depends on the ratio of total assets to annual GDP, which equals $1/\rho$ in the baseline model with log utility. This last ratio equals 25 when $\rho = 0.04$ per year but is sensitive to the choice of ρ.

coefficients of the high- and low-risk-aversion groups; that is, γ_1- γ_2. However, if this gap were roughly constant, then the steady-state mean of the ratio of safe to total assets would be reasonably stable.

A comparison of the results in Table 3 with those in Table 2 indicates that the one-time variations considered in a set of other parameters—the IES, $1/\theta$, the gross replacement rate, ν, the disaster probability, p, and the share of type-1 agents, N_1—do not have large effects on the mean of the steady-state ratio of safe to total assets (for given values of γ_1 and γ_2). Therefore, variations over time in these other parameters are unlikely to be sources of instability in the mean of the steady-state ratio of safe to total assets.

Finally, the results in Figures 1 and 2 show that the ratio of safe to total assets does not vary greatly along a dynamic path that is approaching given steady-state values. For example, in Figure 1, the ratio of safe to total assets varies only from 10.6% to 9.7% over a period of 10 years.

Another issue is that the average ratio of safe to total assets computed by GLM—30-35%—exceeds the steady-state values predicted by our model—which ranged up to 15%. A major reason that GLM's measured ratio of safe to total assets would diverge from our theoretical concept concerns the denominator, total assets. In our theory, total assets comprise the discounted value of the whole of GDP (which equals aggregate consumption in the model without capital). Thus, in effect, the theoretical concept of total assets includes human capital as well as physical capital. In contrast, GLM's concept of total assets corresponds more closely to the value of physical capital, though also including the value of government bonds. These considerations may explain why GLM's measured average ratio of safe to total assets is well above the range predicted by our model. For example, if income from capital constitutes one-third of GDP, then total assets based on the value of capital would be around one-third of the capitalized value of GDP. In this case, if we hold constant the model's predicted level of safe assets, the predicted ratio of safe to total assets would be about 30%, close to the numbers calculated by GLM. However, we have to generalize the model to allow uncertainties associated with human capital to affect the quantity of safe assets.

There are also reasons why GLM's measure of safe assets would diverge from our theoretical concept, which relates to net lending from group 2 (high risk aversion) to group 1 (low risk aversion). One issue is that the GLM measure does not compute a net figure for liabilities of financial institutions; that is, there is no deduction for safe assets held by these institutions. For example, in 2007-2008, Lehman Brothers issued bonds and commercial paper but also held U.S. government securities and liabilities of other financial firms.[22] On this ground, GLM's

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[22]Our model could be extended to account for this kind of borrowing and lending within groups. These patterns might arise because of idiosyncratic shocks that affect individual agents within groups, still defined by coefficients of relative risk aversion.

measured liabilities of government and financial institutions would overstate the net quantity of safe assets.

Another consideration is that an array of financial arrangements—including structured finance, stock options, and insurance contracts—can be used to convert risky assets into relatively safe assets. On this ground, the measured liabilities of governments and financial institutions might understate the quantity of safe assets.

GLM also include government liabilities as safe assets but do not include any portion of capitalized future taxes as "safe liabilities," even at the margin. Although it is true that tax liabilities cannot be directly traded, it is also true that these liabilities—and how they vary along with changes in the quantity of government bonds—affect economic analyses of public debt. To the extent that future taxes are factored in by agents, the gross public debt would overstate a meaningful measure of safe assets.

The net effects from these adjustments to the measured quantity of safe assets are ambiguous, and it is possible that the overall proportionate deviation between GLM's measure and our theoretical construct is not large. That is, the observed average ratio of safe to total assets of 30-35% may match up reasonably well with the predictions of our model, once total assets are adjusted to exclude human capital.

6   Conclusions

We constructed a model with heterogeneity in risk aversion to study the determination of the equilibrium quantity of safe assets. The model achieves tractability and transparency by assuming two types of agents with Epstein-Zin/Weil utility. The agents differ by coefficients of relative risk aversion but have the same intertemporal elasticity of substitution (IES) and rate of time preference. In the baseline model, each agent has log utility, in the sense of IES=1.

