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Mechanism Design with Financially Constrained Agents and Costly Verification*

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Abstract

A principal wishes to distribute an indivisible good to a population of budget-constrained agents. Both valuation and budget are an agent's private information. The principal can inspect an agent's budget through a costly verification process and punish an agent who makes a false statement. I characterize the direct surplus-maximizing mechanism. This direct mechanism can be implemented by a two-stage mechanism in which agents only report their budgets. Specifically, all agents report their budgets in the first stage. The principal then provides budget-dependent cash subsidies to agents and assigns the goods randomly (with uniform probability) at budget-dependent prices. In the second stage, a resale market opens, but is regulated with budget-dependent sales taxes. Agents who report low budgets receive more subsidies in their initial purchases (the first stage), face higher taxes in the resale market (the second stage) and are inspected randomly. This implementation exhibits some of the features of some welfare programs, such as Singapore's housing and development board.

Keywords: Mechanism Design, Budget Constraints, Efficiency, Costly Verification

JEL Classification: D45, D61, D82, H42

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1 Introduction

Governments around the world allocate a variety of valuable resources to agents who are financially constrained. In Singapore, for example, 80% of the population's housing needs are met by the Housing and Development Board (HDB), a government agency founded in 1960 to provide affordable housing.[1] In the United States, Medicaid has provided health care to individuals and families with low income and limited resources since 1965. Medicaid currently accounts for 16.1% of the state general funds[2] and provides health coverage to 80 million low-income people.[3] Similar public housing and social health care programs prevail in many other countries.[4][5] In China, several cities limit the supply of vehicle licenses to curb the growth in private vehicles, and different cities have implemented different mechanisms. For example, Shanghai allocates vehicle licenses through an auction-like mechanism, while Beijing uses a vehicle license lottery (see Rong et al. 2015). The evaluation of existing mechanisms has attracted attention from researchers and policymakers. In comparison to lotteries, an auction-like mechanism is considered more efficient but favors high-income families more.

One justification for this role of a government is that a competitive market outcome will not maximize social surplus if agents are financially constrained. Financial constraints mean that in a competitive market some agents with high valuations will not obtain goods, while agents with low valuations but access to cash will. The natural question arises as to what the surplus-maximizing (or optimal) mechanism is in these circumstances when both valuations and financial constraints are the agents' private information.

The mechanism design literature concerning this question has focused on mechanisms with only monetary transfers and has ignored the possibility of the principal verifying the agents' reported information about their abilities to pay. Indeed, in many instances, the principal relies on agents'

[1] http://www.hdb.gov.sg/fi10/fi10320p.nsf/w/AboutUsPublicHousing?OpenDocument

[2] http://ccf.georgetown.edu/wp-content/uploads/2012/03/Medicaid-state-budgets-2005.pdf

[3] http://www.cbpp.org/research/health/policy-basics-introduction-to-medicaid?fa=view&id=2223

[4] https://en.wikipedia.org/wiki/Public_housing

[5] https://en.wikipedia.org/wiki/Universal_health_coverage_by_country

reports of their ability to pay, and the principal can verify this information and punish an agent who makes a false statement. For example, applicants for HDB flats in Singapore and Medicaid in the United States are subject to a set of eligibility conditions on age, family nucleus, monthly income, and so on. The verification process can be costly, though. First, in some developing countries, verifiable records on household income or wealth are rarely available, and governments lack the administrative capacity to process this information. In such cases, alternative verification methods such as a visit to the household to inspect the visible living conditions are not uncommon but are often costly (see Coady et al. 2004). Second, certain types of income such as tips, side-jobs and cash receipts are costly to verify. Similarly, governments have few ways to verify the income reports by individuals who are self-employed or run small business without performing a costly investigation. Third, agents may be financially constrained due to limited access to the financial market or high expenditures, such as medical expenses or education costs. This information is often costly for governments to verify. Last but not least, even when the verification cost for one individual is low, the total cost can be substantial for a large population.

Hence, it is important to explore how the option of costly verification affects the optimal mechanism. Verification allows the principal to better target low-budget agents and potentially improve their welfare. However, verification is costly and reduces the amount of money available for subsidies. The principal must now trade allocative efficiency for verification cost. The cost of verification also influences whether the principal chooses to use *cash subsidies* or *in-kind subsidies* (the provision of goods at discounted prices). The latter is less efficient because it often involves rationing, but saves verification cost because it only benefits low-budget agents with high valuations. Finally, introducing costly verification also complicates the analysis because it is no longer sufficient to consider "local" incentive compatibility (IC) constraints. Because the IC constraints between distant types can also bind, one cannot anticipate a priori the set of binding IC constraints.

To study these questions, I consider a mechanism design problem in which there is a unit mass of a continuum of agents and a limited supply of indivisible goods. Each agent has two-dimensional private information — his valuation of the good $v \in [\underline{v}, \overline{v}]$ and his exogenous budget constraint b.

The budget constraint is a hard one in the sense that agents cannot be compelled to pay more than their budgets. For simplicity, I assume there are only two possible types of budgets, $b_2 > b_1$. The principal can inspect an agent at a cost, perfectly revealing his budget, and impose a penalty on detected misreporting. The principal is also subject to a budget balance constraint which requires that the revenue from selling the good must exceed the inspection cost. This constraint rules out the possibility that the principal can inject money and relieve all budget constraints. I focus on direct mechanisms in which each agent reports private information directly and is punished if and only if found to have lied about the budget. Given the report, the mechanism specifies for each agent his probability of getting the good, his payment and his probability of being inspected.

I characterize the optimal direct mechanism which maximizes utilitarian efficiency among all mechanisms that are incentive compatible and individually rational, and that satisfy the resource constraint, agents' budget constraints and the principal's budget balance constraint.

Let $u(\underline{v}, b)$ denote the utility of an agent with the lowest valuation $\underline{v}$ and budget b, which is also the amount of cash subsidies received by agents with budget b. There exist three cutoffs $v_1^* \leq v_2^* \leq v_2^{**}$. Firstly, low-budget agents whose valuations are below v_1^* and high-budget agents whose valuations are below v_2^* only receive only cash subsidies. Not surprisingly, these low-budget agents receive higher cash subsidies and are inspected with probability proportional to the difference in cash subsidies $u(\underline{v}, b_1) - u(\underline{v}, b_2)$. Secondly, low-budget agents whose valuations exceed v_1^* receive the good with probability $a^* \leq 1$ and make a payment of $a^* v_1^* - u(\underline{v}, b_1)$. They receive both cash and in-kind subsidies. High-budget agents whose valuations lie in $[v_2^*, v_2^{**}]$ are pooled with low-budget agents whose valuations are above v_1^*. They also receive the good with probability a^*, but they make a payment of $a^* v_2^* - u(\underline{v}, b_2)$. The difference in in-kind subsidies is given by $a^*(v_2^* - v_1^*)$, and these low-budget agents are inspected with probability proportional to the sum of differences in cash and in-kind subsidies $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^*(v_2^* - v_1^*)$. Finally, high-budget agents receive the good for sure and make a payment of $v_2^{**} - u(\underline{v}, b_2)$ if their valuations exceed v_2^{**}.

If budgets are common knowledge, then the principal can without cost target low-budget agents and provides cash subsidies and in-kind subsidies only to low-budget agents. If budgets are agents' private information and cannot be verified, then high-budget agents whose valuations are below v_2^* have incentives to misreport as low-budget types to receive cash subsidies; and high-budget agents whose valuations are slightly above v_2^* have incentives to misreport as low-budget types to receive the good at a lower payment. As a result, in this case, agents with both budgets receive the same amount of cash subsidies ($u(\underline{v}, b_1) = u(\underline{v}, b_2)$) and in-kind subsidies ($v_1^* = v_2^*$).

The optimal direct mechanism can be implemented by a simple two-stage mechanism. Specifically, all agents are asked to report their budgets in the first stage. The principal then provides budget-dependent cash subsidies to agents and assigns the goods randomly (with uniform probability) at budget-dependent prices. Agents who report low budgets receive higher cash subsidies and lower prices. In the second stage, a resale market opens, but is regulated with budget-dependent sales taxes. Agents who report low budgets are subject to higher sales taxes. Only agents who report low budgets are inspected randomly. Unlike the case without inspection, in which all agents are subsidized and regulated equally regardless of their budgets, the two-stage mechanism provides more subsidies to low-budget agents in their initial purchases (the first stage) and imposes more restrictions on them in the resale market (the second stage). Although in my analysis the principal's objective is to maximize social surplus, I conjecture that these features would continue to apply when the principal wants to benefit only low-budget agents.

This implementation exhibits some features of the public housing program in Singapore, as shown in Table 1. In Singapore, buyers of resale HDB flats can apply for additional housing grants. If these flats are purchased with housing grants, these buyers are required to reside in their flats for at least 5 years before they could resell or sublet. In contrast, flats purchased without housing grants are subject to no requirement or a shorter one.

It is interesting to see how verification cost, the supply of goods and other parameters affect the optimal mechanism and welfare. I provide analytic results of comparative statics for extreme cases, such as when verification cost is sufficiently large and the supply of goods is sufficiently large or small, and I explore the intermediate case numerically.

Verification allows the principal to better target low-budget agents and improves their welfare.

Table 1: Minimum occupation periods (MOP) of housing and development board (HDB) flats

Types of HDB flats	MOP	
	Sell	Sublet
Resale flats w/ Grants	5–7 years	5–7 years
Resale flats w/o Grants	0–5 years	3 years

Sources. — Sell: http://www.hdb.gov.sg/cs/infoweb/residential/selling-a-flat/eligibility; and Sublet: http://www.hdb.gov.sg/cs/infoweb/residential/renting-out-a-flat-bedroom/renting-out-your-flat/eligibility.

Intuitively, as verification becomes costly, the principal tends to provide relatively smaller subsidies to low-budget agents and inspect them less frequently. More interestingly, the optimal mechanism makes use of both cash and in-kind subsidies, and the change in verification cost affects that mechanism's reliance on each of them. If verification is cheap, then the principal achieves efficiency mainly by offering more cash subsidies to low-budget agents. As verification becomes costly, the difference in cash subsidies declines but the difference in in-kind subsidies increases. This is because in-kind subsidies are attractive only to high-valuation agents, which is cheaper in terms of verification cost. Eventually, the difference in in-kind subsidies also declines as verification becomes sufficiently costly. Though reducing verification cost improves the welfare of low-budget agents, it may hurt high-budget agents as more subsidies are diverted to low-budget agents.

Another interesting observation is that although an increase in the supply of goods improves the total welfare, its impact on the welfare of each budget type is not monotonic. This is because an increase in the supply has two opposite effects. On the one hand, the principal becomes less budget constrained, and can direct more subsidies to low-budget agents and inspect them more frequently. On the other hand, low-budget agents also become less budget constrained, which reduces the needs to subsidize and inspect them. As a result, the differences in cash and in-kind subsidies and the inspection probability are hump-shaped. Initially, the welfare of both budget types increases as the supply increases. When the supply is large enough that the principal can afford to provide more subsidies to low-budget agents, the welfare of high-budget agents begins to decrease. Eventually, the need to subsidize low-budget agents decreases as the supply increases while the welfare of low-budget agents begins to decrease and that of high-budget agents begins to increase, until they coincide.

Technically, this paper develops a novel method that can potentially be used in solving other mechanism design problems with multidimensional types. If each agent has only one-dimensional private information, i.e., valuation, then it is sufficient to consider adjacent IC constraints; if each agent has two-dimensional private information but the principal cannot inspect budgets, then it is sufficient to consider two one-dimensional deviations. These, however, no longer apply in the case that each agent has two dimensional private information and the principal can inspect budget at a cost. In this case, in addition to downward adjacent IC constraints of misreporting values, one must consider deviations in which an agent can misreport both dimensions of his private information. As a result, the local approach commonly used does not work here.

To overcome this difficulty, I first restrict attention to a class of allocation rules that have enough structures to help me keep track of binding IC constraints, and that are also rich enough to approximate any general allocation rule well. Specifically, I approximate the allocation rule of each budget type using step functions. When restricting attention to step functions, binding IC constraints corresponding to the under-reporting of budgets are between different budget types whose values are the jump discontinuity points of their allocation rules. This structure allows me to write the optimal inspection rule as a function of the possible values and jump discontinuity points of the allocation rule. I then solve a modification of the principal's problem in which the allocation rule of low-budget types are restricted to take at most M distinct values. Because for M sufficiently large step-functions can approximate the optimal allocation rule arbitrarily well, I can obtain a characterization of the optimal mechanism in the limit.

The rest of the paper is organized as follows. Section 1.1 discusses related work. Section 2 presents the model. Section 3 characterizes the direct optimal mechanism when all agents' budget constraints are common knowledge. Section 4 characterizes the direct optimal mechanism when an agent's budget is his private information. Section 5 provides a simple implementation. Section 6 studies the properties of the optimal mechanism. Section 7 considers various extensions of the model. Section 8 concludes. All the proofs are relegated to the appendix.

1.1 Related Literature

This paper is related to two branches of literature. First, it contributes to the literature studying mechanism design problems when agents are financially constrained by incorporating costly verification. Prior work analyzes the revenue or efficiency of a given mechanism or the design of an optimal mechanism when either budgets are common knowledge, or budgets are agents' private information but cannot be verified. See Che and Gale (1998, 2006, 2000), Laffont and Robert (1996), Maskin (2000), Benoit and Krishna (2001), Brusco and Lopomo (2008), Malakhov and Vohra (2008) and Pai and Vohra (2014).

In this first branch of literature, the two closest papers to the current paper are Che et al. (2013) and Richter (2015). In Che et al. (2013) and Richter (2015), like in this paper, there is a unit mass of a continuum of agents and a limited supply of goods. In Richter (2015) agents have linear preferences for an unlimited supply of the goods. He finds that both the revenue-maximizing mechanism and surplus-maximizing mechanism feature a linear price for the good. In addition, the surplus-maximizing mechanism has a uniform cash subsidy. In both Che et al. (2013) and this paper, each agent has a unit demand for an indivisible good, and the surplus-maximizing mechanism can be implemented via a random assignment with a regulated resale and cash subsidy scheme. However, Che et al. (2013) does not consider the possibility that the principal can verify an agent's budget at a cost. This feature also distinguishes the current paper from all the other papers on mechanism design with financially constrained agents. Che et al. (2013) first compare three different methods of assigning the goods when agents have a continuum of possible valuations and a continuum of possible budgets, and then characterize the optimal mechanism in a simple 2×2 model, in which each agent has two possible valuations of the good and two possible budgets. In the presence of costly verification, unlike Che et al. (2013), in which all agents are subsidized and regulated equally regardless of their budgets in an optimal mechanism, I show that an optimal mechanism provides more subsidies to low-budget agents in their initial purchases and imposes more restrictions on them in the resale market.

Second, this paper is related to the costly state verification literature. The first significant contribution to this series is from Townsend (1979), who studies a model of a principal and a single agent. In Townsend (1979) verification is deterministic. Border and Sobel (1987) and Mookherjee and Png (1989) generalize it by allowing random inspection. Gale and Hellwig (1985) consider the effects of costly verification in the context of credit markets. Recently, Ben-Porath et al. (2014) study the allocation problem in the costly state verification framework when there are multiple agents and monetary transfer is not possible. Li (2016) extends Ben-Porath et al. (2014) to environments in which the principal's ability to punish an agent is limited. These models differ from what I consider here in that in their models each agent has only one-dimensional private information.

This paper is also somewhat related to the literature on costless or ex-post verification. Glazer and Rubinstein (2004) can be interpreted as a model of a principal and one agent with limited but costless verification and no monetary transfers. Mylovanov and Zapechelnyuk (2014) study a model of multiple agents with costless verification but limited punishments. This paper differs from these earlier studies in that each agent has two-dimensional private information, verification is costly and there are monetary transfers.

In the literature discussed above, one can anticipate a priori the set of binding IC constraints, which is no longer true here. Instead, I use new techniques for keeping track of binding IC constraints.

2 Model

There is a unit mass of a continuum of agents. There is a mass $S \in (0, 1)$ of indivisible goods.[6] Each agent has a private valuation of the good $v \in V := [\underline{v}, \overline{v}] \subset \mathbb{R}_+$, and a privately known budget $b \in B := \{b_1, b_2\}$. I assume that $b_1 > \underline{v}$ and $b_2 > \overline{v}$.[7] Thus, a high-budget agent is never budget constrained in an individually rational mechanism. The type of an agent is a pair consisting of his valuation and his budget: $t := (v, b)$; and the type space is $T := V \times B$.

[6] The model is also applicable to divisible goods when an agent's per-unit value for the good is constant up to an upper bound.

[7] All the results can be easily extended to any $b_1 \geq 0$. In the paper, I assume $b_1 > \underline{v}$ to make the statement more concise.

I assume v and b are independent. Each agent has a high budget with probability π and a low budget with probability $1 - \pi$. The valuation v is distributed with cumulative distribution function F and strictly positive density f.

The principal can inspect an agent's budget at a cost $k \geq 0$, and can impose a penalty $c > 0$. Inspection perfectly reveals an agent's budget.[8] I assume that the penalty c is large enough that an agent never find it optimal to misreport his budget if he is certain he will be inspected. For the main body of the paper, I assume that the penalty is not transferable. In Section 7.2, I study the case in which penalty is transferable and show that all results hold in that case. For later use, let $\rho := k/c$. As it will become clear, ρ measures the "effective" inspection cost to the principal. The cost to an agent to have his report verified is zero. This assumption is reasonable if the goods are valuable to agents and disclosure costs are negligible. In Section 7.3, I discuss what happens if it is also costly for an agent to have his report verified.

The usual version of the revelation principle (see, e.g., Myerson 1979 and Harris and Townsend 1981) does not apply to models with verification. However, it is not hard to extend the argument to this type of environment.[9] Specifically, I show in Appendix A that it is without loss of generality to restrict attention to direct mechanisms. Furthermore, I assume that the principal can only punish an agent who is inspected and found to have lied about his budget. This assumption, however, is not without loss of generality. I discuss what happens if the principal is allowed to punish an agent without verifying his budget or to punish an agent who is found to have reported his budget truthfully in Section 7.4.

A direct mechanism is a triple (a, p, q), where $a : T \to [0, 1]$ denotes the probability an agent obtains the good, $p : T \to \mathbb{R}$ denotes the payment an agent must make and $q : T \to [0, 1]$ denotes the probability of inspection. In this definition, I implicitly assume that payment rules are deterministic. I discuss random payment rules at the end of this section and show that it is without loss of generality to focus on deterministic payment rules.

[8]The paper's results will not change if the principal cannot detect a lie with some probability.

[9]See Townsend (1988) and Ben-Porath et al. (2014) for more discussion and extension of the revelation principle to various verification models, not including the environment considered in this paper.

The utility of an agent who has type $t := (v, b)$ and reports $\hat{t}$ is

$$u(\hat{t}, t) = \begin{cases} a(\hat{t})v - p(\hat{t}) & \text{if } \hat{b} = b \text{ and } p(\hat{t}) \leq b, \\ a(\hat{t})v - q(\hat{t})c - p(\hat{t}) & \text{if } \hat{b} \neq b \text{ and } p(\hat{t}) \leq b, \\ -\infty & \text{if } p(\hat{t}) > b. \end{cases}$$

An agent has a standard quasi-linear utility up to his budget constraint, and cannot pay more than his budget.

The welfare criterion I use is utilitarian efficiency. For why utilitarian efficiency is a reasonable welfare criterion, see Vickrey (1945) and Harsanyi (1955). Given quasi-linear preferences, the total value realized minus total inspection cost is an equivalent criterion.[10] The principal's problem is[11]

$$\max_{a,p,q} \mathbb{E}_t \left[a(t)v - q(t)k \right], \tag{$\mathcal{P}$}$$

subject to

$$u(t) \equiv u(t,t) \geq 0, \ \forall t \in T, \tag{IR}$$

$$u(t) \geq u(\hat{t}, t), \ \forall t \in T, \hat{t} \in \left\{ \hat{t} \in T \,\middle|\, p(\hat{t}) \leq b \right\}, \tag{IC}$$

$$p(t) \leq b, \ \forall t \in T, \tag{BC}$$

$$\mathbb{E}_t \left[p(t) - q(t)k \right] \geq 0, \tag{BB}$$

$$\mathbb{E}_t \left[a(t) \right] \leq S. \tag{S}$$

The individual rationality (IR) constraint requires that each agent gets a non-negative expected

[10]To see this, consider a feasible mechanism (a, p, q). Note that if (a, p, q) maximizes welfare, then (BB) must hold with equality. Otherwise the principal can improve welfare through lump-sum transfers. Then the principal's objective function becomes

$$\mathbb{E}[u(t)] = \mathbb{E}[va(t) - p(t)] = \mathbb{E}[va(t) - q(t)k],$$

where the last equality holds since (BB) holds with equality.

[11]There are some subtle issues with a continuum of random variables. See Judd (1985). However, if we interpret the continuum model as an approximation of a large economy, then Al-Najjar (2004) makes the limiting argument rigorous.

payoff from participating in the mechanism. The incentive compatibility (IC) constraint requires that it is weakly better for an agent to report his true type than any other type whose transfers he can afford. The budget constraint (BC) states that an agent cannot be ask to make a payment larger than his budget b. To be clear, note that (BC) follows from (IR). This budget constraint is the same as that found in Che and Gale (2000) and Pai and Vohra (2014), but different from Che et al. (2013), who use a per unit price constraint.[12] I discuss the differences of the two frameworks in Section 7.1. The principal's budget balance (BB) constraint requires that the revenue raised from selling the goods must exceed the inspection cost. (BB) rules out the possibility that the principal can inject money and relieve all budget constraints. Finally, the limited supply (S) constraint, which requires that the amount of good assigned cannot exceed the supply. We say a mechanism (a, p, q) is *feasible* if it satisfies constraints (IR), (IC), (BC), (BB) and (S).

Throughout the paper, I assume that $S < 1 - F(b_1)$ since otherwise the first-best can be achieved via a competitive market. I also impose the following two assumptions throughout the paper.

Assumption 1 $\frac{1-F}{f}$ *is non-increasing.*

Assumption 2 f *is non-increasing.*

Assumption 1 is the standard monotone hazard rate condition, which is often adopted in the mechanism design literature. This assumption ensures that allocating more good to agents with higher valuations from those with lower valuations generates higher revenues for the principal. Assumption 2 says that agents are less likely to have higher valuations than to have lower valuations. These two assumptions are also imposed in Richter (2015) and Pai and Vohra (2014). These two assumptions are satisfied by some commonly used distributions such as uniform distributions, exponential distributions and left truncation of a normal distribution.

I conclude this section with a discussion of random payment rules.

[12] This constraint is called ex-post budget constraint in Che et al. (2013).

2.1 Random Payment Rules

When defining a direct mechanism, I implicitly assume that the payment rule is deterministic. I argue that this is without loss of generality. Consider a random payment rule $\tilde{p} : T \to \Delta(\mathbb{R})$. Let $\text{supp}(\tilde{p}(t))$ denote the supremum of payments in the support of $\tilde{p}(t)$. The utility of an agent who has type t and report $\hat{t}$ is

$$u(\hat{t}, t) = \begin{cases} a(\hat{t})v - \mathbb{E}[\tilde{p}(\hat{t})] & \text{if } \hat{b} = b \text{ and } \text{supp}(\tilde{p}(\hat{t})) \leq b, \\ a(\hat{t})v - q(\hat{t})c - \mathbb{E}[\tilde{p}(\hat{t})] & \text{if } \hat{b} \neq b \text{ and } \text{supp}(\tilde{p}(\hat{t})) \leq b, \\ -\infty & \text{if } \text{supp}(\tilde{p}(\hat{t})) > b. \end{cases}$$

In other words, an agent suffers an unbounded dis-utility if his budget constraint is violated with a positive probability. The IC constraints become

$$u(t) \geq u(\hat{t}, t), \ \forall t \in T, \hat{t} \in \left\{ \hat{t} \in T \,\middle|\, \text{supp}(\tilde{p}(\hat{t})) \leq b \right\}, \tag{IC}$$

The principal's objective function and all the other constraints remain intact.

By a similar argument to that used in Pai and Vohra (2014), for any feasible mechanism $(a, \tilde{p}, q)$, one can construct another feasible mechanism $(a, \tilde{p}, q)$ by setting

$$\hat{p}(t) = \begin{cases} \mathbb{E}[\tilde{p}(t)] - \epsilon & \text{with propability } \frac{b - \mathbb{E}[\tilde{p}(t)]}{b - \mathbb{E}[\tilde{p}(t)] + \epsilon}, \\ b & \text{with propability } \frac{\epsilon}{b - \mathbb{E}[\tilde{p}(t)] + \epsilon}, \end{cases}$$

for some $\epsilon > 0$ sufficiently small. Furthermore, both mechanisms have the same welfare. Observe that, under this construction, IC constraints corresponding to over reporting of budget are satisfied "for free". Given these observations, it is not hard to see that one can solve the principal's problem (allowing for random payment rules) by restricting attention to deterministic payment rules but relaxing IC constraints corresponding to the over reporting of budget. As I will show later, in the optimal mechanism of $\mathcal{P}$ no low-budget agent has any incentive to over report his budget. Hence, it is without loss of generality to focus on deterministic payment rules.

3 Common Knowledge Budgets

As a benchmark, I first analyze the case in which all agents' budget constraints are common knowledge. This case can be viewed as the situation in which the principal can inspect an agent's budget for free (i.e., $\rho = k/c = 0$).

Since budgets are common knowledge, the IC constraints hold as long as for each $b \in B$, no agent has incentive to misreport his value:

$$a(v,b)v - p(v,b) \leq a(\hat{v},b)v - p(\hat{v},b),\ \forall v,\hat{v}. \tag{IC-v}$$

The principal's problem becomes

$$\max_{a,p,q} \mathbb{E}_t\left[a(t)v\right], \tag{$\mathcal{P}_{CB}$}$$

subject to (IR), (IC-v), (BC), (S) and

$$\mathbb{E}_t[p(t)] \geq 0, \forall t \in T. \tag{BB}$$

By the standard argument, (IC-v) holds if and only if for all $b \in B$, $a(v,b)$ is non-decreasing in v and $p(v,b) = a(v,b)v - \int_{\underline{v}}^{v} a(\nu,b)\mathrm{d}\nu - u(\underline{v},b)$. Since $a(v,b)$ is non-decreasing in v, the payment $p(v,b)$ is also non-decreasing in v. Hence, (BC) holds if and only if $p(\overline{v},b) \leq b$ for all b.

Let χ denote the characteristic function. The following theorem characterizes the optimal mechanism.

Theorem 1 *Suppose Assumption 2 holds, and budgets are common knowledge. There exist* $v_1^*(0)$,

$v_2^*(0)$, $u_1^*(0)$ and $u_2^*(0)$ such that an optimal mechanism of $\mathcal{P}_{CB}$ is given by

$$\begin{aligned} a(v, b_1) &= \chi_{\{v \geq v_1^*(0)\}} a^*(0), \quad & p(v, b_1) &= \chi_{\{v \geq v_1^*(0)\}} (u_1^*(0) + b_1) - u_1^*(0), \\ a(v, b_2) &= \chi_{\{v \geq v_2^*(0)\}} 1, & p(v, b_2) &= \chi_{\{v \geq v_2^*\}} v_2^*(0), \end{aligned}$$

where $a^*(0) = \left[u_1^*(0) + b_1\right] / v_1^*(0)$, $b_1 < v_1^*(0) \leq v_2^*(0) < \overline{v}$ and $0 = u_2^*(0) < u_1^*(0) \leq v_1^*(0) - b_1$.

In notations $a^*(0)$, $v_i^*(0)$ and $u_i^*(0)$ $(i = 1, 2)$, subscript i indicates the corresponding budget b_i and argument 0 indicates that this can be viewed as an optimal mechanism when $\rho = 0$.

As expected, when budgets are common knowledge, the two budget group can be treated separately. Only low-budget agents receive positive cash subsidies aiming to relax their budget constraints: $u(\underline{v}, b_1) = u_1^*(0) > 0 = u_2^*(0) = u(\underline{v}, b_2)$. There are two cutoffs: $v_1^*(0) \leq v_2^*(0)$. All high-budget agents whose valuations are above $v_2^*(0)$ receive the good with probability one. This allocation can be implemented by posting a price $v_2^*(0)$ for high-budget agents, which is the efficient mechanism when agents are not financially constrained. All low-budget agents whose valuations are above $v_1^*(0)$ receive the good with positive probability but are possibly rationed. The intuition for rationing is familiar from the literature. Increasing allocations to low value agents reduces the payment of high value agents and therefore “relaxes" their budget constraints.

Clearly, a high-budget agent whose value is below $v_1^*(0)$ has a strict incentive to misreport as a low-budget agent since $u(\underline{v}, b_1) > 0 = u(\underline{v}, b_2)$. A high-budget agent whose value is slightly above $v_1^*(0)$ also has strict incentives to misreport as a low-budget agent:

$$\frac{v\left(u(\underline{v}, b_1) + b_1\right)}{v_1^*(0)} - b_1 > \frac{\left(v - v_1^*(0)\right) b_1}{v_1^*(0)} \geq \max\left\{v - v_2^*(0), 0\right\}.$$

The last inequality holds for $v > v_1^*(0)$ sufficiently close to $v_1^*(0)$. As it will become clear in Section 4.1, when budgets are agents' private information and the principal does not inspect, to discourage agents from under reporting their budgets, it must be that $u(\underline{v}, b_1) = u(\underline{v}, b_2)$ and a high-budget agent must receive the good with a probability no less than that of a low-budget agent who has the same valuation.

4 Privately Known Budgets

In this section, I analyze the case in which an agent's budget is his private information. In this case, IC constraints can be separated into two categories:

$$\text{Misreport value: } a(v,b)v - p(v,b) \geq a(\hat{v},b)v - p(\hat{v},b),\ \forall v,\hat{v},b, \tag{IC-v}$$

$$\text{Misreport both: } a(v,b)v - p(v,b) \geq \chi_{\{p(\hat{v},\hat{b})\leq b\}}\left(a(\hat{v},\hat{b})v - q(\hat{v},\hat{b})c - p(\hat{v},\hat{b})\right),\ \forall v,\hat{v},b,\hat{b}. \tag{1}$$

As I stated in the previous section, (IC-v) holds if and only if for all $b \in \boldsymbol{B}$, $a(v,b)$ is non-decreasing in v and $p(v,b) = va(v,b) - \int_{\underline{v}}^{v} a(\nu,b)\mathrm{d}\nu - u(\underline{v},b)$. The difficulty arises from (1). In what follows, I first consider a relaxed problem by replacing (1) with the following constraint:

$$a(v,b_2)v - p(v,b_2) \geq a(\hat{v},b_1)v - q(\hat{v},b_1)c - p(\hat{v},b_1),\ \forall v,\hat{v}. \tag{IC-b}$$

This relaxation formalizes the intuition that the principal's main concern is to prevent high-budget agents from falsely claiming to be low-budget agents. Later, I verify that an optimal mechanism of the relaxed problem automatically satisfies IC constraints corresponding to over-reporting of budgets. In other words, it also solves the original problem.

To summarize, the principal's relaxed problem is

$$\max_{a,p,a} \mathbb{E}_t[a(t)v - q(t)k], \tag{$\mathcal{P}'$}$$

subject to (IR), (IC-v), (IC-b), (BC), (BB) and (S).

4.1 No Verification

In this section, I consider the case in which the principal does not inspect agents, i.e., $q \equiv \mathbf{0}$. In this case, as will become clear in the discussion below, it is sufficient to consider two *one-dimensional deviations*, which greatly simplifies the analysis. Although some of the results may be familiar, it highlights the differences in my approach. Denote the principal's problem in this case by $\mathcal{P}_{NI}$ and the corresponding relaxed problem by $\mathcal{P}'_{NI}$. As will become clear in Section 6, if the inspection cost, k, is sufficiently high relative to the punishment, c, then it is optimal for the principal not to use inspection. In particular, this is the case when the principal's inspection cost is infinity (i.e., $\rho = k/c = \infty$).

Observe first that in this case (IC-b) holds if and only if (IC-v) holds and

$$a(v, b_2)v - p(v, b_2) \geq a(v, b_1)v - p(v, b_1), \ \forall v. \tag{2}$$

To see this, note that if (2) holds, then

$$\begin{aligned} a(v, b_2)v - p(v, b_2) &\geq a(v, b_1)v - p(v, b_1) \\ &\geq a(\hat{v}, b_1)v - p(\hat{v}, b_1), \end{aligned}$$

where the second inequality follows from (IC-v). Thus, it is sufficient to consider the two one-dimensional deviations: only misreport value and only misreport budget. The above inequality says that if a type (v, b_2) agent has no incentive to misreport (v, b_1), then he has no incentive to misreport $(\hat{v}, b_1)$. This argument is not true when there is verification because it is possible that types (v, b_1) and $(\hat{v}, b_1)$ are inspected with different probabilities. Instead, one must identify for each type $(\hat{v}, b_1)$ the high-budget type who benefits most from misreporting $(\hat{v}, b_1)$ in the absence of inspection, which determines the set of binding (IC-b) constraints.

Using the envelope condition, (2) can be rewritten as

$$u(\underline{v}, b_2) + \int_{\underline{v}}^{v} a(\nu, b_2)\mathrm{d}\nu \geq u(\underline{v}, b_1) + \int_{\underline{v}}^{v} a(\nu, b_1)\mathrm{d}\nu, \ \forall v. \tag{3}$$

If $v = \underline{v}$, then (3) implies that $u(\underline{v}, b_2) \geq u(\underline{v}, b_1)$. If $u(\underline{v}, b_2) > u(\underline{v}, b_1)$, then one can construct another feasible mechanism by reducing cash subsidies to high-budget agents while increasing their probabilities of receiving the goods, which generates the same welfare. Hence, it is without loss of generality to assume that $u(\underline{v}, b_1) = u(\underline{v}, b_2)$. This result is summarized in Lemma 1, and a complete proof can be found in the appendix.[13]

Lemma 1 *Suppose Assumption 2 holds, and the principal does not inspect agents. In an optimal mechanism of* $\mathcal{P}'_{NI}$*, it is without loss of generality to assume that* $u(\underline{v}, b_1) = u(\underline{v}, b_2)$.

One implication of Lemma 1 is that in an optimal mechanism agents receive positive cash subsidies regardless of their budgets. This result contrasts the case of common knowledge budgets in which only low-budget agents receive positive cash subsidies.

Next, I show that, for any given v, an optimal mechanism on average allocates weakly more resources to high-budget agents whose valuations are below v than to low-budget agents whose valuations are below v.

Lemma 2 *Suppose Assumptions 1 and 2 hold, and the principal does not inspect agents. In an optimal mechanism of* $\mathcal{P}'_{NI}$*, the allocation rule satisfies*

$$\int_{\underline{v}}^{v} a(\nu, b_2) f(\nu)\mathrm{d}\nu \geq \int_{\underline{v}}^{v} a(\nu, b_1) f(\nu)\mathrm{d}\nu, \ \forall v. \tag{4}$$

Given Lemma 1, (4) follows immediately from (3) if v is uniformly distributed. Lemma 2 shows that the result holds more generally for any distribution with non-increasing density. Using Lemmas 1 and 2, one can prove the following theorem, which characterizes the optimal direct mechanism.

[13] It is immediate that $u(\underline{v}, b_1) = u(\underline{v}, b_2)$ if one also requires that a low-budget agent has no incentive to misreport as a high-budget agent.

Theorem 2 *Suppose Assumptions 1 and 2 hold, and the principal does not inspect agents. There exist* $v_1^*(\infty)$, $v_2^*(\infty)$, $v_2^{**}(\infty)$, $u_1^*(\infty)$ *and* $u_2^*(\infty)$ *such that an optimal mechanism of* $\mathcal{P}_{NI}$ *with no inspection satisfies*

$$
\begin{aligned}
&a(v, b_1) = \chi_{\{v \geq v^*(\infty)\}} a^*(\infty),\ p(v, b_1) = \chi_{\{v \geq v_1^*(\infty)\}} (u_1^*(\infty) + b_1) - u_1^*(\infty), \\
&a(v, b_2) = \chi_{\{v \geq v_2^*(\infty)\}} a^*(\infty) + \chi_{\{v \geq v_2^{**}(\infty)\}} (1 - a^*(\infty)), \\
&p(v, b_2) = \chi_{\{v \geq v_2^*(\infty)\}} (u_2^*(\infty) + b_1) + \chi_{\{v \geq v_2^{**}(\infty)\}} (1 - a^*(\infty))\, v_2^{**}(\infty) - u_2^*(\infty),
\end{aligned}
$$

where

$$a^*(\infty) = \frac{u_1^*(\infty) + b_1}{v_1^*(\infty)},$$

$b_1 < v_1^*(\infty) = v_2^*(\infty) \leq v_2^{**}(\infty) \leq \overline{v}$ *and* $0 < u_1^*(\infty) = u_2^*(\infty) \leq v_1^*(\infty) - b_1$.

In notations $a^*(\infty)$, $v_i^*(\infty)$, $v_2^{**}(\infty)$ and $u_i^*(\infty)$ $(i = 1, 2)$, subscript i indicates the corresponding budget b_i and argument ∞ indicates that this can be viewed as an optimal mechanism when $\rho = \infty$.

Not surprisingly the optimal allocation rule obtained here shares similar features with the one found in Pai and Vohra (2014).There are three cutoffs: $v_1^*(\infty) = v_2^*(\infty) < v_2^{**}(\infty)$. All high-budget agents whose valuations are above $v_2^{**}(\infty)$ receive the good with probability one. All low-budget agents whose valuations are above $v_1^*(\infty)$ receive the good with positive probability but may be rationed. In addition, high-budget agents whose valuations are in $[v_2^*(\infty), v_2^{**}(\infty)]$ are pooled with low-budget agents whose valuations are at least $v_1^*(\infty)(= v_2^*(\infty))$. To understand this pooling, consider two agents with the same valuation v, but different budgets $b_2 > b_1$. Then (IC-b) implies that as long as agent (v, b_2)'s payment is less than b_1, he must receive the good with the same probability as (v, b_1) does.

The proof of Theorem 2 follows a weight-shifting argument similar to that of Lemma 1 in Richter (2015). Consider a feasible mechanism $(a, p, \mathbf{0})$ whose allocation rule is indicated by the two thick dotted curves in Figure 1. One can construct another feasible mechanism $(a^*, p^*, \mathbf{0})$, whose allocation rule is indicated by the thick solid lines, in the following way. Find a v_1^* and shift the allocation mass of low-budget agents from the region to the left of v_1^* to the region to the right

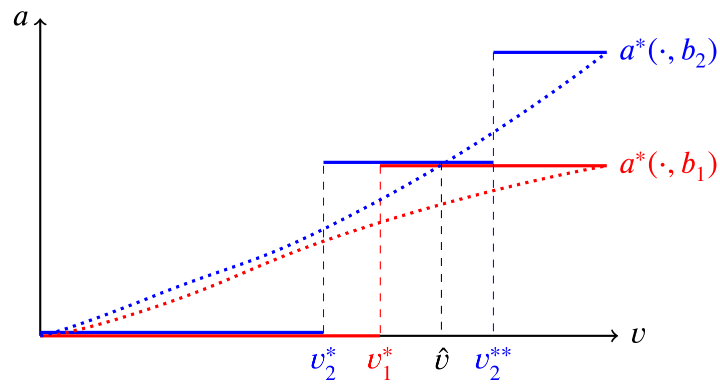


Figure 1: Proof sketch of Theorem 2

of v_1^*. The choice of v_1^* is uniquely determined so that the supply to low-budget agents remains unchanged. Let $\hat{v}$ denote the minimum valuation of high-budget agents who receive the good with a probability of at least $a(\overline{v}, b_1) = a^*(\overline{v}, b_1)$. Find v_2^* and v_2^{**} such that $v_2^* \leq \hat{v} \leq v_2^{**}$. Shift the allocation mass of high-budget agents from the region to the left of v_2^* to $[v_2^*, \hat{v}]$ and from $[\hat{v}, v_2^{**}]$ to the region to the right of v_2^{**}. The choice of v_2^* (and v_2^{**}, respectively) is uniquely determined so that the supply to high-budget agents whose valuations are in $[\underline{v}, \hat{v}]$ (and $[\hat{v}, \overline{v}]$, respectively) remains unchanged. Finally, define the new payment rule using the envelope condition. If f is "regular", i.e., satisfies Assumptions 1 and 2, then the new mechanism improves welfare and revenue while remaining affordable. Lemma 2 guarantees that $v_2^* \leq v_1^*$. Thus, no high-budget agent has incentive to misreport his budget. It is easy to see that one can further improve welfare by increasing v_2^* and reducing v_1^*. Hence, in an optimal mechanism $v_1^*(\infty) = v_2^*(\infty)$.

