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Mechanism Design with Costly Verification and Limited Punishments

by

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Abstract

A principal allocates an object among a finite number of agents, each of whom values the object. Each agent has access to private information about the principal's payoff if he receives the object. There are no monetary transfers. The object is allocated based on the agents' reports. The principal can inspect agents' reports at a cost and punish them, but punishments are sufficiently limited because verification is imperfect or information arrives only after the object has been allocated for a set period of time. If the number of agents is small, a threshold exists such that all agents whose values are below the threshold are pooled and an optimal allocation rule is efficient at the top of value distributions. If the number of agents is large, an optimal allocation rule also involves pooling at the top. If the number of agents is sufficiently large, the pooling areas at the bottom and the top meet and an optimal mechanism can be implemented via a shortlisting procedure. The fact that optimal mechanisms depend on the number of agents implies that small and large organizations should behave differently.

Keywords: Mechanism Design, Costly Verification, Limited Punishments

JEL Classification: D82

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1 Introduction

In many large organizations, scarce resources must be allocated internally without the benefit of prices. Examples include the head of personnel for an organization choosing one of several applicants for a job, venture capital firms choosing which startup to fund and funding agencies allocating a grant among researchers. In these settings, the principal must rely on the verification of agents' claims, which can be costly. For example, the head of personnel can confirm a job applicant's past work experience or monitor their performance once they are hired. A venture capital firm can investigate competing startups or audit the progress of a startup once it is funded. Furthermore, the principal can punish an agent if his claim is found to be false. For example, the head of personnel can reject an applicant, fire an employee or deny a promotion. Venture capitals and funding agencies can cut off funding.

Prior work has examined two extreme cases. In Ben-Porath et al. (2014), verification is costly but punishment is large enough in the sense that an agent can be rejected and does not receive the resource. In Mylovanov and Zapechelnyuk (2017), verification is free but punishment is sufficiently limited. In this paper, I consider a situation with both costly verification *and* sufficiently limited punishment. I interpret verification as acquiring information (such as by requesting documentation, interviewing an agent or monitoring an agent at work), which could be costly. Moreover, punishment can be sufficiently limited because verification is imperfect or information arrives only after an agent has been hired for some time.

This paper has three main contributions. Firstly, despite of the similarity between the problems they study, Ben-Porath et al. (2014) and Mylovanov and Zapechelnyuk (2017) obtain different sets of optimal mechanisms. The first contribution of this paper is to establish a connection between Ben-Porath et al. (2014) and Mylovanov and Zapechelnyuk (2017) in a symmetric environment, and highlights the role played by the number of agents. In the concluding section, I provide a more detailed comparison of the results in this paper with those in previous papers regarding the role played by the number of agents. Secondly, in practice, the principal can often take actions that affect verification cost and punishment level simultaneously. For example, the principal can obtain more precise information by incurring a higher information acquisition cost, which leads in turn to a higher expected punishment. Thus, it is important to understand the interactions between verification cost and punishment level. In the paper, I show that the impact of a change in punishment level on optimal mechanisms is qualitatively different from that in Mylovanov and Zapechelnyuk (2017) in which verification is free. This highlights the importance of considering both costly verification and sufficiently limited punishment. Thirdly, I provide results on optimal mechanisms in a general asymmetric environment. Some of the comparative statics results I obtain are different from those in Ben-Porath et al. (2014), which again highlights the importance of considering both costly verification and sufficiently limited punishment. This analysis also extends that of Mylovanov and Zapechelnyuk (2017), who consider only the symmetric environment.

Specifically, in the model, there is one principal who has to allocate one indivisible object among a finite number of agents. She would like to give the object to the agent who has the highest value to her, but doing this encourages all agents to exaggerate their values. At her disposal, the principal has two devices to discourage agents from exaggeration: firstly, the principal can ration at the bottom or top of the distribution of values, but this reduces allocative efficiency; secondly, the principal can verify an agent's claim and punish him if his claim is found to be false, but verification is a costly procedure. The goal of this paper is to identify the optimal way to provide incentives via these two devices.

In Sections 3 and 4, I focus on the symmetric environment and characterize an optimal symmetric mechanism in this setting. If the number of agents is sufficiently small, then a *one-threshold mechanism* as in Ben-Porath et al. (2014) is optimal. The allocation rule in this mechanism is efficient at the top of the value distribution and involves pooling only at the bottom. For intermediate and large numbers of agents, the allocation rule involves pooling at both the top and the bottom as in Mylovanov and Zapechelnyuk (2017). Specifically, the following *two-threshold mechanism* is optimal. If there are agents whose values are above the upper threshold, then one of them is chosen at random and inspected with probability one. If all agents' values are below the upper threshold but some are above the lower threshold, then the one with the highest value is chosen and inspected with some probability. If all agents' values are below the lower threshold, then one of them is chosen at random and no one is inspected. It should be noted that a one-threshold mechanism can be viewed as a two-threshold mechanism whose upper threshold is equal to the upper-bound of the value support. For a sufficiently large number of agents, the two thresholds coincide, and the two-threshold mechanism can be implemented using a *shortlisting procedure*. In this shortlisting procedure, agents whose values are above a threshold are shortlisted with probability one, and agents whose values are below the threshold are shortlisted with some probability. The principal then chooses one agent from the shortlist at random. The selected agent is inspected if and only if his value is above the threshold. The fact that the optimal mechanism depends on the number of agents implies that small and large organizations should behave differently.