We focused on a stationary version of the model in which agents randomly experience changes in their coefficients of relative risk aversion. In the baseline setting, Ricardian Equivalence holds in that the quantity of government bonds does not affect rates of return or the net quantity of safe assets. The predicted crowding-out coefficient for private bonds with respect to government bonds is around -0.5, in line with some existing empirical evidence.

We generated quantitative implications for the quantity of safe assets by calibrating the model with sufficient disaster risk to get the model's predictions into the right ballpark for the average equity premium and risk-free rate. In a benchmark case, the magnitude of safe assets ranged up to 15% of total assets, which comprised the capitalized value of the full GDP. These results can be reconciled with an existing estimate that found the ratio of safe to total assets in the United States to be roughly stable over time at a value between 30% and 35%.

The basic structure of the model with heterogeneity in coefficients of relative risk aversion can be applied to other economic problems. For example, the framework can incorporate credit-market imperfections, including the necessity for enforcement mechanisms to ensure repayment of private debts. This extension relates to issues concerning collateral, liquidity, and asymmetric information. This type of extension would be important for assessing implications for the magnitude and composition of investment.

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Table 1: Representative-Agent Economy (Single Coefficient of Relative Risk Aversion)

$\gamma_1=\gamma_2=\gamma$	r^e	r^f
1	0.052	0.046
1.5	0.052	0.042
2	0.052	0.038
2.5	0.052	0.032
3	0.052	0.025
3.5	0.052	0.017
4	0.052	0.007
4.5	0.052	-0.005
5	0.052	-0.019
5.5	0.052	-0.035
6	0.052	-0.055

When the coefficients of relative risk aversion are the same for the two agents, $\gamma_1=\gamma_2=\gamma$, the equilibrium quantities of bonds, B_1 and B_2, are zero and the ownership of equity is evenly distributed, $K_1 = K_2 = 0.5$. The table shows the equilibrium risk-free rate, r^f, for each value of γ. The calculations assume that the growth-rate parameter is $g = 0.025$ per year, the rate of time preference is $\rho = 0.04$ per year, the disaster probability is $p = 0.04$ per year (corresponding in the historical data to contractions of per capita GDP by at least 10%), and the effective disaster size is $b = 0.32$. The expected growth rate is $g^* = g - p * b = 0.0122$ per year. The reciprocal of the IES is $\theta = 1$. The expected rate of return on equity, given $\theta = 1$, is $r^e = \rho + g^* = 0.052$ per year, which is independent of γ. The price of equity is $P = Y/\rho = 25 * Y$. In this representative-agent case, the equilibrium risk-free rate can be written in closed form, if $\gamma \neq 1$, as:

$$r^f = \rho + \theta g + p \left(\frac{\theta - 1}{\gamma - 1}\right) - p\left(1 - b\right)^{-\gamma} + p \left(\frac{\gamma - \theta}{\gamma - 1}\right) \left(1 - b\right)^{1-\gamma}.$$

If $\theta = 1$, as γ approaches 1, r^f approaches $\rho + g - pb/(1 - b)$.

Table 2: Steady-State Equity Ownership, Wealth Share, and Safe Assets Alternative values of γ_1 and γ_2 and γ_2 that generate $r^e = 0.052$ and $r^f = 0.010$

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
γ_1	γ_2	r^e	r^f	K_1	W_1/W	$|B_1|/W$	$|B_1|/Y$
3.86	3.86	0.052	0.010	0.500	0.500	0.000	0.00
3.6	4.25	0.052	0.010	0.593	0.560	0.034	0.84
3.4	4.9	0.052	0.010	0.695	0.625	0.071	1.76
3.3	5.6	0.052	0.010	0.767	0.670	0.097	2.40
3.2	7.0	0.052*	0.010*	0.857*	0.729*	0.128*	3.19*
3.1	8.7	0.052*	0.010*	0.916*	0.769*	0.147*	3.70*

This analysis assumes a gross replacement rate for agents of $\nu = 0.02$ per year, so that the net replacement rate is $\nu/2 = 0.01$ per year. The coefficients of relative risk aversion for the two agents, γ_1 and γ_2, are values that generate a steady-state rate of return on equity, r^e, of 0.052 and a risk-free interest rate, r^f, of 0.010. The other parameters, other than the rate of time preference, ρ, are the same as those assumed in Table 1. The value of ρ, needed to target the two rates of return, is 0.0400 (as in Table 1) for the first four rows, 0.0397 for row 5, and 0.0387 for the last row. The other columns show the steady-state means of agent 1's share of equity ownership, K_1, and total assets, W_1/W, and the ratio of the magnitude of safe assets, B_1, to total assets and GDP.