4.2 The General Case

I now turn to the general problem of the principal. Using the envelope condition, (IC-b) becomes the following: For all v and $\hat{v}$,

$$u(\underline{v}, b_2) + \int_{\underline{v}}^{v} a(\nu, b_2)\mathrm{d}\nu \geq u(\underline{v}, b_1) + a(\hat{v}, b_1)(v - \hat{v}) - q(\hat{v}, b_1)c + \int_{\underline{v}}^{\hat{v}} a(\nu, b_1)\mathrm{d}\nu. \tag{IC-b}$$

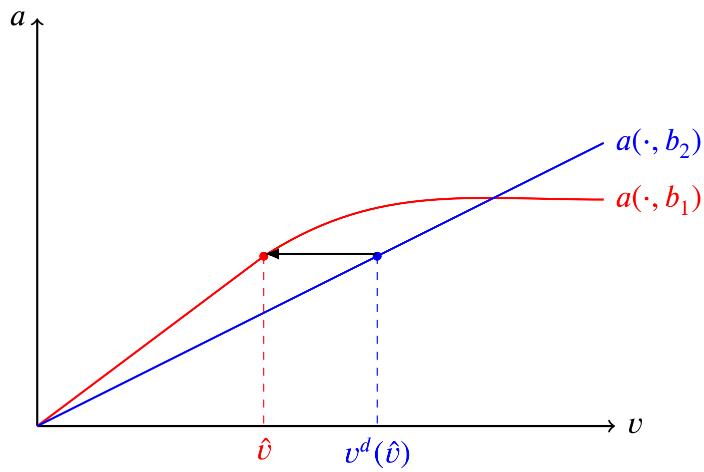


Figure 2: The set of binding (IC-b) constraints

First, for each $\hat{v}$, I identify the type of high-budget agents whose gains from falsely claiming to be a type $(\hat{v}, b_1)$ agent are the largest. (IC-b) holds if and only if for each $\hat{v} \in V$, $q(\hat{v}, b_1)c \geq \sup_v \Delta(v, \hat{v})$, where

$$\Delta(v, \hat{v}) \equiv u(\underline{v}, b_1) - u(\underline{v}, b_2) - \int_{\underline{v}}^{v} a(\nu, b_2)\mathrm{d}\nu + a(\hat{v}, b_1)(v - \hat{v}) + \int_{\underline{v}}^{\hat{v}} a(\nu, b_1)\mathrm{d}\nu.$$

Since $\partial\Delta(v, \hat{v})/\partial v = -a(v, b_2) + a(\hat{v}, b_1)$ is non-increasing in v, $\Delta(v, \hat{v})$ is concave in v and achieves its maximum at $v = v^d(\hat{v})$, where

$$v^d(\hat{v}) \equiv \inf \left\{ v | a(v, b_2) \geq a(\hat{v}, b_1) \right\}. \tag{5}$$

Suppose the allocation rules for both budget types are continuous in value v. Then the high-budget agents who benefit most from falsely claiming to be $(\hat{v}, b_1)$ are those who get the goods with the same probability as type $(\hat{v}, b_1)$ agents do. This point is illustrated by Figure 2, which plots an allocation rule for high-budget agents, $a(\cdot, b_2)$, and an allocation rule for low-budget agents, $a(\cdot, b_1)$, as a function of their valuations v.

Since the principal's objective function is strictly decreasing in q, the optimal inspection rule satisfies

$$q(\hat{v}, b_1) = \frac{1}{c} \max\left\{0, \Delta(v^d(\hat{v}))\right\}. \tag{6}$$

Note that $v^d(\cdot)$ is defined using the allocation rule. As a result, one cannot anticipate, a priori, which (IC-b) constraint binds. Furthermore, (IC-b) constraints are frequently binding not only among local types. These difficulties are inherent in all multidimensional problems, and as a result the existing approaches in the mechanism literature do not apply to this problem.[14]

In order to keep track of the binding (IC-b) constraints, we solve the principal's problem by approximating the allocation rule using step functions. Fix $M \geq 2$. Let $\underline{v} = v_1^0 < v_1^1 < \cdots < v_1^M = \overline{v}$ and $0 = a^0 \leq a^1 < a^2 < \cdots < a^M \leq a^{M+1} = 1$. Suppose the allocation rule for type b_1 agents takes M distinct values: $a(v, b_1) = a^m$ if $v \in (v_1^{m-1}, v_1^m)$ for $m = 1, \ldots, M$. The next lemma shows that the optimal allocation rule for type b_2 agents can take at most $M + 2$ distinct values: $a^0, a^1, \ldots, a^{M+1}$.

Lemma 3 *Suppose Assumptions 1 and 2 hold. Suppose $a(v, b_1) = a^m$ if $v \in (v_1^{m-1}, v_1^m)$ for $m = 1, \ldots, M$. Then there exists $\underline{v} \leq v_2^0 \leq v_2^1 \leq \cdots \leq v_2^M \leq \overline{v}$ such that an optimal allocation rule for b_2 satisfies $a(v, b_2) = a^m$ if $v \in (v_2^{m-1}, v_2^m)$ for $m = 1, \ldots, M$, $a(v, b_2) = 0$ if $v < v_2^0$ and $a(v, b_2) = 1$ if $v > v_2^M$.*

The proof of Lemma 3 is similar to that of Theorem 2 and illustrated by Figure 3, where the allocation rule for low-budget agents (the solid red line) takes three distinctive values: $a^1 < a^2 < a^3$. Consider a feasible allocation rule for high-budget agents indicated by the dotted blue curve. Suppose there exist a payment rule and an inspection to be used in conjunction with the allocation rule so that the resulting mechanism is feasible. For ease of exposition, suppose $a(\cdot, b_2)$ is continuous and let $\hat{v}_2^m$ be such that $a(\hat{v}_2^m, b_2) = a^m$ for $m = 1, 2, 3$. For each $m = 1, 2, 3$, find v_2^m and move the allocation mass of high-budget agents from $[\hat{v}_2^m, v_2^m]$ to $[v_2^m, \hat{v}_2^{m+1}]$, where $\hat{v}_2^4 = \overline{v}$. The choice of v_2^m is uniquely determined so that the supply to high-budget agents whose value is in $[\hat{v}_2^m, \hat{v}_2^{m+1}]$ remains unchanged. Redefine the payment rule using the envelope condition and let the inspection

[14] See Rochet and Stole (2003) for a survey on multidimensional mechanism design problem.

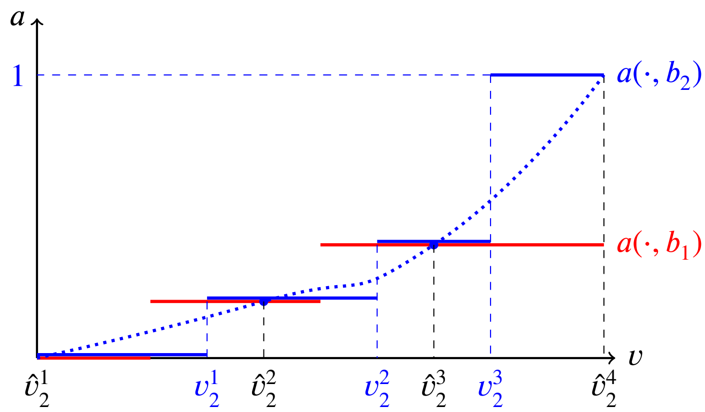


Figure 3: Proof Sketch of Lemma 3

rule remain the same. One can verify that the new mechanism is feasible and clearly improves welfare.

We say an allocation rule a is an M*-step allocation rule* if there exist $\underline{v} = v_1^0 < v_1^1 < \cdots < v_1^M = \overline{v}$, $\underline{v} \le v_2^0 \le v_2^1 \le \cdots \le v_2^M \le \overline{v}$ and $0 = a^0 \le a^1 < a^2 < \cdots < a^M \le a^{M+1} = 1$ for some $M \ge 2$ such that $a(v, b_1) = a^m$ if $v \in (v_1^{m-1}, v_1^m)$ for $m = 1, \ldots, M$ and $a(v, b_2) = a^m$ if $v \in (v_2^{m-1}, v_2^m)$ for $m = 0, 1, \ldots, M+1$. Lemma 3 shows that it is without loss of generality to focus on M-step-allocation rules among all step allocation rules.

Consider a mechanism using a M-step allocation rule. It is easy to see that for $v \in (v_1^{m-1}, v_1^m)$, the type b_2 agents who benefit most from falsely claiming to be type (v, b_1) have valuations $v^d(v) = v_2^{m-1}$. Hence, we can keep track of the binding (IC-b) constraints by keeping track of the jump points of the allocation rule. In this case, the optimal inspection rule satisfies $q(v, b_1) = q^m$ for all $v \in (v_1^{m-1}, v_1^m)$ and

$$q^m = \frac{1}{c} \max \left\{ 0, u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m} (a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) \right\} \tag{7}$$

for $m = 1, \ldots, M$.

Consider the principal's problem $(\mathcal{P}')$ with two modifications:

$$\max_{a,p,q} \mathbb{E}_t[a(t)v - q(t)k], \tag{$\mathcal{P}'(M,d)$}$$

subject to (IR), (IC-v), (IC-b), (BC), (S),

$$a \text{ is a } M'\text{-step allocation rule for some } M' \leq M,$$

$$\mathbb{E}[p(t) - q(t)k] \geq -d. \tag{BB-d}$$

The second modification is to relax the government's budget balance constraint by $d \geq 0$. As it will become clear later, any feasible mechanism of $\mathcal{P}'$ can be approximated arbitrarily well by a feasible mechanism of $\mathcal{P}'(M, d)$ for M sufficiently large and d sufficiently small.

Next, I show that in an optimal mechanism of $\mathcal{P}'(M, d)$, in the absence of verification, either no high-budget agent has incentives to misreport as low budget, or all high-budget agents weakly prefer to misreport as low budget.

Lemma 4 *Suppose Assumptions 1 and 2 hold. An optimal mechanism of $\mathcal{P}'(M, d)$ satisfies one of the following two conditions:*

(C1) For all $m = 1, \ldots, M$,

$$u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) \geq 0. \tag{8}$$

(C2) For all $m = 1, \ldots, M$,

$$u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) \leq 0. \tag{9}$$

The basic intuition underlying Lemma 4 is as follows: As long as a mechanism satisfies neither (C1) nor (C2), one can strictly improve welfare by adjusting the allocation rule in regions in which high-budget agents find it strictly optimal to report their budgets truthfully. I provide only a proof sketch of Lemma 4 here. The full proof can be found in the appendix.

Proof Sketch. The proof is by contradiction. Let (a, p, q) be a feasible mechanism, where a is a M-step allocation rule. Suppose (a, p, q) satisfies neither (C1) nor (C2). I show that one can construct another feasible mechanism (a^*, p^*, q^*), which strictly improves welfare and satisfies one of the two conditions. Furthermore, a^* is a M'-step function for some $M' \leq M$. I break the proof into two steps.

Step 1. I show that it is without loss of generality to assume that (8) holds for $m = 1$. Suppose, on the contrary, that $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1 v_2^0 < 0$. Then there exists $m > 1$ such that $v_2^{m'-1} - v_1^{m'-1} \leq 0$ for all $m' < m$ and $v_2^{m-1} - v_1^{m-1} > 0$. One can construct another feasible mechanism by redirecting cash subsidies from high-budget agents to low-budget agents, and shifting the allocation mass from low-budget agents in $[v_1^{m-1}, \tilde{v}_1^{m-1}]$ to high-budget agents in $[\tilde{v}_2^{m-1}, v_2^{m-1}]$ for some $v_1^{m-1} \leq \tilde{v}_1^{m-1} \leq \tilde{v}_2^{m-1} \leq v_2^{m-1}$.

Step 2. Suppose $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1 v_2^0 \geq 0$. There exists $m > 1$ such that (8) holds for all $m' < m$ and

$$u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m} (a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) < 0.$$

It must be the case that $v_2^{m-1} < v_1^{m-1}$. For ease of exposition, assume that $v_2^m > v_2^{m-1}$.[15] One can construct another feasible mechanism by either shifting the allocation mass from high-budget agents in $[v_2^{m-1}, \hat{v}]$ to high-budget agents in $[\hat{v}, v_2^m]$ for some $v_2^{m-1} < \hat{v} < v_2^m$, or shifting the allocation mass from high-budget agents in $[v_2^{m-1}, \tilde{v}_2^{m-1}]$ to low-budget agents in $[\tilde{v}_1^{m-1}, v_1^{m-1}]$ for some $v_2^{m-1} \leq \tilde{v}_2^{m-1} \leq \tilde{v}_1^{m-1} \leq v_1^{m-1}$. ■

If (C2) holds, then the optimal inspection rule is $q \equiv \mathbf{0}$. The optimal mechanism of $\mathcal{P}'$ in this case, which is characterized in Section 4.1, is a feasible mechanism of $\mathcal{P}'(M, d)$ and satisfies (C1) with equality. Thus, I can conclude that an optimal mechanism of $\mathcal{P}'(M, d)$ satisfies (C1).

Corollary 1 *Suppose Assumptions 1 and 2 hold. An optimal mechanism of $\mathcal{P}'(M, d)$ satisfies (C1).*

Hence, an optimal inspection rule satisfies $q(v, b_1) = q^m$ for all $v \in (v_1^{m-1}, v_1^m)$, where

$$q^m = \frac{1}{c}\left[u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m} (a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1})\right] \tag{10}$$

[15] In the appendix, I break the proof in three steps. I consider the case in which $v_2^{m-1} < v_2^M$ in Step 2 and the case $v_2^{m-1} = v_2^M$ in Step 3.

for $m = 1, \ldots, M$. Now the principal's problem $\mathcal{P}'(M, d)$ can be written as follows, where the Greek letters in parentheses denote the corresponding Lagrangian multipliers.

$$\max_{\substack{u(\underline{v},b_1),u(\underline{v},b_2),\\ \{a^m\}_{m=1}^M,\{v_1^m\}_{m=1}^{M-1},\{v_2^m\}_{m=0}^M}} \pi \sum_{m=1}^{M+1} \int_{v_2^{m-1}}^{v_2^m} a^m v f(v)\mathrm{d}v + (1-\pi) \sum_{m=1}^{M} \int_{v_1^{m-1}}^{v_1^m} a^m v f(v)\mathrm{d}v$$

$$-(1-\pi)\frac{k}{c} \sum_{m=1}^{M} \int_{v_1^{m-1}}^{v_1^m} \left[u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m} (a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) \right] f(v)\mathrm{d}v,$$

subject to

$$\pi \sum_{m=1}^{M+1} a^m [F(v_2^m) - F(v_2^{m-1})] + (1-\pi) \sum_{m=1}^{M} a^m [F(v_1^m) - F(v_1^{m-1})] \leq S, \tag{β}$$

$$a^M v_1^{M-1} - \sum_{j=1}^{M-1} a^j (v_1^j - v_1^{j-1}) - u(\underline{v}, b_1) \leq b_1, \tag{η}$$

$$\begin{aligned} &- (1-\pi)u(\underline{v}, b_1) + (1-\pi) \sum_{m=1}^{M} \int_{v_1^{m-1}}^{v_1^m} a^m \left[v - \frac{1-F(v)}{f(v)} \right] f(v)\mathrm{d}v \\ &- (1-\pi)\frac{k}{c} \sum_{m=1}^{M} \int_{v_1^{m-1}}^{v_1^m} \left[u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m} (a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) \right] f(v)\mathrm{d}v \\ &- \pi u(\underline{v}, b_2) + \pi \sum_{m=1}^{M+1} \int_{v_2^{m-1}}^{v_2^m} a^m \left[v - \frac{1-F(v)}{f(v)} \right] f(v)\mathrm{d}v \geq -d, \end{aligned} \tag{λ}$$

$$u(\underline{v}, b_1) \geq 0, u(\underline{v}, b_2) \geq 0, \tag{ξ_1, ξ_2}$$

$$u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m} (a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) \geq 0, \ m = 1, \ldots, M, \tag{μ^m}$$

$$0 = a^0 \leq a^1 \leq a^2 \leq \cdots \leq a^M \leq a^{M+1} = 1, \tag{$\alpha^1, \ldots, \alpha^{M+1}$}$$

$$\underline{v} = v_1^0 \leq v_1^1 \leq \cdots \leq v_1^M = \overline{v}, \tag{$\gamma_1^1, \ldots, \gamma_1^M$}$$

$$\underline{v} \leq v_2^0 \leq v_2^1 \leq \cdots \leq v_2^M \leq \overline{v}. \tag{$\gamma_2^0, \ldots, \gamma_2^{M+1}$}$$

To solve this problem, I first show that in an optimal mechanism of $\mathcal{P}'(M, d)$, the inspection probability is non-decreasing in a low-budget agent's reported value:

Lemma 5 *Suppose Assumptions 1 and 2 hold. In an optimal mechanism of* $\mathcal{P}'(M, d)$, $v_2^1 - v_1^1 \geq 0$.

Suppose in addition that $V(M,d) > V(M-1,d)$ for $M \geq 3$, then

$$v_2^{M-1} - v_1^{M-1} > \cdots > v_2^1 - v_1^1 \geq 0.$$

As a result, the inspection probability in an optimal mechanism of $\mathcal{P}'(M,d)$ is non-decreasing in reported value, i.e., $q^M \geq \cdots \geq q^1 \geq 0$.

To understand the intuition behind the monotonicity of inspection probability, consider a low-budget agent and a high-budget agent both receiving the good with probability a^m. Let p_1^m and p_2^m denote their payments respectively. The difference in their payments, to which the inspection probability is proportional, is

$$p_2^m - p_1^m = u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m} (a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}).$$

Clearly, this difference is non-decreasing in m since $v_2^{m-1} - v_1^{m-1} \geq 0$. Suppose, on the contrary, that $q^{m-1} > q^m$. Then the principal can shift allocation from low-budget agents in $[v_1^{m-2}, v_1^{m-1}]$ to low-budget agents in $[v_1^{m-1}, v_1^m]$, which clearly improves allocation efficiency and revenue. This shift also strictly reduces inspection cost because more low-budget agents are inspected with probability q^m rather than q^{m-1} and $q^{m-1} > q^m$.

The inequality constraints corresponding to μ^m's in $\mathcal{P}'(M,d)$ are non-negativity constraints on inspection probabilities. As shown in Lemma 5, in an optimal mechanism of $\mathcal{P}'(M,d)$, the inspection probability is non-decreasing in a low-budget agent's reported value. As a result, it is sufficient to consider the inequality constraint corresponding to μ^1:

$$u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) \geq 0.$$

Note that for fixed jump discontinuity points v_i^m's, the principal's problem $\mathcal{P}'(M,d)$ is linear in $u(\underline{v}, b_1)$, $u(\underline{v}, b_2)$ and a^m's. Hence, an optimal solution can be obtained at an extreme point of the feasible region. The monotonicity of inspection probability implies that in addition to the monotonicity constraints on a^m's there are only finitely many other constraints binding. As a result, for an M sufficiently large, there are finitely many distinct a^m's in an optimal mechanism. More formally, let $V(M,d)$ denote the value of $\mathcal{P}'(M,d)$. Then $V(M,d) = V(M-1,d)$ for M sufficiently large. This result still holds if I replace (BC) with a per-unit price constraint, as shown in Section 7.1. If I impose only (BC), then I can further prove that in an optimal mechanism of $\mathcal{P}'(M,d)$ the allocation rule is a 2-step allocation rule, i.e., $V(M,d) = V(M-1,d)$ for $M \geq 3$.

Lemma 6 *Suppose Assumptions 1 and 2 hold. Then $V(M,d) = V(2,d)$ for all $M \geq 2$ and $d \geq 0$. Furthermore, for all $M \geq 2$, in an optimal mechanism of $\mathcal{P}'(M,d)$ the allocation rule is a 2-step allocation rule.*

Proof Sketch. I provide a proof sketch of Lemma 6. Assume, for ease of exposition, that

$$u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) > 0, \qquad (\mu^1)$$

$$\underline{v} = v_1^0 < v_1^1 < \cdots < v_1^M = \overline{v}, \qquad (\gamma_1^1, \ldots, \gamma_1^M)$$

$$0 \leq v_2^0 < v_2^1 < \cdots < v_2^M < \overline{v}. \qquad (\gamma_2^0, \ldots, \gamma_2^{M+1})$$

Then $\mu^1 = \cdots = \mu^M = 0$, $\gamma_1^1 = \cdots = \gamma_1^M = 0$ and $\gamma_2^1 = \cdots = \gamma_2^{M+1} = 0$. The first-order conditions for v_1^m and v_2^m ($m = 1, \ldots, M-1$) are

$$(1-\pi)\left[(\beta - v_1^m - \lambda\varphi(v_1^m))f(v_1^m) + (1+\lambda)\rho[1-F(v_1^m)] + (1+\lambda)\rho(v_2^m - v_1^m)f(v_1^m)\right] - \eta = 0,$$

$$\pi(\beta - v_2^m - \lambda\varphi(v_2^m))f(v_2^m) - (1-\pi)(1+\lambda)\rho[1-F(v_1^m)] = 0,$$

where $\varphi(v) := v - [1-F(v)]/f(v)$ denotes the virtual value function. I show in the appendix that If f is "regular", which is to say that it satisfies Assumptions 1 and 2, then the above system of equations has at most one solution. This result is illustrated by Example 1. Hence, I can conclude that $V(M,d) = V(2,d)$. ■

Example 1 *Let v be uniformly distributed on $[0,1]$ and $\rho < \frac{\pi+\sqrt{\pi}}{1-\pi}$. Then the first-order conditions* *for* v_1^m *and* v_2^m $(m = 1, \dots, M-1)$

$$(1-\pi)\left[\beta + \lambda + (1+\lambda)\rho - (1+2\lambda+2(1+\lambda)\rho)v_1^m + (1+\lambda)\rho v_2^m\right] - \eta = 0, \tag{11}$$

$$\pi(\beta+\lambda) - (1-\pi)(1+\lambda)\rho + (1-\pi)(1+\lambda)\rho v_1^m - \pi(1+2\lambda)v_2^m = 0. \tag{12}$$

Given β, η *and* λ, (11) *and* (12) *define* v_2^m *as functions of* v_1^m, *denoted by* g_1 *and* g_2, *respectively. Then*

$$g_1'(v_1^m) = 2 + \frac{1+2\lambda}{\rho(1+\lambda)} > \frac{(1-\pi)(1+\lambda)\rho}{\pi(1+2\lambda)} = g_2'(v_1^m).$$

This inequality implies that g_1 *can cross* g_2 *at most once from below. Hence,* (11) *and* (12) *have at most one solution.*

The main result of this section is Theorem 3, which characterizes an optimal mechanism of the original problem $\mathcal{P}$. In particular, I show that an optimal mechanism of $\mathcal{P}'(2,0)$ is also an optimal mechanism of $\mathcal{P}$. In other words, in an optimal mechanism of $\mathcal{P}$, the allocation rule is a 2-step allocation rule.

Let V denote the value of $\mathcal{P}'$. We prove Theorem 3 by first showing that for any $d > 0$ there exists $\overline{M}(d) > 0$ such that for all $M > \overline{M}(d)$

$$V - V(M,d) \leq (1-\pi)(1+\rho)\frac{\mathbb{E}[v]}{M}.$$

The proof is by construction. Fix $d > 0$ and an integer $M > \overline{M}(d)$. I can construct a feasible mechanism of $\mathcal{P}'(M,d)$ that possibly violates (BB) by at most d and generates welfare which is at least $V - (1-\pi)(1+\rho)\mathbb{E}[v]/M$. By Lemma 5, $V - V(M,d) = V - V(2,d) \leq (1-\pi)(1+k/c)\mathbb{E}[v]/M$ for all $d > 0$ and $M > \overline{M}(d)$. Fixing $d > 0$ and taking M to infinity yields $V \leq V(2,d)$ for all $d > 0$. By definition, $V \geq V(2,0)$. Hence, $V = V(2,0)$ by the continuity of $V(2,\cdot)$. Thus, an optimal mechanism of $\mathcal{P}'$ also solves $\mathcal{P}'$. It is easy to verify that an optimal solution to $\mathcal{P}'(2,0)$ satisfies (IC) constraints corresponding to agents over reporting their budgets and therefore solves $\mathcal{P}$. Finally, I show that $v_2^0 = \underline{v}$ and $a^2 = 0$ in an optimal mechanism. Let $a^*(\rho) = a^2$, $v_1^*(\rho) = v_1^1$, $v_2^*(\rho) = v_2^1$, $v_2^{**}(\rho) = v_2^2$, $u_1^*(\rho) = u(\underline{v}, b_1)$ and $u_2^*(\rho) = u(\underline{v}, b_2)$, then an optimal mechanism is characterized by the following Theorem 3.

Theorem 3 *Suppose Assumptions 1 and 2 hold. There exist $a^*(\rho)$, $v_1^*(\rho)$, $v_2^*(\rho)$, $v_2^{**}(\rho)$, $u_1^*(\rho)$ and $u_2^*(\rho)$ such that an optimal mechanism of $\mathcal{P}$ is given by*

$$
\begin{aligned}
&a(v, b_1) = \chi_{\{v \geq v_1^*(\rho)\}} a^*(\rho), \ \ p(v, b_1) = \chi_{\{v \geq v_1^*(\rho)\}} a^*(\rho) v_1^*(\rho) - u_1^*(\rho), \\
&q(v, b_1) = \frac{1}{c} \left[\chi_{\{v \geq v_1^*(\rho)\}} a^*(\rho) \left(v_2^*(\rho) - v_1^*(\rho) \right) + u_1^*(\rho) - u_2^*(\rho) \right], \\
&a(v, b_2) = \chi_{\{v \geq v_2^*(\rho)\}} a^*(\rho) + \chi_{\{v \geq v_2^{**}(\rho)\}} \left(1 - a^*(\rho) \right), \\
&p(v, b_2) = \chi_{\{v \geq v_2^*(\rho)\}} a^*(\rho) v_2^*(\rho) + \chi_{\{v \geq v_2^{**}(\rho)\}} \left(1 - a^*(\rho) \right) v_2^{**}(\rho) - u_2^*(\rho), \\
&q(v, b_2) = 0,
\end{aligned}
$$

where $a^*(\rho) = \left[u_1^*(\rho) + b_1 \right] / v_1^*(\rho)$, $\underline{v} < v_1^*(\rho) \leq v_2^*(\rho) \leq v_2^{**}(\rho) \leq \overline{v}$, $0 < a^*(\rho) \leq 1$ and $u_1^*(\rho) \geq u_2^*(\rho)$.

In notations $a^*(\rho)$, $v_i^*(\rho)$, $v_2^{**}(\rho)$ and $u_i^*(\rho)$ $(i = 1, 2)$, subscript i indicates the corresponding budget b_i and argument ρ indicates their dependence on ρ. In an optimal mechanism, low-budget agents receive more cash subsidies (as in the case of common knowledge budgets), but high-budget agents may also receive strictly positive cash subsidies (as in the case of private budgets without inspection). There are three cutoffs: $v_1^*(\rho) \leq v_2^*(\rho) \leq v_2^{**}(\rho)$. All high-budget agents whose valuations are above $v_2^{**}(\rho)$ receive the good with probability 1. All low-budget agents whose valuations are above $v_1^*(\rho)$ receive the good with positive probability but may be rationed. Similar to the case of private budgets without inspection, some high-budget agents (whose valuations are in $[v_2^*(\rho), v_2^{**}(\rho)]$) are pooled with low-budget agents. However, $v_1^*(\rho) \leq v_2^*(\rho)$. This difference between $v_1^*(\rho)$ and $v_2^*(\rho)$, together with budget dependent cash subsidies, creates an incentive for high-budget agents to under report their budgets. All agents who report low budgets are inspected with non-negative probability and those who receive the goods are more likely to be inspected.

I note here that if $\rho = 0$, then $u_2^*(0) = 0$ and $v_2^*(0) = v_2^{**}(0)$, which is the case in Theorem 1. If $\rho = \infty$, then $u_1^*(\infty) = u_2^*(\infty)$ and $v_1^*(\infty) = v_2^*(\infty)$, which is the case in Theorem 2. To simplify notation, in what follows, I suppress the dependence of u_1^*, u_2^*, v_1^*, v_2^*, v_2^{**} and a^* on ρ whenever it is clear.

Theorem 3 also greatly simplifies the analysis. Now the principal's problem can be reduced to:

$$\max_{\substack{u(\underline{v},b_1),u(\underline{v},b_2),\\ a^2,v_1^1,v_2^1,v_2^2}} \pi\left[\int_{v_2^1}^{v_2^2} a^2 v f(v)\mathrm{d}v + \int_{v_2^2}^{\overline{v}} v f(v)\mathrm{d}v\right] + (1-\pi)\int_{v_1^1}^{\overline{v}} a^2 v f(v)\mathrm{d}v$$
$$-(1-\pi)\frac{k}{c}\left[u(\underline{v},b_1) - u(\underline{v},b_2)\right]F(v_1^1) - (1-\pi)\frac{k}{c}\int_{v_1^1}^{\overline{v}}\left[u(\underline{v},b_1) - u(\underline{v},b_2) + a^2(v_2^1 - v_1^1)\right]f(v)\mathrm{d}v,$$

subject to

$$\pi a^2\left[F(v_2^2) - F(v_2^1)\right] + \pi\left[1 - F(v_2^2)\right] + (1-\pi)a^2[1 - F(v_1^1)] \leq S, \tag{β}$$
$$a^2 v_1^1 - u(\underline{v},b_1) \leq b_1, \tag{η}$$
$$\begin{aligned}&-(1-\pi)u(\underline{v},b_1) + (1-\pi)\int_{v_1^1}^{v_1^2} a^2\left[v - \frac{1-F(v)}{f(v)}\right]f(v)\mathrm{d}v\\ &-(1-\pi)\frac{k}{c}\left[u(\underline{v},b_1) - u(\underline{v},b_2)\right]F(v_1^1) - (1-\pi)\frac{k}{c}\int_{v_1^1}^{\overline{v}}\left[u(\underline{v},b_1) - u(\underline{v},b_2) + a^2(v_2^1 - v_1^1)\right]f(v)\mathrm{d}v\\ &-\pi u(\underline{v},b_2) + \pi\int_{v_2^1}^{v_2^2} a^2\left[v - \frac{1-F(v)}{f(v)}\right]f(v)\mathrm{d}v + \pi\int_{v_2^2}^{\overline{v}}\left[v - \frac{1-F(v)}{f(v)}\right]f(v)\mathrm{d}v \geq 0,\end{aligned} \tag{λ}$$
$$u(\underline{v},b_1) \geq 0, u(\underline{v},b_2) \geq 0, \tag{ξ_1,ξ_2}$$
$$u(\underline{v},b_1) - u(\underline{v},b_2) \geq 0, \tag{μ^1}$$
$$u(\underline{v},b_1) - u(\underline{v},b_2) + a^2(v_2^1 - v_1^1) \geq 0, \tag{μ^2}$$
$$0 \leq a^2 \leq a^3 = 1, \tag{α^2,α^3}$$
$$\underline{v} \leq v_1^1 \leq \overline{v}, \tag{γ_1^1,γ_1^2}$$
$$\underline{v} \leq v_2^1 \leq v_2^2 \leq \overline{v}. \tag{$\gamma_2^1,\gamma_2^2,\gamma_2^3$}$$

Furthermore, the optimal mechanism is *unique*. Suppose, on the contrary, that there are two optimal mechanism. Since $\mathcal{P}'$ is linear in (a, p, q),[16] the convex combination of these two optimal mechanisms is also optimal. However, the convex combination of two 2-step allocation rules is not a 2-step allocation rule in general, which cannot be optimal by Lemma 3. Hence, there exists a unique optimal mechanism.

Corollary 2 *Suppose Assumptions 1 and 2 hold. There exists a unique optimal mechanism of $\mathcal{P}$. Furthermore, u_1^*, u_2^*, v_1^*, v_2^*, v_2^{**} and a^* are continuous in k, c, π, b_1 and S.*

4.3 Subsidies in cash and in kind

I complete this section by a discussing subsidies in cash and in kind. In the optimal mechanism, compared with the high-budget agents who do not receive the goods, high-budget agents whose valuations exceed v_2^{**} receive the good with probability 1 by making an additional payment $a^* v_2^* + (1-a^*)v_2^{**}$. All high-budget agents whose valuations lie in $[v_2^*, v_2^{**}]$ receive the good with probability a^* by making an additional payment $a^* v_2^*$, which is an *in-kind subsidy*. In the literature, the value of an in-kind subsidy is often measured by its market value. In this paper, I do not model the private market explicitly, so I use the additional payment, $a^* v_2^* + (1-a^*)v_2^{**}$, made by high-budget high-value agents as a measure of "price". Then the amount of in-kind subsidies offered to a high-budget agent is $a^* \left[a^* v_2^* + (1-a^*)v_2^{**}\right] - a^* v_2^*$. Note that high-budget agents do not receive any in-kind subsidies if $v_2^* = v_2^{**}$, as in the case when budgets are common knowledge. Similarly, the amount of in-kind subsidies offered to a low-budget agent is $a^* \left[a^* v_2^* + (1-a^*)v_2^{**}\right] - a^* v_1^*$. The difference in in-kind subsidies offered to the two budget types is $a^*(v_2^* - v_1^*)$.

In-kind subsidies are widespread around the world. The conventional wisdom rationalizing the prevalence of in-kind subsidies is paternalism. A more recent justification is based on the idea that agents have private information about their financial constraints, and governments cannot accurately identify low-budget agents in need of help. As a result, in-kind subsidies will be part of a surplus-maximizing mechanism as it is less susceptible to mimicking by high-budget agents. One difficulty with this justification is that, in many transfer programs, governments first "verify

[16] $\mathcal{P}'(2, 0)$ is not linear in $u(\underline{v}, b_1)$, $u(\underline{v}, b_2)$, a^2, v_1^1, v_2^1 and v_2^2.

income, and then give benefits in kind, which would seem to rule out self-targeting as the primary reason for supplying benefits in-kind".[17] Moreover, governments "generally expend considerable resources determining eligibility".[18] In this paper, I formalize the idea that governments can verify agents' private information about their financial constraints via a costly procedure, and show that in such an environment the optimal mechanism still makes use of both cash and in-kind subsidies.

5 Implementation

In this section, I provide one simple implementation of the direct optimal mechanism characterized in Section 4. This implementation exhibits some of the features of Singapore's housing and development board (HDB).

Consider the following *random assignment with regulated resale and cash subsidy* (RwRRC) scheme, which consists of two stages.

1. In the first stage, each agent reports his budget. Agents who report low budget are inspected with a probability of $(u_1^* - u_2^*)/c$. The principal offers cash subsidies u_1^* to low-budget agents and u_2^* to high-budget agents. The principal also offers low-budget agents the choice of participating in a lottery at price $p_1^* := a^* v_1^*$ and high budget agents the choice of participating in the same lottery at price $p_2^* := a^* v_2^*$. The principal distributes the goods randomly with uniform probability among all participants of the lottery. Each participant receives one unit of good with a probability no more than a^*.

2. In the second stage, the resale market opens, in which agents can purchase goods from each other and the principal if not all the goods are distributed in the first stage. The per-unit sales taxes are $\tau_1^* := v_2^{**} - v_1^*$ for low-budget sellers and $\tau_2^* := v_2^{**} - v_2^*$ for high-budget sellers. Agents who report low budget in the first stage and choose not to sell the good in the second stage are inspected with probability $(v_2^* - v_1^*)/c$.

[17] Currie and Gahvari (2008)
[18] Currie and Gahvari (2008)

Let a denote a lottery participant's expected probability of receiving the good in the first stage, and p^s denote the expected price a buyer pays in the second stage. Assume without loss of generality that $p^s > b_1$ so that a low-budget agent cannot afford it. Consider a low-budget agent whose type is (v, b_1) and who reports his budget truthfully. Then his payoff is u_1^* if he does not enter the lottery. If he buys the lottery, there are two possibilities. If he keeps the good when he receives it in the first stage, then his payoff is $u_1^* + av - a^* v_1^*$; otherwise his payoff is

$$u_1^* - a^* v_1^* + a(p^s - v_2^{**} + v_1^*).$$

Clearly, in the second stage, it is optimal for him to keep the good if and only if $v \geq p^s - v_2^{**} + v_1^*$. In the first stage, it is optimal for him to purchase the lottery if and only if

$$a \max \left\{ v, p^s - v_2^{**} + v_1^* \right\} \geq a^* v_1^*.$$

Similarly, consider a high-budget agent whose type is (v, b_2) and who reports his budget truthfully. It is easy to see that if it is optimal for an agent not to buy the lottery in the first stage, then it is also optimal for him not to buy the good in the second stage. If it is optimal for an agent to sell the good he receives in the first stage, then it is optimal for him not to buy the good in the second stage when he does not receive it in the first stage. Then his payoff is u_2^* if he does not buy the lottery. If he buys the lottery, there are three possibilities. If he buys the lottery, keeps the good when he receives it and buys it when he does not receive it, his payoff is

$$u_2^* - a^* v_2^* + av + (1 - a)(v - p^s);$$

if he buys the lottery, keeps the good when he receives it and does not buy when he does not receive it, his payoff is $u_1^* + a^*(v - v_1^*)$; if he buys the lottery and sells the good when he receives it, then his payoff is

$$u_2^* - a^* v_2^* + a(p^s - v_2^{**} + v_2^*).$$

Clearly, in the second stage, it is optimal for him to keep the good if and only if $v \geq p^s - v_2^{**} + v_2^*$ and buy the good if and only if $v \geq p^s$. In the first stage, it is optimal for him to purchase the lottery if and only if

$$a \max \left\{v, p^s - v_2^{**} + v_2^*\right\} \geq a^* v_2^*.$$

Hence, in the second stage, the demand of the goods is $\pi(1-a)\left[1 - F(p^s)\right]$ and the supply of the goods is

$$S - a(1-\pi)\left[1 - F\left(\max\left\{p^s - v_2^{**} + v_1^*, \frac{a^* v_1^*}{a}\right\}\right)\right] - a\pi\left[1 - F\left(\max\left\{p^s - v_2^{**} + v_2^*, \frac{a^* v_2^*}{a}\right\}\right)\right].$$

It is not hard to verify that $a = a^*$ and $p^s = v_2^{**}$ is the unique equilibrium.[19] Note that in this equilibrium, an agent is indifferent between not buying the lottery, and buying the lottery but selling the good when he receives it. All low-budget agents whose valuations are above v_1^* strictly prefer to participate in the lottery and keep the good they receive. All high-budget agents whose valuations are above v_2^* strictly prefer to participate in the lottery and keep the goods they receive. In addition, all high-budget agents whose valuations are above v_2^{**} will buy the goods in the second stage if they do not receive any in the first stage. These arguments prove the following result.

Proposition 1 *Suppose Assumptions 1 and 2 hold. The optimal mechanism is implemented by RwRRC with* $\underline{v} \leq v_1^* \leq v_2^* \leq v_2^{**} \leq \overline{v}$, $u_1^* \geq u_2^*$ *and* $0 \leq a^* \leq 1$ *given by Theorem 3.*

If inspection is sufficiently costly or the principal cannot inspect agents, then in the RwRRC scheme agents receive the same amount of cash subsidies $u_1^* = u_2^*$ and the same price $p_1^* = p_2^*$ in the first stage and face the same sales taxes $\tau_1^* = \tau_2^*$ in the second stage regardless of their budgets. This is consistent with the findings in Che et al. (2013). If inspection is not too costly, then the principal provides financial aids to low-budget agents ($u_1^* \leq u_2^*$, $p_1^* \geq p_2^*$) in the first stage and discourages them from reselling by imposing a higher sales tax in the second stage.

[19] Clearly, for each a, there is a unique p^s such that demand is equal to supply in the second stage. By construction, $a \leq a^*$. Suppose $a < a^*$, then the market clearing condition in the second stage implies that $p^s < v_2^{**}$. This implies that a low-budget agent buys the lottery only if $v > v_1^*$ and a high-budget agent buys the lottery only if $v > v_2^*$, which in turn implies that $a = a^*$, a contradiction. Hence, $a = a^*$ and $p^s = v_2^{**}$.

Table 1: Minimum occupation periods (MOP) of housing and development board (HDB) flats

Types of HDB flats	MOP	
	Sell	Sublet
Resale flats w/ Grants	5–7 years	5–7 years
Resale flats w/o Grants	0–5 years	3 years

Sources. — Sell: http://www.hdb.gov.sg/cs/infoweb/residential/selling-a-flat/eligibility; and Sublet: http://www.hdb.gov.sg/cs/infoweb/residential/renting-out-a-flat-bedroom/renting-out-your-flat/eligibility.

This implementation exhibits some of the features of Singapore's HDB. HDB develops new flats and sells them to eligible buyers.[20] Buyers can purchase new flats directly from HDB or resale flats from existing owners in the open market. Buyers must have resided in their flats for a period of time, referred to as the minimum occupation period (MOP), before they are eligible to resell or sublet their flats. Buyers of resale HDB flats can apply for a CPF housing grant, which is a housing subsidy to help eligible households. HDB flats purchased with CPF housing grants are subject to longer MOPs as illustrated by Table 1.

6 Properties of the Optimal Mechanism

Having derived the optimal mechanism, I would like to investigate the following questions. Is it optimal for the principal to limit the supply of goods? When can the first-best outcome be achieved? What is the effect of a decrease in verification cost as, for example, a government's bureaucratic efficiency improves? What is the effect of an increase in the supply as, for example, a government builds more houses? What if agents become less budget-constrained? This loosening of constraints could happen as more agents are admitted into the formal financial system (π increases) or if their wealth increases as the economy grows (b_1 increases). What if the principal becomes less budget-constrained as a government increases expenditures on transfer programs? In what follows, I investigate each of these questions in turn.

Firstly, I show that it is not optimal for the principal to limit the supply of goods.

[20] 90% of HDB flats are owned by their residents. The remainder are rental flats for people who cannot afford to purchase the cheapest form of HDB flats despite financial aid.

Proposition 2 *Suppose Assumptions 1 and 2 hold. In an optimal mechanism, (S) holds with equality.*

This result is straightforward if agents are unconstrained. However, it is not immediate from the principal's concern for efficiency if agents are budget-constrained. Recall that the principal also has a budget constraint, and this constraint may cause her to restrict supply. To see why, consider the extreme case in which low-budget agents have no money, i.e., $b_1 = 0$. In this case, the principal needs to raise all money from selling to high-budget agents. On the one hand, as she increases the amount of goods sold to high-budget agents, the revenue will start declining at some point. On the other hand, increasing the amount of goods allocated to low-budget agents raises the inspection cost. Thus, it is not obvious that in an optimal mechanism all the goods are distributed to agents. In the proof of Proposition 2, I show that if not all the goods are distributed to agents yet, then the principal can increase the amount of goods allocated to high-budget and low-budget agents simultaneously. For an appropriately chosen allocation rule, the resulting mechanism is feasible and strictly improves welfare.[21]

Secondly, I give a necessary and sufficient condition under which the first-best is achieved.

Proposition 3 *Suppose Assumptions 1 and 2 hold. The first-best is achieved if and only if $S \geq \hat{S}(b_1)$, where $\hat{S}(b_1)$ is the solution to*

$$b_1 - v^* F(v^*) = 0$$

with $v^* = F^{-1}(1 - S)$. Furthermore, $\hat{S}(b_1)$ is strictly decreasing in b_1.

Intuitively, the first-best is achieved if the supply of the good is abundant or agents have ample budgets. Note that the condition given in Proposition 3 is independent of inspection cost k, punishment c and the percentage of high-budget agents π. This is because when the first-best is achieved, agents of both budget types receive the same amount of cash subsidies and the same allocation rule,

[21] I thank Michael Richter for suggesting this proof.

and the inspection probability is zero. For the rest of this section, I assume that the first-best cannot be achieved, i.e., $S < \hat{S}(b_1)$.

Thirdly, I study the impact of changes in effective inspection cost ($\rho = k/c$), supply (S), budget (b_1) and the percentage of high-budget agents (π) on the optimal mechanism as well as welfare. The optimal mechanism is characterized by u_1^*, u_2^*, v_1^*, v_2^*, v_2^{**} and a^*, which (together with the corresponding Lagrangian multipliers) are solutions to a system of non-linear equations. As a result, it is hard to perform all comparative statics analysis analytically. In what follows, I give some analytic results for extreme cases such as when effective inspection cost is sufficiently large and explore the intermediate case numerically.

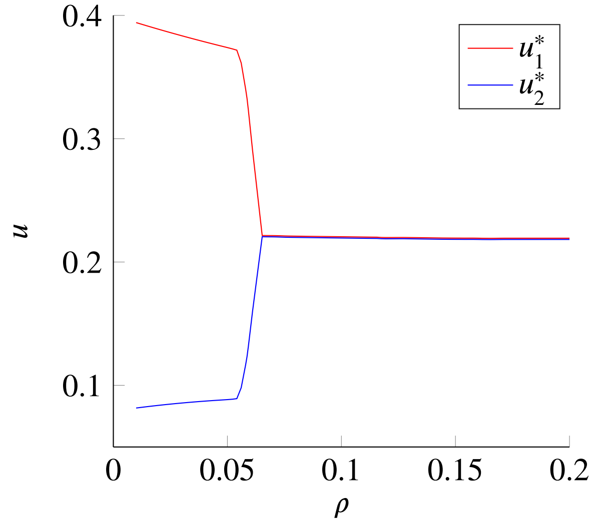
Effective Verification Cost (ρ). Intuitively, as verification becomes more costly (ρ increases), the principal tends to inspect agents less frequently in the optimal mechanism. To maintain incentive compatibility, the principal needs to reduce the differences in cash and in-kind subsidies offered to agents with different budgets. Proposition 4 shows that, for a large ρ, agents of both budget types receive the same amount of cash subsidies. Eventually, for ρ sufficiently large, verification is never used. The two lower bounds given in Proposition 4 are not tight, as illustrated in the numerical example in Figure 4. If v is uniformly distributed, then one can further prove that, for fixed punishment c, the verification probability is non-increasing in verification cost k. However, the change in total verification cost may not be monotonic as illustrated by Figure 4c.

Proposition 4 *Suppose Assumptions 1 and 2 hold.*

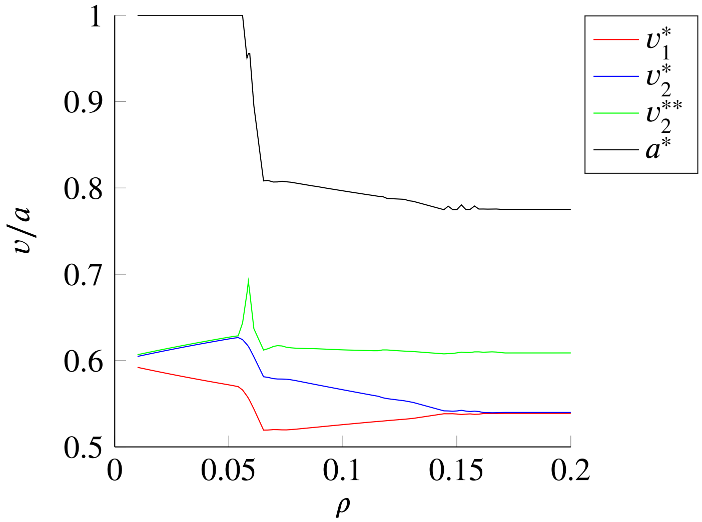
1. *If $\rho \geq \frac{\pi}{1-\pi}$, then agents of both budget types receive the same amount of cash subsidies, i.e., $u_1^* = u_2^*$.*

2. *There exists $\overline{\rho} \leq \frac{\pi}{S(1-\pi)}$ such that the verification probability in an optimal mechanism is zero, i.e., $q(v, b) = 0$ for all v and b, if and only if $\rho \geq \overline{\rho}$. Furthermore, the total welfare is strictly decreasing in ρ over $[0, \overline{\rho}]$ and constant in ρ over $[\overline{\rho}, \infty)$.*

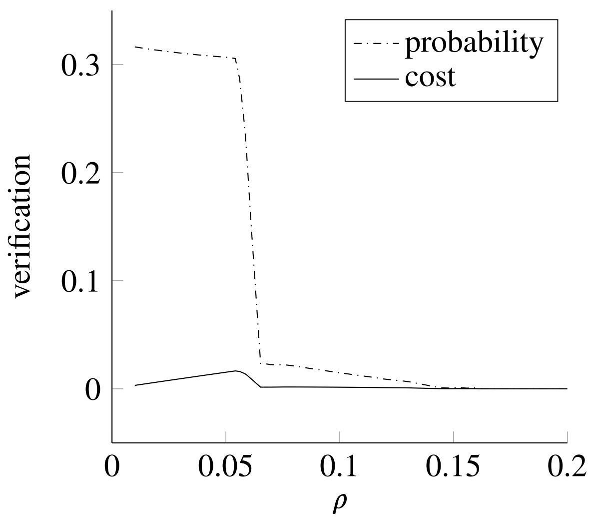
3. *If v is uniformly distributed, then the verification probability is non-increasing in k.*



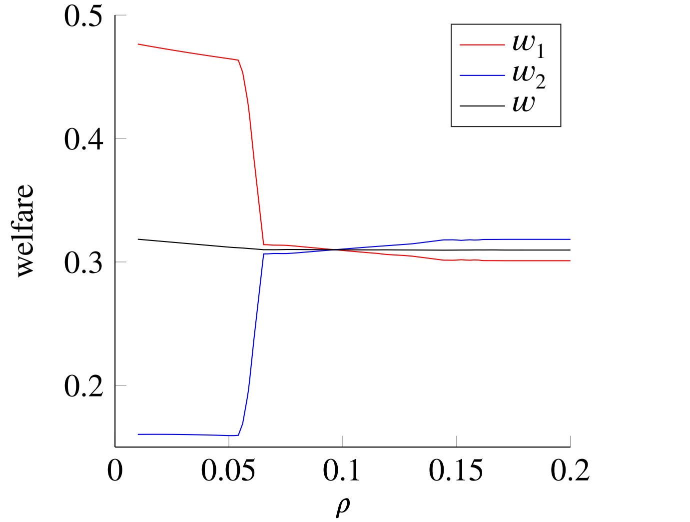
(a) Cash Subsidies



(b) Allocation



(c) Verification



(d) Welfare

Figure 4: The impact of an increase in effective inspection cost (ρ) on cash subsidies, allocation, inspection and welfare. In this numerical example, v is uniformly distributed on $[0, 1]$, $S = 0.4$, $b_1 = 0.2$, $\pi = 0.5$ and $\rho \in [0, 0.2]$.

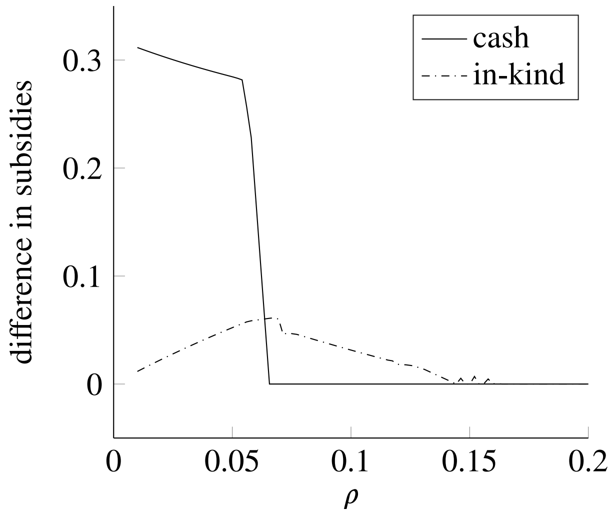
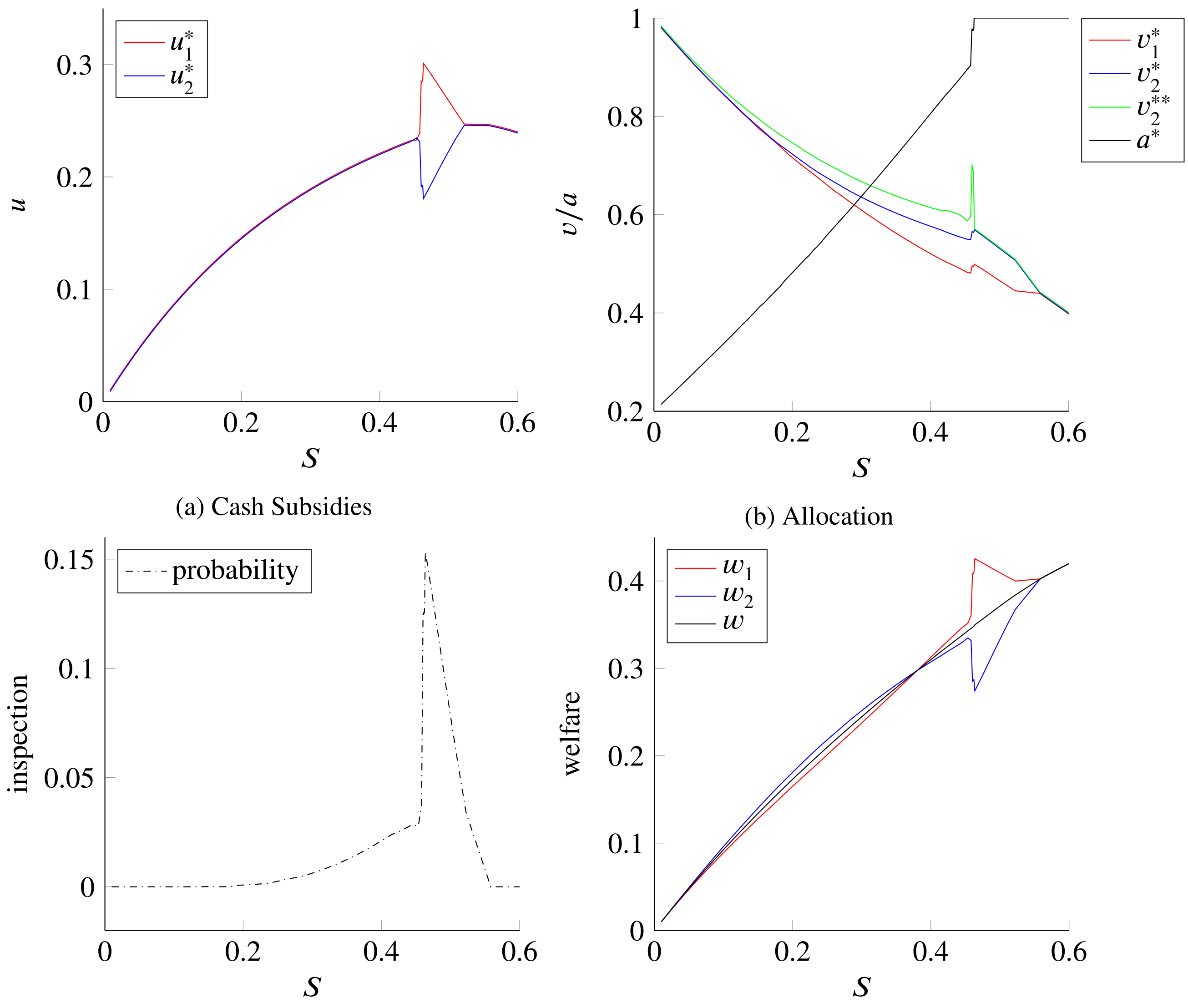


Figure 5: The impact of an increase in effective inspection cost (ρ) on the differences in cash and in-kind subsidies. In this numerical example, v is uniformly distributed on [0, 1], $S = 0.4$, $b_1 = 0.2$, $\pi = 0.5$ and $\rho \in [0, 0.2]$.

Figure 4 plots the impact of an increase in effective verification cost (ρ) on cash subsidies, allocation, verification and welfare in a numerical example. It is straightforward that an increase in ρ reduces the total welfare but its impacts on different budget types are different. Verification allows the principal to more accurately target low-budget agents and improves their welfare. As a result, as ρ increases, the welfare of low-budget agents declines while that of high-budget agents rises, as seen in Figure 4d.

More interestingly, the optimal mechanism makes use of both cash and in-kind subsidies, and the change in verification cost affects that mechanism's reliance on each of them as shown in Figure 5. If ρ is sufficiently small, then the principal helps low-budget agents mainly by offering them more cash subsidies. As ρ increases, the difference in cash subsidies declines but the difference in in-kind subsidies increases. This is because even though cash subsidy is more efficient in the sense that it does not introduce any distortion in allocation, it is more expensive in terms of verification cost. Cash subsidy is attractive to everyone regardless of their valuations. In contrast, in-kind subsidy is attractive only to agents whose valuations are high enough. Eventually, the difference in in-kind subsidies also declines as verification becomes sufficiently costly.



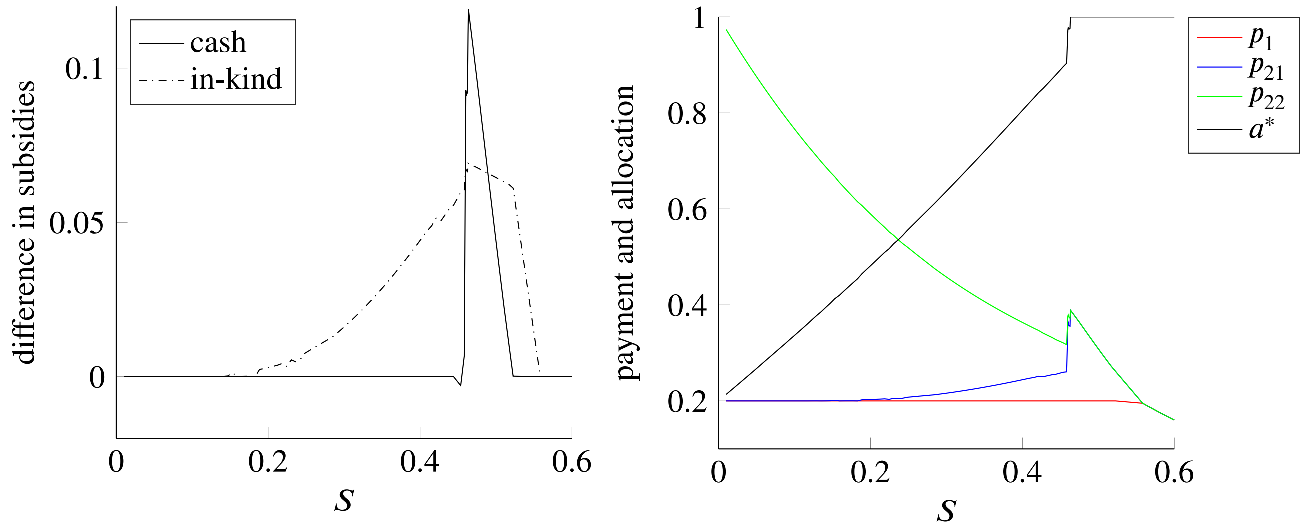
(a) Cash Subsidies

(b) Allocation

(c) Verification Probability

(d) Welfare

Figure 6: The impact of an increase in the supply (S) on cash subsidies, allocation, verification and welfare. In this numerical example, v is uniformly distributed on $[0, 1]$, $\rho = 0.08$, $b_1 = 0.2$, $\pi = 0.5$ and $S \in [0, 0.6]$.



(a) Differences in Cash and In-kind Subsidies

(b) Payment and Allocation

Figure 7: The impact of an increase in the supply (S) on the differences in cash and in-kind subsidies, allocation and payment. In the right panel, the red line (p_1) denotes the payment by a low-budget agent who receives the good with probability a^*, the blue line (p_{21}) denotes the payment by a high-budget agent who receives the good with probability a^*, and the red line (p_{22}) denotes the payment by a high-budget agent who receives the good with probability one. In this numerical example, v is uniformly distributed on $[0, 1]$, $\rho = 0.08$, $b_1 = 0.2$, $\pi = 0.5$ and $S \in [0, 0.6]$.

Supply (S). The impact of an increase in the supply (S) on the optimal mechanism is complicated. On the one hand, the principal becomes less budget constrained, and can direct more subsidies to low-budget agents and inspect them more frequently. On the other hand, low-budget agents also become less budget constrained as S increases,[22] which reduces the needs to subsidize and inspect them. As shown in Propositions 4 and 5, for sufficiently large and small S, agents of both budget types receive the same amount of cash subsidies.

Proposition 5 *Suppose Assumptions 1 and 2 hold. If S is sufficiently small, then agents of both budget types receive the same amount of cash subsidies, i.e.,* $u_1^* = u_2^*$.

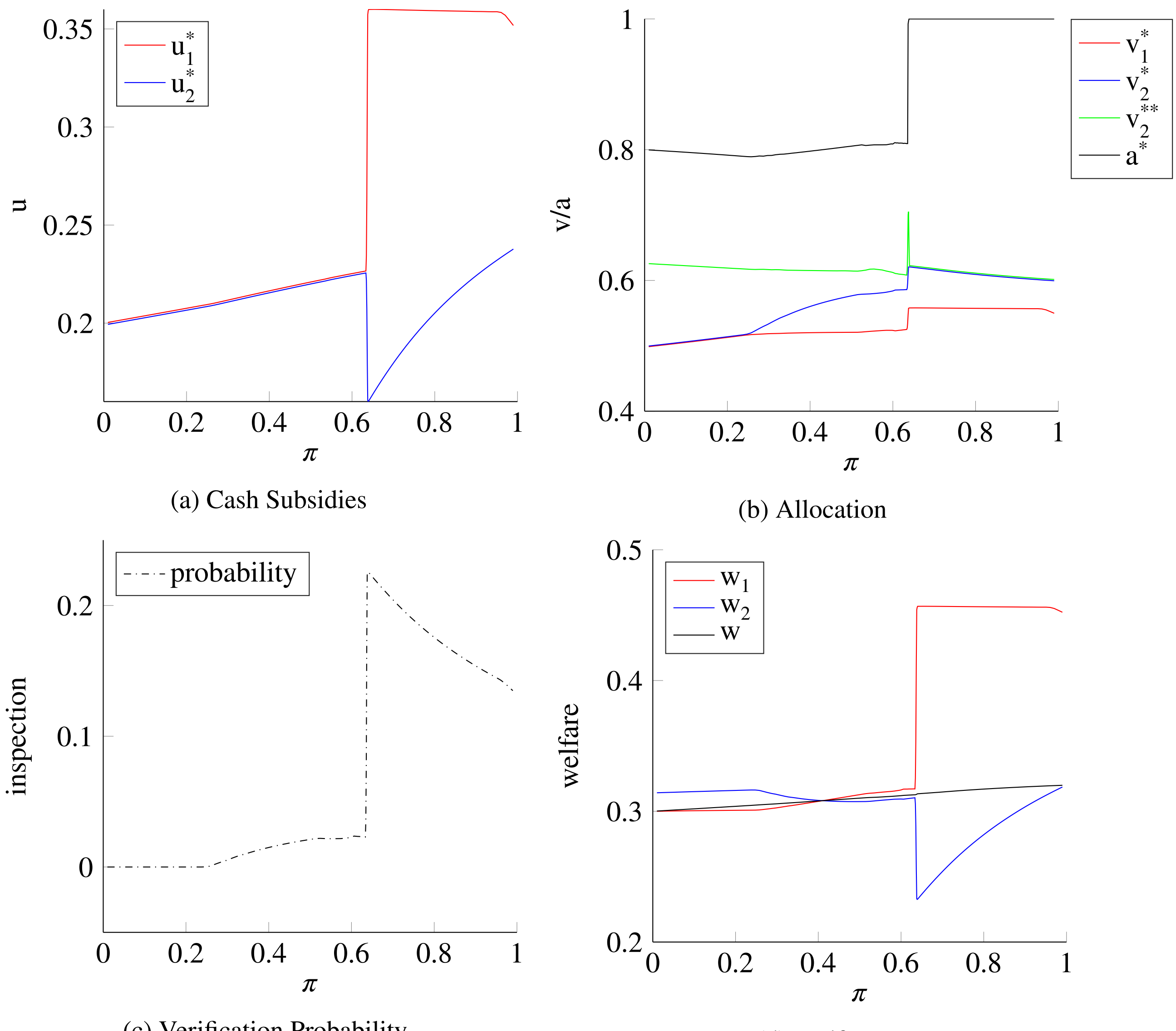
These effects can also be seen in Figures 6 and 7, which plot the impact of an increase in the supply (S) on cash subsidies, allocation, verification and welfare in a numerical example. Specifically,

[22] As in Section 4.3, I use the additional payment $a^* v_2^* + (1 - a^*) v_2^{**}$ made by a high-budget high-valuation agent a measure of "price". Then this price generally declines as S increases and low-budget agents become less budget constrained in the sense that the gap between this price and their budgets shrinks.

Figure 7a plots the differences in cash and in-kind subsidies between high-budget and low-budget agents. If S is sufficiently small, then agents receive the same amount of subsidies regardless of their budgets. As S increases, the principal raises first the difference in in-kind subsidies and then that in cash subsidies. This order occurs because it is less expensive to target only low-budget high-valuation agents than all low-budget agents. Eventually, the differences in both cash and in-kind subsidies decline as the need to subsidize low-budget agents declines. As a result, the verification probability is hump-shaped as shown in Figure 6c.

Intuitively, the total welfare is strictly increasing in S. More interestingly, the welfare of each type is not monotonic in S. Figure 6d plots the total welfare and the average utility of each budget type as a function of S. Initially, the average utilities of both budget types increase as S increases. When S is large enough that the principal begins to divert more cash subsidies and goods to low-budget agents, the average utility of high-budget agents begins to decrease as S increases. Eventually, the need to subsidize low-budget agents decreases as S increases, and the average utility of low-budget agents begins to decrease while that of high-budget agents begins to increase, until they coincide. Specifically, low-valuation agents of both budget types can get worse off as they receive less cash subsidies. Interestingly, high-budget high-valuation agents can also get worse off because their payments can increase as S increases (see Figure 7b). These increases in payments occur because disproportionately more goods will be allocated to low-budget agents and there will be less pooling when S increases.

Percentage of high-budget agents (π). Proposition 4 also proves that for small π, agents of both budget types receive the same amount of cash subsidies. Eventually, for π sufficiently small, verification is never used. This result is intuitive because a smaller π means a larger population of low-budget agents and therefore higher total verification cost given the same mechanism. Hence, the principal tends to inspect agents less frequently as π decreases. However, this change in verification probability is not monotonic in π, because an increase in π not only makes verification less costly but also makes the economy wealthier. If π is sufficiently large, then the principal becomes



(a) Cash Subsidies

(b) Allocation

(c) Verification Probability

(d) Welfare

Figure 8: The impact of an increase in the percentage of high-budget agents (π) on cash subsidies, allocation, verification and welfare. In this numerical example, v is uniformly distributed on $[0, 1]$, $\rho = 0.08$, $b_1 = 0.2$, $S = 0.4$ and $\pi \in [0, 1]$.

less budget-constrained and can afford to maintain incentive compatibility by subsidizing high-budget agents directly rather than inspect low-budget agents. This is illustrated by the numerical example in Figure 8.

The total welfare as well as the welfare of low-budget agents are strictly increasing in π, but the welfare of high-budget agents is not monotonic in π. Initially, as π increases, the welfare of high-budget agents declines as the principal provides more subsidies to low-budget agents. Eventually, the welfare of high-budget agents rises as the principal subsidizes high-budget agents rather than inspecting low-budget agents.

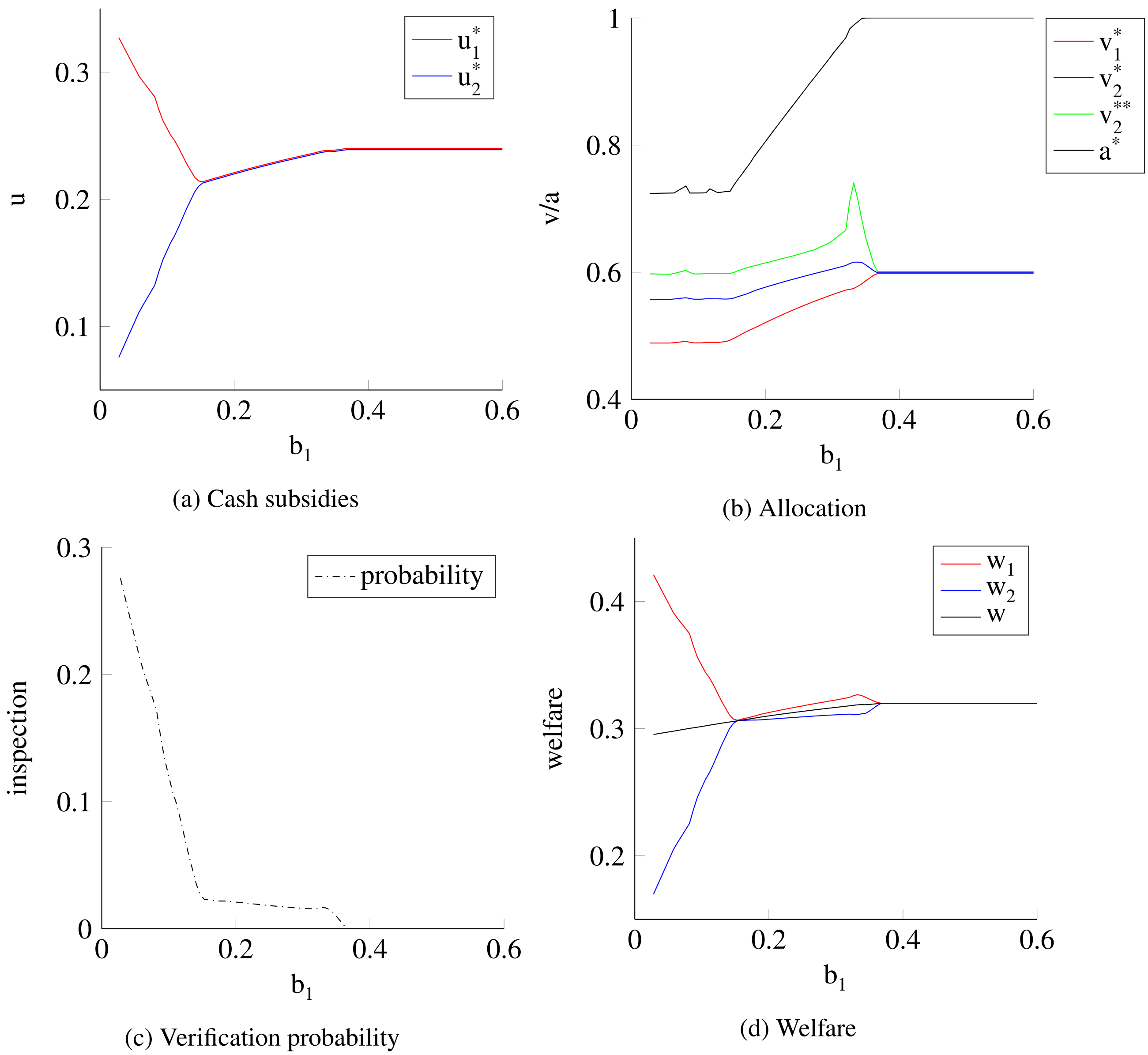
Budget (b_1) Low-budget agents become less budget constrained as b_1 increases. This change reduces the need for subsidies, which leads to a decline in verification probability. Proposition 6 proves that for large b_1, agents of both budget types receive the same amount of cash subsidies. Figure 9 plots the impact of an increase in b_1 on cash subsidies, allocation, verification and welfare in a numerical example. In this numerical example, the total verification probability is non-increasing in b_1 and zero for b_1 sufficiently large.

Proposition 6 *Suppose Assumptions 1 and 2 hold. If b_1 is sufficiently large, then agents of both budget types receive the same amount of cash subsidies, i.e., $u_1^* = u_2^*$.*

The total welfare, as well as, the welfare of high-budget agents is strictly increasing in b_1, but the welfare of low-budget agents is not monotonic in b_1. On the one hand, low-budget agents become less budget-constrained as b_1 increases. On the other hand, the principal provides lower cash and in-kind subsidies to low-budget agents as b_1 increases. As shown in Figure 9d, either effect can dominate the other. Hence, the welfare of low-budget agents may either increase or decrease as b_1 increases.

Lastly, I study the impact of relaxing the principal's budget-balanced constraint on the optimal mechanism and welfare. Specifically, I reformulate the principal's budget constraint as follows:

$$\mathbb{E}_t\left[p(t) - q(t)k\right] \geq -d. \tag{BB}$$



(a) Cash subsidies

(b) Allocation

(c) Verification probability

(d) Welfare

Figure 9: The impact of relaxing low-budget agents' budget constraint (b_1) on cash subsidies, allocation, verification and welfare. In this numerical example, υ is uniformly distributed on $[0, 1]$, $\rho = 0.08$, $S = 0.4$, $\pi = 0.5$ and $b_1 \in [0, 0.6]$.

In the main part of the paper, I assume $d = 0$. But it is easy to see that all the results in Sections 3 and 4 extend to the case of $d \geq 0$.

Figure 10 plots the impact of an increase in the principal's budget (d) on cash subsidies, allocation, verification and welfare in a numerical example. Note that an increase in d leads to an increase in the total cash subsidies by more than one-hundred percent. This is easy to see when there is no verification. An increase in d raises cash subsidies to low-budget agents, which relax their budget constraints and improve allocation efficiency. Under Assumption 1, this in turn improves the principal's revenue and allows her to further raise cash subsidies. The numerical example suggests this is still true when verification is possible.

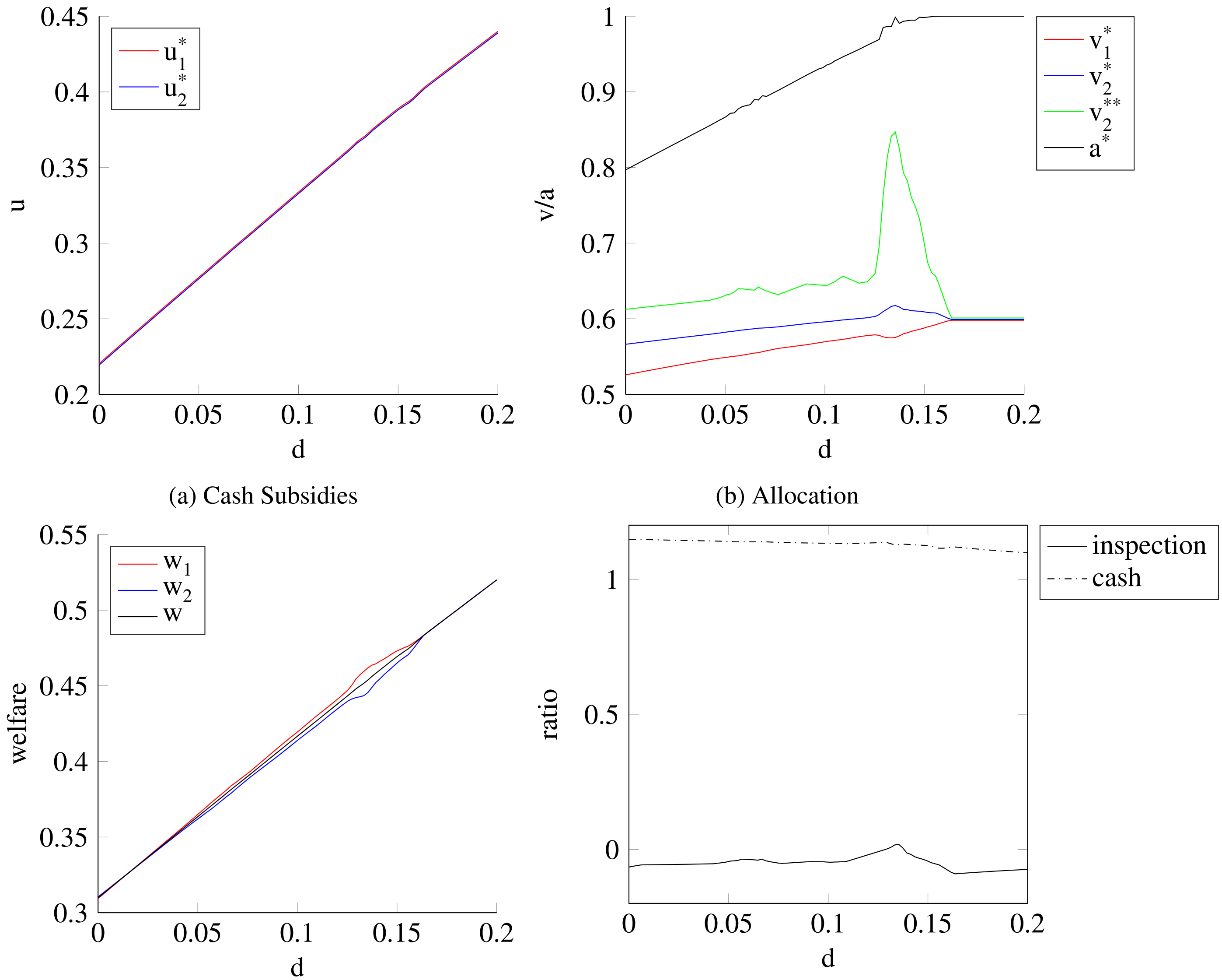
7 Extensions and Discussions

In this section, I discuss several issues. Section 7.1 shows that some of the analysis extends if I replace the budget constraint by a more stringent per-unit price constraint. Section 7.2 shows that the analysis extends to the case where punishment is transferable. Sections 7.3 and 7.4 discuss the robustness of my analysis to weakening the assumptions on verification and punishment, respectively. Section 7.5 discusses why it is necessary to explicitly model budget constraints.

7.1 Per-unit Price Constraint

In the optimal direct mechanism, agents make payments to the principal regardless of whether they receive the goods,[23] which some may consider unrealistic. The question, then, is whether this direct mechanism can be implemented by a mechanism in which agents pay if and only if they receive the goods and their payments are within their budgets. Such an implementation is impossible if $a^* < 1$. I can guarantee that such an implementation always exists if I replace (BC)

[23] For a finite number of agents, this is similar to an all-pay auction.



(a) Cash Subsidies

(b) Allocation

(c) Welfare

(d) Change in Verification Cost/Cash Subsidies

Figure 10: The impact of an increase in the principal's budget (d) on cash subsidies, allocation, verification and welfare. In this numerical example, v is uniformly distributed on $[0, 1]$, $\rho = 0.04$, $b_1 = 0.2$, $S = 0.4$, $\pi = 0.5$ and $d \in [0.0.2]$.

by the following *per-unit price constraint*:

$$p(t) \leq a(t)b, \ \forall t = (v, b). \tag{PC}$$

(BC) is the same as that found in Che and Gale (2000) and Pai and Vohra (2014), but different from Che et al. (2013), which uses (PC).

Nevertheless, I assume (BC) in the main body of the paper for the following reasons. First, as will become clear, the optimal mechanisms in these two settings share qualitatively similar features. Second, for some parameter values (e.g., verification cost is low, resources are relatively abundant or the percentage of budget constrained agents is small), there is no rationing in the optimal mechanism ($a^*(\rho) = 1$). Third, rationing is realistic if b_1 is close to zero. For example, families with very low income may receive free coverage from Medicaid.

In the rest of this subsection, I consider the principal's problem in which (BC) is replaced by (PC), denoted by $\mathcal{P}_{PC}$. I first make the observation that if (PC) holds for v' then it holds for all $v < v'$. This is trivial if $a(v, b) = 0$. If $a(v, b) > 0$, then by the envelope condition we have

$$\begin{aligned}\frac{p(v', b)}{a(v', b)} - \frac{p(v, b)}{a(v, b)} =& \int_{v}^{v'} \left(1 - \frac{a(\nu, b)}{a(v', b)}\right) \mathrm{d}\nu - \int_{\underline{v}}^{v} \left(\frac{a(\nu, b)}{a(v', b)} - \frac{a(\nu, b)}{a(v, b)}\right) \mathrm{d}\nu - \frac{u(\underline{v}, b)}{a(v', b)} + \frac{u(\underline{v}, b)}{a(v, b)} \\ \geq& 0,\end{aligned}$$

where the last inequality holds since $a(v, b)$ is non-decreasing in v. Hence, (PC) holds if and only if $p(\overline{v}, b) \leq a(\overline{v}, b)b$ for all b.

Given this observation, it is straightforward to extend the results of Theorems 1 and 2 to the current setting. Theorem 4 characterizes an optimal mechanism of $\mathcal{P}_{PC}$ when budgets are common knowledge ($\rho = 0$). Theorem 5 characterizes an optimal mechanism of $\mathcal{P}_{PC}$ when budget is an agent's private information and the principal cannot verify this information ($\rho = \infty$). The latter theorem extends the results in Section 3 of Che et al. (2013) to the setting of a continuum of values under the regularity conditions.

Theorem 4 *Suppose Assumption 2 holds, and budgets are common knowledge. There exists $v_1^*(0)$, $v_2^*(0)$, $u_1^*(0)$ and $u_2^*(0)$ such that an optimal mechanism of $\mathcal{P}_{PC,CB}$ is given by*

$$
\begin{aligned}
a(v, b_1) &= \chi_{\{v \geq v_1^*(0)\}} a^*(0), \quad & p(v, b_1) &= \chi_{\{v \geq v_1^*(0)\}} \left(u_1^*(0) + a^*(0) b_1\right) - u^*(0), \\
a(v, b_2) &= \chi_{\{v \geq v_2^*(0)\}}, & p(v, b_2) &= \chi_{\{v \geq v_2^*(0)\}} v_2^*(0),
\end{aligned}
$$

where $a^*(0) = u_1^*(0)/\left[v_1^*(0) - b_1\right]$, $b_1 < v_1^*(0) \leq v_2^*(0) < \bar{v}$ and $0 = u_2^*(0) < u_1^*(0) \leq v_1^*(0) - b_1$.

Theorem 5 *Suppose Assumptions 1 and 2 hold, and the principal does not inspect agents. There exists $v_1^*(\infty)$, $v_2^*(\infty)$, $v_2^{**}(\infty)$, $u_1^*(\infty)$ and $u_2^*(\infty)$such that an optimal mechanism of $\mathcal{P}_{PC,NI}$ with no verification satisfies*

$$
\begin{aligned}
a(v, b_1) &= \chi_{\{v \geq v_1^*(\infty)\}} a^*(\infty), \;\; p(v, b_1) = \chi_{\{v \geq v_1^*(\infty)\}} a^*(\infty) v_1^*(\infty) - u_1^*(\infty), \\
a(v, b_2) &= \chi_{\{v \geq v_1^*(\infty)\}} a^*(\infty) + \chi_{\{v \geq v_2^{**}(\infty)\}} \left(1 - a^*(\infty)\right), \\
p(v, b_2) &= \chi_{\{v \geq v_2^*(\infty)\}} a^*(\infty) v_2^*(\infty) + \chi_{\{v \geq v_2^{**}\}} \left(1 - a^*(\infty)\right) v_2^{**}(\infty) - u_2^*(\infty),
\end{aligned}
$$

where $a^*(\infty) = u_1^*(\infty)/\left[v_1^*(\infty) - b_1\right]$, $b_1 < v_1^*(\infty) = v_2^*(\infty) \leq v_2^{**}(\infty) \leq \bar{v}$ and $0 < u_1^*(\infty) = u_2^*(\infty) \leq v_1^*(\infty) - b_1$.

The analysis is more complex if budget is an agent's private information and the principal can verify this information at a cost. As before, I first consider the principal's relaxed problem $\mathcal{P}'_{PC}$ in which I relax (IC) corresponding to over-reporting of budgets. One can show that Lemmas 3 and 4 and Corollary 1 still hold. Next, I consider the principal's relaxed problem with two modifications: (i) The allocation rule is an M'-step allocation rule for some integer $M' \leq M$ and $M \geq 2$ is a fixed integer; and (ii) the principal's budget balance constraint is relaxed by a constant $d \geq 0$. Denote this problem by $\mathcal{P}'_{PC}(M, d)$ and its value by $V_{PC}(M, d)$. Then $\mathcal{P}'_{PC}(M, d)$ is identical to $\mathcal{P}'(M, d)$ if I replace (BC) by the following (PC) constraint:

$$
a^M v_1^{M-1} - \sum_{j=1}^{M-1} a^j (v_1^j - v_1^{j-1}) - u(\underline{v}, b_1) \leq b_1 a^M. \tag{η}
$$

One can readily extend the results of Lemma 5 to the current setting, which says that, in an optimal mechanism of $\mathcal{P}'_{PC}(M,d)$, the verification probability is non-decreasing in a low-budget agent's reported value. Using the monotonicity of verification probability and the linearity of $\mathcal{P}'_{PC}(M,d)$ in $u(\underline{v},b_1)$, $u(\underline{v},b_2)$ and a^m's, we have $V_{PC}(M,d) = V_{PC}(M-1,d)$ for M sufficiently large. By a similar approximation argument to that in the proof of Theorem 3, one can prove the following theorem, which characterizes an optimal mechanism of $\mathcal{P}_{PC}$.