Numerical estimates with a * are subject to significantly higher measurement error than the other estimates.

Table 3: Steady-State Equity Ownership, Wealth Share, and Safe Assets. Alternative Parameter Values

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
γ_1	γ_2	r^e	r^f	K_1	$W_1/\mathbf{W}$	B_1/W	B_1/Y
				$\theta = 0.5$			
3.6	4.25	0.054	0.0118	0.635	0.602	0.033	0.79
3.4	4.9	0.053	0.0128	0.750	0.687	0.063	1.52
3.2	7.0	0.053	0.0138	0.879	0.778	0.100	2.46
				$\theta = 2.0$			
3.6	4.25	0.049	0.0069	0.571	0.537	0.034	0.92
3.4	4.9	0.049	0.0064	0.661	0.584	0.077	2.08
3.2	7.0	0.049*	0.0002*	0.777*	0.616*	0.161*	4.40*
				$\nu = 0.05$ per year			
3.6	4.25	0.052	0.0095	0.558	0.524	0.034	0.85
3.4	4.9	0.052	0.0080	0.631	0.556	0.076	1.88
3.2	7.0	0.053*	0.0041*	0.802*	0.635*	0.166*	4.13*
				$p = 0.02$ per year			
3.6	4.25	0.059	0.0368	0.564	0.530	0.034	0.84
3.4	4.9	0.059	0.0362	0.641	0.568	0.074	1.83
3.2	7.0	0.058	0.0345	0.798	0.648	0.149	3.74
				$N_1{=}0.25$			
3.6	4.25	0.052	0.0065	0.339	0.308	0.030	0.75
3.4	4.9	0.052	0.0033	0.490	0.413	0.078	1.93
3.2	7.0	0.053*	0.0027*	0.792*	0.618*	0.174*	4.30*

These results use the parameter values from Table 2, except for the change in the indicated parameter value. The first three lines use the reciprocal of the IES $\theta = 0.5$ (instead of 1.0), the next three use $\theta = 2.0$, the next three use the gross replacement rate $\nu = 0.05$ per year (instead of 0.02), the next three use the disaster probability $p = 0.02$ per year (instead of 0.04), and the last three use the population share $N_1 = 0.25$ for the low-risk-aversion group (instead of 0.5).