Theorem 6 *Suppose Assumptions 1 and 2 hold. There exists an integer* $2 \le M \le 5$, $\underline{v} < v_1^1 < \cdots < v_1^{M-1} < \bar{v}$, $\underline{v} \le v_2^0 \le v_2^1 < \cdots < v_2^{M-1} \le v_2^M < \bar{v}$ *and* $0 \le a^1 < a^2 < \ldots a^M \le 1$ *such that an optimal mechanism of* $\mathcal{P}_{PC}$ *is given by*

$$a(v,b_1) = \sum_{m=1}^{M} \chi_{\{v_1^{m-1} < v \le v_1^m\}} a^m,$$

$$p(v,b_1) = \sum_{m=1}^{M-1} \chi_{\{v \ge v_1^m\}} \left(a^{m+1} - a^m\right) v_1^m - u(\underline{v},b_1),$$

$$q(v,b_1) = \frac{1}{c}\left[u(\underline{v},b_1) - u(\underline{v},b_2) + a^1\left(v_2^0 - \underline{v}\right) + \sum_{m=1}^{M-1} \chi_{\{v \ge v_1^m\}} \left(a^{m+1} - a^m\right)\left(v_2^m - v_1^m\right)\right]$$

$$a(v,b_2) = \sum_{m=1}^{M} \chi_{\{v_2^{m-1} < v \le v_2^m\}} a^m + \chi_{\{v \ge v_2^M\}},$$

$$p(v,b_2) = \sum_{m=0}^{M-1} \chi_{\{v \ge v_2^m\}} \left(a^{m+1} - a^m\right) v_2^m + \chi_{\{v \ge v_2^M\}} \left(1 - a^M\right) - u(\underline{v},b_2).$$

However, it is hard to further improve this result, as in Section 4.2 when we require only the weaker (BC) constraint. In particular, the proof of Lemma 6 does not apply here. It holds if we also make the following assumption.

Assumption 3 $a(v,b) = 0$ *for all* $v < b_1$.

Assumption 3 requires that an agent whose valuation is too low (lower than b_1) receives the good with probability zero. Note that optimal mechanisms in Theorem 4 and Theorem 5 satisfy this condition. I conjecture this condition also holds in the general case, although I cannot prove it. Under this additional assumption, we have

Lemma 7 *Suppose Assumptions 1, 2 and 3 hold. Then $V_{PC}(M,d) = V_{PC}(2,d)$ for all $M \geq 2$ and $d \geq 0$.*

Given Lemma 7, it is easy to extend the result of Theorem 3 to this setting.

Theorem 7 *Suppose Assumptions 1, 2 and 3 hold. There exist $a^*(\rho)$, $v_1^*(\rho)$, $v_2^*(\rho)$, $v_2^{**}(\rho)$, $u_1^*(\rho)$ and $u_2^*(\rho)$ such that an optimal mechanism of $\mathcal{P}_{PC}$ is given by*

$$
\begin{aligned}
a(v,b_1) &= \chi_{\{v \geq v_1^*(\rho)\}} a^*(\rho), \;\; p(v,b_1) = \chi_{\{v \geq v_1^*(\rho)\}} a^*(\rho) v_1^*(\rho) - u_1^*(\rho), \\
q(v,b_1) &= \frac{1}{c}\left[\chi_{\{v \geq v_1^*(\rho)\}} a^*(\rho)\left(v_2^*(\rho) - v_1^*(\rho)\right) + u_1^*(\rho) - u_2^*(\rho)\right], \\
a(v,b_2) &= \chi_{\{v \geq v_2^*(\rho)\}} a^*(\rho) + \chi_{\{v \geq v_2^{**}(\rho)\}}\left(1 - a^*(\rho)\right), \\
p(v,b_2) &= \chi_{\{v \geq v_2^*(\rho)\}} a^*(\rho) v_2^*(\rho) + \chi_{\{v \geq v_2^{**}(\rho)\}}\left(1 - a^*(\rho)\right) v_2^{**}(\rho) - u_2^*(\rho), \\
q(v,b_2) &= 0,
\end{aligned}
$$

where $a^*(\rho) = u_1^*(\rho)/\left[v_1^*(\rho) - b_1\right]$, $\underline{v} \leq v_1^*(\rho) \leq v_2^*(\rho) \leq v_2^{**}(\rho) \leq \overline{v}$, $0 < a^*(\rho) \leq 1$ and $u_1^*(\rho) \geq u_2^*(\rho)$.

7.2 Monetary Penalty

In this subsection, I discuss what happens if penalty is transferable. Specifically, the principal can inspect an agent's budget at a cost $k > 0$, and can impose a monetary penalty of up to $c \geq 0$. I also allow the principal to punish an innocent agent and an agent without verification. Nonetheless, as I will show later, it is optimal for the principal to punish an agent if and only if he is found to have lied about his budget. Using this result, the principal's problem can be reduced to the one stated in Section 2, when penalty is not transferable. Hence, all results in previous sections also hold in the case of monetary penalty.

I also relax the assumption that an agent is punished if and only if he is found to have lied. In this case, a direct mechanism is a quadruple (a, p, q, θ), where a, p and q are defined as before and $\theta : T \times \{b_1, b_2, n\} \to [0, c]$ denotes the penalty imposed on an agent. In particular, $\theta(\hat{t}, n) \in$

$[0, c]$ denotes the penalty imposed on an agent who reports $\hat{t}$ and is not inspected, and $\theta(\hat{t}, b) \in [0, c]$ denotes the penalty imposed on an agent who reports $\hat{t}$ and is inspected, and whose budget is revealed to be b. The utility of an agent who has type $t := (v, b)$ and reports $\hat{t}$ is

$$u(\hat{t}, t) = \begin{cases} a(\hat{t})v - p(\hat{t}) - (1 - q(\hat{t}))\theta(\hat{t}, n) - q(\hat{t})\theta(\hat{t}, b) & \text{if } p(\hat{t}) + \theta(\hat{t}, b) \leq b \text{ and } p(\hat{t}) + \theta(\hat{t}, n) \leq b, \\ -\infty & \text{otherwise.} \end{cases}$$

The principal's problem is

$$\max_{a,p,q,\theta} \mathbb{E}_t \left[a(t)v - q(t)k \right], \tag{$\mathcal{P}$}$$

subject to

$$u(t) \equiv u(t, t) \geq 0, \ \forall t \in T, \tag{IR}$$

$$u(t) \geq u(\hat{t}, t), \ \forall t \in T, \hat{t} \in \left\{ \hat{t} \in T \left| p(\hat{t}) + \max\{\theta(\hat{t}, n), \theta(\hat{t}, b)\} \leq b \right. \right\}, \tag{IC}$$

$$p(t) + \max\{\theta(t, n), \theta(t, b)\} \leq b, \ \forall t \in T, \tag{BC}$$

$$\mathbb{E}_t \left[p(t) + (1 - q(t))\theta(t, n) + q(t)\theta(t, b) - q(t)k \right] \geq 0, \tag{BB}$$

$$\mathbb{E}_t \left[a(t) \right] \leq S. \tag{S}$$

Note that (BC) requires that an agent must be able to afford the payment and the punishment. I show that it is without loss of generality to focus on mechanisms in which an agent is penalized if and only if he is found to have lied about his budget, and whenever he is found to have lied he has the maximum possible monetary penalty c imposed upon him.

Lemma 8 *It is without loss of generality to focus on mechanisms in which $\theta(\hat{t}, n) = 0$, $\theta(\hat{t}, b) = 0$ if $\hat{b} = b$ and $\theta(\hat{t}, b) = c$ if $\hat{b} \neq b$.*

Using Lemma 8, the principal's problem can be reduced to the one stated in Section 2 when penalty is not transferable. Hence, all results in previous sections also hold in the case of monetary penalty.

7.3 Costly Disclosure

In this subsection, I study what happens if it is also costly for an agent to have his report verified. For example, agents may bear some costs of providing documentation. Assume that an agent incurs a non-monetary cost only when his report is verified. Let $c^T \geq 0$ denote the cost incurred by an agent from being verified by the principal if he reported his type truthfully, and let $c^F \geq c^T$ be his cost if he lied.[24] As I will show below, disclosure costs have three effects. Firstly, similar to monetary transfers, disclosure costs can also be used to screen agents with different valuations and help relax agents' budget constraints. Secondly, it is more costly for an agent to lie about his budget because $c^F \geq c^T$. Finally, disclosure costs make verification more costly for the principal whose concern is welfare. Even though it is difficult to solve the optimal mechanism, I show that if the difference between c^F and c^T is sufficiently large, then the first two effects dominate and introducing disclosure costs improves welfare.

The utility of an agent who has type $t = (v, b)$ and reports $\hat{t}$ is

$$u(\hat{t}, t) = \begin{cases} a(\hat{t})v - p(\hat{t}) - q(\hat{t})c^T & \text{if } \hat{b} = b \text{ and } p(\hat{t}) \leq b, \\ a(\hat{t})v - p(\hat{t}) - q(\hat{t})\left(c^F + c\right) & \text{if } \hat{b} \neq b \text{ and } p(\hat{t}) \leq b, \\ -\infty & \text{if } p(\hat{t}) > b. \end{cases}$$

The principal's problem is

$$\max_{a,p,q} \mathbb{E}_t \left[a(t)v - q(t)k - q(t)c^T\right], \tag{$\mathcal{P}_{DC}$}$$

subject to (IR), (IC), (BC), (BB) and (S). Note that if $c^T = 0$, then ($\mathcal{P}_{DC}$) is equivalent to the original problem ($\mathcal{P}$) in which the punishment is $c^F + c$.

Consider the more general case in which $c^T \geq 0$. Define $p^e(t) := p(t) + q(t)c^T$, $k^e := k + c^T$ and $c^e := c + c^F - c^T$. As in Section 4, I separate (IC) into two categories and ignore those corresponding to over-reporting of budgets. Then the principal's relaxed problem can be written

[24]The analysis goes through as long as $c^F + c \geq c^T$.

as:

$$\max_{a,p,q} \mathbb{E}_t \left[a(t)v - q(t)k^e \right], \tag{$\mathcal{P}'_{DC}$}$$

subject to

$$a(t)v - p^e(t) \geq 0, \ \forall t \in T, \tag{IR}$$

$$a(v,b)v - p^e(v,b) \geq a(\hat{v},b)v - p^e(\hat{v},b), \ \forall v, \hat{v}, b, \tag{IC-v}$$

$$a(v,b_2)v - p^e(v,b_2) \geq a(\hat{v},b_1)v - q(\hat{v},b_1)c^e - p^e(\hat{v},b_1), \ \forall v, \hat{v}. \tag{IC-b}$$

$$p^e(t) \leq b + q(t)c^T, \ \forall t \in T, \tag{BC}$$

$$\mathbb{E}_t \left[p^e(t) - q(t)k^e \right] \geq 0,, \tag{BB}$$

$$\mathbb{E}_t \left[a(t) \right] \leq S. \tag{S}$$

Compare $\mathcal{P}'_{DC}$ with $\mathcal{P}'$. It is easy to see that the two problems are identical except for the (BC) constraint. In $\mathcal{P}'_{DC}$, a low-budget agent faces a less stringent budget constraint if he expects to be inspected with a non-zero probability. This is because in the presence of disclosure cost the *effective payment* made by an agent who reports his type truthfully is $p^e(t) = p(t) + q(t)c^T$. In addition to the monetary transfer $p(t)$, disclosure cost $q(t)c^T$ can also be used to screen agents with different valuations. Intuitively, an agent with a higher valuation is also willing to bear a higher disclosure cost. Though disclosure cost can be used to relax an agent's budget constraint, it reduces an agent's utility which makes verification more costly from the principal's perspective, i.e., $k^e = k + c^T \geq k$. As a result, the total welfare effect of introducing c^T is ambiguous.

The *effective punishment* perceived by an agent is now $c + c^F - c^T$, the original punishment plus the additional disclosure cost one must incur when lying about his budget. Hence, an increase in c^F is always welfare-enhancing, as it discourages agents from misreporting their budgets.

Though solving $\mathcal{P}_{DC}$ is beyond the scope of this paper, Proposition 7 provides a sufficient condition under which introducing disclosure costs c^T and c^F improve the total welfare. Let $V(k, c, b_1)$ denote the value of the principal's original problem, in which verification cost is k, punishment is

c and low-budget agent's budget is b_1; and let $V_{DC}(k, c, b_1, c^T, c^F)$ denote the value of the principal's problem in which verification cost is k, punishment is c, low-budget agent's budget is b_1 and disclosure costs are c^T and c^F. Then

Proposition 7 *Suppose Assumptions 1 and 2 hold. If $k/c \geq c^T/(c^F - c^T)$, then $V_{DC}(k, c, b_1, c^F, c^T) \geq V(k, c, b_1)$. Furthermore, if $q(\overline{v}, b_1) > 0$ in the optimal mechanism of $\mathcal{P}(k + c^T, c + c^F - c^T, b_1)$, then $V_{DC}(k, c, b_1, c^F, c^T) > V(k, c, b_1)$.*

7.4 Punishing the Innocent or without Verification

In Appendix A, I show that it is without loss of generality to focus on a direct mechanism (a, p, q, θ), where $a : T \to [0, 1]$ denotes the probability an agent obtains the good, $p : T \to \mathbb{R}$ denotes the payment an agent must make, $q : T \to [0, 1]$ denotes the probability of inspecting and $\theta : T \times \{b_1, b_2, n\} \to [0, 1]$ denotes the probability of punishment. In particular, $\theta(\hat{t}, n) \in [0, 1]$ denotes the probability of punishing an agent who reports $\hat{t}$ and is not inspected, and $\theta(\hat{t}, b) \in [0, 1]$ denotes the probability of punishing an agent who reports $\hat{t}$ and is inspected and whose budget is revealed to be b. In the main part of the paper, I assume that $\theta((v, b), n) = \theta((v, b), b) = 0$. In other words, the principal is not allowed to punish an agent without verifying his budget or an agent who is found to have reported his budget truthfully. This assumption is not without loss of generality.

In this case, the utility of an agent who has type $t = (v, b)$ and reports $\hat{t} = (\hat{v}, \hat{b})$ is

$$u(\hat{t}, t) = \begin{cases} a(\hat{t})v - p(\hat{t}) - (1 - q(\hat{t}))\theta(\hat{t}, n)c - q(\hat{t})\theta(\hat{t}, b)c & \text{if } p(\hat{t}) \leq b \\ -\infty & \text{if } p(\hat{t}) > b \end{cases}$$

Then the principal's problem is

$$\max_{a,p,q,\theta} \mathbb{E}_t \left[a(t)v - q(t)k - (1 - q(t))\theta(t, n)c - q(t)\theta(t, b)c \right], \qquad (\mathcal{P}_{PI})$$

subject to (IR), (IC), (BC), (BB) and (S). Lemma 9 shows that the principal finds it optimal to *always* punish an agent who is found to have lied about his budget and *never* punish an agent who is found to have reported his budget truthfully.

Lemma 9 *An optimal mechanism of $\mathcal{P}_{PI}$ satisfies that (i) $\theta((\hat{v}, \hat{b}), b) = 1$ if $\hat{b} \neq b$ and (ii) $\theta((v, b), b) = 0$ for almost all (v, b).*

Define $p^e(t) := p(t) + (1 - q(t))\theta(t, n)c$, which is the *effective payment* made by an agent who reports his type truthfully. As in Section 4, I separate (IC) constraints into two categories and ignore those corresponding to over-reporting of budgets. Using Lemma 9, the principal's problem can be written as:

$$\max_{a,p,q,\theta} \mathbb{E}_t \left[a(t)v - q(t)k - (1 - q(t))\theta(t, n)c\right], \tag{$\mathcal{P}'_{PI}$}$$

subject to

$$a(t)v - p^e(t) \geq 0, \ \forall t \in T, \tag{IR}$$

$$a(v, b)v - p^e(v, b) \geq a(\hat{v}, b)v - p^e(\hat{v}, b), \ \forall v, \hat{v}, b, \tag{IC-v}$$

$$a(v, b_2)v - p^e(v, b_2) \geq a(\hat{v}, b_1)v - q(\hat{v}, b_1)c - p^e(\hat{v}, b_1), \ \forall v, \hat{v}. \tag{IC-b}$$

$$p^e(t) \leq b + (1 - q(t))\theta(t, n)c, \ \forall t \in T, \tag{BC}$$

$$\mathbb{E}_t \left[p^e(t) - q(t)k - (1 - q(t))\theta(t, n)c\right] \geq 0, \tag{BB}$$

$$\mathbb{E}_t \left[a(t)\right] \leq S. \tag{S}$$

Compare $\mathcal{P}'_{PI}$ with $\mathcal{P}'$. Note that by punishing an agent without verifying his budget, the principal relaxes the agent's budget constraint. However, it is costly, as reflected in the principal's objective function and (BB). Hence, in general, it is unclear whether it is optimal for the principal to do so.

7.5 Modified Type

In the standard environment, when agents are not budget-constrained, an agent's valuation is defined as the maximum amount of money he is willing to pay for the good. When agents are budget constrained, the natural analogue is the minimum of an agent's valuation v and budget b. I follow Pai and Vohra (2014) and redefine $t := \min\{v, b\}$ as an agent's *modified type*. In this subsection, I show why it is necessary to explicitly model budget constraint rather than accommodate budgets in the above way.

Let G denote the distribution of the modified type. Then

$$G(t) = \begin{cases} F(t) & \text{if } t < b_1, \\ \pi F(b_1) + 1 - \pi & \text{if } t = b_1, \\ \pi F(t) & \text{if } t > b_1. \end{cases}$$

The principal's ability to inspect an agent's budget implies that she can perfectly learn a low-budget agent's modified type if his valuation exceeds his budget. I first solve the principal's problem by assuming common-knowledge budgets and then verify that no agent has any incentive to misreport his modified type. In other words, there is no inspection in the optimal mechanism. Denote the principal's problem by $\mathcal{P}_{MT}$.

Proposition 8 *Suppose an agent's budget is common knowledge. (i) If $\pi\left[1 - F(b_1)\right] \leq S < 1 - F(b_1)$, then the optimal mechanism of $\mathcal{P}_{MT}$ is given by*

$$a(t) = \chi_{\{t=b_1\}} \frac{S - \pi\left[1 - F(b_1)\right]}{1 - \pi} + \chi_{\{t>b_1\}}, \quad p(t) = \chi_{\{t=b_1\}} \frac{S - \pi\left[1 - F(b_1)\right]}{1 - \pi} b_1 + \chi_{\{t \geq b_1\}} b_1.$$

(ii) If $S < \pi\left[1 - F(b_1)\right]$, then the optimal mechanism is given by

$$a(t) = \chi_{\{t>t^*\}}, \quad p(t) = \chi_{\{t \geq b_1\}} t^*,$$

where t^* is such that $\pi\left[1 - F(t^*)\right] = S$.

The following corollary is a straightforward corollary of Proposition 8.

Corollary 3 *Suppose an agent's budget is his private information. The mechanism given in Proposition 8 is incentive compatible and therefore optimal in $\mathcal{P}_{MT}$.*

Compared with Theorem 3, the mechanism given in Proposition 8 is sub-optimal because (i) it allocates too many resources to high-budget agents; and (ii) it has "too little" rationing for high-budget agents but "too much" rationing for low-budget agents.

What went wrong here? First, consider a low-budget agent with modified type $t = b_1$ and a high-budget agent with modified type $t = b_1 + \epsilon$ for some $\epsilon > 0$. Then the low-budget agent's expected valuation is higher than the high-budget agent's valuation, i.e., $\mathbb{E}[v|t = b_1, b = b_1] > b_1 + \epsilon$, for $\epsilon > 0$ sufficiently small. This implies that the low-budget agent should receive the good with higher probability as in Theorem 3, i.e., $v_1^* \leq v_2^*$. However, in the current mechanism it is the high-budget agent who receives the good with higher probability. Second, consider two low-budget agents with valuations $v = b_1$ and $v' = b_1 + \epsilon$ for $\epsilon > 0$ sufficiently small, respectively. In the current mechanism, they are pooled. However, their payments are $p(b_1) < b_1$, which suggests that they should be separated as in Theorem 3, i.e., $v_1^* > b_1$. The second observation is also made in Pai and Vohra (2014) in which the principal's objective is maximizing revenue.

8 Conclusion

In this paper, I study the problem of a principal who wishes to distribute an indivisible good to a population of budget-constrained agents. Both valuation and budget are an agent's private information. The principal can inspect an agent's budget through a costly verification process and punish an agent who makes a false statement. I characterize the direct surplus-maximizing mechanism. This direct mechanism can be implemented by a two-stage mechanism that exhibits some of the features of Singapore's housing and development board.

Throughout the paper, I impose two regularity assumptions on the distribution of an agent's valuation: monotone hazard rate condition and decreasing density condition. These two assumptions are commonly used in the literature studying mechanism design problem with financially constrained agents. Their primary role is to rule out complicated pooling regions in an optimal mechanism, which greatly simplifies analysis. Several of the paper's results (Lemmas 3, 4 and 5) still hold if I replace these two assumptions by weaker conditions. However, Lemma 6 may not hold anymore as an optimal mechanism is expected to involve more complicated pooling regions.

Another simplifying assumption I make in the paper is that valuation and budget are independent. In some environments, this assumption is reasonable. For example, an individual's valuation of health insurance is largely affected by his or her health risk, which is relatively independent of his or her wealth. In general, an individual's valuation and budget can be either positively or negatively correlated, depending on whether the goods are considered normal goods or inferior goods. The independence assumption is much harder to relax. As Pai and Vohra (2014) show, if valuation and budget are correlated, an optimal mechanism may involve more complicated pooling regions.

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A The Revelation Principle

Consider a general mechanism that consists of a message space $\mathscr{M}$ and a quadruplet (a, p, q, θ), where $a : \mathscr{M} \to [0, 1]$ denotes the probability an agent obtains the good, $p : \mathscr{M} \to [0, 1]$ denotes the payment an agent must make, $q : \mathscr{M} \to [0, 1]$ denotes the probability of inspecting and $\theta : \mathscr{M} \times \{n, b_1, b_2\} \to [0, 1]$ denotes the probability an agent is penalized. In particular, $\theta(m, n)$ denotes the probability an agent is penalized if he is not inspected and $\theta(m, b)$ denotes the probability an agent is penalized if he is inspected and his budget is revealed to be b.

Given a mechanism, an agent of type $t = (v, b)$ chooses $m \in \mathscr{M}$ to maximize

$$a(m)v - p(m) - (1 - q(m))\theta(m, n)c - q(m)\theta(m, b)c$$

subject to the constraint that $p(m) \leq b$. Let $m^*(t)$ denote the solution to the agent's problem. For simplicity, I assume $m^*(t)$ is deterministic, but it is easy to accommodate mixed strategies. If the agent's problem has multiple solutions, then some deterministic selection rule is used. Consider a new mechanism with message space T. Let $a^*(t) = a(m^*(t))$, $p^*(t) = p(m^*(t))$, $q^*(t) = a(m^*(t))$ and $\theta^*(t, \cdot) = \theta(m^*(t), \cdot)$. Then the new mechanism is incentive compatible. Clearly, an agent has no incentive to report $\hat{t}$ such that $p^*(\hat{t}) > b$. For $\hat{t}$ such that $p^*(\hat{t}) \leq b$, we have

$$\begin{aligned}
&a(m^*(t))v - p(m^*(t)) - (1 - q(m^*(t)))\theta(m^*(t), n)c - q(m^*(t))\theta(m^*(t), b)c \\
\geq &a(m^*(\hat{t}))v - p(m^*(\hat{t})) - (1 - q(m^*(\hat{t})))\theta(m^*(\hat{t}), n)c - q(m^*(\hat{t}))\theta(m^*(\hat{t}), b)c.
\end{aligned}$$

The inequality simply follows from the fact that $m^*(t)$ is the solution to a type t agent's problem in the original mechanism. Clearly, the principal's payoff in the truthtelling equilibrium is as same as that in the original mechanism.

B Common Knowledge Budgets

Proof of Theorem 1. Let (a,p) be a feasible mechanism. For each $b \in B$, $a(\cdot,b)$ is non-decreasing and $p(v,b) = va(v,b) - \int_{\underline{v}}^{v} a(\nu,b)\mathrm{d}\nu - u(\underline{v},b)$. Consider another mechanism (a^*,p^*). Let $a^*(\cdot,b)$ be defined by

$$a^*(v,b) = \begin{cases} a(\overline{v},b) & \text{if } v \geq v_b^* \\ 0 & \text{otherwise} \end{cases},$$

where v_b^* is such that

$$\int_{\underline{v}}^{\overline{v}} a(v,b) f(v)\mathrm{d}v = a(\overline{v},b)(1-F(v_b^*)). \tag{13}$$

Let $p^*(v,b) = va^*(v,b) - \int_{\underline{v}}^{v} a^*(\nu,b)\mathrm{d}\nu - u(\underline{v},b)$. Clearly, (a^*,p^*) satisfies constraints (IR), (IC) and (S) and improves welfare. The revenue obtained by (a^*,p^*) is

$$\mathbb{E}_t[p^*(t)] = -(1-\pi)u(\underline{v},b_1) - \pi u(\underline{v},b_2) + \int_{\underline{v}}^{\overline{v}} \left[v - \frac{1-F(v)}{f(v)}\right]\left[(1-\pi)a^*(v,b_1) + \pi a^*(v,b_2)\right]\mathrm{d}v.$$

By Assumption 1, $v - [1-F(v)]/f(v)$ is strictly increasing. Thus, (a^*,p^*) also improves revenue, and therefore satisfies the (BB) constraint. Finally, we show that the (BC) constraint holds:

$$\begin{aligned} p^*(\overline{v},b) =& \overline{v}a(\overline{v},b) - \int_{\underline{v}}^{\overline{v}} a^*(v,b)\mathrm{d}v - u(\underline{v},b) \\ \leq& \overline{v}a(\overline{v},b) - \int_{\underline{v}}^{\overline{v}} a(v,b)\mathrm{d}v - u(\underline{v},b) \leq b. \end{aligned}$$

The inequality holds if and only if

$$\begin{aligned} &\int_{\underline{v}}^{\overline{v}} [a^*(v,b) - a(v,b)]\mathrm{d}v \geq 0, \\ \iff &\int_{v_b^*}^{\overline{v}} [a^*(v,b) - a(v,b)]\mathrm{d}v \geq \int_{\underline{v}}^{v_b^*} [a(v,b) - a^*(v,b)]\mathrm{d}v. \end{aligned}$$

The inequality holds since

$$
\begin{aligned}
\int_{v_b^*}^{\overline{v}} [a^*(v,b) - a(v,b)]\mathrm{d}v &= \int_{v_b^*}^{\overline{v}} [a^*(v,b) - a(v,b)] f(v) \frac{1}{f(v)} \mathrm{d}v \\
&\geq \int_{v_b^*}^{\overline{v}} [a^*(v,b) - a(v,b)] f(v) \frac{1}{f(v_b^*)} \mathrm{d}v \\
&= \int_{\underline{v}}^{v_b^*} [a(v,b) - a^*(v,b)] f(v) \frac{1}{f(v_b^*)} \mathrm{d}v \\
&\geq \int_{\underline{v}}^{v_b^*} [a(v,b) - a^*(v,b)] f(v) \frac{1}{f(v)} \mathrm{d}v \\
&= \int_{\underline{v}}^{v_b^*} [a(v,b) - a^*(v,b)]\mathrm{d}v,
\end{aligned}
$$

where the second and fourth line holds since f is non-increasing by Assumption 2 and the third line holds by (13). Hence there exists v_1^* and v_2^* such that the optimal allocation rule satisfies $a(v,b_1) = \chi_{\{v \geq v_1^*\}}(v) \min\left\{\frac{u(\underline{v},b_1)+b_1}{v_1^*}, 1\right\}$ and $a(v,b_2) = \chi_{\{v \geq v_2^*\}}(v)$. ■

C Privately Known Budgets

C.1 No Verification

Proof of Lemma 1. If $v = \underline{v}$, (3) reduces to $u(\underline{v},b_2) \geq u(\underline{v},b_1)$. Suppose $u(\underline{v},b_2) > u(\underline{v},b_1)$. Let

$$u^*(\underline{v},b_1) = u^*(\underline{v},b_2) = (1-\pi)u(\underline{v},b_1) + \pi u(\underline{v},b_2).$$

Let v^* be such that

$$v^* := \sup\left\{ v \;\middle|\; \begin{array}{c} \int_{\underline{v}}^{v} a(\nu,b_1)\mathrm{d}\nu + u(\underline{v},b_1) - \int_{\underline{v}}^{v} \min\{a(\nu,b_1), a(\nu,b_2)\}\mathrm{d}\nu \\ -(1-\pi)u(\underline{v},b_1) - \pi u(\underline{v},b_2) \leq 0 \end{array} \right\}.$$

Let $v^- := \sup\{v \leq v^* | a(v,b_2) \geq a(v,b_1)\}$ and $v^+ := \inf\{v \geq v^* | a(v,b_2) \geq a(v,b_1)\}$. Note that if $v^* = \overline{v}$, then $v^+ = v^* = \overline{v}$. Note also that if $a(v^*,b_2) \geq a(v^*,b_1)$, then $v^+ = v^- = v^*$. Clearly,

$a(v, b_1) > a(v, b_2)$ for all $v \in (v^-, v^+)$. There exists $\alpha \in (0, 1)$ such that

$$\int_{\underline{v}}^{v^+} a(\nu, b_1)\mathrm{d}\nu + u(\underline{v}, b_1) - \int_{\underline{v}}^{v^-} \min\{a(\nu, b_1), a(\nu, b_2)\}\mathrm{d}\nu$$
$$- \int_{v^-}^{v^+} [\alpha a(\nu, b_1) + (1-\alpha)a(\nu, b_2)]\mathrm{d}\nu - (1-\pi)u(\underline{v}, b_1) - \pi u(\underline{v}, b_2) = 0.$$

Assume without loss of generality that $a(v^-, b_1) = a(v^-, b_2)$ and $a(v^+, b_1) = a(v^+, b_2)$. Let

$$a^*(v, b_1) = \begin{cases} \min\{a(v, b_1), a(v, b_2)\} & \text{if } v < v^- \\ \alpha a(v, b_1) + (1-\alpha)a(v, b_2) & \text{if } v^- < v < v^+ \\ a(v, b_1) & \text{if } v > v^+ \end{cases},$$

and

$$a^*(v, b_2) = \begin{cases} \frac{(1-\pi)\left[a(v,b_1) - \min\{a(v,b_1), a(v,b_2)\}\right]}{\pi} + a(v, b_2) & \text{if } v < v^- \\ \frac{(1-\pi)\left[a(v,b_1) - \alpha a(v,b_1) - (1-\alpha)a(v,b_2)\right]}{\pi} + a(v, b_2) & \text{if } v^- < v < v^+ \\ a(v, b_2) & \text{if } v > v^+ \end{cases}.$$

Clearly, $a^*(v, b)$ is feasible and non-decreasing in v. By construction, we have $(1-\pi)u(\underline{v}, b_1) + \pi u(\underline{v}, b_2) = (1-\pi)u^*(\underline{v}, b_1) + \pi u^*(\underline{v}, b_2)$, $(1-\pi)a(v, b_1) + \pi a(v, b_2) = (1-\pi)a^*(v, b_1) + \pi a^*(v, b_2)$ for all v, and

$$u(\underline{v}, b_1) + \int_{\underline{v}}^{v^+} a(v, b_1)\mathrm{d}v = u^*(\underline{v}, b_1) + \int_{\underline{v}}^{v^+} a^*(v, b_1)\mathrm{d}v. \tag{14}$$

Hence

$$u(\underline{v}, b_2) + \int_{\underline{v}}^{v^+} a(v, b_2)\mathrm{d}v = u^*(\underline{v}, b_2) + \int_{\underline{v}}^{v^+} a^*(v, b_2)\mathrm{d}v. \tag{15}$$

Let $p^*(v,b) = va^*(v,b) - \int_{\underline{v}}^{v} a(\nu,b)\mathrm{d}\nu - u^*(\underline{v},b)$. Then

$$\begin{aligned}
p^*(\overline{v},b_1) &= \overline{v}a^*(\overline{v},b_1) - \int_{\underline{v}}^{\overline{v}} a^*(v,b_1)\mathrm{d}v - u^*(\underline{v},b_1) \\
&= \overline{v}a(\overline{v},b_1) - \int_{\underline{v}}^{v^+} a^*(v,b_1)\mathrm{d}v - \int_{v^+}^{\overline{v}} a(v,b_1)\mathrm{d}v - u^*(\underline{v},b_1) \\
&= \overline{v}a(\overline{v},b_1) - \int_{0}^{v^+} a(v,b_1)\mathrm{d}v - \int_{v^+}^{\overline{v}} a(v,b_1)\mathrm{d}v - u(\underline{v},b_1) \leq b_1,
\end{aligned}$$

where the third line follows from (14). Hence the (BC) constraint holds. For $v < v^-$, we have $a^*(v,b_2) \geq a^*(v,b_1)$ and $u^*(\underline{v},b_1) = u^*(\underline{v},b_2)$. Hence (3) holds. For $v > v^+$, we have

$$\begin{aligned}
u^*(\underline{v},b_1) + \int_{\underline{v}}^{v} a^*(\nu,b_1)\mathrm{d}\nu &= u(\underline{v},b_1) + \int_{\underline{v}}^{v} a(\nu,b_1)\mathrm{d}\nu \\
&\leq u(\underline{v},b_2) + \int_{\underline{v}}^{v} a(\nu,b_2)\mathrm{d}\nu \\
&= u^*(\underline{v},b_2) + \int_{\underline{v}}^{v} a^*(\nu,b_2)\mathrm{d}\nu,
\end{aligned}$$

where the first line follows from (14) and the third line follows from (15). Finally, consider $v \in [v^-, v^+]$. Suppose $\alpha \leq 1-\pi$, then $a^*(v,b_1) \leq a^*(v,b_2)$ for $v \in (v^-, v^+)$ and we have

$$\begin{aligned}
u^*(\underline{v},b_1) + \int_{\underline{v}}^{v} a^*(\nu,b_1)\mathrm{d}\nu &= u^*(\underline{v},b_1) + \int_{\underline{v}}^{v^-} a^*(\nu,b_1)\mathrm{d}\nu + \int_{v^-}^{v} a^*(\nu,b_1)\mathrm{d}\nu \\
&\leq u^*(\underline{v},b_2) + \int_{\underline{v}}^{v^-} a^*(\nu,b_2)\mathrm{d}\nu + \int_{v^-}^{v} a^*(\nu,b_2)\mathrm{d}\nu \\
&= u^*(\underline{v},b_2) + \int_{\underline{v}}^{v} a^*(\nu,b_2)\mathrm{d}\nu.
\end{aligned}$$

Suppose $\alpha > \pi$, then $a^*(v, b_1) > a^*(v, b_2)$ for $v \in [v^-, v^+]$ and we have

$$
\begin{aligned}
u^*(\underline{v}, b_1) + \int_{\underline{v}}^{v} a^*(\nu, b_1)\mathrm{d}\nu &= u^*(\underline{v}, b_1) + \int_{\underline{v}}^{v^+} a^*(\nu, b_1)\mathrm{d}\nu - \int_{v}^{v^+} a^*(\nu, b_1)\mathrm{d}\nu \\
&\leq u^*(\underline{v}, b_2) + \int_{\underline{v}}^{v^+} a^*(\nu, b_2)\mathrm{d}\nu - \int_{v}^{v^+} a^*(\nu, b_2)\mathrm{d}\nu \\
&= u^*(\underline{v}, b_2) + \int_{\underline{v}}^{v} a^*(\nu, b_2)\mathrm{d}\nu.
\end{aligned}
$$

Hence the (IC-b) constraint holds. Clearly, (a^*, p^*) also satisfies constraints (IR), (IC-v), (S) and (BB), and does not change welfare. ■

Proof of Lemma 2. Given Lemma 1, (3) becomes

$$
\int_{\underline{v}}^{v} a(\nu, b_2)\mathrm{d}\nu \geq \int_{\underline{v}}^{v} a(\nu, b_1)\mathrm{d}\nu, \ \forall v. \tag{16}
$$

For each $b \in B$, we have

$$
\begin{aligned}
\int_{\underline{v}}^{v} a(\nu, b) f(\nu)\mathrm{d}\nu &= \int_{\underline{v}}^{v} f(\nu)\mathrm{d} \int_{\underline{v}}^{\nu} a(\nu', b)\mathrm{d}\nu' \\
&= f(v) \int_{\underline{v}}^{v} a(\nu', b)\mathrm{d}\nu' - \int_{\underline{v}}^{v} \left[\int_{\underline{v}}^{\nu} a(\nu', b)\mathrm{d}\nu' \right] f'(\nu)\mathrm{d}\nu.
\end{aligned}
$$

Since $f \geq 0$ and $-f' \geq 0$, (4) follows from (16). ■

Proof of Theorem 2. We first solve the optimal mechanism of $\mathcal{P}'$ and then verify that the optimal mechanism also satisfies the (IC) constraint of low-budget agents. Let (a, p) be a feasible mechanism. For each $b \in B$, $a(\cdot, b)$ is non-decreasing and $p(v, b) = va(v, b) - \int_{\underline{v}}^{v} a(\nu, b)\mathrm{d}\nu - u(\underline{v}, b)$. Consider another mechanism (a^*, p^*).

Let $\hat{v} := \inf\{v | a(v, b_2) \geq a(\overline{v}, b_1)\}$. Note that $\hat{v} = \overline{v}$ if $a(\overline{v}, b_1) > a(\overline{v}, b_2)$ and $\hat{v} = \underline{v}$ if

$a(\overline{v}, b_1) \leq a(\underline{v}, b_2)$. Let a^* be defined by

$$a^*(v, b_1) = \begin{cases} a(\overline{v}, b_1) & \text{if } v \geq v_1^*, \\ 0 & \text{otherwise,} \end{cases}$$

where v_1^* satisfies $a(\overline{v}, b_1)[1 - F(v_1^*)] = \int_{\underline{v}}^{\overline{v}} a(v, b_1) f(v) \mathrm{d}v$, and

$$a^*(v, b_2) = \begin{cases} 1 & \text{if } v \geq v_2^{**}, \\ a(\overline{v}, b_1) & \text{if } v_2^* \leq v < v_2^{**}, \\ 0 & \text{otherwise,} \end{cases}$$

where $v_2^* \leq \hat{v}$ satisfies $a(\overline{v}, b_1)[F(\hat{v}) - F(v_2^*)] = \int_{\underline{v}}^{\hat{v}} a(v, b_2) f(v) \mathrm{d}v$ and $v_2^{**} \geq \hat{v}$ satisfies $1 - F(v_2^{**}) + a(\overline{v}, b_1)[F(v_2^{**}) - F(\hat{v})] = \int_{\hat{v}}^{\overline{v}} a(v, b_2) f(v) \mathrm{d}v$. Let $p^*(v, b) = v a^*(v, b) - \int_{\underline{v}}^{v} a^*(\nu, b) \mathrm{d}\nu - u(\underline{v}, b)$.

We show that $v_1^* \geq v_2^*$. If $v_1^* \geq \hat{v}$, then $v_1^* \geq v_2^*$. If $v_1^* < \hat{v}$, then

$$\begin{aligned} a(\overline{v}, b_1)[F(\hat{v}) - F(v_1^*)] &= \int_{\underline{v}}^{\hat{v}} a(v, b_1) f(v) \mathrm{d}v + \int_{\hat{v}}^{\overline{v}} [a(v, b_1) - a(\overline{v}, b_1)] f(v) \mathrm{d}v \\ &\leq \int_{\underline{v}}^{\hat{v}} a(v, b_1) f(v) \mathrm{d}v \\ &\leq \int_{\underline{v}}^{\hat{v}} a(v, b_2) f(v) \mathrm{d}v \\ &= a(\overline{v}, b_1)[F(\hat{v}) - F(v_2^*)], \end{aligned}$$

where the third line holds by Lemma 2. In this case, it must be that $a(\overline{v}, b_1) > 0$ since otherwise $a(\overline{v}, b_1) = 0 \leq a(0, b_2)$, which implies $\hat{v} = \underline{v} \leq v_1^*$. Hence, $v_2^* \leq v_1^*$. Thus, (a^*, p^*) satisfies the (IC-b) constraint.

Clearly, (a^*, p^*) also satisfies constraints (BC), (IR), (IC-v), (S) and (BB) and strictly improves welfare. Suppose $v_2^* < v_1^*$, then it is welfare improving to increase v_2^* and reduce v_1^* without affecting any constraint. Hence, it is optimal to set $v_1^* = v_2^* = v^*$. Let $u^* = u(\underline{v}, b_1) = u(\underline{v}, b_2)$. Then the optimal allocation rule satisfies $a(v,b_1)=\chi_{\{v\geq v^*\}}\min\left\{\frac{u^*+b_1}{v^*},1\right\}$ and $a(v,b_2)=\chi_{\{v\geq v^*\}}\min\left\{\frac{u^*+b_1}{v^*},1\right\}+\chi_{\{v\geq v_2^{**}\}}\left(1-\min\left\{\frac{u^*+b_1}{v^*},1\right\}\right)$.

Clearly, if $u^*+b_1>v^*$, we can reduce u^* such that $u^*+b_1=v^*$ without affecting any constraint or the principal's objective function. This completes the characterization of the optimal mechanism of $\mathcal{P}'$. Finally, it is easy to see that the (IC) constraint of low-budget types is satisfied. This completes the proof. ■

C.2 The General Case

Proof of Lemma 3. Suppose not. Then one can construct another feasible mechanism (a^*,p^*,q^*), which strictly improves welfare.

Let $\hat{v}_2^m=\inf\{v|a(v,b_2)\geq a^m\}$ for $m=1,\ldots,M$, $\hat{v}_2^0=0$ and $\hat{v}_2^{M+1}=\overline{v}$. Given a, the optimal verification rule satisfies $q(v,b_1)=q^m$ if $v\in(v_1^{m-1},v_1^m)$ for $m=1,\ldots,M$, where

$$q^m=\frac{1}{c}\max\left\{0,u(\underline{v},b_1)-u(\underline{v},b_2)-\int_{\underline{v}}^{\hat{v}_2^m}a(\nu,b_2)\mathrm{d}\nu+a^m(\hat{v}_2^m-v_1^{m-1})+\int_{\underline{v}}^{v_1^{m-1}}a(\nu,b_1)\mathrm{d}\nu\right\}.$$

For each $m=1,\ldots,M+1$, there exists $v_2^{m-1}\in[\hat{v}_2^{m-1},\hat{v}_2^m]$ such that

$$\int_{\hat{v}_2^{m-1}}^{\hat{v}_2^m}a(v,b_2)f(v)\mathrm{d}v=a^{m-1}[F(v_2^{m-1})-F(\hat{v}_2^{m-1})]+a^m[F(\hat{v}_2^m)-F(v_2^{m-1})].\tag{17}$$

Consider $a^*(v,b_2)$ such that $a^*(v,b_2)=a^m$ if $v\in(v_2^{m-1},v_2^m)$ for $m=1,\ldots,M$, $a^*(v,b_2)=0$ if $v<v_2^0$ and $a^*(v,b_2)=1$ if $v>v_2^M$. Note that if $a^1=0$, then $v_2^0=\underline{v}$. If $a^M=1$, then v_2^M is in-determined and we assume $v_2^M=v_2^{M-1}$. Let $a^*(v,b_1)=a(v,b_1)$.

Let $p^*(v,b)=va^*(v,b)-\int_{\underline{v}}^{v}a^*(\nu,b)\mathrm{d}\nu-u(\underline{v},b)$. Let $q^*(v,b_1)=q(v,b_1)$. We show that the (IC-b) constraint is satisfied. That is, for $m=1,\ldots,M$,

$$q^mc\geq u(\underline{v},b_1)-u(\underline{v},b_2)-\int_{\underline{v}}^{v_2^{m-1}}a^*(\nu,b_2)\mathrm{d}\nu+a^m(v_2^{m-1}-v_1^{m-1})+\int_{\underline{v}}^{v_1^{m-1}}a(\nu,b_1)\mathrm{d}\nu.$$

Since $a^*(v, b_2) = a^m$ for $v \in (v_2^{m-1}, \hat{v}_2^m)$, we have

$$\begin{aligned}
&u(\underline{v}, b_1) - u(\underline{v}, b_2) - \int_{\underline{v}}^{v_2^{m-1}} a^*(\nu, b_2)\mathrm{d}\nu + a^m(v_2^{m-1} - v_1^{m-1}) + \int_{\underline{v}}^{v_1^{m-1}} a(\nu, b_1)\mathrm{d}\nu \\
=&u(\underline{v}, b_1) - u(\underline{v}, b_2) - \int_{\underline{v}}^{\hat{v}_2^m} a^*(\nu, b_2)\mathrm{d}\nu + a^m(\hat{v}_2^m - v_1^{m-1}) + \int_{\underline{v}}^{v_1^{m-1}} a(\nu, b_1)\mathrm{d}\nu \\
\leq&u(\underline{v}, b_1) - u(\underline{v}, b_2) - \int_{\underline{v}}^{\hat{v}_2^m} a(\nu, b_2)\mathrm{d}\nu + a^m(\hat{v}_2^m - v_1^{m-1}) + \int_{\underline{v}}^{v_1^{m-1}} a(\nu, b_1)\mathrm{d}\nu,
\end{aligned}$$

where the last inequality holds if and only if

$$\int_{\underline{v}}^{\hat{v}_2^m} [a^*(\nu, b_2) - a(\nu, b_2)]\mathrm{d}\nu \geq 0.$$

To prove this, we prove that for $m = 1, \dots, M$

$$\int_{\hat{v}_2^{m-1}}^{\hat{v}_2^m} [a^*(\nu, b_2) - a(\nu, b_2)]\mathrm{d}\nu \geq 0. \tag{18}$$

(18) holds if and only if

$$\int_{v_2^{m-1}}^{\hat{v}_2^m} [a^*(\nu, b_2) - a(\nu, b_2)]\mathrm{d}\nu \geq \int_{\hat{v}_2^{m-1}}^{v_2^{m-1}} [a(\nu, b_2) - a^*(\nu, b_2)]\mathrm{d}\nu. \tag{19}$$

(19) follows from the construction of a^* and Assumption 2:

$$\begin{aligned}
\int_{v_2^{m-1}}^{\hat{v}_2^m} [a^*(\nu, b_2) - a(\nu, b_2)]\mathrm{d}\nu &\geq \int_{v_2^{m-1}}^{\hat{v}_2^m} [a^*(\nu, b_2) - a(\nu, b_2)]f(\nu)\frac{1}{f(v_2^{m-1})}\mathrm{d}\nu \\
&= \int_{\hat{v}_2^{m-1}}^{v_2^{m-1}} [a(\nu, b_2) - a^*(\nu, b_2)]f(\nu)\frac{1}{f(v_2^{m-1})}\mathrm{d}\nu \\
&\geq \int_{\hat{v}_2^{m-1}}^{v_2^{m-1}} [a(\nu, b_2) - a^*(\nu, b_2)]\mathrm{d}\nu.
\end{aligned}$$

By Assumption 1, $\mathbb{E}_t[p^*(t)] \geq \mathbb{E}_t[p(t)]$. Hence, constraint (BB) is satisfied. It is also clear that (a^*, p^*, q^*) satisfies constraints (IR), (IC-v), (BC) and (S), and strictly improves welfare. ■

Proof of Lemma 4. The proof is by contradiction. Let (a, p, q) be a feasible mechanism, where a is a M-step allocation rule, p satisfies the envelope condition and q is given by (7). Suppose (a, p, q) satisfies neither (C1) nor (C2). I show that one can construct another feasible mechanism (a^*, p^*, q^*), which strictly improves welfare and satisfies one of the two conditions. Furthermore, a^* is a M'-step function for some $M' \leq M$. I break the proof into three steps.

Step 1. Suppose $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1 \left(v_2^0 - v_1^0\right) < 0$. Let $m > 1$ be such that $v_2^{m'-1} - v_1^{m'-1} \leq 0$ for all $m' < m$ and $v_2^{m-1} - v_1^{m-1} > 0$. If there is no such m, then (a, p, q) satisfies (C2). Let $\hat{v}$ be defined by $F(\hat{v}) = \pi F(v_2^{m-1}) + (1-\pi)F(v_1^{m-1})$ if $F(v_1^m) > \pi F(v_2^{m-1}) + (1-\pi)F(v_1^{m-1})$ and $\hat{v} = v_1^m$ otherwise. Consider two different cases.

Case 1

Suppose $(a^m - a^{m-1})(\hat{v} - v_1^{m-1}) \geq \pi[u(\underline{v}, b_2) - u(\underline{v}, b_1) - a^1 \left(v_2^0 - v_1^0\right)]$, let $\tilde{v}_1^{m-1} \in [v_1^{m-1}, \hat{v}]$ be such that

$$(a^m - a^{m-1})(\tilde{v}_1^{m-1} - v_1^{m-1}) = \pi[u(\underline{v}, b_2) - u(\underline{v}, b_1) - a^1 \left(v_2^0 - v_1^0\right)].$$

Let $\tilde{v}_2^{m-1} \in [\hat{v}, v_2^{m-1}]$ be such that $\pi[F(v_2^{m-1}) - F(\tilde{v}_2^{m-1})] = (1-\pi)[F(\tilde{v}_1^{m-1}) - F(v_1^{m-1})]$. Let $\tilde{v}_i^{m'} = v_i^{m'}$ for $i = 1, 2$ and $m' \neq m-1$. Let $a^*(v, b_1) = a^{m-1}$ if $v \in (v_1^{m-1}, \tilde{v}_1^{m-1})$ and $a^*(v, b_1) = a(v, b_1)$ otherwise. Let $a^*(v, b_2) = a^m$ if $v \in (\tilde{v}_2^{m-1}, v_2^{m-1})$ and $a^*(v, b_2) = a(v, b_2)$ otherwise. Let $u^*(\underline{v}, b_1) = (1-\pi)u(\underline{v}, b_1) + \pi u(\underline{v}, b_2) - \pi a^1 \left(v_2^0 - v_1^0\right)$ and $u^*(\underline{v}, b_2) = (1-\pi)u(\underline{v}, b_1) + \pi u(\underline{v}, b_2) + (1-\pi)a^1 \left(v_2^0 - v_1^0\right)$. Let $p^*(v, b) = va^*(v, b) - \int_{\underline{v}}^{v} a^*(\nu, b)\mathrm{d}\nu - u^*(\underline{v}, b)$. By construction, $p^*(\overline{v}, b_1) = p(\overline{v}, b_1) \leq b_1$. Hence, the (BC) constraint is satisfied. Let $q^*(v, b_1) = q(v, b_1)$. By Assumption 1, the (BB) constraint holds. For $v \in (\tilde{v}_1^{m'-1}, \tilde{v}_1^{m'})$, $m' = 1, \ldots, m-1$, (IC-b) holds since

$$u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) + \sum_{j=1}^{m'} (a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}) \leq 0 \leq q^*(v, b_1)c.$$

For $v \in (\tilde{v}_1^{m'-1}, \tilde{v}_1^{m'})$, $m' = m, \ldots, M$, we have $q^*(v, b_1) = q^m$. Then (IC-b) holds since

$$
\begin{aligned}
& u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) + \sum_{j=1}^{m'} (a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}) \\
=& \sum_{j=1}^{m'} (a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) + (a^m - a^{m-1})(\tilde{v}_2^{m-1} - \tilde{v}_1^{m-1} - v_2^{m-1} + v_1^{m-1}) - a^1 \left(v_2^0 - v_1^0\right) \\
\leq& \sum_{j=1}^{m'} (a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) + \frac{(a^m - a^{m-1})(v_1^{m-1} - \tilde{v}_1^{m-1})}{\pi} - a^1 \left(v_2^0 - v_1^0\right) \\
=& \sum_{j=1}^{m'} (a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) + u(\underline{v}, b_1) - u(\underline{v}, b_2) \\
=& q^{m'} c,
\end{aligned}
$$

where the third line holds since by Assumption 2

$$
\begin{aligned}
v_2^{m-1} - \tilde{v}_2^{m-1} \geq& \frac{1}{f(\tilde{v}_2^{m-1})} [F(v_2^{m-1}) - F(\tilde{v}_2^{m-1})] \\
\geq& \frac{1-\pi}{\pi} \frac{1}{f(\tilde{v}_1^{m-1})} [F(\tilde{v}_1^{m-1}) - F(v_1^{m-1})] \\
\geq& \frac{1-\pi}{\pi} \left(\tilde{v}_1^{m-1} - v_1^{m-1}\right).
\end{aligned}
$$

Clearly, (a^*, p^*, q^*) also satisfies constraints (IR), (IC-v) and (S) and strictly increases welfare. Note also that the new mechanism satisfies $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1 \left(v_2^0 - v_1^0\right) = 0$.

Suppose $\tilde{v}_1^{m-1} < v_1^m$, then continue with the argument in step 2.

Suppose $\tilde{v}_1^{m-1} = v_1^m < \tilde{v}_2^{m-1}$, then by the arguments in Lemma 3, we can construct a new mechanism which is feasible and strictly increases welfare, and whose allocation rule is a $(M-1)$-step allocation rule. Continue with the argument in step 2.

Case 2

Suppose $(a^m - a^{m-1})(\hat{v} - v_1^{m-1}) < \pi[u(\underline{v}, b_2) - u(\underline{v}, b_1) + a^1 \left(v_2^0 - v_1^0\right)]$. Let $\tilde{v}_1^{m-1} = \hat{v}$. Let $\tilde{v}_2^{m-1} \in [\hat{v}, v_2^{m-1}]$ be such that $\pi[F(v_2^{m-1}) - F(\tilde{v}_2^{m-1})] = (1-\pi)[F(\tilde{v}_1^{m-1}) - F(v_1^{m-1})]$. Let

$\tilde{v}_i^{m'} = v_i^{m'}$ for $i = 1, 2$ and $m' \neq m - 1$. Let $a^*(v, b_1) = a^{m-1}$ if $v \in (v_1^{m-1}, \tilde{v}_1^{m-1})$, and $a^*(v, b_1) = a(v, b_1)$ otherwise. Let $a^*(v, b_2) = a^m$ if $v \in (\tilde{v}_2^{m-1}, v_2^{m-1})$ and $a^*(v, b_2) = a(v, b_2)$ otherwise. Let $u^*(\underline{v}, b_1) = u(\underline{v}, b_1) + (a^m - a^{m-1})(\hat{v} - v_1^{m-1})$ and $u^*(\underline{v}, b_2) = u(\underline{v}, b_2) - (1 - \pi)(a^m - a^{m-1})(\hat{v} - v_1^{m-1})/\pi$. Then $u^*(\underline{v}, b_2) > u^*(\underline{v}, b_1) + a^1 \left(v_2^0 - v_1^0\right) \geq 0$. Let $p^*(v, b) = va^*(v, b) - \int_{\underline{v}}^{v} a^*(\nu, b)\mathrm{d}\nu - u^*(\underline{v}, b)$. By construction, $p^*(\overline{v}, b_1) = p(\overline{v}, b_1) \leq b_1$. Hence, the (BC) constraint is satisfied. Let $q^*(v, b_1) = q(v, b_1)$. By Assumption 1, the (BB) constraint is satisfied. By the same argument in Case 1, the (IC-b) constraint is satisfied. Clearly, (a^*, p^*, q^*) also satisfies constraints (IR), (IC-v) and (S) and strictly increases welfare.

In this case, by construction, we have $\tilde{v}_1^{m-1} = \min\left\{\tilde{v}_2^{m-1}, v_1^m\right\}$.

Suppose $\tilde{v}_1^{m-1} = \tilde{v}_2^{m-1} < v_1^m$, then let $m^* > m$ be such that $\tilde{v}_2^{m'-1} - \tilde{v}_1^{m'-1} \leq 0$ for all $m' < m^*$ and $\tilde{v}_2^{m^*-1} - \tilde{v}_1^{m^*-1} > 0$. If there is no such m^*, (a^*, p^*, q^*) then satisfies (C2). Otherwise repeat the argument in step 1 for m^*.

Suppose $\tilde{v}_1^{m-1} = v_1^m \leq \tilde{v}_2^{m-1}$, then by the argument in Lemma 3, we can construct a new mechanism which is feasible and strictly increases welfare, and whose allocation rule is a $(M - 1)$-step allocation rule. Repeat the arguments in step 1 for m.

Since M is finite, in finite steps we can construct a feasible mechanism (a, p, q) that either satisfies (C2) or $u(0, b_1) - u(0, b_2) + a^1 v_2^0 \geq 0$. In the latter case, continue with the argument in step 2.

Step 2. Suppose $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1 \left(v_2^0 - v_1^0\right) \geq 0$. Consider $m \geq 2$. Suppose (8) holds for all $m' < m$ and

$$u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) < 0.$$

If there is no such m, then (a, p, q) satisfies (C1). It must be the case that $v_2^{m-1} < v_1^{m-1}$. Suppose $v_2^{m-1} < v_2^M$. Let $m^* \geq m$ be the smallest m' such that $v_2^{m'} > v_2^{m-1}$. That is, $v_2^{m^*} > v_2^{m-1}$ and $v_2^{m'} = v_2^{m-1}$ for $m' = m, \ldots, m^* - 1$. Let $\hat{v} \in [v_2^{m-1}, v_1^{m-1}]$ be such that

$$u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m-1}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) + (a^m - a^{m-1})(\hat{v} - v_1^{m-1}) = 0.$$

We consider two different cases.

Case 1

Suppose $(a^{m^*} - a^{m-1})[F(\hat{v}) - F(v_2^{m-1})] \leq (a^{m^*+1} - a^{m^*})[F(v_2^{m^*}) - F(\hat{v})]$. Let $\tilde{v}_2^{m^*} \in [\hat{v}, v_2^{m^*})$ be such that

$$(a^{m^*} - a^{m-1})[F(\tilde{v}_2^{m-1}) - F(v_2^{m-1})] = (a^{m^*+1} - a^{m^*})[F(v_2^{m^*}) - F(\tilde{v}_2^{m^*})]. \tag{20}$$

Let $\tilde{v}_2^{m'} = \hat{v}$ for $m' = m-1, \ldots, m^*-1$ and $\tilde{v}_2^{m'} = v_2^{m'}$ if $m' < m-1$ or $m' > m^*$. Let $\tilde{v}_1^{m'} = v_1^{m'}$ for all m'. Let $a^*(v, b_1) = a(v, b_1)$. Let $a^*(v, b_2) = a^{m-1}$ if $v \in (v_2^{m-1}, \tilde{v}_2^{m-1})$, $a^*(v, b_2) = a^{m^*+1}$ if $v \in (\tilde{v}_2^{m^*}, v_2^{m^*})$ and $a^*(v, b_2) = a(v, b_2)$ otherwise. Let $p^*(v, b) = va^*(v, b) - \int_{\underline{v}}^{v} a^*(\nu, b)\mathrm{d}\nu - u(\underline{v}, b)$. Clearly, $p^*(\overline{v}, b_1) = p(\overline{v}, b_1) \leq b_1$. Hence, the (BC) constraint is satisfied. Let $q^*(v, b_1) = q(v, b_1)$. By Assumption 1, the (BB) constraint holds.

Finally, we show that (a^*, p^*, q^*) satisfies the (IC-b) constraint. That is, for $m' = 1, \ldots, M$

$$u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m'}(a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}) \leq q^{m'}c.$$

This is trivial for $m' \leq m$. For $m' = m+1, \ldots, m^*$, we have $\tilde{v}_2^{m'-1} = \tilde{v}_2^{m-1} \leq v_1^{m-1} < v_1^{m'-1}$. Hence

$$u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) + \sum_{j=m+1}^{m'}(a^j - a^{j-1})(\tilde{v}_2^{j-1} - v_1^{j-1}) < 0 \leq q^{m'}c.$$

Finally, consider $m' \geq m^* + 1$. It suffices to show that

$$\begin{aligned}
&u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m^*+1}(a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}) \\
\leq &u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m^*+1}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}),
\end{aligned}$$

which holds if and only if

$$(a^{m^*} - a^{m-1})(\tilde{v}_2^{m-1} - v_2^{m-1}) \le (a^{m^*+1} - a^{m^*})(v_2^{m^*} - \tilde{v}_2^{m^*}).$$

The last inequality holds by (20) and Assumption 2. Clearly, (a^*, p^*, q^*) also satisfies constraints (IR), (IC-v) and (S) and strictly increases welfare. Let $m'' > m$ be such that (8) holds for all $m' < m''$ and is violated for m''. If there is no such m'', then (a^*, p^*, q^*) satisfies (C1). Otherwise repeat the argument in step 2 for m''.

Case 2

Suppose $(a^{m^*} - a^{m-1})[F(\hat{v}) - F(v_2^{m-1})] > (a^{m^*+1} - a^{m^*})[F(v_2^{m^*}) - F(\hat{v})]$. Let $\tilde{v}_2^{m-1}$ be such that

$$(a^{m^*} - a^{m-1})[F(\tilde{v}_2^{m-1}) - F(v_2^{m-1})] = (a^{m^*+1} - a^{m^*})[F(v_2^{m^*}) - F(\tilde{v}_2^{m-1})].$$

Let $\tilde{v}_2^{m'} = \tilde{v}_2^{m-1}$ for $m' = m, \dots, m^*$ and $\tilde{v}_2^{m'} = v_2^{m'}$ if $m' < m-1$ or $m' > m^*$. Let $\tilde{v}_1^{m'} = v_1^{m'}$ for all m'. Let $a^*(v, b_1) = a(v, b_1)$. Let $a^*(\cdot, b_2)$ such that $a^*(v, b_2) = a^{m-1}$ if $v \in (v_2^{m-1}, \tilde{v}_2^{m-1})$, $a^*(v, b_2) = a^{m^*+1}$ if $v \in (\tilde{v}_2^{m-1}, v_2^{m^*})$ and $a^*(v, b_2) = a(v, b_2)$ otherwise. Let $p^*(v, b) = va^*(v, b) - \int_{\underline{v}}^{v} a^*(\nu, b)\mathrm{d}\nu - u(\underline{v}, b)$. Clearly, $p^*(\overline{v}, b_1) = p(\overline{v}, b_1) \le b_1$. Hence, the (BC) constraint is satisfied. Let $q^*(v, b_1) = q(v, b_1)$. By Assumption 1, the (BB) constraint holds. By the same argument in Case 1, the (IC-b) constraint is satisfied. Clearly, (a^*, p^*, q^*) also satisfies constraints (IR), (IC-v) and (S) and strictly increases welfare. Note that for (a^*, p^*, q^*) we have $\tilde{v}_2^{m^*} = \tilde{v}_2^{m-1}$. Repeat the argument in step 2 for m with m^* replaced by $m^* + 1$.

Since M is finite, in finite steps we can construct a feasible mechanism (a, p, q) that either satisfies (C1), or $v_2^M = v_2^{m-1} < v_1^{m-1}$ and

$$u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m} (a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) < 0.$$

In the latter case, continue with the argument in step 3.

Step 3. Suppose $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1 \left(v_2^0 - v_1^0\right) \geq 0$,

$$u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m'} (a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) \geq 0$$

for all $m' = 1, \ldots, m-1$, $v_2^M = v_2^{m-1} < v_1^{m-1}$, and $u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m} (a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) < 0$.

Let $\tilde{v}_1^{m-1} = v_1^{m-1} - \varepsilon$ for some $\varepsilon > 0$ and $\tilde{v}_2^{m'} = v_2^{m-1} + \delta$ for $m' = m-1, \ldots, M$, where $\delta > 0$ is such that

$$(1 - \pi)(a^m - a^{m-1}) \left[F(v_1^{m-1}) - F(\tilde{v}_1^{m-1})\right] = \pi(1 - a^{m-1}) \left[F(v_2^{m-1}) - F(\tilde{v}_2^{m-1})\right]. \tag{21}$$

Let $\tilde{v}_i^{m'} = v_i^{m'}$ if $m' \neq m-1$ for $i = 1, 2$. Let $\varepsilon > 0$ be such that

$$\min \left\{ \tilde{v}_1^{m-1} - v_1^{m-2}, u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m} (a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}) \right\} = 0. \tag{22}$$

Since $\sum_{j=1}^{m-1} (a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}) \geq 0$, we have $\tilde{v}_2^{m'} \leq \tilde{v}_1^{m'}$ for all $m' \geq m-1$. Let $a^*(v, b_i) = a^m$ if $v \in (\tilde{v}_i^{m-1}, \tilde{v}_i^m)$ for $i = 1, 2$ and $m = 1, \ldots, M$, $a^*(v, b_2) = 0$ if $v < \tilde{v}_2^0$ and $a^*(v, b_2) = 1$ if $v > \tilde{v}_2^M$. Let $p^*(v, b) = v a^*(v, b) - \int_{\underline{v}}^{v} a^*(\nu, b) \mathrm{d}\nu - u(\underline{v}, b)$. Since $a^*(\overline{v}, b_1) = a(\overline{v}, b_1)$ and $a^*(v, b_1) \geq a(v, b_1)$ for all v, we have $p^*(\overline{v}, b_1) \leq p(\overline{v}, b_1) \leq b_1$. Hence, the (BC) constraint is satisfied. Let $q^*(v, b_1) = q^m$ if $v \in (\tilde{v}_1^{m-1}, \tilde{v}_1^m)$ for $m = 1, \ldots, M$. Then the change of the verification cost is

$$k(q^m - q^{m-1})[F(v_1^{m-1}) - F(\tilde{v}_1^{m-1})].$$

Since $q^m = 0 \leq q^{m-1}$, the verification cost is reduced. Furthermore, by Assumption 1, the revenue increases. Hence, the (BB) constraint holds. Finally, we show that the (IC-b) constraint is satisfied. That is, for $m' = 1, \ldots, M$

$$u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m'} (a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}) \leq q^{m'} c.$$

This is trivial for $m' < m$. For $m' \geq m$, this holds since

$$u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m'} (a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}) \leq 0 = q^{m'} c.$$

Clearly, (a^*, p^*, q^*) also satisfies constraints (IR), (IC-v) and (S) and strictly increases welfare.

If the first term of (22) reaches zero first, then by the argument in Lemma 3, we can construct a new mechanism which is feasible and strictly increases welfare, and whose allocation rule is a $(M-1)$-step allocation rule. Then repeat the argument in step 3 for $m-1$. If the second term of (22) reaches zero first and $m < M$, then repeat the argument in step 3 for $m+1$. If the second term of (22) reaches zero first and $m = M$, then (a^*, p^*, q^*) satisfies (C1).

Since M is finite, in finite steps we can construct a feasible mechanism (a^*, p^*, q^*), which strictly improves welfare and satisfies (C1). Furthermore, a^* is a M'-step allocation rule for some $M' \leq M$.

■

Lemma 10 *Suppose Assumptions 1 and 2 hold. An optimal mechanism of $\mathcal{P}'(M, d)$ satisfies that $v_2^1 \geq v_1^1$.*

Proof of Lemma 10. Assume without loss of generality that $a^2 > a^1$. Suppose, on the contrary, that $v_2^1 < v_1^1$. Since (8) holds for $m = 2$, it must be that $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_0^2 - v_0^1) > 0$. Hence, it is either (i) $u(\underline{v}, b_1) > u(\underline{v}, b_2) \geq 0$, or (ii) $a^1 > 0$ and $v_2^0 > v_0^1$.

Suppose $u(\underline{v}, b_1) > u(\underline{v}, b_2) \geq 0$.

We construct another feasible mechanism (a^*, p^*, q^*), which strictly improves welfare. Let $\varepsilon > 0$ be sufficiently small. Let $\tilde{v}_1^1 = v_1^1 - \pi\varepsilon/(1-\pi)$ and $\tilde{v}_2^1 > v_2^1$ be such that $(1 -$

$\pi)\left[F(v_1^1)-F(\tilde{v}_1^1)\right]=\pi\left[F(\tilde{v}_2^1)-F(v_2^1)\right]$. By Assumption 2,

$$\begin{aligned}\tilde{v}_2^1-v_2^1 &\le \left[F(\tilde{v}_2^1)-F(v_2^1)\right]\frac{1}{f(\tilde{v}_2^1)}\\ &\le \frac{1-\pi}{\pi}\left[F(v_1^1)-F(\tilde{v}_1^1)\right]\frac{1}{f(\tilde{v}_1^1)}\\ &\le \frac{1-\pi}{\pi}(v_1^1-\tilde{v}_1^1)=\varepsilon.\end{aligned}$$

For $\varepsilon>0$ sufficiently small, $\tilde{v}_2^1\le\tilde{v}_1^1$. Let $\tilde{v}_i^m=v_i^m$ for $i=1,2$ and $m\neq 1$. Let $u^*(\underline{v},b_2)=u(\underline{v},b_2)+(a^2-a^1)\varepsilon$ and $u^*(\underline{v},b_1)=u(\underline{v},b_1)-\pi(a^2-a^1)\varepsilon/(1-\pi)$. For $\varepsilon>0$ sufficiently small, $u^*(\underline{v},b_1)\ge u^*(\underline{v},b_2)>0$. Let $a^*(v,b_i)=a^m$ for $v\in(\tilde{v}_i^{m-1},\tilde{v}_i^m)$ for $i=1,2$ and $m=1,\dots,M$, $a^*(v,b_2)=0$ if $v<\tilde{v}_2^0$ and $a^*(v,b_1)=1$ if $v>\tilde{v}_2^M$. Let $p^*(v,b)=va^*(v,b)-\int_{\underline{v}}^{v}a^*(\nu,b)\mathrm{d}\nu-u^*(\underline{v},b)$. By construction, $p^*(\overline{v},b_1)=p(\overline{v},b_1)$. Hence, the (BC) constraint is satisfied. Let $q^*(v,b)=q(v,b)$. Clearly, (a^*,p^*,q^*) satisfies constraints (IR), (IC-v) and (S), and strictly improves welfare. (a^*,p^*,q^*) satisfies (BB) by Assumption 1.

Finally, we show that (a^*,p^*,q^*) satisfies the (IC-b) constraint. If $v<\tilde{v}_1^1$, then

$$u^*(\underline{v},b_1)-u^*(\underline{v},b_2)+a^1(\tilde{v}_2^0-\tilde{v}_1^0)=u(\underline{v},b_1)-u(\underline{v},b_2)+a^1(v_2^0-v_1^0)-\frac{(a^2-a^1)\varepsilon}{1-\pi}\le q^1c.$$

If $v\in(\tilde{v}_1^1,v_1^2)$, then

$$\begin{aligned}&u^*(\underline{v},b_1)-u^*(\underline{v},b_2)+a^1(\tilde{v}_2^0-\tilde{v}_1^0)+(a^2-a^1)(\tilde{v}_2^1-\tilde{v}_1^1)\\ =&u(\underline{v},b_1)-u(\underline{v},b_2)+a^1(v_2^0-v_1^0)+(a^2-a^1)(v_2^1-v_1^1)-\frac{(a^2-a^1)\varepsilon}{1-\pi}+(a^2-a^1)(\tilde{v}_2^1-v_2^1+v_1^1-\tilde{v}_1^1)\\ \le&u(\underline{v},b_1)-u(\underline{v},b_2)+a^1(v_2^0-v_1^0)+(a^2-a^1)(v_2^1-v_1^1)-\frac{(a^2-a^1)\varepsilon}{1-\pi}+(a^2-a^1)\left(\varepsilon+\frac{\pi\varepsilon}{1-\pi}\right)\\ =&u(\underline{v},b_1)-u(\underline{v},b_2)+a^1(v_2^0-v_1^0)+(a^2-a^1)(v_2^1-v_1^1)\\ \le&\min\{q^2c,q^1c\},\end{aligned}$$

where the first inequality holds since $\tilde{v}_2^1-v_2^1\le\varepsilon$ and the last inequality holds since $v_2^1<v_1^1$.

If $v \in (v_1^{m-1}, v_1^m)$ for $m \geq 3$, then

$$
\begin{aligned}
&u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) + a^1(\tilde{v}_2^0 - \tilde{v}_1^0) + (a^2 - a^1)(\tilde{v}_2^1 - \tilde{v}_1^1) + \sum_{j=3}^{m}(a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}) \\
\leq & u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) + (a^2 - a^1)(v_2^1 - v_1^1) + \sum_{j=3}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) \\
\leq & q^m c.
\end{aligned}
$$

Hence, (IC-b) constraint is satisfied. This contradicts to that (a, p, q) is optimal.

Suppose $a^1 > 0$ and $v_2^0 > v_1^0$.

We construct another feasible mechanism (a^*, p^*, q^*), which strictly improves welfare. Let $\varepsilon \in (0, a^1]$ be sufficiently small. Let

$$
\tilde{v}_1^1 = \frac{(a^2 - a^1)v_1^1 + \varepsilon v_1^0}{a^2 - a^1 + \varepsilon} < v_1^1.
$$

By Assumption 2, we have

$$
\begin{aligned}
&(a^2 - a^1)\left[F(v_1^1) - F(\tilde{v}_1^1)\right] \\
\leq & (a^2 - a^1)(v_1^1 - \tilde{v}_1^1) f(\tilde{v}_1^1) \\
= & \varepsilon(\tilde{v}_1^1 - v_1^0) f(\tilde{v}_1^1) \\
\leq & \varepsilon\left[F(\tilde{v}_1^1) - F(v_1^0)\right].
\end{aligned}
$$

Let $\Delta := (a^2 - a^1)\left[F(v_1^1) - F(\tilde{v}_1^1)\right] - \varepsilon\left[F(\tilde{v}_1^1) - F(v_1^0)\right] \geq 0$. If $v_2^1 > v_2^0$, then let $\tilde{v}_2^0 = v_2^0$ and $\tilde{v}_2^1$ be such that

$$
\pi(a^2 - a^1)\left[F(v_2^1) - F(\tilde{v}_2^1)\right] = \pi\varepsilon\left[F(\tilde{v}_2^1) - F(v_2^0)\right] + (1 - \pi)\Delta.
$$

For $\varepsilon > 0$ sufficiently small, $\tilde{v}_2^1 \geq \tilde{v}_2^0 \geq v_1^0$. If $v_2^1 = v_2^0$, then let $\tilde{v}_2^1 = \tilde{v}_2^0$ be such that

$$\pi(a^2 - a^1)\left[F(v_2^1) - F(\tilde{v}_2^1)\right] = (1 - \pi)\Delta.$$

For $\varepsilon > 0$ sufficiently small, $\tilde{v}_2^1 = \tilde{v}_2^0 \geq v_1^0$. Let $\tilde{v}_i^m = v_i^m$ for $i = 1, 2$ and $m \geq 2$. For $i = 1, 2$, let $a^*(v, b_i) = a^1 - \varepsilon$ if $v \in (\tilde{v}_i^0, \tilde{v}_i^1)$, $a^*(v, b_i) = a^2$ if $v \in (\tilde{v}_i^1, v_i^1)$, and $a^*(v, b_i) = a(v, b_i)$ otherwise. Let $u^*(\underline{v}, b) = u(\underline{v}, b)$ and $p^*(v, b) = va^*(v, b) - \int_{\underline{v}}^{v} a^*(\nu, b)\mathrm{d}\nu - u^*(\underline{v}, b)$. By construction, $p^*(\overline{v}, b_1) = p(\overline{v}, b_1)$. Hence, the (BC) constraint is satisfied. Let $q^*(v, b) = q(v, b)$. Clearly, (a^*, p^*, q^*) satisfies constraints (IR), (IC-v) and (S), and strictly improves welfare. (a^*, p^*, q^*) satisfies (BB) by Assumption 1.

Finally, we show that (a^*, p^*, q^*) satisfies the (IC-b) constraint. Suppose $v_2^1 > v_2^0$. If $v < \tilde{v}_1^1$, then

$$u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) + (a^1 - \varepsilon)(v_2^0 - v_1^0) < u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) = q^1 c.$$

If $v \in (\tilde{v}_1^1, \tilde{v}_1^2)$, then

$$\begin{aligned}
&u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) + (a^1 - \varepsilon)(v_2^0 - v_1^0) + (a^2 - a^1 + \varepsilon)(\tilde{v}_2^1 - \tilde{v}_1^1) \\
=&u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) + (a^2 - a^1)(v_2^1 - v_1^1) \\
&+ (a^2 - a^1 + \varepsilon)(\tilde{v}_2^1 - \tilde{v}_1^1) - \varepsilon(v_2^0 - v_1^0) - (a^2 - a^1)(v_2^1 - v_1^1) \\
=&u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) + (a^2 - a^1)(v_2^1 - v_1^1) \\
&+ (a^2 - a^1 + \varepsilon)\tilde{v}_2^1 - \varepsilon v_2^0 - (a^2 - a^1)v_2^1 \\
\leq&u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) + (a^2 - a^1)(v_2^1 - v_1^1) \leq \min\{q^1 c, q^2 c\},
\end{aligned}$$

where the last inequality holds since $v_2^1 < v_1^1$. To see that the first inequality holds, note that by Assumption 2,

$$
\begin{aligned}
(a^2 - a^1)(v_2^1 - \tilde{v}_2^1) &\geq (a^2 - a^1)\left[F(v_2^1) - F(\tilde{v}_2^1)\right]\frac{1}{f(\tilde{v}_2^1)} \\
&\geq \varepsilon\left[F(\tilde{v}_2^1) - F(v_2^0)\right] \\
&\geq \varepsilon(\tilde{v}_2^1 - v_2^0).
\end{aligned}
$$

Hence, $(a^2 - a^1 + \varepsilon)\tilde{v}_2^1 \leq (a^2 - a^1)v_2^1 + \varepsilon v_2^0$. Furthermore, $\tilde{v}_i^m = v_i^m$ for $i = 1, 2$ and $m \geq 2$. Hence, the (IC-b) constraint is satisfied. Suppose $v_2^0 = v_2^1$. If $v < \tilde{v}_1^1$, then

$$
u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) + (a^1 - \varepsilon)(v_2^0 - v_1^0) < u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) = q^1 c.
$$

If $v \in (\tilde{v}_1^1, \tilde{v}_1^2)$, then

$$
\begin{aligned}
&u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) + (a^1 - \varepsilon)(v_2^0 - v_1^0) + (a^2 - a^1 + \varepsilon)(\tilde{v}_2^1 - \tilde{v}_1^1) \\
=&u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) + (a^2 - a^1)(v_2^1 - v_1^1) + a^2(\tilde{v}_2^1 - v_2^1) \\
\leq&u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) + (a^2 - a^1)(v_2^1 - v_1^1) \leq \min\{q^1 c, q^2 c\},
\end{aligned}
$$

where the first inequality holds since $\tilde{v}_2^1 \leq v_2^1$ and the second inequality holds since $v_2^1 < v_1^1$. Furthermore, $\tilde{v}_i^m = v_i^m$ for $i = 1, 2$ and $m \geq 2$. Hence, the (IC-b) constraint is satisfied. This contradicts to that (a, p, q) is optimal.

Hence, $v_2^1 \geq v_1^1$. ■

Let $M \geq 3$ be an integer. We note that if a mechanism is a feasible solution to $\mathcal{P}'(M-1, d)$, then it is also a feasible solution to $\mathcal{P}'(M, d)$. Clearly, $V(M-1, d) \leq V(M, d)$. Suppose $V(M-1, d) < V(M, d)$, then in an optimal solution to $\mathcal{P}'(M, d)$ the allocation rule must be a M-step allocation rule, i.e.,

$$0 = a^0 \le a^1 < a^2 < \cdots < a^M \le a^{M+1} = 1,$$
$$\underline{v} = v_1^0 < v_1^1 < \cdots < v_1^M = \overline{v}.$$

Hence $\alpha^2 = \cdots = \alpha^M = 0$ and $\gamma_1^1 = \cdots = \gamma_1^M = 0$. Let $\rho := k/c$. Then the first-order conditions of

$\mathcal{P}'(M,d)$ are

$$\pi\left[\int_{v_2^{m-1}}^{v_2^m}\left[(1+\lambda)v-\lambda\frac{1-F(v)}{f(v)}\right]\mathrm{d}v-\beta[F(v_2^m)-F(v_2^{m-1})]\right]$$
$$+(1-\pi)\left[\int_{v_1^{m-1}}^{v_1^m}\left[(1+\lambda)v-\lambda\frac{1-F(v)}{f(v)}\right]f(v)\mathrm{d}v\right]$$
$$-(1-\pi)(1+\lambda)\rho(v_2^{m-1}-v_1^{m-1})[F(v_1^m)-F(v_1^{m-1})]-(1-\pi)\beta[F(v_1^m)-F(v_1^{m-1})]$$
$$+(1-\pi)(1+\lambda)\rho(v_2^m-v_1^m-v_2^{m-1}+v_1^{m-1})[1-F(v_1^m)]+\eta(v_1^m-v_1^{m-1})+\mu^m(v_2^{m-1}-v_1^{m-1})$$
$$-(v_2^m-v_1^m-v_2^{m-1}+v_1^{m-1})\sum_{j=m+1}^{M}\mu^j+\alpha^m-\alpha^{m+1}=0, \qquad (a^m, 1\le m\le M-1)$$

$$\pi\left[\int_{v_2^{M-1}}^{v_2^M}\left[(1+\lambda)v-\lambda\frac{1-F(v)}{f(v)}\right]f(v)\mathrm{d}v-\beta[F(v_2^M)-F(v_2^{M-1})]\right]$$
$$+(1-\pi)\int_{v_1^{M-1}}^{v_1^M}\left[(1+\lambda)v-\lambda\frac{1-F(v)}{f(v)}\right]f(v)\mathrm{d}v$$
$$-(1-\pi)(1+\lambda)\rho(v_2^{M-1}-v_1^{M-1})[F(v_1^M)-F(v_1^{M-1})]-(1-\pi)\beta[F(v_1^M)-F(v_1^{M-1})]$$
$$-\eta v_1^{M-1}+\mu^M(v_2^{M-1}-v_1^{M-1})+\alpha^M-\alpha^{M+1}=0, \qquad (a^M)$$

$$(a^{m+1}-a^m)\Big\{(1-\pi)\left[(\beta-(1+\lambda)v_1^m)f(v_1^m)+(\lambda+\rho+\lambda\rho)\left[1-F(v_1^m)\right]+(1+\lambda)\rho(v_2^m-v_1^m)f(v_1^m)\right]$$
$$-\sum_{j=m+1}^{M}\mu^j-\eta\Big\}=0, \qquad (v_1^m, 1\le m\le M-1)$$

$$a^1\left\{\pi\left[(\beta-(1+\lambda)v_2^0)f(v_2^0)+\lambda[1-F(v_2^0)]\right]-(1-\pi)(1+\lambda)\rho+\sum_{j=1}^{M}\mu^j\right\}+\gamma_2^0-\gamma_2^1=0, \quad (v_2^0)$$

$$(a^{m+1}-a^m)\left\{\pi\left[(\beta-(1+\lambda)v_2^m)f(v_2^m)+\lambda[1-F(v_2^m)]\right]-(1-\pi)(1+\lambda)\rho[1-F(v_1^m)]+\sum_{j=m+1}^{M}\mu^j\right\}$$
$$+\gamma_2^m-\gamma_2^{m+1}=0, \qquad (v_2^m, 1\le m\le M-1)$$

$$\pi(a^{M+1}-a^M)\left[(\beta-(1+\lambda)v_2^M)f(v_2^M)+\lambda[1-F(v_2^M)]\right]+\gamma_2^M-\gamma_2^{M+1}=0, \qquad (v_2^M)$$

$$\eta+\sum_{m=1}^{M}\mu^m-(1-\pi)(\lambda+\rho+\lambda\rho)+\xi_1=0, \qquad (u(\underline{v},b_1))$$

$$-\sum_{m=1}^{M}\mu^m-\pi\lambda+(1-\pi)(1+\lambda)\rho+\xi_2=0. \qquad (u(\underline{v},b_2))$$

The variables in the parentheses denote with respect to which variables the first-order conditions are taken.