Numerical estimates with a * are subject to significantly higher measurement error than the other estimates.
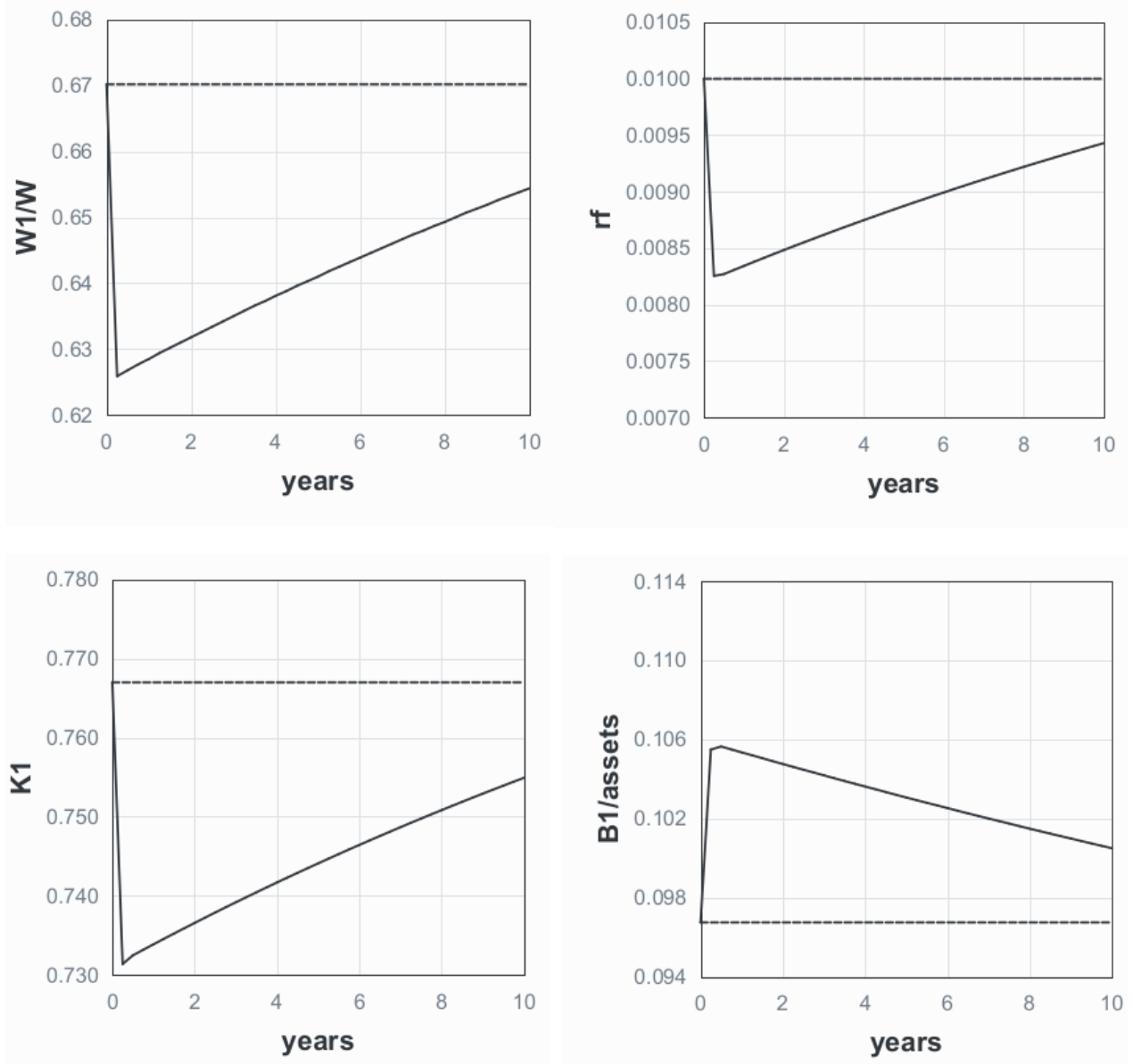
Table 4: Changes in Safe Assets when the Government Issues Bonds

	Agent 1	Agent 2	Total
Case 1: Government bonds up by 100, held 50-50			
Changes in:			
Private bond holdings, B	0	0	0
Government bond holdings, B^g	+50	+50	+100
Taxes (present value)	+50	+50	+100
Net safe assets in model	0	0	0
Net safe assets as measured by Gorton, et al. (2012)	+50	+50	+100
Case 2: Government bonds up by 100, all held by agent 2			
Changes in:			
Private bond holdings, B	+50*	-50	0
Government bond holdings, B^g	0	+100	+100
Taxes (present value)	+50	+50	+100
Net safe assets in model	0	0	0
Net safe assets as measured by Gorton, et al. (2012)	0	+50	+50