Lemma 11 *Suppose Assumptions 1 and 2 hold and $V(M,d) > V(M-1,d)$ for some $M \geq 3$. An optimal mechanism of $\mathcal{P}'(M,d)$ satisfies that $v_2^m - v_1^m$ is strictly increasing in $m = 1, \dots M-1$.*

Proof of Lemma 11. Since $a^{m+1} > a^m$ for $m = 1, \dots M-1$, the FOCs of v_1^m become

$$(1-\pi)\left[(\beta - (1+\lambda)v_1^m)f(v_1^m) + (\lambda + \rho + \lambda\rho)\left[1 - F(v_1^m)\right] + (1+\lambda)\rho(v_2^m - v_1^m)f(v_1^m)\right] \\ - \sum_{j=m+1}^{M} \mu^j - \eta = 0,$$

for $m = 1, \dots, M-1$. Then for $m = 1, \dots, M-1$

$$v_2^m - v_1^m = \frac{1}{\rho}v_1^m - \frac{\lambda + \rho + \lambda\rho}{(1+\lambda)\rho}\frac{1 - F(v_1^m)}{f(v_1^m)} - \frac{\beta}{(1+\lambda)\rho} + \frac{\eta + \sum_{j=m+1}^{M} \mu^j}{(1-\pi)(1+\lambda)\rho f(v_1^m)},$$

which is strictly increasing in v_1^m by Assumptions 1 and 2. If $\mu^{m+1} = 0$, then $v_2^{m+1} - v_1^{m+1} > v_2^m - v_1^m$ since $v_1^{m+1} > v_1^m$.

If $\mu^{m+1} > 0$, then $v_2^{m+1} \geq v_1^{m+1} > v_1^m \geq v_2^m$ since (8) holds for m and $m+2$ and (8) holds with equality for $m+1$. Hence, $v_2^{m+1} - v_1^{m+1} \geq 0 \geq v_2^m - v_1^m$. By Lemma 10, $v_2^1 \geq v_1^1$. Hence, if there exists $m \geq 1$ such that $v_2^{m+1} - v_1^{m+1} \geq 0 \geq v_2^m - v_1^m$, then it must be the case that $v_2^{m+1} - v_1^{m+1} = v_2^m - v_1^m = \dots = v_2^1 - v_1^1 = 0$. In particular, $v_2^2 - v_1^2 = v_2^1 - v_1^1$. Then we can construct another feasible mechanism (a^*, p^*, q^*), which strictly improves welfare. Let $\hat{v} \in (v_1^1, v_1^2)$ be such that

$$(a^3 - a^2)\left[F(v_1^2) - F(\hat{v})\right] = (a^2 - a^1)\left[F(\hat{v}) - F(v_1^1)\right].$$

Let $a^*(v,b) = a^1$ if $v \in (v_1^1, \hat{v})$, $a^*(v,b) = a^3$ if $v \in (\hat{v}, v_1^2)$ and $a^*(v,b) = a(v,b)$ otherwise. Let $p^*(v,b) = va^*(v,b) - \int_{\underline{v}}^{v} a^*(\nu,b)\mathrm{d}\nu - u(\underline{v},b)$. Then the (BC) constraint is satisfied by Assumption 2. Let $q^*(v,b_1) = q(v,b_1)$. Clearly, (a^*, p^*, q^*) satisfies constraints (IR), (IC-v), (IC-b), (S) and (BB), and strictly improves welfare. A contradiction. ■

Lemma 12 *Suppose Assumptions 1 and 2 hold. Then* $V(M,d) = V(5,d)$ *for all* $M \geq 5$ *and* $d \geq 0$.

Proof of Lemma 12. Fix $d \geq 0$ and $M \geq 6$ be an integer. We show that $V(M-1,d) = V(M,d)$. Suppose, on the contrary, that $V(M-1,d) < V(M,d)$, then in an optimal solution to $\mathcal{P}'(M,d)$ the allocation rule must be a M-step allocation rule. In particular, $0 = a^0 \leq a^1 < a^2 < \cdots < a^M \leq a^{M+1} = 1$. By Lemmas 10 and 11, an optimal solution to $\mathcal{P}'(M,d)$ must satisfies

$$v_2^{M-1} - v_1^{M-1} > v_2^{M-2} - v_1^{M-2} > \cdots > v_2^1 - v_1^1 \geq 0. \tag{23}$$

Fix $\underline{v} = v_1^0 < v_1^1 < \cdots < v_1^M = \overline{v}$ and $0 \geq v_2^0 \leq v_2^1 \leq \cdots \leq v_2^M \leq v_2^{M+1} \leq \overline{v}$ such that (23) holds. Then $\mathcal{P}'(M,d)$ is linear in $u(\underline{v},b_1)$, $u(\underline{v},b_2)$ and a^m for $m = 1,\ldots,M$. Then an optimal solution can be obtained at an extreme point of the feasible region. By (23), inequalities corresponding to μ^m for $m = 2,\ldots M$ holds if the inequality corresponding to μ^1 holds. Hence, the feasible set is characterized by (S), (BC), (BB) and the following inequalities:

$$\begin{aligned}
&u(\underline{v},b_1) \geq 0, u(\underline{v},b_2) \geq 0,\\
&u(\underline{v},b_1) - u(\underline{v},b_2) + a^1(v_2^0 - v_1^0) \geq 0, \\
&0 \leq a^0 \leq a^1 \leq \cdots \leq a^M \leq a^{M+1} = 1.
\end{aligned} \tag{24}$$

Note that if $a^1 = 0$, then $u(\underline{v},b_1) \geq 0$ is redundant. Hence, in addition to (S), (BC), (BB) and $a^M \leq 1$, at most three of the following four inequalities are active at the same time: $u(\underline{v},b_1) \geq 0$, $u(\underline{v},b_2) \geq 0$, $a^1 \geq 0$ and (24). Since $M \geq 6$, at least one of the following constraints hold with equality: $a^1 \leq a^2 \cdots \leq a^{M-1} \leq a^M$, a contradiction. ■

Lemma 13 *Suppose Assumptions 1 and 2 hold. For any* $d > 0$, *there exists* $\overline{M}(d)$ *such that for all* $M > \overline{M}(d)$,

$$V - V(M,d) \leq (1-\pi)\left(1 + \frac{k}{c}\right)\frac{\mathbb{E}[v]}{M}.$$

Proof of Lemma 13. Let (a, p, q) denote an optimal mechanism of $\mathcal{P}'$. Then $p(v, b) = va(v, b) - \int_{\underline{v}}^{v} a(\nu, b)\mathrm{d}\nu - u(\underline{v}, b)$ for all $(v, b) \in T$ and q is defined by (6). Fix $M \geq 2$. Let $a^0 = 0$, $a^{M+1} = 1$ and $a^m = (m-1)a(\overline{v}, b_1)/M$ for $m = 1, \ldots, M$. Let $v_1^0 = \underline{v}$, $v_1^M = \overline{v}$ and for $m = 0, \ldots, M-1$

$$v_1^m = \inf\left\{ v \,\middle|\, a(v, b_1) \geq a^{m+1} \right\}.$$

Then $\underline{v} = v_1^0 \leq v_1^1 \leq \cdots \leq v_1^M = \overline{v}$ and $0 = a^0 \leq a^1 < a^2 < \cdots < a^M \leq a^{M+1} = 1$. Let $a^*(v, b_1) = a^m$ if $v \in (v_1^{m-1}, v_1^m)$ for $m = 1, \ldots, M$. Then $a(v, b_1) - 1/M \leq a^*(v, b_1) \leq a(v, b_1)$. Let $\hat{v}_2^m = \inf\{v | a(v, b_2) \geq a^m\}$ for $m = 1, \ldots, M$, $\hat{v}_2^0 = 0$ and $\hat{v}_2^{M+1} = \overline{v}$. For each $m = 1, \ldots, M+1$, there exists $v_2^{m-1} \in [\hat{v}_2^{m-1}, \hat{v}_2^m]$ such that

$$\int_{\hat{v}_2^{m-1}}^{\hat{v}_2^m} a(v, b_2) f(v)\mathrm{d}v = a^{m-1}[F(v_2^{m-1}) - F(\hat{v}_2^{m-1})] + a^m[F(\hat{v}_2^m) - F(v_2^{m-1})]. \tag{25}$$

Consider $a^*(v, b_2)$ such that $a^*(v, b_2) = a^m$ if $v \in (v_2^{m-1}, v_2^m)$ for $m = 1, \ldots, M$, $a^*(v, b_2) = 0$ if $v < v_2^0$ and $a^*(v, b_2) = 1$ if $v > v_2^M$. Note that since $a^1 = 0$, we have $v_2^0 = \underline{v}$. Clearly, a^* satisfies constraint (S). Let $p^*(v, b) = va^*(v, b) - \int_{\underline{v}}^{v} a^*(\nu, b)\mathrm{d}\nu - u(\underline{v}, b)$ for $b \in B$. Let q^* be such that

$$cq^*(v, b_1) = cq(v, b_1) + \frac{v}{M}.$$

We show that the (IC-b) constraint is satisfied, i.e., for all $v \in (v_1^{m-1}, v_1^m)$, $m = 1, \ldots, M$,

$$cq^*(v, b_1) \geq u(\underline{v}, b_1) - u(\underline{v}, b_2) - \int_{\underline{v}}^{v_2^{m-1}} a(\nu, b_2)\mathrm{d}\nu + a^m(v_2^{m-1} - v) + \int_{\underline{v}}^{v} a^*(\nu, b_1)\mathrm{d}\nu.$$

Recall that for $v \in (v_1^{m-1}, v_1^m)$, we have

$$cq(v, b_1) \geq u(\underline{v}, b_1) - u(\underline{v}, b_2) - \int_{\underline{v}}^{\hat{v}_2^m} a(\nu, b_2)\mathrm{d}\nu + a(v, b_1)(\hat{v}_2^m - v) + \int_{\underline{v}}^{v} a(\nu, b_1)\mathrm{d}\nu.$$

Then for $v \in (v_1^{m-1}, v_1^m)$

$$
\begin{aligned}
cq^*(v, b_1) &= cq(v, b_1) + \frac{v}{M} \\
&\geq u(\underline{v}, b_1) - u(\underline{v}, b_2) - \int_{\underline{v}}^{\hat{v}_2^m} a(\nu, b_2)\mathrm{d}\nu + a(v, b_1)\hat{v}_2^m - \left(a(v, b) - \frac{1}{M}\right) v + \int_{\underline{v}}^{v} a(\nu, b_1)\mathrm{d}\nu \\
&\geq u(\underline{v}, b_1) - u(\underline{v}, b_2) - \int_{\underline{v}}^{\hat{v}_2^m} a(\nu, b_2)\mathrm{d}\nu + a^m(\hat{v}_2^m - v) + \int_{\underline{v}}^{v} a(\nu, b_1)\mathrm{d}\nu \\
&\geq u(\underline{v}, b_1) - u(\underline{v}, b_2) - \int_{\underline{v}}^{\hat{v}_2^m} a^*(\nu, b_2)\mathrm{d}\nu + a^m(\hat{v}_2^m - v) + \int_{\underline{v}}^{v} a^*(\nu, b_1)\mathrm{d}\nu \\
&= u(\underline{v}, b_1) - u(\underline{v}, b_2) - \int_{\underline{v}}^{v_2^{m-1}} a^*(\nu, b_2)\mathrm{d}\nu + a^m(v_2^{m-1} - v) + \int_{\underline{v}}^{v} a^*(\nu, b_1)\mathrm{d}\nu,
\end{aligned}
$$

where the third line holds since $a(v, b) - 1/M \leq a^*(v, b) \leq a(v, b)$ and the fourth line holds by the same argument in the proof of Lemma 3. Then

$$
\begin{aligned}
&\mathbb{E}_t[p^*(t) - q^*(t)k] - \mathbb{E}_t[p(t) - q(t)k] \\
=&\pi \int_{\underline{v}}^{\bar{v}} \left[v - \frac{1 - F(v)}{f(v)}\right] [a^*(v, b_2) - a(v, b_2)] f(v)\mathrm{d}v \\
&+ (1 - \pi) \int_{\underline{v}}^{\bar{v}} \left[v - \frac{1 - F(v)}{f(v)}\right] [a^*(v, b_1) - a(v, b_1)] f(v)\mathrm{d}v - (1 - \pi) \int_{\underline{v}}^{\bar{v}} k[q^*(v, b_1) - q(v, b_1)] f(v)\mathrm{d}v \\
\geq& - \frac{\mathbb{E}[v]}{M} - (1 - \pi)\frac{\mathbb{E}[v]}{M}\frac{k}{c}.
\end{aligned}
$$

For any $d > 0$, there exists $\overline{M}(d)$ such that for all $M > \overline{M}(d)$, we have $\frac{\mathbb{E}[v]}{M} + (1 - \pi)\frac{\mathbb{E}[v]}{M}\frac{k}{c} < d$. Then (a^*, p^*, q^*) is a feasible solution to $\mathcal{P}'(M, d)$ for $M > \overline{M}(d)$. Hence,

$$
\begin{aligned}
&V - V(M, d) \\
\leq&(1 - \pi) \left[\int_{\underline{v}}^{\bar{v}} v[a(v, b_1) - a^*(v, b_1)] f(v)\mathrm{d}v - \int_{\underline{v}}^{\bar{v}} \left[q(v, b_1) - q^*(v, b_1)\right] k f(v)\mathrm{d}v\right] \\
\leq&(1 - \pi)\left(1 + \frac{k}{c}\right)\frac{\mathbb{E}[v]}{M}.
\end{aligned}
$$

■

Proof of Theorem 3. By Lemmas 6 and 13, we have

$$V - V(2,d) = V - V(M,d) \leq (1-\pi)\left(1 + \frac{k}{c}\right)\frac{\mathbb{E}[v]}{M}.$$

Let M goes to infinity and we have $V(2,0) \leq V \leq V(2,d)$ for all $d > 0$. By Lemma 14, $\lim_{d\to 0} V(2,d) = V(2,0)$. Hence, $V = V(2,0)$.

Hence, there exists $u(\underline{v}, b_1) \geq 0$, $u(\underline{v}, b_2) \geq 0$, $\underline{v} \leq v_1^1 \leq \bar{v}$, $\underline{v} \leq v_2^0 \leq v_2^1 \leq v_2^2 \leq \bar{v}$ and $0 \leq a^1 \leq a^2 \leq \bar{v}$ the optimal mechanism of $\mathcal{P}'$ is given by

$$\begin{aligned}
a(v, b_1) &= a^1 + \chi_{\{v \geq v_1^1\}}\left(a^2 - a^1\right),\\
a(v, b_2) &= \chi_{\{v \geq v_2^0\}} a^1 + \chi_{\{v \geq v_2^1\}}\left(a^2 - a^1\right) + \chi_{\{v \geq v_2^2\}}(1 - a^2),\\
p(v, b_1) &= -u(\underline{v}, b_1) + \chi_{\{v \geq v_1^1\}}\left(a^2 - a^1\right) v_1^1,\\
p(v, b_2) &= -u(\underline{v}, b_2) + \chi_{\{v \geq v_2^0\}} a^1 v_2^0 + \chi_{\{v \geq v_2^1\}}\left(a^2 - a^1\right) v_2^1 + \chi_{\{v \geq v_2^2\}}(1 - a^2) v_2^2,\\
q(v, b_1) &= \frac{1}{c}\left[u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1\left(v_2^0 - v_1^0\right) + \chi_{\{v \geq v_1^1\}}\left(a^2 - a^1\right)(v_2^1 - v_1^1)\right],\\
q(v, b_2) &= 0.
\end{aligned}$$

By Lemma 10, $v_2^1 \geq v_1^1$. We show below that $v_2^0 = \underline{v}$ and $a^1 = 0$.

First, we show that $v_2^0 = \underline{v}$. We consider two different cases: $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1\left(v_2^0 - v_1^0\right) = 0$ and $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1\left(v_2^0 - v_1^0\right) > 0$.

Suppose $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1\left(v_2^0 - v_1^0\right) = 0$.

Suppose to the contradiction that $v_2^0 > \underline{v}$. Then we can construct another feasible mechanism (a^*, p^*, q^*), which strictly improves welfare. Since $v_2^0 > \underline{v} = v_1^0$, we have $u(\underline{v}, b_2) > u(\underline{v}, b_1)$ and, by construction, $a^1 > 0$ and $v_1^1 > \underline{v}$.

Let $\varepsilon > 0$ be sufficiently small. Let $\tilde{v}_1^0 = \underline{v} + \varepsilon$ and $\tilde{v}_2^0 < v_2^0$ be such that $\pi[F(v_2^0) - F(\tilde{v}_2^0)] = (1-\pi)F(\underline{v} + \varepsilon)$. For $\varepsilon > 0$ sufficiently small, $\tilde{v}_1^0 < \min\{v_1^1, \tilde{v}_2^0\}$. Let $\tilde{v}_i^1 = v_i^1$ for $i = 1, 2$. Let $u^*(\underline{v}, b_1) = u(\underline{v}, b_1) + a^1\varepsilon$ and $u^*(\underline{v}, b_2) = u(\underline{v}, b_2) - (1-\pi)a^1\varepsilon/\pi$. For $\varepsilon > 0$ sufficiently small, $u^*(\underline{v}, b_2) \geq u^*(\underline{v}, b_1) > 0$. Let $a^*(v, b_1) = 0$ if $v < \tilde{v}_1^0$ and $a^*(v, b_1) =$

$a(v, b_1)$ otherwise. Let $a^*(v, b_2) = a^1$ if $v \in (\tilde{v}_2^0, v_2^0)$ and $a^*(v, b_2) = a(v, b_2)$ otherwise. Let $p^*(v, b) = va^*(v, b) - \int_{\underline{v}}^{v} a^*(\nu, b)\mathrm{d}\nu - u^*(\underline{v}, b)$. Since $u^*(\underline{v}, b_1) - a^1\tilde{v}_1^0 = u(\underline{v}, b_1) - a^1 v_1^0$, we have $p^*(\overline{v}, b_1) = p(\overline{v}, b_1) \leq b_1$. Hence, the (BC) constraint is satisfied. Let $q^*(v, b_1) = q(v, b_1)$. Clearly, (a^*, p^*, q^*) satisfies constraints (IR), (IC-v) and (S), and strictly improves welfare. (a^*, p^*, q^*) satisfies (BB) by Assumption 1.

Finally, we show that (a^*, p^*, q^*) satisfies the (IC-b) constraint. First, for $v < \underline{v} + \varepsilon$, we have $u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) \leq 0 \leq q^*(v, b_1)c$. Next, we show that for $m = 1, 2$

$$q^m c \geq u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}).$$

Since

$$\begin{aligned}
v_2^0 - \tilde{v}_2^0 &= \int_{\tilde{v}_2^0}^{v_2^0} f(v)\frac{1}{f(v)}\mathrm{d}v \\
&\geq \frac{1}{f(\tilde{v}_2^0)}[F(v_2^0) - F(\tilde{v}_2^0)] \\
&\geq \frac{1-\pi}{\pi}\frac{F(\underline{v}+\varepsilon)}{f(\underline{v}+\varepsilon)} \\
&\geq \frac{1-\pi}{\pi}\varepsilon,
\end{aligned}$$

where the inequalities hold by Assumption 2, we have

$$\begin{aligned}
&u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) + \sum_{j=1}^{1}(a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}) \\
=&u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1 v_2^0 + \frac{a^1\varepsilon}{\pi} + a^1(\tilde{v}_2^0 - v_2^0) - a^1(v_1^0 + \varepsilon) \\
\leq&u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1\left(v_2^0 - v_1^0\right).
\end{aligned}$$

Furthermore, $\tilde{v}_i^m = v_i^m$ for $i = 1, 2$ and $m \geq 1$. Hence, the (IC-b) constraint is satisfied. This contradicts to that (a, p, q) is optimal. Hence $v_2^0 = \underline{v}$.

Suppose $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1 \left(v_2^0 - v_1^0\right) > 0$.

Suppose to the contradiction that $v_2^0 > \underline{v}$. In this case, $\gamma_2^0 = 0$. By construction, we have $a^1 > 0$ and $v_1^1 > \underline{v}$. Hence, $\alpha_1 = 0$. Furthermore, since $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1 \left(v_2^0 - v_1^0\right) > 0$ and $v_2^1 \geq v_1^1$, we have $\mu^1 = \mu^2 = 0$. Then v_2^0 satisfies

$$\pi \left[(\beta - (1+\lambda)v_2^0)f(v_2^0) + \lambda[1 - F(v_2^0)]\right] - (1-\pi)(1+\lambda)\rho[1 - F(v_1^0)] = 0, \tag{26}$$

$$\begin{aligned}
&\pi \int_{v_2^0}^{v_2^M} [v + \lambda\varphi(v) - \beta] f(v)\mathrm{d}v \\
&+ (1-\pi)\left[\int_{v_1^0}^{v_1^M} [v + \lambda\varphi(v) - \beta] f(v)\mathrm{d}v - (1+\lambda)\rho(v_2^0 - v_1^0)[1 - F(v_1^0)]\right] \\
&- \eta v_1^0 - \alpha^{M+1} = 0.
\end{aligned} \tag{27}$$

Since $v_2^0 \geq v_1^0$, it follows from Claims 3 and (27) that $\int_{v_1^0}^{v_1^M} [v + \lambda\varphi(v)] f(v)\mathrm{d}v \geq \beta[1 - F(v_1^0)]$, i.e., $\hat{v}(\beta) = \underline{v}$.

Given β, η and λ, (34) and (35) define v_2^1 as functions of v_1^1, denoted by g_1 and g_2, respectively. By a similar argument in Claim 6, $g_1'(v) > 1$, and $g_2'(v) < 1$ if $v > \hat{v}(\beta)$ and $g_2(v) \geq v$. Let Δ_3 denote the left-hand side of (30) or (27), then

$$\begin{aligned}
\frac{\partial \Delta_3}{\partial v_1^1} &= (1-\pi)\left[(\beta - v_1^1 - \lambda\varphi(v_1^1))f(v_1^1) + (1+\lambda)\rho(v_2^1 - v_1^1)f(v_1^1) + (1+\lambda)\rho[1 - F(v_1^1)]\right] - \eta, \\
\frac{\partial \Delta_3}{\partial v_2^1} &= \pi(\beta - v_2^1 - \lambda\varphi(v_2^1))f(v_2^1) - (1-\pi)(1+\lambda)\rho[1 - F(v_1^1)].
\end{aligned}$$

Clearly, $\partial\Delta_3(v_1, g_2(v_1))/\partial v_2 = 0$ by (33). Since $v_2^1 \geq v_1^1$, then $g_2(v) > g_1(v)$ for all $v < v_1^1$. Then $\partial\Delta_3(v_1, g_2(v_1))/\partial v_1 > \Delta_3(v_1, g_1(v_1))/\partial v_1 = 0$ for all $v_1 < v_1^1$. Then

$$0 = \Delta_3(v_1^1, v_2^1) = \Delta_3(v_1^0, v_2^0) + \int_{v_1^0}^{v_1^1} \frac{\partial\Delta_3(v_1, g_2(v_1))}{\partial v_1}\mathrm{d}v_1 > \Delta_3(v_1^0, v_2^0) = 0,$$

a contradiction. Hence, $v_2^0 = \underline{v}$.

Next, we show that $a^1 = 0$. Suppose $a^1 > 0$, then $\alpha^1 = 0$. Then v_2^0 satisfies

$$a^1 \left\{ \pi \left[\beta - v_2^0 - \lambda\varphi(v_2^0)\right] f(v_2^0) - (1-\pi)(1+\lambda)\rho + \sum_{j=1}^{2} \mu^j \right\} + \gamma_2^0 = 0, \tag{28}$$

$$\pi \int_{v_2^0}^{v_2^M} [v + \lambda\varphi(v) - \beta] f(v)\mathrm{d}v + (1-\pi) \int_{v_1^0}^{v_1^M} [v + \lambda\varphi(v) - \beta] f(v)\mathrm{d}v - \eta v_1^0 - \alpha^{M+1} = 0. \tag{29}$$

By Claims 3, it follows from (29) that $\int_{v_1^0}^{v_1^M} [v + \lambda\varphi(v)] f(v)\mathrm{d}v - \beta \geq 0$, i.e., $\hat{v}(\beta) = \underline{v}$. Since $g_2'(v) \leq 1$ if $v \geq \hat{v}(\beta)$ and $g_2(v) \geq v$, and $g_2(v_1^1) = v_2^1 \geq v_1^1$, we have $v_2^0 = g_2(v_1^0) > v_1^0 = \underline{v}$, a contradiction. Hence, $a^1 = 0$.

Let $a^* = a^2$, $v_1^* = v_1^1$, $v_2^* = v_2^1$ and $v_2^{**} = v_2^2$. Let $u_1^* = u(\underline{v}, b_1)$ and $u_2^* = u(\underline{v}, b_2)$. This completes the proof. ■

Proof of Corollary 2. This results holds trivially if the first-best can be achieved. For the rest of the proof, I assume that the first-best can be achieved. Suppose there are two optimal mechanisms (a, p, q) and $(\hat{a}, \hat{p}, \hat{q})$. By Theorem 3, there exist $(u_1^*, u_2^*, a^*, v_1^*, v_2^*, v_2^{**})$ and $(\hat{u}_1^*, \hat{u}_2^*, \hat{a}^*, \hat{v}_1^*, \hat{v}_2^*, \hat{v}_2^{**})$ that characterize the two different optimal mechanisms, respectively.

First, I show that the convex combination of the two mechanisms $(\kappa a + (1-\kappa)\hat{a}, \kappa p + (1-\kappa)\hat{p}, \kappa + (1-\kappa)\hat{q})$, where $\kappa \in (0,1)$, is also optimal. Clearly, it satisfies (IR), (BC), (BB) and (S):

$$[\kappa a(t) + (1-\kappa)\hat{a}(t)]v - [\kappa p(t) + (1-\kappa)\hat{p}(t)] = \kappa[a(t)v - p(t)] + (1-\kappa)[\hat{a}(t)v - \hat{p}(t)] \geq 0,$$

$$\kappa p(t) + (1-\kappa)\hat{p}(t) \leq b,$$

$$\mathbb{E}_t\left[\kappa p(t) + (1-\kappa)\hat{p}(t) - [\kappa q(t) + (1-\kappa)\hat{q}(t)]k\right] = \kappa\mathbb{E}_t[p(t) - q(t)k] + (1-\kappa)\mathbb{E}_t[\hat{p}(t) - \hat{q}(t)k] \geq 0,$$

$$\mathbb{E}_t[\kappa a(t) + (1-\kappa)\hat{a}(t)] = \kappa\mathbb{E}_t[a(t)] + (1-\kappa)\mathbb{E}[\hat{a}(t)] \leq S.$$

It satisfies (IC-v) since $\kappa a(v,b)+(1-\kappa)\hat{a}(v,b)$ is non-decreasing in v and

$$
\begin{aligned}
&\kappa p(v,b)+(1-\kappa)\hat{p}(v,b)\\
=&\kappa\left[a(v,b)v-\int_{\underline{v}}^{v}a(\nu,b)\mathrm{d}\nu-u(\underline{v},b)\right]+(1-\kappa)\left[\hat{a}(v,b)v-\int_{\underline{v}}^{v}\hat{a}(\nu,b)\mathrm{d}\nu-\hat{u}(\underline{v},b)\right]\\
=&\left[\kappa a(v,b)+(1-\kappa)\right]v-\int_{\underline{v}}^{v}[\kappa a(\nu,b)+(1-\kappa)\hat{a}(\nu,b)]\mathrm{d}\nu-[\kappa u(\underline{v},b)+(1-\kappa)\hat{u}(\underline{v},b)].
\end{aligned}
$$

Finally, it satisfies (IC-b) since

$$
\begin{aligned}
&[\kappa a(v,b_2)+(1-\kappa)\hat{a}(v,b_2)]v-[\kappa p(v,b_2)+(1-\kappa)\hat{p}(v,b_2)]\\
=&\kappa[a(v,b_2)v-p(v,b_2)]+(1-\kappa)[\hat{a}(v,b_2)v-\hat{p}(v,b_2)]\\
\geq&\kappa[a(\hat{v},b_1)v-p(\hat{v},b_1)-q(\hat{v},b_1)c]+(1-\kappa)[\hat{a}(\hat{v},b_1)v-\hat{p}(\hat{v},b_1)-\hat{q}(\hat{v},b_1)c]\\
=&[\kappa a(v,b_1)+(1-\kappa)\hat{a}(v,b_1)]v-[\kappa p(\hat{v},b_1)+(1-\kappa)\hat{p}(\hat{v},b_1)]-[\kappa q(\hat{v},b_1)c+(1-\kappa)q(\hat{v},b_1)]c.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
&\mathbb{E}_t\left[[\kappa a(t)+(1-\kappa)\hat{a}(t)]v-[\kappa q(t)+(1-\kappa)\hat{q}(t)]k\right]\\
=&\kappa\mathbb{E}_t[a(t)v-q(t)k]+(1-\kappa)\mathbb{E}_t[\hat{a}(t)v-\hat{q}(t)k]\\
=&V.
\end{aligned}
$$

Hence, $(\kappa a+(1-\kappa)\hat{a},\kappa p+(1-\kappa)\hat{p},\kappa+(1-\kappa)\hat{q})$ is an optimal mechanism of $\mathcal{P}$.

Second, I show that $v_1^*=\hat{v}_1^*$. Suppose, on the contrary, that $v_1^*<\hat{v}_1^*$. Then

$$
\kappa a(v,b_1)+(1-\kappa)\hat{a}(v,b_1)=\chi_{\{v\geq v_1^*\}}\kappa a^*+\chi_{v\geq\hat{v}_1^*}(1-\kappa)\hat{a}^*,
$$

which is a 3-step function, a contradiction.

Third, I show that $v_2^*=\hat{v}_2^*$, $v_2^{**}=\hat{v}_2^{**}$ and $a^*=\hat{a}^*$. Suppose $a^*=\hat{a}^*=1$. By Proposition 2, (S) holds with equality in an optimal mechanism. Hence, $v_2^*=v_2^{**}=\hat{v}_2^*=\hat{v}_2^{**}$.

Suppose $a^* < 1$ and $\hat{a}^* = 1$. Since (S) holds with equality in both mechanisms, it must be that $v_2^* < \hat{v}_2^*$. In this case, $a(v, b_1) = \chi_{\{v \geq v_1^*\}}[\kappa a^* + (1-\kappa)]$. If $v \in \left(v_2^*, \min\{v_2^{**}, \hat{v}_2^{**}\}\right)$, then $a(v, b_2) = \kappa a^* < \kappa a^* + (1-\kappa)$, which is a contradiction to Lemma 3.

Suppose $a^* < 1$ and $\hat{a}^* < 1$. In this case, $a(v, b_1) = \chi_{\{v \geq v_1^*\}}[\kappa a^* + (1-\kappa)\hat{a}^*]$. Suppose, on the contrary, that $v_2^* < \hat{v}_2^*$. If $v \in \left(v_2^*, \min\{v_2^{**}, \hat{v}_2^*\}\right)$, then $a(v, b_2) = \kappa a^* < \kappa a^* + (1-\kappa)\hat{a}^*$, which is a contradiction to Lemma 3. Hence, $v_2^* = \hat{v}_2^*$. Suppose, on the contrary, that $v_2^{**} < \hat{v}_2^{**}$. If $v \in \left(v_2^{**}, \hat{v}_2^{**}\right)$, then $a(v, b_2) = \kappa + (1-\kappa)\hat{a}^* > \kappa a^* + (1-\kappa)\hat{a}^*$, a contradiction to Lemma 3. Hence, $v_2^{**} = \hat{v}_2^{**}$. Finally, since (S) holds with equality in both mechanisms, it must be the case $a^* = \hat{a}^*$.

Lastly, I show that $u_i^* = \hat{u}_i^*$ for $i = 1, 2$. Proposition 9 shows that if the first-best cannot be achieved then both (BC) and (BB) hold with equality in an optimal mechanism. Hence, $u_1^* = a^* v_1^* - b_1 = \hat{a}^* \hat{v}_1^* - b_1 = \hat{u}_1^*$. If $\rho \geq \pi/(1-\pi)$, then by Proposition 4 $u_2^* = u_1^* = \hat{u}_1^* = \hat{u}_2^*$. If $\rho < \pi/(1-\pi)$, then $u_2^* = \hat{u}_2^*$ by (BB). ■

C.3 Proof of Lemma 6

Let $M \geq 3$ be an integer. We want to show that $V(M-1, d) = V(M, d)$. Suppose to the contradiction that $V(M-1, d) < V(M, d)$, then an optimal solution to $\mathcal{P}'(M, d)$ satisfies the first-order conditions given before the proof of Lemma 11 in Appendix 4.2.

For later use, we note here that the summation of FOCs of $a^{m'}$, $m+1 \leq m' \leq M$, $m = 0, \ldots, M-1$, gives:

$$\pi \left[\int_{v_2^m}^{v_2^M} \left[(1+\lambda)v - \lambda \frac{1-F(v)}{f(v)} \right] f(v) \mathrm{d}v - \beta[F(v_2^M) - F(v_2^m)] \right]$$
$$+ (1-\pi) \left[\int_{v_1^m}^{v_1^M} \left[(1+\lambda)v - \lambda \frac{1-F(v)}{f(v)} \right] f(v) \mathrm{d}v - (1+\lambda)\rho(v_2^m - v_1^m)[1-F(v_1^m)] - \beta[1-F(v_1^m)] \right]$$
$$- \eta v_1^m + (v_2^m - v_1^m) \sum_{j=m+1}^{M} \mu^j + \alpha^{m+1} - \alpha^{M+1} = 0. \tag{30}$$

Recall that $\alpha^2 = \cdots = \alpha^M = 0$. We break the proof into several claims. In all claims, we assume, without explicitly repeating this, that Assumptions 1 and 2 hold, $u(\underline{v}, b_1)$, $u(\underline{v}, b_2)$, $\{a^m\}_{m=1}^{M}$, $\{v_1^m\}_{m=1}^{M-1}$ and $\{v_2^m\}_{m=0}^{M}\}$ define an optimal mechanism of $\mathcal{P}'(M, d)$ and β, η, λ, ξ_1, ξ_2, $\{\mu^m\}_{m=1}^{M}$, $\{\alpha^m\}_{m=1}^{M+1}$, $\{\gamma_1^m\}_{m=1}^{M}$ and $\{\gamma_2^m\}_{m=0}^{M+1}$ are the associated Lagrangian multipliers.

Claim 1 $\gamma_2^m = 0$ *for* $m = 2, \dots, M-1$.

Proof. Since $a^{m+1} > a^m$ for $m = 1, \dots M-1$, the FOCs of v_1^m become

$$(1-\pi)\left[(\beta - (1+\lambda)v_1^m)f(v_1^m) + (\lambda+\rho+\lambda\rho)[1-F(v_1^m)] + (1+\lambda)\rho(v_2^m - v_1^m)f(v_1^m)\right] - \sum_{j=m+1}^{M}\mu^j - \eta = 0,$$

for $m = 1, \dots, M-1$. Then for $m = 1, \dots, M-1$

$$v_2^m = \frac{1+\rho}{\rho}v_1^m - \frac{\lambda+\rho+\lambda\rho}{(1+\lambda)\rho}\frac{1-F(v_1^m)}{f(v_1^m)} - \frac{\beta}{(1+\lambda)\rho} + \frac{\eta + \sum_{j=m+1}^{M}\mu^j}{(1-\pi)(1+\lambda)\rho f(v_1^m)}, \tag{31}$$

which is strictly increasing in v_1^m by Assumptions 1 and 2. Let $m = 1, \dots, M-2$. If $\mu^{m+1} = 0$, then $v_2^{m+1} > v_2^m$ since $v_1^{m+1} > v_1^m$ and (31). If $\mu^{m+1} > 0$, then $v_2^{m+1} \geq v_1^{m+1} > v_1^m \geq v_2^m$ since (8) holds for m and $m+2$ and (8) holds with equality for $m+1$. Hence, $\gamma_2^m = 0$ for $m = 2, \dots, M-1$. ■

Let

$$\varphi(v) := v - \frac{1-F(v)}{f(v)},$$

denote the "virtual" value, which is strictly increasing in v by Assumption 1. By Lemmas 10 and 11, we have $v_2^{M-1} > v_1^{M-1}$. In this case, $\mu^M = 0$.

Claim 2 *Suppose* $v_2^{M-1} > v_1^{M-1}$*, then* $\overline{v} + \lambda\varphi(\overline{v}) > \beta \geq v_2^{M-1} + \lambda\varphi(v_2^{M-1})$.

Proof. Since $\mu^M = 0$, the FOC of v_2^{M-1} implies that $\beta \geq v_2^{M-1} + \lambda\varphi(v_2^{M-1})$. Since $v_2^{M-1} > v_1^{M-1}$ and $\mu^M = 0$, the FOC of a^M implies that

$$\pi\int_{v_2^{M-1}}^{v_2^M}[v + \lambda\varphi(v) - \beta]\,f(v)\mathrm{d}v + (1-\pi)\int_{v_1^{M-1}}^{v_1^M}[v + \lambda\varphi(v) - \beta]\,f(v)\mathrm{d}v \geq 0.$$

Hence, it must be the case that $\beta < \overline{v} + \lambda\varphi(\overline{v})$. ■

Claim 3 *Suppose $v_2^{M-1} > v_1^{M-1}$, then $\gamma_2^M = \gamma_2^{M+1} = 0$ and $v_2^M + \lambda\varphi(v_2^M) \le \beta$.*

Proof. Suppose $v_2^M + \lambda\varphi(v_2^M) > \beta \ge v_2^{M-1} + \lambda\varphi(v_2^{M-1})$, then $v_2^M > v_2^{M-1}$ and therefore $\gamma_2^M = 0$. Suppose $v_2^M + \lambda\varphi(v_2^M) \le \beta < \overline{v} + \lambda\varphi(\overline{v})$, then $v_2^M < \overline{v}$ and therefore $\gamma_2^{M+1} = 0$. Since $\gamma_2^{M+1} = 0$ and $v_2^M + \lambda\varphi(v_2^M) \le \beta$, the FOC of v_2^M implies that $\gamma_2^M = 0$. Hence, $\gamma_2^M = 0$.

Suppose $a^{M+1} > a^M$, then the FOC of v_2^M implies that $\beta \ge v_2^M + \lambda\varphi(v_2^M)$. Suppose $a^{M+1} = a^M$, then by construction $v_2^M = v_2^{M-1}$ and therefore $v_2^M + \lambda\varphi(v_2^M) \le \beta$. Hence, $v_2^M + \lambda\varphi(v_2^M) \le \beta < \overline{v} + \lambda\varphi(\overline{v})$, which implies that $v_2^M < \overline{v}$ and therefore $\gamma_2^{M+1} = 0$. ■

In what follows, we consider two cases: $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1\left(v_2^0 - v_1^0\right) > 0$ and $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1\left(v_2^0 - v_1^0\right) = 0$.

Case 1. $v_2^{M-1} > v_1^{M-1}$ and $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1\left(v_2^0 - v_1^0\right) = 0$.

Claim 4 *Suppose $v_2^{M-1} > v_1^{M-1}$ and $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1\left(v_2^0 - v_1^0\right) = 0$, then $\gamma_2^1 = 0$.*

Proof. Suppose $\gamma_2^0 > 0$, then $v_2^0 = \underline{v}$. Since (8) holds for $m = 2$, we have $v_2^1 \ge v_1^1 > \underline{v} = v_2^0$. Hence, $\gamma_2^1 = 0$.

Suppose $\gamma_2^0 = 0$. Suppose $a^1 = 0$, then the FOC of v_2^0 implies that $\gamma_2^1 = 0$. Suppose $a^1 > 0$. Suppose to the contradiction that $\gamma_2^1 > 0$, then we can construct another feasible mechanism (a^*, p^*, q^*), which strictly improves welfare. Since $\gamma_2^1 > 0$, we have $v_2^0 = v_2^1 \ge v_1^1$. We consider two different cases: (1) $v_2^0 = v_2^1 = v_1^1$ and (2) $v_2^0 = v_2^1 > v_1^1$.

Suppose $v_2^0 = v_2^1 = v_1^1$.

Let $\tilde{v}_1^1$ be such that $a^2(v_1^1 - \tilde{v}_1^1) = a^1(v_1^1 - \underline{v})$. Then, by Assumption 2, we have

$$
\begin{aligned}
a^2\left[F(v_1^1) - F(\tilde{v}_1^1)\right] &= (a^2 - a^1 + a^1)\left[F(v_1^1) - F(\tilde{v}_1^1)\right] \\
&\le a^1\left[F(v_1^1) - F(\tilde{v}_1^1)\right] + (a^2 - a^1) f(\tilde{v}_1^1)(v_1^1 - \tilde{v}_1^1) \\
&= a^1\left[F(v_1^1) - F(\tilde{v}_1^1)\right] + a^1 f(\tilde{v}_1^1)\tilde{v}_1^1 \\
&\le a^1 F(v_1^1).
\end{aligned}
$$

Let $\tilde{v}_2^0 = \underline{v}$ and $\tilde{v}_2^1$ be such that $\pi\left[F(v_2^1) - F(\tilde{v}_2^1)\right] = (1-\pi)\left[a^1 F(v_1^1) - a^2\left[F(v_1^1) - F(\tilde{v}_1^1)\right]\right]$. Let $\tilde{v}_1^m = v_1^m$ and $\tilde{v}_2^m = v_2^m$ for all $m \geq 1$. Let $a^*(v, b_i) = a^m$ if $v \in (\tilde{v}_i^{m-1}, \tilde{v}_i^m)$ for $m \geq 2$ and $i = 1, 2$ and $a^*(v, b_i) = 0$ if $v \in (\underline{v}, \tilde{v}_i^1)$ for $i = 1, 2$. Let $p^*(v, b) = va^*(v, b) - \int_{\underline{v}}^{v} a^*(\nu, b)\mathrm{d}\nu - u(\underline{v}, b)$. Then, by construction, $p^*(\overline{v}, b_1) = p(\overline{v}, b_1) \leq b_1$. Hence, the (BC) constraint is satisfied. Let $q^*(v, b_1) = q(v, b_1)$. By Assumption 1, (a^*, p^*, q^*) improves revenue and therefore satisfies the (BB) constraint. Clearly, (a^*, p^*, q^*) satisfies constraints (IR), (IC-v) and (S), and strictly improves welfare.

Finally, we show that (a^*, p^*, q^*) satisfies the (IC-b) constraint. For $v \in (\underline{v}, \tilde{v}_1^1)$, we have

$$u(\underline{v}, b_1) - u(\underline{v}, b_2) < u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1\left(v_2^0 - v_1^0\right) = q^1 c.$$

For $v \in (\tilde{v}_1^1, v_1^1)$, we have

$$u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^2(\tilde{v}_2^1 - \tilde{v}_1^1) \leq u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1\left(v_2^0 - v_1^0\right) = q^1 c.$$

The first inequality holds since $a^2(\tilde{v}_2^1 - \tilde{v}_1^1) \leq a^2(v_1^1 - \tilde{v}_1^1) = a^1(v_1^1 - \underline{v}) = a^1\left(v_2^0 - v_1^0\right)$. For $v \in (v_1^{m-1}, v_1^{m-2})$, $m \geq 2$, we have

$$\begin{aligned}
&u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^2(\tilde{v}_2^1 - \tilde{v}_1^1) + \sum_{j=3}^{m}(a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}) \\
=&u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) + a^2(\tilde{v}_2^1 - \tilde{v}_1^1) - (a^2 - a^1)(v_2^1 - v_1^1) - a^1\left(v_2^0 - v_1^0\right) \\
=&u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) + a^2\tilde{v}_2^1 - a^2\tilde{v}_1^1 - a^2 v_2^1 + (a^2 - a^1)v_1^1 + a^1 v_1^0 \\
\leq&u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) = q^m c,
\end{aligned}$$

where the last inequality holds by construction. Hence, the (IC-b) constraint is satisfied. Thus, (a^*, p^*, q^*) is feasible. However, this contradicts to that (a, p, q) is optimal.

Suppose $v_2^0 = v_2^1 > v_1^1$.

Let $a^*(v, b_1) = a^1 - \varepsilon$ for some $\varepsilon > 0$ sufficiently small if $v < v_1^1$ and $a^*(v, b_1) = a(v, b_1)$ otherwise. Let $\tilde{v}_2^0 < v_2^0$ be such that $\pi(a^1 - \varepsilon)\left[F(v_2^0) - F(\tilde{v}_2^0)\right] = (1 - \pi)\varepsilon F(v_1^1)$. For $\varepsilon > 0$ sufficiently small, $v_1^1 < \tilde{v}_2^0$. Let $\tilde{v}_2^m = v_2^m$ for $m \geq 1$. Let $a^*(v, b_2) = a^1 - \varepsilon$ if $v \in (\tilde{v}_2^0, \tilde{v}_2^1)$ and $a^*(v, b_2) = a(v, b_2)$ otherwise. Let $u^*(\underline{v}, b_1) = u(\underline{v}, b_1) + \varepsilon(v_1^1 - v_1^0)$ and $u^*(\underline{v}, b_2) = u(\underline{v}, b_2) - (1 - \pi)\varepsilon(v_1^1 - v_1^0)/\pi$. For $\varepsilon > 0$ sufficiently small, $u^*(\underline{v}, b_2) \geq u^*(\underline{v}, b_1) > 0$. Let $p^*(v, b) = va^*(v, b) - \int_{\underline{v}}^{v} a(\nu, b)\mathrm{d}\nu - u(\underline{v}, b)$. Then, by construction, we have $p^*(\overline{v}, b_1) = p(\overline{v}, b_1) \leq b_1$. Hence, the (BC) constraint is satisfied. Let $q^*(v, b_1) = q(v, b_1)$. Then (a^*, p^*, q^*) satisfies (BB) by Assumption 1. Clearly, (a^*, p^*, q^*) satisfies constraints (IR), (IC-v) and (S), and strictly improves welfare.

Finally, we show that (a^*, p^*, q^*) satisfies the (IC-b) constraint. Note that, by Assumption 2, we have

$$
\begin{aligned}
(a^1 - \varepsilon)(v_2^0 - \tilde{v}_2^0) &= (a^1 - \varepsilon)\int_{\tilde{v}_2^0}^{v_2^0} f(v)\frac{1}{f(v)}\mathrm{d}v \\
&\geq (a^1 - \varepsilon)\frac{1}{f(\tilde{v}_2^0)}[F(v_2^0) - F(\tilde{v}_2^0)] \\
&\geq \frac{1-\pi}{\pi}\varepsilon\frac{1}{f(v_1^1)}F(v_1^1) \\
&\geq \frac{1-\pi}{\pi}\varepsilon(v_1^1 - v_1^0).
\end{aligned}
$$

Then, for $v < v_1^1$, we have

$$
\begin{aligned}
&u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) + (a^1 - \varepsilon)(\tilde{v}_2^0 - v_1^0) \\
=&u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1 v_2^0 + \frac{\varepsilon(v_1^1 - v_1^0)}{\pi} + (a^1 - \varepsilon)(\tilde{v}_2^0 - v_2^0) - \varepsilon v_2^0 - (a^1 - \varepsilon)v_1^0 \\
\leq&u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1 v_2^0 + \frac{\varepsilon(v_1^1 - v_1^0)}{\pi} - \frac{(1-\pi)\varepsilon(v_1^1 - v_1^0)}{\pi} - \varepsilon v_2^0 - (a^1 - \varepsilon)v_1^0 \\
\leq&u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1\left(v_2^0 - v_1^0\right) + \varepsilon(v_1^1 - v_2^0) \\
<&u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1\left(v_2^0 - v_1^0\right) = q^1 c.
\end{aligned}
$$

For $v \in (v_1^{m-1}, v_1^m)$ for $m = 2, \dots, M$, we have

$$
\begin{aligned}
& u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) + (a^1 - \varepsilon)(\tilde{v}_2^0 - v_1^0) + (a^2 - a^1 + \varepsilon)(v_2^1 - v_1^1) + \sum_{j=3}^{m}(a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}) \\
\leq & u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) + \varepsilon(v_1^1 - v_2^0) + \varepsilon(v_2^1 - v_1^1), \\
= & u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) = q^m c.
\end{aligned}
$$

Hence, the (IC-b) constraint is satisfied. Thus, (a^*, p^*, q^*) is feasible. However, this contradicts to that (a, p, q) is optimal.

Hence, it must be that $\gamma_2^1 = 0$. ■

By Claims 1, 3 and 4, we have $\gamma_2^m = 0$ for $m = 1, \dots, M+1$. Thus, for $m = 1, \dots, M-1$, v_1^m and v_2^m satisfy

$$
\begin{aligned}
& (1-\pi)\left[(\beta - v_1^m - \lambda\varphi(v_1^m))f(v_1^m) + (1+\lambda)\rho[1 - F(v_1^m)] + (1+\lambda)\rho(v_2^m - v_1^m)f(v_1^m)\right] \\
& - \sum_{j=m+1}^{M} \mu^j - \eta = 0, \qquad (32) \\
& \pi(\beta - v_2^m - \lambda\varphi(v_2^m))f(v_2^m) - (1-\pi)(1+\lambda)\rho[1 - F(v_1^m)] + \sum_{j=m+1}^{M} \mu^j = 0. \qquad (33)
\end{aligned}
$$

Recall that (32) and (33) are the first-order conditions of v_1^m and v_2^m, respectively, for $m = 1, \dots, M-1$.

Claim 5 *Suppose $v_2^{M-1} > v_1^{M-1}$ and $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1\left(v_2^0 - v_1^0\right) = 0$, then $\mu^m = 0$ for $m = 3, \dots, M$.*

Proof. The result follows directly from Lemmas 10 and 11. ■

Claim 6 *Suppose $v_2^{M-1} > v_1^{M-1}$ and $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1\left(v_2^0 - v_1^0\right) = 0$, then $M \leq 2$.*

Proof. Let $\hat{m} = 1$ if $\mu^2 = 0$ and $\hat{m} = 2$ if $\mu^2 > 0$. For $m = \hat{m}, \dots, M-1$, (32) and (33) become

$$(1-\pi)\left[(\beta - v_1^m - \lambda\varphi(v_1^m))f(v_1^m) + (1+\lambda)\rho[1-F(v_1^m)] + (1+\lambda)\rho(v_2^m - v_1^m)f(v_1^m)\right] - \eta = 0, \tag{34}$$

$$\pi(\beta - v_2^m - \lambda\varphi(v_2^m))f(v_2^m) - (1-\pi)(1+\lambda)\rho[1-F(v_1^m)] = 0, \tag{35}$$

Given β, η and λ, (34) and (35) define v_2^m as functions of v_1^m, denoted by g_1 and g_2, respectively. Clearly, by Assumptions 1 and 2, $g_1'(v_1^m) > 1$. Since $\mu^{\hat{m}} > 0$, (8) holds by equality for $\hat{m}$, which implies that $v_2^{\hat{m}} \geq v_1^{\hat{m}}$. Furthermore, since $g_1'(v_1^m) > 1$, $v_2^m \geq v_1^m$ for all $\hat{m} \leq m \leq M-1$. Since $v + \lambda\varphi(v) < \beta$ for all $v < v_2^M$, $v_2^m \geq v_1^m \geq \underline{v} \geq 0$, $\sum_{j=m+1}^{M} \mu^j = 0$, $\eta \geq 0$, $\alpha^{m+1} = 0$ and $\alpha^{M+1} \geq 0$, (30) implies that

$$\int_{v^m}^{\bar{v}} [v + \lambda\varphi(v) - \beta]\, f(v)\mathrm{d}v \geq 0,$$

which holds if and only if $v^m \geq \hat{v}(\beta)$, where

$$\hat{v}(\beta) := \inf\left\{ \hat{v} \,\middle|\, \int_{v^m}^{\bar{v}} [v + \lambda\varphi(v) - \beta]\, f(v)\mathrm{d}v \geq 0 \right\}.$$

By the implicit function theorem, we have

$$g_2'(v_1^m) = \frac{1-\pi}{\pi} \frac{(1+\lambda)\rho f(v_1^m)}{-(\beta - (1+\lambda)v_2^m)f'(v_2^m) + (1+2\lambda)f(v_2^m)} > 0. \tag{36}$$

To see that the last inequality holds, note that $(\beta - v - \lambda\varphi(v))f(v)$ is strictly decreasing in v for $v < v_2^M$. Taking derivative with respect to v yields $(\beta - (1+\lambda)v)f'(v) - (1+2\lambda)f(v) < 0$ for $v < v_2^M$. Note that Assumption 1 implies that for all $v \geq v_1^m$, we have

$$f(v) \geq f(v_1^m)\frac{1-F(v)}{1-F(v_1^m)}. \tag{37}$$

Then for $v_1^m \geq \hat{v}(\beta)$ we have

$$
\begin{aligned}
1 - F(v_1^m) &\geq \frac{f(v_1^m)}{1 - F(v_1^m)} \int_{v_1^m}^{\bar{v}} (1 - F(v))\mathrm{d}v \\
&= \frac{f(v_1^m)}{1 - F(v_1^m)} \left[(1+\lambda) \int_{v_1^m}^{\bar{v}} (1 - F(v))\mathrm{d}v - \lambda \int_{v_1^m}^{\bar{v}} (1 - F(v))\mathrm{d}v \right] \\
&= \frac{f(v_1^m)}{1 - F(v_1^m)} \left[-(1+\lambda) v_1^m [1 - F(v_1^m)] + \int_{v_1^m}^{\bar{v}} \left[(1+\lambda)v - \lambda \frac{1 - F(v)}{f(v)} \right] f(v)\mathrm{d}v \right] \\
&\geq (\beta - (1+\lambda)v_1^m) f(v_1^m),
\end{aligned}
$$

where the first line holds by (37), the third line holds by integration by parts, and the last line holds since $v_1^m \geq \hat{v}(\beta)$. Combining this and (35) yields

$$
\begin{aligned}
(\beta - v_2^m - \lambda\varphi(v_2^m)) f(v_2^m) &= \frac{1-\pi}{\pi} (1+\lambda)\rho[1 - F(v_1^m)] \\
&= \frac{1-\pi}{\pi} \rho \left[[1 - F(v_1^m)] + \lambda[1 - F(v_1^m)] \right] \\
&\geq \frac{1-\pi}{\pi} \rho \left[(\beta - (1+\lambda)v_1^m) f(v_1^m) + \lambda[1 - F(v_1^m)] \right] \\
&= \frac{1-\pi}{\pi} \rho \left[\beta - v_1^m - \lambda\varphi(v_1^m) \right] f(v_1^m).
\end{aligned}
$$

Hence,

$$
\frac{\rho f(v_1^m)}{f(v_2^m)} \leq \frac{\pi}{1-\pi} \frac{\beta - v_2^m - \lambda\varphi(v_2^m)}{\beta - v_1^m - \lambda\varphi(v_1^m)}.
$$

Furthermore,

$$
\begin{aligned}
& -(\beta - (1+\lambda)v_2^m) f'(v_2^m) + (1+2\lambda) f(v_2^m) \\
=& -(\beta - v_2^m - \lambda\varphi(v_2^m)) f'(v_2^m) + \lambda \left\{ \frac{[1 - F(v_2^m)] f'(v_2^m)}{f(v_2^m)} + f(v_2^m) \right\} + (1+\lambda) f(v_2^m) \\
\geq& (1+\lambda) f(v_2^m),
\end{aligned}
$$

where the last inequality holds since $\beta - v_2^m - \lambda\varphi(v_2^m) > 0$, $f' \le 0$ by Assumption 2 and $[1 - F(v_2^m)]f'(v_2^m) + f^2(v_2^m) \ge 0$ by Assumption 1. Finally, since $v_2^m \ge v_1^m \ge \hat{v}(\beta)$, we have

$$g_2'(v_1^m) = \frac{1-\pi}{\pi}\frac{(1+\lambda)\rho f(v_1^m)}{-(\beta-(1+\lambda)v_2^m)f'(v_2^m)+(1+2\lambda)f(v_2^m)} \le \frac{\beta - v_2^m - \lambda\varphi(v_2^m)}{\beta - v_1^m - \lambda\varphi(v_1^m)} \le 1.$$

Note that $g_2'(v_1^m) < 1$ if $v_1^m > \hat{v}(\beta)$ or $v_1^m < v_2^m$.

Thus, there exists at most one $v_1^m \ge \hat{v}(\beta)$ such that $g_1(v_1^m) = g_2(v_1^m) \ge v_1^m$, i.e., (34) and (35) has at most one solution such that $v_2^m \ge v_1^m \ge \hat{v}(\beta)$. Hence, $M \le \hat{m} + 1 \le 3$.

Suppose $M = 3$. By Claim 3, $v + \lambda\varphi(v) < \beta$ for all $v \le v_2^M$. Furthermore, $\eta \ge 0$ and $\alpha^{M+1} \ge 0$. Hence, it follows from (30) that

$$\int_{v^1}^{\overline{v}} [v + \lambda\varphi(v) - \beta] f(v)\mathrm{d}v \ge 0,$$

i.e., $v^1 \ge \hat{v}(\beta)$. Then we have $v_1^{\hat{m}} > v^{\hat{m}-1} \ge \hat{v}(\beta)$, and $g_2(v_1^{\hat{m}}) = v_2^{\hat{m}} \ge v_1^{\hat{m}}$ since $\mu^{\hat{m}} > 0$. Since $g_2'(v) < 1$ if $v > \hat{v}(\beta)$ and $g_2(v) \ge v$, we have $g_2(v) > v$ for all $v < v_1^{\hat{m}}$. Hence, $v^{\hat{m}-1} = g_2(v^{\hat{m}-1}) > v^{\hat{m}-1}$, a contradiction. Hence, $M = 2$ and $v_2^1 \ge v_1^1$. ■

Case 2. $v_2^{M-1} > v_1^{M-1}$ **and** $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1\left(v_2^0 - v_1^0\right) > 0$. In this case, by Lemmas 10 and 11, $\mu^m = 0$ for $m = 1, \dots, M$.

Claim 7 *Suppose* $v_2^{M-1} > v_1^{M-1}$ *and* $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1\left(v_2^0 - v_1^0\right) > 0$, *then* $\gamma_2^1 = 0$.

Proof. Since $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1\left(v_2^0 - v_1^0\right) > 0$, $\mu^1 = 0$. Suppose, on the contrary, that $\gamma_2^1 > 0$. Then $v_2^1 = v_2^0$.

Suppose $\gamma_2^0 > 0$, then $v_2^1 = v_2^0 = \underline{v} = v_1^0$. Hence, $u(\underline{v}, b_1) > u(\underline{v}, b_2)$. Let $\tilde{v}_2^1 = \underline{v} + \varepsilon$ for some $\varepsilon > 0$ sufficiently small. Let $\tilde{v}_1^1$ be such that $\pi F(\varepsilon) = (1-\pi)\left[F(v_1^1) - F(\tilde{v}_1^1)\right]$. For $\varepsilon > 0$ sufficiently small, $\tilde{v}_2^1 < \tilde{v}_1^1$. Let $\tilde{v}_i^m = v_i^m$ and for $i = 1, 2$ and $m \neq 1$. Let $a^*(v, b_2) = a^1$ for all $v \in (\underline{v}, \tilde{v}_2^1)$ and $a^*(v, b_2) = a(v, b_2)$ otherwise. Let $a^*(v, b_1) = a^2$ for $v \in (\tilde{v}_1^1, v_1^1)$ and $a^*(v, b_1) = a(v, b_1)$ otherwise. Let $u^*(\underline{v}, b_1) = u(\underline{v}, b_1) - (a^2 - a^1)(v_1^1 - \tilde{v}_1^1)$ and $u^*(\underline{v}, b_2) = u(\underline{v}, b_2) + \frac{1-\pi}{\pi}(a^2 - a^1)(v_1^1 - \tilde{v}_1^1)$. For $\varepsilon > 0$

sufficiently small, $u^*(\underline{v}, b_1) \geq u^*(\underline{v}, b_2) > 0$. Let $p^*(v, b) = va^*(v, b) - \int_{\underline{v}}^{v} a^*(\nu, b)\mathrm{d}\nu - u^*(\underline{v}, b)$. By construction, $p^*(\overline{v}, b_1) = p(\overline{v}, b_1) \leq b_1$. Hence, the (BC) constraint holds. Let $q^*(v, b_1) = q(v, b_1)$. By Assumption 1, the (BB) constraint holds. For $v \in (\underline{v}, \tilde{v}_1^1)$, (IC-b) holds since

$$u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) + a^1(\tilde{v}_2^0 - \tilde{v}_1^0) = u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) - \frac{(a^2 - a^1)(v_1^1 - \tilde{v}_1^1)}{\pi} \leq q^1 c.$$

For $v \in (\tilde{v}_1^1, v_1^1)$, (IC-b) holds since

$$\begin{aligned}
&u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) + a^1(\tilde{v}_2^0 - \tilde{v}_1^0) + (a^2 - a^1)(\tilde{v}_2^1 - \tilde{v}_1^1) \\
=&u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{2}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) \\
&- \frac{(a^2 - a^1)(v_1^1 - \tilde{v}_1^1)}{\pi} + (a^2 - a^1)(v_1^1 - \tilde{v}_1^1 + \tilde{v}_2^1 - v_2^1) \\
=&u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{2}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) + (a^2 - a^1)(\tilde{v}_2^1 - v_2^1) - \frac{(1-\pi)(a^2 - a^1)(v_1^1 - \tilde{v}_1^1)}{\pi} \\
\leq&u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{2}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) \\
\leq&u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) = q^1 c,
\end{aligned}$$

where the last inequality holds since $v_2^1 = \underline{v} < v_1^1$, and the first inequality holds since by Assumption 2 we have

$$\begin{aligned}
\tilde{v}_2^1 - v_2^1 &\leq \frac{F(\varepsilon)}{f(\tilde{v}_2^1)} \\
&\leq \frac{1}{f(\tilde{v}_1^1)} \frac{1-\pi}{\pi} \left[F(v_1^1) - F(\tilde{v}_1^1)\right] \\
&\leq \frac{(1-\pi)(v_1^1 - \tilde{v}_1^1)}{\pi}.
\end{aligned}$$

For $v \in (v_1^{m-1}, v_1^m)$, $m = 2, \dots, M$, (IC-b) holds since

$$
\begin{aligned}
& u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) + \sum_{j=1}^{m} (a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}) \\
=& u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m} (a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) + (a^2 - a^1)(\tilde{v}_2^1 - v_2^1) - \frac{(1-\pi)(a^2 - a^1)(v_1^1 - \tilde{v}_1^1)}{\pi} \\
\leq& u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m} (a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) = q^m c.
\end{aligned}
$$

Clearly, (a^*, p^*, q^*) also satisfies constraints (IR), (IC-v) and (S), and strictly improves welfare. This contradicts to the optimality of (a, p, q). Hence, $\gamma_2^1 = 0$.

Suppose $\gamma_2^0 = 0$. Suppose $a^1 = 0$, then the FOC of v_2^0 implies that $\gamma_2^1 = 0$. Suppose $a^1 > 0$. Then

$$
\begin{aligned}
& \pi(\beta - (1+\lambda)v_2^0) f(v_2^0) + \pi\lambda[1 - F(v_2^0)] \\
\geq& (1-\pi)(1+\lambda)\rho - \sum_{j=2}^{M} \mu^j \\
>& (1-\pi)(1+\lambda)\rho[1 - F(v_1^1)] - \sum_{j=2}^{M} \mu^j \\
\geq& \pi(\beta - (1+\lambda)v_2^1) f(v_2^1) + \pi\lambda[1 - F(v_2^1)].
\end{aligned}
$$

Since $(\beta - (1+\lambda)v) f(v) + \lambda[1 - F(v)]$ is strictly decreasing in v when $v + \lambda\varphi(v) < \beta$, we have $v_2^1 > v_2^0$ and therefore $\gamma_2^1 = 0$. ■

By Claims 1, 3 and 7, we have $\gamma_2^m = 0$ for $m = 1, \dots, M$. Thus, for $m = 1, \dots, M-1$, v_1^m and v_2^m satisfies (34), (35) and (30).

Claim 8 *Suppose $v_2^{M-1} > v_1^{M-1}$ and $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1 \left(v_2^0 - v_1^0\right) > 0$, then $M \leq 2$.*

Proof. Suppose, on the contrary, that $M \geq 3$. Then there exists $1 \leq m < M-1$ such that $v_2^m \geq v_1^m$. It follows from (30) that $\int_{v_1^m}^{v_1^M} [v + \lambda\varphi(v)] f(v) \mathrm{d}v \geq \beta[1 - F(v_1^m)]$, i.e., $v_1^m \geq \hat{v}(\beta)$. Both (v_1^m, v_2^m) and (v_1^{M-1}, v_2^{M-1}) are solutions to (34) and (35), and satisfy $v_2 \geq v_1 \geq \hat{v}(\beta)$. However, by a similar argument in Claim 6, (34) and (35) have at most one solution satisfying $v_2 \geq v_1 \geq \hat{v}(\beta)$, a contradiction. Hence, it must be $M \leq 2$. ■

To summarize, we have shown in both cases that $M \leq 2$. However, this contradicts to the assumption that $M \geq 3$. Hence, it must be that $V(M, d) = V(M-1, d)$ for all $M \geq 3$. This completes the proof of Lemma 6.

C.4 Continuity

Let $\rho = k/c$. I abuse notation and let $\mathcal{P}'(2, \rho, \pi, S, b_1, d)$ denote the principal's problem $\mathcal{P}'(2, d)$ when verification cost is k, punishment is c, the percentage of high-budget agents is π, supply is S and low-budget agent's budget is b_1. Define $V : R_+ \times (0,1)^2 \times [\underline{v}, \overline{v}] \times \mathbb{R}_+ \to \mathbb{R}_+$ and $\Gamma^* : R_+ \times (0,1)^2 \times [\underline{v}, \overline{v}] \times R_+ \to R_+^2 \times [0,1] \times [\underline{v}, \overline{v}]^3$ as follows. Let $V(\rho, \pi, S, b_1, d)$ denote the value of $\mathcal{P}'(2, \rho, \pi, S, b_1, d)$ and $\Gamma^*(\rho, \pi, S, b_1, d)$ the set of optimal solutions.

Lemma 14 *Suppose Assumption 1 holds. Then $V(\rho, \pi, S, b_1, d)$ is continuous and $\Gamma^*(\rho, \pi, S, b_1, d)$ is upper hemicontinuous.*

Proof of Lemma 14. Let correspondence $\Gamma : R_+ \times (0,1)^2 \times [\underline{v}, \overline{v}] \times R_+ \to R_+^2 \times [0,1] \times [\underline{v}, \overline{v}]^3$ be defined as follows. For each (ρ, π, S, b_1, d), let $(u(\underline{v}, b_1), u(\underline{v}, b_2), a^2, v_1^1, v_2^1, v_2^2) \in \Gamma(\rho, \pi, S, b_1, d)$ if and only if it is a feasible solution to $\mathcal{P}(2, d)$. To simplify notation, let $u_1 = u(\underline{v}, b_1)$ and $u_2 = u_2(\underline{v}, b_2)$. Clearly, Γ is compact-valued and upper hemicontinuous. I show that it is also lower hemicontinuous.

Fix (ρ, π, S, b_1, d), $(u_1, u_2, a^2, v_1^1, v_2^1, v_2^2) \in \Gamma(\rho, \pi, S, b_1, d)$ and a sequence $(\rho(n), \pi(n), S(n), b_1(n), d(n)) \to (\rho, \pi, S, b_1, d)$ as $n \to \infty$. Let $\varphi(v) := v - \frac{1-F(v)}{f(v)}$ and r be such that $\varphi(r) = 0$. I show that after taking a subsequence there exist $\left(u_1(n), u_2(n), a^2(n), v_1^1(n), v_2^1(n), v_2^2(n)\right) \in \Gamma\left(\rho(n), \pi(n), S(n), b_1(n), d(n)\right)$ for all n and $\left(u_1(n), u_2(n), a^2(n), v_1^1(n), v_2^1(n), v_2^2(n)\right) \to (u_1, u_2, a^2, v_1^1, v_2^1, v_2^2)$ as $n \to \infty$.

Case 1: Suppose $a^2 = 0$. Then (S) and (BB) become:

$$\pi\left[1 - F(v_2^2)\right] \leq S,$$
$$-u_1 - [(1-\pi)\rho - \pi](u_1 - u_2) + \pi\int_{v_2^2}^{\overline{v}} \varphi(v)f(v)\mathrm{d}v \geq -d.$$

Case 1.1: Suppose $v_2^2 < r$.

After taking a subsequence, I can assume that for all n, $F^{-1}\left(\frac{\pi(n)-S(n)}{\pi(n)}\right) < r$ and

$$-(1-\pi(n))(1+\rho(n))u_1 + [(1-\pi(n))\rho(n) - \pi(n)]\,u_2 + \pi(n)\int_r^{\overline{v}} \varphi(v)f(v)\mathrm{d}v \geq -d(n).$$

Let $u_1(n) = u_1$, $u_2(n) = u_2$, $a^2(n) = a^2$, $v_1^1(n) = v_1^1$, $v_2^1(n) = v_2^1$ and

$$v_2^2(n) = \inf\left\{\begin{array}{c} v \geq \max\left\{v_2^2, F^{-1}\left(\frac{\pi(n)-S(n)}{\pi(n)}\right)\right\}, \\ -(1-\pi(n))(1+\rho(n))u_1 + [(1-\pi(n))\rho(n) - \pi(n)]\,u_2 + \pi(n)\int_v^{\overline{v}} \varphi(\nu)f(\nu)\mathrm{d}\nu \geq -d(n) \end{array}\right\}.$$

Clearly, $\left(u_1(n), u_2(n), a^2(n), v_1^1(n), v_2^1(n), v_2^2(n)\right) \in \Gamma\left(\rho(n), \pi(n), S(n), b_1(n), d(n)\right)$ for all n and $\left(u_1(n), u_2(n), a^2(n), v_1^1(n), v_2^1(n), v_2^2(n)\right) \to (u_1, u_2, a^2, v_1^1, v_2^1, v_2^2)$ as $n \to \infty$.

Case 1.2

Suppose $v_2^2 \geq r$. Suppose $v_2^2 = \overline{v}$, then (BB) implies that $u_1 = u_2 = 0$. Let $u_1(n) = u_1$, $u_2(n) = u_2$, $a^2(n) = a^2$, $v_1^1(n) = v_1^1$, $v_2^1(n) = v_2^1$ and $v_2^2(n) = v_2^2$. Clearly, $\left(u_1(n), u_2(n), a^2(n), v_1^1(n), v_2^1(n), v_2^2(n)\right) \in \Gamma\left(\rho(n), \pi(n), S(n), b_1(n), d(n)\right)$ for all n and $\left(u_1(n), u_2(n), a^2(n), v_1^1(n), v_2^1(n), v_2^2(n)\right) \to (u_1, u_2, a^2, v_1^1, v_2^1, v_2^2)$ as $n \to \infty$.

Suppose $v_2^2 < \overline{v}$. After taking a subsequence, I can assume that for all n, $F^{-1}\left(\frac{\pi(n)-S(n)}{\pi(n)}\right) < \overline{v}$. Let $a^2(n) = a^2$, $v_1^1(n) = v_1^1$, $v_2^1(n) = v_2^1$ and

$$v_2^2(n) = \max\left\{v_2^2, F^{-1}\left(\frac{\pi(n)-S(n)}{\pi(n)}\right)\right\}.$$

For n sufficiently large, $\int_{v_2^2(n)}^{\bar{v}} \varphi(v) f(v) \mathrm{d}v > 0$. If $u_2 > 0$, then let $u_1(n) = u_1 + \min\{\Delta(n), 0\}$ and $u_2(n) = u_2 + \min\{\Delta(n), 0\}$; otherwise let $u_1(n) = u_1 + \min\left\{\frac{\Delta(n)}{(1-\pi(n))(1+\rho(n))}, 0\right\}$ and $u_2(n) = u_2$, where

$$\Delta(n) = -(1-\pi(n))(1+\rho(n))u_1 + [(1-\pi(n))\rho(n) - \pi(n)]\, u_2 + \pi(n) \int_{v_2^2(n)}^{\bar{v}} \varphi(\nu) f(\nu) \mathrm{d}\nu + d(n).$$

Clearly, $\left(u_1(n), u_2(n), a^2(n), v_1^1(n), v_2^1(n), v_2^2(n)\right) \in \Gamma\left(\rho(n), \pi(n), S(n), b_1(n), d(n)\right)$ for all n and $\left(u_1(n), u_2(n), a^2(n), v_1^1(n), v_2^1(n), v_2^2(n)\right) \to (u_1, u_2, a^2, v_1^1, v_2^1, v_2^2)$ as $n \to \infty$.

Case 2: Suppose $a^2 > 0$.

Case 2.1: Suppose $v_2^2 < r$.

Let $a^2(n) = \min\left\{\frac{b_1(n)+u_1}{v_1^1}, a^2\right\}$. After taking a subsequence, I can assume that for all n,

$$F^{-1}\left(\frac{\pi(n) - S(n) + (1-\pi)a^2(n)\left[1 - F(v_1^1)\right] - \pi a^2 F(v_2^1)}{1 - \pi(n)a^2(n)}\right) < r$$

and

$$\begin{aligned}
&-(1-\pi(n))(1+\rho(n))u_1 + [(1-\pi(n))\rho(n) - \pi(n)]\, u_2 + (1-\pi(n)) \int_{v_1^1}^{\bar{v}} a^2(n)\varphi(v) f(v) \mathrm{d}v \\
&-(1-\pi(n))\rho(n)a^2(n)(v_2^1 - v_1^1)\left[1 - F(v_1^1)\right] + \pi(n) \int_{v_2^1}^{r} a^2(n)\varphi(v) f(v) \mathrm{d}v + \pi(n) \int_{r}^{\bar{v}} \varphi(v) f(v) \mathrm{d}v \geq -d(n).
\end{aligned}$$

Let $u_1(n) = u_1$, $u_2(n) = u_2$, $v_1^1(n) = v_1^1$, $v_2^1(n) = v_2^1$ and

$$v_2^2(n) = \inf\left\{\begin{array}{c}
v \geq \max\left\{v_2^2, F^{-1}\left(\frac{\pi(n)-S(n)+(1-\pi)a^2(n)\left[1-F(v_1^1)\right]-\pi a^2 F(v_2^1)}{1-\pi(n)a^2(n)}\right)\right\}, \\
-(1-\pi(n))(1+\rho(n))u_1 + [(1-\pi(n))\rho(n) - \pi(n)]\, u_2 \\
+(1-\pi(n)) \int_{v_1^1}^{\bar{v}} a^2(n)\varphi(\nu) f(\nu) \mathrm{d}\nu - (1-\pi(n))\rho(n)a^2(n)(v_2^1 - v_1^1)\left[1 - F(v_1^1)\right] \\
+\pi(n) \int_{v_2^1}^{v} a^2(n)\varphi(\nu) f(\nu) \mathrm{d}\nu + \pi(n) \int_{v}^{\bar{v}} \varphi(\nu) f(\nu) \mathrm{d}\nu \geq -d(n)
\end{array}\right\}.$$

Clearly, $\left(u_1(n), u_2(n), a^2(n), v_1^1(n), v_2^1(n), v_2^2(n)\right) \in \Gamma\left(\rho(n), \pi(n), S(n), b_1(n), d(n)\right)$ for all n

and $\left(u_1(n), u_2(n), a^2(n), v_1^1(n), v_2^1(n), v_2^2(n)\right) \to (u_1, u_2, a^2, v_1^1, v_2^1, v_2^2)$ as $n \to \infty$.

Case 2.2: Suppose $v_2^2 \geq r$.

Suppose $v_1^1 = \overline{v}$ and $v_2^1 = v_2^2$, then the proof follows that of Case 1.2. Assume for the rest of the proof that $v_1^1 < \overline{v}$ or $v_2^1 < v_2^2$. Let

$$A := (1-\pi) \int_{v_1^1}^{v_1^2} \varphi(v) f(v) \mathrm{d}v - (1-\pi)\rho \int_{v_1^1}^{\overline{v}} (v_2^1 - v_1^1) f(v) \mathrm{d}v + \pi \int_{v_2^1}^{v_2^2} \varphi(v) f(v) \mathrm{d}v.$$

and

$$A(n) := (1-\pi(n)) \int_{v_1^1}^{v_1^2} \varphi(v) f(v) \mathrm{d}v - (1-\pi(n))\rho(n) \int_{v_1^1}^{\overline{v}} (v_2^1 - v_1^1) f(v) \mathrm{d}v + \pi(n) \int_{v_2^1}^{v_2^2} \varphi(v) f(v) \mathrm{d}v.$$

Suppose $A < 0$. After taking a subsequence, I can assume that for all n, $S(n) - \pi(n)\left[1 - F(v_2^2)\right] > 0$, $A(n) < 0$ and

$$-(1-\pi(n))(1+\rho(n))u_1 + [(1-\pi(n))\rho(n) - \pi(n)]\, u_2 + \pi(n) \int_{v_2^2}^{\overline{v}} \varphi(v) f(v) \mathrm{d}v > -d(n).$$

Let $u_1(n) = u_1$, $u_2(n) = u_2$, $v_1^1(n) = v_1^1$, $v_2^1(n) = v_2^1$, $v_2^2(n) = v_2^2$ and

$$a^2(n) = \min \left\{ \begin{array}{c} a^2, \frac{S(n) - \pi(n)\left[1 - F(v_2^2)\right]}{\pi(n)[F(v_2^2) - F(v_2^1)] + (1-\pi(n))[1 - F(v_1^1)]}, \frac{b_1(n) + u_1}{v_1^1}, \\ \frac{-(1-\pi(n))(1+\rho(n))u_1 + [(1-\pi(n))\rho(n) - \pi(n)]u_2 + \pi(n) \int_{v_2^2}^{\overline{v}} \varphi(v) f(v) \mathrm{d}v + d(n)}{-A(n)} \end{array} \right\}.$$

Clearly, $\left(u_1(n), u_2(n), a^2(n), v_1^1(n), v_2^1(n), v_2^2(n)\right) \in \Gamma\left(\rho(n), \pi(n), S(n), b_1(n), d(n)\right)$ for all n and $\left(u_1(n), u_2(n), a^2(n), v_1^1(n), v_2^1(n), v_2^2(n)\right) \to (u_1, u_2, a^2, v_1^1, v_2^1, v_2^2)$ as $n \to \infty$.

Suppose $A = 0$.

Suppose $d = 0$ and $v_2^2 = \overline{v}$. Then $u_1 = u_2 = 0$ and $v_2^1 \geq v_1^1$. Define

$$g_n(v) := \begin{cases} (1-\pi(n)) \int_v^{\overline{v}} \varphi(\nu) f(\nu) \mathrm{d}\nu - (1-\pi(n))\rho(n)(v_2^1 - v)[1 - F(v)] + \pi(n) \int_{v_2^1}^{\overline{v}} \varphi(v) f(v) \mathrm{d}v & \text{if } v < v_2^1 \\ \int_v^{\overline{v}} \varphi(\nu) f(\nu) \mathrm{d}\nu & \text{if } v \geq v_2^1 \end{cases}.$$

Then

$$g_n'(v) := \begin{cases} -(1-\pi(n))\varphi(v)f(v) + (1-\pi(n))\rho(n)(v_2^1 - v)f(v) + (1-\pi(n))\rho(n)[1-F(v)] & \text{if } v < v_2^1 \\ -\varphi(v)f(v) & \text{if } v \geq v_2^1 \end{cases}.$$

Let g_∞ and g_∞' denote the case in which $\pi(n) = \pi$ and $\rho(n) = \rho$.

Suppose $v_1^1 < v_2^1$. Then

$$g_n(v_2^1) = \int_{v_2^1}^{\overline{v}} \varphi(\nu)f(\nu)\mathrm{d}\nu > 0.$$

Let $u_1(n) = u_1$, $u_2(n) = u_2$, $v_2^1(n) = v_2^1$, $v_2^2(n) = v_2^2$,

$$v_1^1(n) = \inf\left\{v \geq v_1^1 \,\middle|\, g_n(v) \geq 0\right\} < v_2^1,$$
$$\text{and } a^2(n) = \min\left\{a^2, \frac{S(n) - \pi(n)\left[1 - F(v_2^2)\right]}{\pi(n)[F(v_2^2) - F(v_2^1)] + (1-\pi(n))[1 - F(v_1^1(n))]}, \frac{b_1(n)}{v_1^1(n)}\right\}.$$

If $v \in (v_1^1, v_2^1)$, then $g_n'(v)/f(v)$ is strictly decreasing. Since $g_\infty(v_1^1) = A = 0$ and $g_\infty(v_2^1) > 0$, $g_\infty(v) > 0$ for all $v \in (v_1^1, v_2^1)$. Hence, $v_1^1(n) \to v_1^1$ as $n \to \infty$. Clearly, $\left(u_1(n), u_2(n), a^2(n), v_1^1(n), v_2^1(n), v_2^2(n)\right) \in \Gamma\left(\rho(n), \pi(n), S(n), b_1(n), d(n)\right)$ for all n and $\left(u_1(n), u_2(n), a^2(n), v_1^1(n), v_2^1(n), v_2^2(n)\right) \to (u_1, u_2, a^2, v_1^1, v_2^1, v_2^2)$ as $n \to \infty$.

Suppose $v_1^1 = v_2^1 < \overline{v}$. Let $u_1(n) = u_1$, $u_2(n) = u_2$, $v_1^1(n) = v_1^1$, $v_2^1(n) = v_2^1$, $v_2^2(n) = v_2^2$ and

$$a^2(n) = \min\left\{ a^2, \frac{S(n)-\pi(n)\left[1-F(v_2^2)\right]}{\pi(n)[F(v_2^2)-F(v_2^1)]+(1-\pi(n))[1-F(v_1^1(n))]}, \frac{b_1(n)}{v_1^1(n)} \right\}.$$

Clearly, $\left(u_1(n), u_2(n), a^2(n), v_1^1(n), v_2^1(n), v_2^2(n)\right) \in \Gamma\left(\rho(n), \pi(n), S(n), b_1(n), d(n)\right)$ for all n and $\left(u_1(n), u_2(n), a^2(n), v_1^1(n), v_2^1(n), v_2^2(n)\right) \to (u_1, u_2, a^2, v_1^1, v_2^1, v_2^2)$ as $n \to \infty$.