Note: In all cases, the government issues 100 of bonds, B^g, and transfers these bonds 50-50 to agents 1 and 2. The present value of taxes rises by 100, divided 50-50 between agents 1 and 2. In case 1, the added government bonds are held 50-50 by agents 1 and 2. In case 2, all of the added government bonds are held by agent 2.
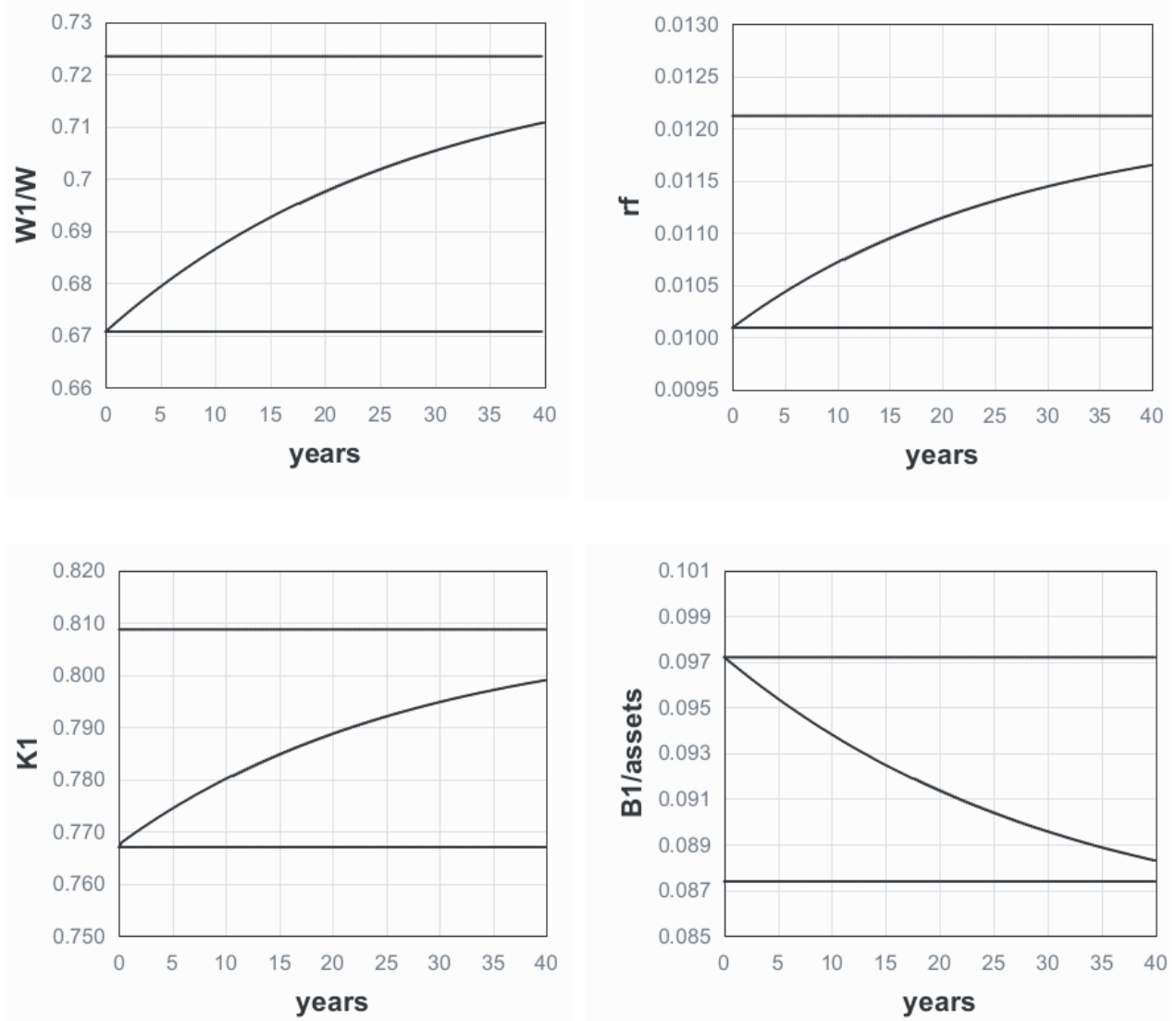
In the case signaled by a *, borrowing by agent 1 goes down by 50.

Figure 1: Dynamic Paths Following a Disaster

This analysis corresponds to the case where $\gamma_1 = 3.3$ and $\gamma_2 = 5.6$ in Table 2. The simulated paths start from the steady state value of W_1/W, 0.670, then assume that a disaster of proportionate size 0.32 materializes in period 1. Subsequently, no further disasters occur. The panels show the dynamic paths after period 1 for agent 1's wealth share, W_1/W, the risk-free interest rate, r^f, agent 1's share of total equity, K_1, and the ratio of the magnitude of safe assets, B_1, to total assets.

Figure 2: Dynamic Paths for 40 Years of Tranquility

This analysis corresponds to the case where $\gamma_1 = 3.3$ and $\gamma_2 = 5.6$ in Table 2. The simulated paths start from the steady state value of W_1/W, 0.670, then assume that no disasters occur over the next 40 years. The panels show the dynamic paths after period 1 for agent 1's wealth share, W_1/W, the risk-free interest rate, r^f, agent 1's share of total equity, K_1, and the ratio of the magnitude of safe assets, B_1, to total assets. The lines marked as "steady states" are values that would be approached asymptotically conditional on disasters never happening.