Suppose $v_2^2 < \overline{v}$ or $d > 0$. After taking a subsequence, I can assume there exists $\varepsilon > 0$ such that for all n, $S(n) - \pi(n)[1 - F(n)] > \varepsilon$, $b_1(n) > \varepsilon$ and

$$\int_{v_2^2}^{\overline{v}} \varphi(v) f(v) \mathrm{d}v + d(n) > \varepsilon.$$

Note that $(0, 0, 0, v_1^1, v_2^1, v_2^2) \in \Gamma(\rho(n), \pi(n), S(n), b_1(n), d(n))$. Define $\kappa(n) \in (0, 1]$ as follows:

$$\kappa(n) = \sup \left\{ \kappa \leq 1 \;\middle|\; \begin{gathered} \pi(n)\kappa a^2[F(v_2^2) - F(v_2^1)] + \pi(n)[1 - F(v_2^2)] + (1 - \pi(n))\kappa a^2[1 - F(v_1^1)] \geq S(n), \\ \kappa a^2 v_1^1 - \kappa u_1 \leq b_1(n), \\ -(1 - \pi(n))(1 + \rho(n))\kappa u_1 + [(1 - \pi(n))\rho(n) - \pi(n)]\, \kappa u_2 \\ +(1 - \pi(n)) \int_{v_1^1}^{\overline{v}} \kappa a^2 \varphi(\nu) f(\nu) \mathrm{d}\nu - (1 - \pi(n))\rho(n)\kappa a^2 (v_2^1 - v_1^1) \left[1 - F(v_1^1)\right] \\ +\pi(n) \int_{v_2^1}^{v_2^2} \kappa a^2 \varphi(\nu) f(\nu) \mathrm{d}\nu + \pi(n) \int_{v_2^2}^{\overline{v}} \varphi(\nu) f(\nu) \mathrm{d}\nu \geq -d(n) \end{gathered} \right\}.$$

Since at $(0, 0, 0, v_1^1, v_2^1, v_2^2)$ constraints (S), (BC) and (BB) hold with strict inequality by a gap at least ε, it is not hard to see that $\kappa(n) \to 1$ as $n \to \infty$. Let $u_1(n) = \kappa(n)u_1$, $u_2(n) = \kappa(n)u_2$, $a^2(n) = \kappa(n)a^2$, $v_1^1(n) = v_1^1$, $v_2^1(n) = v_2^1$ and $v_2^2(n) = v_2^2$. Clearly, $\left(u_1(n), u_2(n), a^2(n), v_1^1(n), v_2^1(n), v_2^2(n)\right) \in \Gamma\left(\rho(n), \pi(n), S(n), b_1(n), d(n)\right)$ for all n and $\left(u_1(n), u_2(n), a^2(n), v_1^1(n), v_2^1(n), v_2^2(n)\right) \to (u_1, u_2, a^2, v_1^1, v_2^1, v_2^2)$ as $n \to \infty$.

Suppose $A > 0$. After taking a subsequence, I can assume that there exists $\varepsilon > 0$ such that for all n, $S(n) - \pi(n)\left[1 - F(v_2^2)\right] > \varepsilon$, $b_1(n) > \varepsilon$ and $A(n) > \varepsilon$.

Suppose $v_2^2 = \overline{v}$ and $d = 0$. Let

$$\hat{a}^2 = \min \left\{ a^2, \frac{\varepsilon}{2\left[F(v_2^2) - F(v_2^1) + 1 - F(v_1^1)\right]}, \frac{\varepsilon}{2v_1^1} \right\} > 0.$$

Then $(0, 0, \hat{a}^2, v_1^1, v_2^1, v_2^2) \in \Gamma(\rho(n), \pi(n), S(n), b_1(n), d(n))$. Define $\kappa(n) \in (0, 1]$ as follows:

$$\kappa(n) = \sup\left\{ \kappa \leq 1 \;\middle|\; \begin{array}{c} \pi(n)[\kappa a^2 + (1-\kappa)\hat{a}^2][F(v_2^2) - F(v_2^1)] + \pi(n)[1 - F(v_2^2)] \\ +(1-\pi(n))[\kappa a^2 + (1-\kappa)\hat{a}^2][1 - F(v_1^1)] \geq S(n), \\ [\kappa a^2 + (1-\kappa)\hat{a}^2]v_1^1 - \kappa u_1 \leq b_1(n), \\ -(1-\pi(n))(1+\rho(n))\kappa u_1 + [(1-\pi(n))\rho(n) - \pi(n)]\,\kappa u_2 \\ +(1-\pi(n))\int_{v_1^1}^{\overline{v}}[\kappa a^2 + (1-\kappa)\hat{a}^2]\varphi(v)f(v)\mathrm{d}v \\ -(1-\pi(n))\rho(n)[\kappa a^2 + (1-\kappa)\hat{a}^2](v_2^1 - v_1^1)\left[1 - F(v_1^1)\right] \\ +\pi(n)\int_{v_2^1}^{v_2^2}[\kappa a^2 + (1-\kappa)\hat{a}^2]\varphi(v)f(v)\mathrm{d}v + \pi(n)\int_{v_2^2}^{\overline{v}}\varphi(v)f(v)\mathrm{d}v \geq -d(n) \end{array} \right\}.$$

Since at $(0, 0, \hat{a}^2, v_1^1, v_2^1, v_2^2)$ constraints (S), (BC) and (BB) hold with strict inequality by a gap at least $\min\{\hat{a}^2\varepsilon, \varepsilon/2\}$, it is not hard to see that $\kappa(n) \to 1$ as $n \to \infty$. Let $u_1(n) = \kappa(n)u_1$, $u_2(n) = \kappa(n)u_2$, $a^2(n) = \kappa(n)a^2$, $v_1^1(n) = v_1^1$, $v_2^1(n) = v_2^1$ and $v_2^2(n) = v_2^2$. Clearly, $\left(u_1(n), u_2(n), a^2(n), v_1^1(n), v_2^1(n), v_2^2(n)\right) \in \Gamma\left(\rho(n), \pi(n), S(n), b_1(n), d(n)\right)$ for all n and $\left(u_1(n), u_2(n), a^2(n), v_1^1(\right.$
$(u_1, u_2, a^2, v_1^1, v_2^1, v_2^2)$ as $n \to \infty$.

Suppose $v_2^2 < \overline{v}$ or $d > 0$. Then by a similar argument to that of $A = 0$, there exist $\left(u_1(n), u_2(n), a^2(n), v_1^1(n), v_2^1(n), v_2^2(n)\right) \in \Gamma\left(\rho(n), \pi(n), S(n), b_1(n), d(n)\right)$ for all n and $\left(u_1(n), u_2(n), a^2(n), v_1^1(\right.$
$(u_1, u_2, a^2, v_1^1, v_2^1, v_2^2)$ as $n \to \infty$.

Hence, Γ is hemicontinuous. By Berge's Maximum Theorem, V is continuous and Γ^* is upper hemicontinuous. ■

D Properties of the Optimal Mechanism

Let $a^* = a^2$, $v_1^* = v_1^1$, $v_2^* = v_2^1$, $v_2^{**} = v_2^2$, $u_1^* = u(\underline{v}, b_1)$ and $u_2^* = u(\underline{v}, b_2)$ denote an solution to $\mathcal{P}'(2, 0)$. Let $\beta, \eta, \lambda, \mu^1, \mu^2, \alpha^3, \xi_1$ and ξ_2 denote the corresponding Lagrangian multipliers.

Proof of Proposition 3. First-best is achieved if the allocation rule satisfies $v^* := v_1^* = v_2^* = F^{-1}(1 - S)$ and $a^* = 1$, and verification is zero. Hence, $u_1^* = u_2^* = v^* - b_1$ and (BB) holds if and only if

$$b_1 - v^* F(v^*) \geq 0. \tag{38}$$

Since $v^* = F^{-1}(1-S)$, there exists $\hat{S}(b_1) < 1$ such that (38) holds if and only if $S \geq \hat{S}(b_1)$. Clearly, $\hat{S}(b_1)$ is strictly decreasing in b_1. ■

Proof of Proposition 2. Let $S' := (1-\pi)a^* \left[1 - F(v_1^*)\right] + \pi a^*[F(v_2^{**}) - F(v_2^*)] + \pi[1 - F(v_2^{**})]$. Suppose to the contradiction that $S' < S$. Let $\kappa \in (0,1)$ be such that $\kappa + (1-\kappa)S' = S$. Consider a new mechanism (a^*, p^*, q^*). Let $a^*(v,b) = \kappa + (1-\kappa)a(v,b)$ and $p^*(v,b) = va^*(v,b) - \int_{\underline{v}}^{v} a^*(\nu, b)\mathrm{d}\nu - (1-\kappa)u(\underline{v}, b)$ for all v and b. Finally, let $q(v, b_2) = 0$ for all v, $q(v, b_1) = (1-\kappa)\left[u(\underline{v}, b_1) - u(\underline{v}, b_2)\right]/c$ if $v < v_1^*$ and $q(v, b_1) = (1-\kappa)\left[u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^*(v_2^* - v_1^*)\right]/c$ if $v > v_1^*$. Clearly, (a^*, p^*, q^*) strictly improves welfare upon (a, p, q). Now we show that (a^*, p^*, q^*) is also feasible. By construction, (IR) and (IC-v) hold. Note that

$$\begin{aligned}
p^*(v,b) &= va^*(v,b) - \int_{\underline{v}}^{v} a^*(\nu, b)\mathrm{d}\nu - (1-\kappa)u(\underline{v}, b) \\
&= (1-\kappa)va(v,b) + \kappa v - \int_{\underline{v}}^{v} [\kappa + (1-\kappa)a(\nu, b)]\, \mathrm{d}\nu - (1-\kappa)u(\underline{v}, b) \\
&= (1-\kappa)va(v,b) - (1-\kappa)\int_{\underline{v}}^{v} a(\nu, b)\mathrm{d}\nu - (1-\kappa)u(\underline{v}, b) \\
&= (1-\kappa)p(v,b).
\end{aligned}$$

Hence, $\mathbb{E}\left[p^*(v,b) - kq^*(v,b)\right] = (1-\kappa)\mathbb{E}\left[p(v,b) - kq(v,b)\right] \geq 0$. That is, (BB) holds. Since $p^*(\overline{v}, b_1) = (1-\kappa)p(\overline{v}, b_1) \leq b_1$, (BC) holds. Since $\mathbb{E}[a^*(v,b)] = \kappa + (1-\kappa)\mathbb{E}[a(v,b)] = \kappa + (1-\kappa)S' = S$, (S) holds. Finally, we show that (IC-b) holds. If $v \leq v_1^*$, then

$$(1-\kappa)\left[u(\underline{v}, b_1) - u(\underline{v}, b_2)\right] + \kappa(\underline{v} - \underline{v}) \leq q(v, b_1)c.$$

If $v > v_1^*$, then

$$(1-\kappa)\left[u(\underline{v}, b_1) - u(\underline{v}, b_2)\right] + \kappa(\underline{v} - \underline{v}) + (\kappa + (1-\kappa)a^* - \kappa)(v_2^* - v_1^*) \leq q(v, b_1)c.$$

Thus, we can conclude that (a^*, p^*, q^*) is feasible. However, this contradicts to that (a, p, q) is optimal. Hence, (S) holds with equality. ■

Proposition 9 *Suppose Assumptions 1 and 2 hold. Suppose also that $S < \hat{S}(b_1)$, i.e., the first-best cannot be achieved. In an optimal mechanism of $\mathcal{P}$, (S), (BB) and (BC) hold with equality.*

Proof of Proposition 9. First, it follows from Proposition 2 that (S) holds with equality. Second, we show that (BC) holds with equality. Suppose to the contradiction that (BC) holds with strict inequality. We consider four different cases: (1) $v_2^* > v_1^*$, (2) $v_2^{**} > v_2^* = v_1^*$, (3) $v_2^{**} = v_2^* = v_1^*$ and $a^* < 1$ and (4) $v_2^{**} = v_2^* = v_1^*$ and $a^* = 1$.

Suppose $v_2^* > v_1^*$.

Let $\varepsilon > 0$ and $\delta > 0$ be such that $(1-\pi)\left[F(v_1^* + \varepsilon) - F(v_1^*)\right] = \pi[F(v_2^*) - F(v_2^* - \delta)]$. For $\varepsilon > 0$ sufficiently small, we have $v_2^* - v_1^* - \varepsilon - \delta \geq 0$. Consider a new mechanism (a^*, p^*, q^*) that satisfies

$$
\begin{aligned}
& a^*(v, b_1) = \chi_{\{v \geq v_1^* + \varepsilon\}} a^*, \ \ p(v, b_1) = \chi_{\{v \geq v_1^*\}} a^*(v_1^* + \varepsilon) - u_1^*, \\
& q(v, b_1) = \frac{1}{c}\left[\chi_{\{v \geq v_1^*\}} a^*(v_2^* - v_1^* - \varepsilon - \delta) + u_1^* - u_2^*\right], \\
& a(v, b_2) = \chi_{\{v \geq v_2^* - \delta\}} a^* + \chi_{\{v \geq v_2^{**}\}} (1 - a^*), \\
& p(v, b_2) = \chi_{\{v \geq v_2^* - \delta\}} a^*(v_2^* - \delta) + \chi_{\{v \geq v_2^{**}\}} (1 - a^*) v_2^{**} - u_2^*, \\
& q(v, b_2) = 0.
\end{aligned}
$$

Clearly, for $\varepsilon > 0$ sufficiently small, (a^*, p^*, q^*) is feasible and strictly improves welfare upon (a, p, q), a contradiction.

Suppose $v_2^{**} > v_2^* = v_1^*$.

Let $\varepsilon > 0$ and $\delta > 0$ be such that $a^*\left[F(v_1^* + \varepsilon) - F(v_1^*)\right] = \pi[F(v_2^{**}) - F(v_2^{**} - \delta)]$. For $\varepsilon > 0$ sufficiently small, we have $v_2^{**} - v_1^* - \varepsilon - \delta \geq 0$. Consider a new mechanism (a^*, p^*, q^*)

that satisfies

$$
\begin{aligned}
&a^*(v,b_1)=\chi_{\{v\geq v_1^*+\varepsilon\}}a^*,\ p(v,b_1)=\chi_{\{v\geq v_1^*\}}a^*(v_1^*+\varepsilon)-u_1^*,\\
&q(v,b_1)=\frac{1}{c}\left[\chi_{\{v\geq v_1^*\}}a^*(v_2^*-v_1^*)+u_1^*-u_2^*\right],\\
&a(v,b_2)=\chi_{\{v\geq v_2^*+\varepsilon\}}a^*+\chi_{\{v\geq v_2^{**}-\delta\}}(1-a^*),\\
&p(v,b_2)=\chi_{\{v\geq v_2^*+\varepsilon\}}a^*(v_2^*+\varepsilon)+\chi_{\{v\geq v_2^{**}-\delta\}}(1-a^*)(v_2^{**}-\delta)-u_2^*,\\
&q(v,b_2)=0.
\end{aligned}
$$

Clearly, for $\varepsilon>0$ sufficiently small, (a^*,p^*,q^*) is feasible and strictly improves welfare upon (a,p,q), a contradiction.

Suppose $v_2^{**}=v_2^*=v_1^*$ and $a^*<1$.

Let $\varepsilon>0$ and $\delta>0$ be such that $[(1-\pi)a^*+\pi]\left[F(v_1^*+\varepsilon)-F(v_1^*)\right]=(1-\pi)\delta[1-F(v_1^*+\varepsilon)]$. For $\varepsilon>0$ sufficiently small, we have $\delta\leq 1-a^*$. Consider a new mechanism (a^*,p^*,q^*) that satisfies

$$
\begin{aligned}
&a^*(v,b_1)=\chi_{\{v\geq v_1^*+\varepsilon\}}(a^*+\delta),\ p(v,b_1)=\chi_{\{v\geq v_1^*\}}(a^*+\delta)(v_1^*+\varepsilon)-u_1^*,\\
&q(v,b_1)=\frac{1}{c}\left(u_1^*-u_2^*\right),\\
&a(v,b_2)=\chi_{\{v\geq v_2^*+\varepsilon\}},\ p(v,b_2)=\chi_{\{v\geq v_2^*+\varepsilon\}}-u_2^*,\\
&q(v,b_2)=0.
\end{aligned}
$$

Clearly, for $\varepsilon>0$ sufficiently small, (a^*,p^*,q^*) is feasible and strictly improves welfare upon (a,p,q), a contradiction.

Suppose $v_2^{**}=v_2^*=v_1^*$ and $a^*=1$.

In this case, the first-best allocation rule is achieved. Hence, it must be the case that the total verification cost is strictly positive, i.e., $u_1^*>u_2^*\geq 0$. Let $u_2^*-u_1^*\geq\varepsilon>0$. Consider a new mechanism (a^*, p^*, q^*) that satisfies

$$
\begin{aligned}
& a^*(v, b_1) = \chi_{\{v \geq v_1^*\}}, \ p(v, b_1) = \chi_{\{v \geq v_1^*\}} v_1^* - u_1^* + \varepsilon, \\
& q(v, b_1) = \frac{1}{c} \left(u_1^* - u_2^* - \varepsilon \right), \\
& a(v, b_2) = \chi_{\{v \geq v_2^*\}}, \ p(v, b_2) = \chi_{\{v \geq v_2^*\}} - u_2^*, \\
& q(v, b_2) = 0.
\end{aligned}
$$

Clearly, for $\varepsilon > 0$ sufficiently small, (a^*, p^*, q^*) is feasible and strictly improves welfare upon (a, p, q), a contradiction.

Lastly, I show that (BB) holds with equality. Suppose not. Then we can increase u_1^* and u_2^* by the same amount. The resulting new mechanism is feasible and gives the same welfare. In particular, (BC) holds with strict inequality in the new mechanism. Then we can repeat the above argument and construct another feasible mechanism which strictly improves welfare upon (a, p, q), a contradiction. ■

By Theorem 3, $v_1^*, v_2^*, v_2^{**}, a^2, u_1^*, u_2^*, \beta, \eta, \lambda, \mu^1, \mu^2, \alpha^3, \xi_1$ and ξ_2 satisfy the following first-order conditions:

$$(1-\pi)\left[(\beta-v_1^*-\lambda\varphi(v_1^*))f(v_1^*)+(1+\lambda)\rho[1-F(v_1^*)]+(1+\lambda)\rho(v_2^*-v_1^*)f(v_1^*)\right]$$

$$-\eta-\mu^2=0, \tag{39}$$

$$\pi(\beta-v_2^*-\lambda\varphi(v_2^*))f(v_2^*)-(1-\pi)(1+\lambda)\rho[1-F(v_1^*)]+\mu^2=0, \tag{40}$$

$$(1-a^*)(\beta-v_2^{**}-\lambda\varphi(v_2^{**}))f(v_2^{**})=0, \tag{41}$$

$$\pi\int_{v_2^*}^{v_2^{**}}[v+\lambda\varphi(v)-\beta]\,f(v)\mathrm{d}v$$

$$+(1-\pi)\left[\int_{v_1^*}^{\bar{v}}[v+\lambda\varphi(v)-\beta]\,f(v)\mathrm{d}v-(1+\lambda)\rho(v_2^*-v_1^*)[1-F(v_1^*)]\right]$$

$$-\eta v_1^*+\mu^2(v_2^*-v_1^*)-\alpha^3=0, \tag{42}$$

$$\eta+\mu^1+\mu^2-(1-\pi)(\lambda+\rho+\lambda\rho)+\xi_1=0, \tag{43}$$

$$-\mu^1-\mu^2-\pi\lambda+(1-\pi)(1+\lambda)\rho+\xi_2=0. \tag{44}$$

Furthermore, (S) and (BB) become:

$$(1-\pi)a^*[1-F(v_1^*)]+\pi a^*[F(v_2^{**})-F(v_2^*)]+\pi[1-F(v_2^{**})]=S, \tag{45}$$

$$-(1-\pi)u_1^*+(1-\pi)a^*v_1^*[1-F(v_1^*)]-\pi u_2^*+\pi a^*v_2^*[1-F(v_2^*)]+\pi(1-a^*)v_2^{**}[1-F(v_2^{**})]$$

$$-(1-\pi)\rho(u_1^*-u_2^*)-(1-\pi)\rho a^*(v_2^*-v_1^*)[1-F(v_1^*)]=0. \tag{46}$$

Proof of Proposition 4.

1. Suppose, on the contrary, that $u_1^* > u_2^* \geq 0$. In this case, $\xi_1 = \mu^1 = \mu^2 = 0$. (43) implies that $\eta = (1-\pi)(\lambda+\rho+\lambda\rho)$. (44) implies $\xi_2 = \pi\lambda - (1-\pi)(1+\lambda)\rho$. Since $\xi_2 \geq 0$, we have $\lambda[\pi-\rho(1-\pi)] \geq \rho(1-\pi)$ which implies that $\rho < \pi/(1-\pi)$, a contradiction.

2. Since $S < 1$, we have $u_1^* = u_2^*$ by the first result of Proposition 4. It suffices to show that $v_1^* = v_2^*$. Suppose, on the contrary, that $v_2^* > v_1^*$. In this case, $\mu^2 = 0$. Combining (43) and (44) yields $\eta - \lambda + \xi_1 + \xi_2 = 0$. Since $\xi_1, \xi_2 \geq 0$, we have $\eta \leq \lambda$. Taking the difference of

(39) divided by $(1-\pi)f(v_1^*)$ and (40) divided by $\pi f(v_2^*)$ gives

$$\begin{aligned}&[1+(1+\lambda)\rho]\left(v_2^*-v_1^*\right)+\lambda\left[\varphi(v_2^*)-\varphi(v_1^*)\right]+(1+\lambda)\rho\frac{1-F(v_1^*)}{f(v_1^*)}\\&+(1+\lambda)\frac{\rho(1-\pi)}{\pi}\frac{1-F(v_1^*)}{f(v_2^*)}-\frac{\eta}{(1-\pi)f(v_1^*)}=0.\end{aligned}\tag{47}$$

Since $v_2^*>v_1^*$, $f(v_2^*)\le f(v_1^*)$ and $\eta\le\lambda$, we have

$$\begin{aligned}0\ge&[1+(1+\lambda)\rho]\left(v_2^*-v_1^*\right)+\lambda\left[\varphi(v_2^*)-\varphi(v_1^*)\right]+(1+\lambda)\frac{\rho}{\pi}\frac{1-F(v_1^*)}{f(v_1^*)}-\frac{\lambda}{(1-\pi)f(v_1^*)}\\>&\frac{\rho}{\pi}\frac{1-F(v_1^*)}{f(v_1^*)}+\lambda\left[\frac{\rho}{\pi}\frac{1-F(v_1^*)}{f(v_1^*)}-\frac{1}{(1-\pi)f(v_1^*)}\right]\ge 0,\end{aligned}$$

where the last inequality holds since $1-F(v_1^*)\ge S$ and $\rho\ge\pi/[S(1-\pi)]$. A contradiction. Hence, $v_1^*=v_2^*$.

■

Proof of Proposition 5. Suppose, on the contrary, that $u_1^*>u_2^*\ge 0$. In this case, $\xi_1=\mu^1=\mu^2=0$. (43) implies that $\eta=(1-\pi)(\lambda+\rho+\lambda\rho)$. Taking the difference of (39) divided by $(1-\pi)f(v_1^*)$ and (40) divided by $\pi f(v_2^*)$ gives

$$\begin{aligned}&[1+(1+\lambda)\rho]\left(v_2^*-v_1^*\right)+\lambda\left[\varphi(v_2^*)-\varphi(v_1^*)\right]+(1+\lambda)\rho\frac{1-F(v_1^*)}{f(v_1^*)}\\&+(1+\lambda)\frac{\rho(1-\pi)}{\pi}\frac{1-F(v_1^*)}{f(v_2^*)}-\frac{\eta}{(1-\pi)f(v_1^*)}=0.\end{aligned}\tag{47}$$

Suppose $S\le(1-\pi)\left[1-F(b_1)\right]$. Since (BC) holds with equality and $u_1^*\ge 0$, we have $a^*\ge b_1/v_1^*$. By (S), we have

$$(1-\pi)\frac{b_1}{v_1^*}\left[1-F(v_1^*)\right]\le S.$$

Since $S\le(1-\pi)\left[1-F(b_1)\right]$, there exists a unique Let $\hat{v}(S,b_1,\pi)\in[b_1,\overline{v}]$ such that the above inequality holds with equality when $v_1^*=\hat{v}(S,b_1,\pi)$, where $\hat{v}$ is strictly decreasing in S and π and strictly increasing in b_1. Then $v_1^*\ge\hat{v}(S,b_1,\pi)$. Hence, $v_2^*-v_1^*\le\varphi(v_2^*)-\varphi(v_1^*)\le\overline{v}-\varphi\left(\hat{v}(S,b_1,\pi)\right)$.

Since $v_2^* \geq v_1^*$, $f(v_2^*) \leq f(v_1^*)$ and $\eta = (1-\pi)(\lambda + \rho + \lambda\rho)$, we have

$$\begin{aligned}
0 \leq &[1 + (1+\lambda)\rho] \left(v_2^* - v_1^*\right) + \lambda \left[\varphi(v_2^*) - \varphi(v_1^*)\right] + (1+\lambda)\frac{\rho}{\pi}\frac{1 - F(v_1^*)}{f(v_2^*)} - \frac{\lambda + \rho + \lambda\rho}{f(v_1^*)} \\
<&(1+\lambda)(1+\rho) \left[\overline{v} - \varphi\left(\hat{v}(S, b_1, \pi)\right)\right] + (1+\lambda)\frac{\rho}{\pi}\frac{1 - F(v_1^*)}{f(v_2^*)} - \frac{(1+\lambda)\rho}{f(v_1^*)} \\
\leq&(1+\lambda) \left\{ (1+\rho) \left[\overline{v} - \varphi\left(\hat{v}(S, b_1, \pi)\right)\right] + \frac{\rho}{\pi}\frac{1 - F\left(\hat{v}(S, b_1, \pi)\right)}{f(\overline{v})} - \frac{\rho}{f\left(\hat{v}(S, b_1, \pi)\right)} \right\}.
\end{aligned}$$

Note that the term in the braces is strictly increasing in S and converges to $-\rho/f(\overline{v}) < 0$ as S goes to zero. Hence, there exists $\hat{S}$ such that $u_1^* = u_2^*$ if $S < \hat{S}$. ■

Proof of Proposition 6. Suppose, on the contrary, that $u_1^* > u_2^* \geq 0$. In this case, $\xi_1 = \mu^1 = \mu^2 = 0$. (43) implies that $\eta = (1-\pi)(\lambda + \rho + \lambda\rho)$. Taking the difference of (39) divided by $(1-\pi)f(v_1^*)$ and (40) divided by $\pi f(v_2^*)$ gives

$$\begin{aligned}
&[1 + (1+\lambda)\rho] \left(v_2^* - v_1^*\right) + \lambda \left[\varphi(v_2^*) - \varphi(v_1^*)\right] + (1+\lambda)\rho\frac{1 - F(v_1^*)}{f(v_1^*)} \\
&+ (1+\lambda)\frac{\rho(1-\pi)}{\pi}\frac{1 - F(v_1^*)}{f(v_2^*)} - \frac{\eta}{(1-\pi)f(v_1^*)} = 0. \qquad (47)
\end{aligned}$$

Suppose $S \geq (1-\pi)\left[1 - F(b_1)\right]$. Then by (S),

$$\pi[1 - F(v_2^*)] \geq (1-\pi)\left[1 - F(b_1)\right] - S. \qquad (48)$$

Since $S \geq (1-\pi)\left[1 - F(b_1)\right]$, there exists a unique $\hat{v}(S, b_1, \pi) \in [b_1, \overline{v}]$ such that (48) holds with equality, where $\hat{v}$ is strictly decreasing in b_1, S and π. Then $v_2^* \leq \hat{v}(S, b_1, \pi)$. Hence, $v_2^* - v_1^* \leq \varphi(v_2^*) - \varphi(v_1^*) \leq \varphi\left(\hat{v}(S, b_1, \pi)\right) - b_1$.

Since $v_2^* \geq v_1^*$, $f(v_2^*) \leq f(v_1^*)$ and $\eta = (1-\pi)(\lambda+\rho+\lambda\rho)$, we have

$$\begin{aligned}
0 \leq &[1+(1+\lambda)\rho]\left(v_2^* - v_1^*\right) + \lambda\left[\varphi(v_2^*) - \varphi(v_1^*)\right] + (1+\lambda)\frac{\rho}{\pi}\frac{1-F(v_1^*)}{f(v_2^*)} - \frac{\lambda+\rho+\lambda\rho}{f(v_1^*)} \\
<&(1+\lambda)(1+\rho)\left[\varphi\left(\hat{v}(S,b_1,\pi) - b_1\right)\right] + (1+\lambda)\frac{\rho}{\pi}\frac{1-F(v_1^*)}{f(v_2^*)} - \frac{(1+\lambda)\rho}{f(v_1^*)} \\
\leq&(1+\lambda)\left\{(1+\rho)\left[\varphi\left(\hat{v}(S,b_1,\pi)\right) - b_1\right] + \frac{\rho}{\pi}\frac{1-F\left(b_1\right)}{f\left(\hat{v}(S,b_1,\pi)\right)} - \frac{\rho}{f\left(b_1\right)}\right\}.
\end{aligned}$$

Note that the term in the braces is strictly decreasing in b_1 and converges to $-\rho/f(\overline{v}) < 0$ as b_1 goes to $\overline{v}$. Hence, there exists $\hat{b}_1$ such that $u_1^* = u_2^*$ if $b_1 > \hat{b}_1$. ■

E Extensions and Discussions

E.1 Per-unit Price Constraint

Proof of Theorem 4. The proof of Theorem 1 can easily modified to prove Theorem 4. It suffices to show that (a^*, p^*) satisfies (PC) (instead of (BC)):

$$\begin{aligned}
p^*(\overline{v}, b) =&\overline{v}a(\overline{v}, b) - \int_{\underline{v}}^{\overline{v}} a^*(v,b)\mathrm{d}v - u(\underline{v}, b) \\
\leq&\overline{v}a(\overline{v}, b) - \int_{\underline{v}}^{\overline{v}} a(v,b)\mathrm{d}v - u(\underline{v}, b) \\
\leq&a(\overline{v}, b)b \\
=&a^*(\overline{v}, b)b,
\end{aligned}$$

where the third line holds by the same argument used in the proof of Theorem 1 and the last line holds since $a^*(\overline{v}, b) = a(\overline{v}, b)$ by construction. Hence, there exists v_1^* and v_2^* such that the optimal allocation rule satisfies $a(v, b_1) = \chi_{\{v \geq v_1^*\}} \min\left\{\frac{u^*}{v_1^* - b_1}, 1\right\}$ and $a(v, b_2) = \chi_{\{v \geq v_2^*\}}$. ■

Lemma 15 *Suppose Assumption 2 holds, and the principal does not inspect agents. In an optimal mechanism of $\mathcal{P}'_{PC}$, it is without loss of generality to assume that $u(\underline{v}, b_1) = u(\underline{v}, b_2)$.*

Proof. The proof of Lemma 1 can easily modified to prove Lemma 15. It suffices to show that (a^*, p^*) satisfies (PC) (instead of (BC)). Note that $a^*(\overline{v}, b) = a(\overline{v}, b)$ by construction and the rest of the proof follows from a similar argument used in the proof of Theorem 1. ■

Lemma 16 *Suppose Assumptions 1 and 2 hold, and the principal does not inspect agents. In an optimal mechanism of $\mathcal{P}'_{PC}$, the allocation rule satisfies*

$$\int_{\underline{v}}^{v} a(v, b_2) f(v) \mathrm{d}v \geq \int_{\underline{v}}^{v} a(v, b_1) f(v) \mathrm{d}v, \ \forall v. \tag{49}$$

Proof. The proof of Lemma 2 applies. ■

Proof of Theorem 5. The proof of Theorem 2 can easily modified to prove Theorem 5. It suffices to show that (a^*, p^*) satisfies (PC) (instead of (BC)). Note that $a^*(\overline{v}, b) = a(\overline{v}, b)$ by construction and the rest of the proof follows from a similar argument used in the proof of Theorem 1. ■

E.2 Monetary Penalty

Proof of Lemma 8. Consider types $t := (v, b)$ and $\hat{t}$ such that $p(\hat{t}) + \max\{\theta(\hat{t}, n), \theta(\hat{t}, b)\} \leq b$. Then (IC) requires that

$$\begin{aligned} & a(t)v - p(t) - (1 - q(t))\theta(t, n) - q(t)\theta(t, b) \\ \geq & a(\hat{t})v - p(\hat{t}) - (1 - q(\hat{t}))\theta(\hat{t}, n) - q(\hat{t})\theta(\hat{t}, b) \end{aligned}$$

Consider an alternative mechanism $(a^*, p^*, q^*, \theta^*)$ with $a^* = a$ and $q^* = q$. Let $\theta^*(t, n) = \theta^*(t, b) = 0$ for all t and $\theta^*(\hat{t}, b) = c$ for all $\hat{t}$ such that $\hat{b} \neq b$. Let $p^*(t) = p(t) + (1 - q(t))\theta(t, n) + q(t)\theta(t, b)$. Since $p(t) + \max\{\theta(t, n), \theta(t, b)\} \leq b$, we have $p^*(t) \leq b$, i.e., (BC) holds. It is easy to see that the new mechanism also satisfies (IR), (BB) and (S) and does not affect the welfare.

Finally, I show that (IC) holds. Consider types $t := (v, b)$ and $\hat{t}$ such that $p^*(\hat{t}) + c \leq b$. If $\hat{b} = b$, then (BC) in the old mechanism implies that $p(\hat{t}) + \max\{\theta(\hat{t}, n), \theta(\hat{t}, b)\} \leq b$. Hence,

$$
\begin{aligned}
& a^*(t)v - p^*(t) \\
=& a(t)v - p(t) - (1 - q(t))\theta(t, n) - q(t)\theta(t, b) \\
\geq& a(\hat{t})v - p(\hat{t}) - (1 - q(\hat{t}))\theta(\hat{t}, n) - q(\hat{t})\theta(\hat{t}, b) \\
=& a^*(\hat{t})v - p^*(\hat{t}) + q(\hat{t})\theta(\hat{t}, \hat{b}) - q(\hat{t})\theta(\hat{t}, b) \\
=& a^*(\hat{t})v - p^*(\hat{t}).
\end{aligned}
$$

If $\hat{b} \neq b$, then $b \geq p^*(\hat{t}) + c = p(\hat{t}) + (1 - q(\hat{t}))\theta(\hat{t}, n) + q(\hat{t})\theta(\hat{t}, \hat{b}) + c \geq p(\hat{t}) + \max\{\theta(\hat{t}, n), \theta(\hat{t}, b)\}$. Hence,

$$
\begin{aligned}
& a^*(t)v - p^*(t) \\
=& a(t)v - p(t) - (1 - q(t))\theta(t, n) - q(t)\theta(t, b) \\
\geq& a(\hat{t})v - p(\hat{t}) - (1 - q(\hat{t}))\theta(\hat{t}, n) - q(\hat{t})\theta(\hat{t}, b) \\
=& a^*(\hat{t})v - p^*(\hat{t}) + q(\hat{t})\theta(\hat{t}, \hat{b}) - q(\hat{t})\theta(\hat{t}, b) \\
\geq& a^*(\hat{t})v - p^*(\hat{t}) - q^*(\hat{t})\theta^*(\hat{t}, b).
\end{aligned}
$$

The last inequality holds since $\theta(\hat{t}, \hat{b}) \geq 0$ and $\theta^*(\hat{t}, b) = c \geq \theta(\hat{t}, b)$. ■

E.3 Punishing the Innocent or without Verification

Lemma 17 *An optimal mechanism of $\mathcal{P}_{PI}$ satisfies (i) $\theta(t, \hat{b}) = 1$ for $\hat{b} \neq b$, (ii) $p(t) < b$ implies that $\theta(t, n) = \theta(t, b) = 0$ and (iii) $(1 - \theta(t, n))\theta(t, b) = 0$ for almost all t.*

Proof. By the standard argument, (IC-v) implies that a is non-decreasing and

$$
p(t) + (1 - q(t))\theta(t, n)c + q(t)\theta(t, b)c = a(t)v - \int_0^v a(\nu, b)\mathrm{d}\nu - u(0, b).
$$

Consider a new mechanism $(a^*, p^*, q^*, \theta^*)$. Let $a^* = a$. Thus, (S) holds. Let $\theta^*(\hat{t}, b) = 1$ and

$q^*(\hat{t}) = q(\hat{t})\theta(\hat{t}, b)$ for $b \neq \hat{b}$. Let $p^*(t) = \min\{p(t) + (1 - q(t))\theta(t, n)c + q(t)\theta(t, b)c, b\}$. Thus, (BC) holds. Since $p^*(t) \geq p(t)$ and $q^*(t) \leq q(t)$, (BB) holds. Let $\theta^*(t, n) = 0$ if $q^*(t) = 1$ and otherwise

$$\theta^*(t, n) = \min\left\{\frac{p(t) + (1 - q(t))\theta(t, n)c + q(t)\theta(t, b)c - p^*(t)}{(1 - q^*(t))c}, 1\right\}.$$

Finally, let $\theta^*(t, b) = 0$ if $q^*(t) = 0$ and otherwise

$$\theta^*(t, b) = \frac{p(t) + (1 - q(t))\theta(t, n)c + q(t)\theta(t, b)c - p^*(t) - (1 - q^*(t))\theta^*(t, n)c}{q^*(t)c} \geq 0.$$

Then, $\theta^*(t, b) > 0$ if and only if $\theta^*(t, n) = 1$. Furthermore, $p^*(t) \geq p(t)$. Hence, $\theta^*(t, b) > 0$ implies that

$$\theta^*(t, b) \leq \frac{(1 - q(t))\theta(t, n) + q(t)\theta(t, b) - 1 + q^*(t)}{q^*(t)} \leq 1.$$

Note also that, by construction, $p(t) < b$ implies that $\theta^*(t, n) = \theta^*(t, b) = 0$. By construction,

$$p(t) + (1 - q(t))\theta(t, n)c + q(t)\theta(t, b)c = p^*(t) + (1 - q^*(t))\theta^*(t, n)c + q^*(t)\theta^*(t, b)c.$$

Hence (IR) holds. Consider a type $t = (v, b)$ and $\hat{t}$ such that $p(\hat{t}) \leq p^*(\hat{t}) \leq b$. Then

$$\begin{aligned}
& a^*(t)v - p^*(t) - (1 - q^*(t))\theta^*(t, n)c - q^*(t)\theta^*(t, b)c \\
=& a(t)v - p(t) - (1 - q(t))\theta(t, n)c - q(t)\theta(t, b)c \\
\geq& a(\hat{t})v - p(\hat{t}) - (1 - q(\hat{t}))\theta(\hat{t}, n)c - q(\hat{t})\theta(\hat{t}, b)c \\
\geq& a^*(\hat{t})v - p^*(\hat{t}) - (1 - q^*(\hat{t}))\theta^*(\hat{t}, n)c - q^*(\hat{t})c.
\end{aligned}$$

If $\hat{b} = b$, the last inequality holds trivially. If $\hat{b} \neq b$, the last inequality holds since $p^*(\hat{t}) + (1 - q^*(\hat{t}))\theta(\hat{t}, n)c \geq p(\hat{t}) + (1 - q(\hat{t}))\theta(\hat{t}, n)c$ and $q^*(\hat{t}) = q(\hat{t})\theta(\hat{t}, b)$. Hence, (IC) holds. Thus, we have verified that $(a^*, p^*, q^*, \theta^*)$ is feasible. Since $p^*(t) \geq p(t)$, we have $(1 - q(t))\theta(t, n)c + q(t)\theta(t, b)c \geq (1 - q^*(t))\theta^*(t, n)c + q^*(t)\theta^*(t, b)c$. Furthermore, $q^*(t) \leq q(t)$. For a positive measure set of t, one of the above two inequalities holds strictly. Hence, $(a^*, p^*, q^*, \theta^*)$ strictly improves welfare. ■

Lemma 18 *An optimal mechanism of $\mathcal{P}_{PI}$ satisfies $\theta(t,b)=0$ for almost all t.*

Proof. Fix a mechanism (a,p,q,θ). Suppose $\theta(t,b)>0$ on a positive measure set of t. Consider a new mechanism (a^*,p^*,q^*,θ^*) with $a^*=a$ and $p^*=p$. If $\theta(t,b)>0$, let $q^*(t)=q(t)[1-\theta(t,b)]<q(t)$, $\theta^*(t,b)=0$ and

$$\theta^*(t,n)=\frac{(1-q(t))\theta(t,n)+q(t)\theta(t,b)}{1-q^*(t)}=1.$$

If $\theta(t,b)=0$, let $q^*(t)=q(t)$, $\theta^*(t,b)=0$ and $\theta^*(t,n)=\theta(t,n)$. By construction,

$$(1-q(t))\theta(t,n)c+q(t)\theta(t,b)c=(1-q^*(t))\theta^*(t,n)c+q^*(t)\theta^*(t,b)c.$$

Clearly, the new mechanism satisfies (IR), (BC), (BB) and (S) and strictly improves welfare. Consider types $t=(v,b)$ and $\hat{t}$ such that $p(\hat{t})=p^*(\hat{t})\leq b$. Then

$$\begin{aligned}
&a^*(t)v-p^*(t)-(1-q^*(t))\theta^*(t,n)c-q^*(t)\theta^*(t,b)c\\
=&a(t)v-p(t)-(1-q(t))\theta(t,n)c-q(t)\theta(t,b)c\\
\geq&a(\hat{t})v-p(\hat{t})-(1-q(\hat{t}))\theta(\hat{t},n)c-q(\hat{t})c\\
=&a^*(\hat{t})v-p^*(\hat{t})-(1-q^*(\hat{t}))\theta^*(\hat{t},n)c-q^*(\hat{t})c.
\end{aligned}$$

If $\theta(\hat{t},\hat{b})=0$, the last equality holds trivially. If $\theta(\hat{t},\hat{b})>0$, the last equality holds since $\theta(\hat{t},n)=\theta^*(\hat{t},n)=1$. ■