To understand the intuition behind these results, consider an agent with the lowest possible value to the principal. Intuitively, as the number of agents increases, this agent becomes worse off and has stronger incentives to exaggerate his value in a one-threshold mechanism as it is now more likely that another agent whose value is above the threshold exists. When punishments are sufficiently limited, the principal can make exaggeration less attractive only by introducing distortions to the allocation rule at the top of the value distribution.

This distinction between small and intermediate numbers of agents is important because it allows us to establish a connection between Ben-Porath et al. (2014) and Mylovanov and Zapechelnyuk (2017). Note that this distinction is *absent* if either verification is free or punishment is large enough. In Ben-Porath et al. (2014), an optimal mechanism never involves pooling at the top of the value distribution because punishment is large enough. If punishment is sufficiently limited, then pooling at the top is part of the optimal mechanism for a sufficiently large number of agents. In Mylovanov and Zapechelnyuk (2017), an optimal mechanism always involves pooling at the top because verification is free. If verification is costly, then pooling at the top disappears for a sufficiently small number of agents.

As an effort to understand the trade-off between verification (or information) cost and punishment level (or information quality), I provide some comparative statics results with respect to verification cost and punishment level in Section 4. An increase in verification cost has two opposite effects on the size of the pooling areas. Firstly, when verification becomes costlier, the optimal threshold mechanism involves more pooling at the bottom to save verification cost. Secondly, the enlarging pooling area at the bottom benefits agents with very low values and reduces their incentives to exaggerate their values, which leads to less or no pooling at the top. In the paper, I show that the second effect dominates and that one-threshold or two-threshold mechanisms consequently remain optimal for a larger number of agents as verification becomes costlier.

The impact of a change in punishment level is ambiguous and more interesting. On the one hand, a reduction in punishment effectively makes verification costlier as the principal must inspect agents more frequently to maintain incentive compatibility. Then the above analysis implies that one-threshold or two-threshold mechanisms remain optimal for a larger number of agents as punishment becomes less severe. On the other hand, a reduction in punishment level makes it more difficult to prevent agents from exaggeration through punishments, which leads to larger pooling areas at both the bottom and the top to restore incentive compatibility. This in turn implies that one-threshold or two-threshold mechanisms remain optimal for a smaller number of agents as punishment becomes less severe. In general, the impact of a change in punishment level is not monotonic. In particular, the top pooling area can either increase or decrease as the penalty decreases. This result is in contrast to the findings in Mylovanov and Zapechelnyuk (2017). In Mylovanov and Zapechelnyuk (2017), the top pooling area increases as the penalty decreases because the first effect is absent if verification is free.[1]

In Section 5, I study a general model with asymmetric agents. In this setting, threshold mechanisms remain optimal. The analysis, however, is much more complex. Although there is still a unique lower threshold for all agents, different agents may face different upper thresholds. This analysis also extends that of Mylovanov and Zapechelnyuk (2017), who consider only the symmetric environment, to the asymmetric environment.

Because of the complication of the pooling areas at the top, it is generally extremely difficult to fully characterize the set of optimal mechanisms. Thus, I provide results only for some important special cases. In Section 5.1, I study the environment in which different agents may have different value distributions, but they are identical otherwise. For simplicity, assume that there are two groups of agents: H and L. I consider two cases. First, I assume that the value distribution of group H agents are "better" than that of group L agents in the sense of first-order stochastic dominance. If the number of agents is small, then a one-threshold mechanism in which group H agents are favored is optimal. Group H agents are favored in the sense that if all agents' reported values are below the threshold, then a group H agents is more likely to be selected. If the number of agents is large, then it is optimal for the principal to ignore group L agents and chooses an optimal mechanism as if she faces only group H agents. One surprising implication of this result is that group H agents can actually get better off when the number of competitors increases. This is true even if additional competitor all come from group H.

Second, I consider the case in which the value distribution of group L agents are "more risky" than that of group H agents in the sense of mean-preserving spread. If the number of agents is small, then a one-threshold mechanism in which group H agents are favored is optimal. This result is consistent with that of Ben-Porath et al. (2014). Intuitively, the principal benefits less from checking an agent if there is less uncertainty about his type. However, in contrast to Ben-Porath et al. (2014), if the number of agents is sufficiently large, a shortlisting procedure is optimal and the "more risky" group L is favored. This is because in this case if a group is favored, the agents in that group are more likely to be selected both when their values are above and below the threshold. Since it is more likely to select an agent with high type from the right tail if his value distribution is "more risky", group L is favored.

In Section 5.2, I revisit the symmetric environment and characterize the set of all optimal threshold mechanisms. First, I show that in an optimal mechanism all agents must share the same upper threshold. This result is consistent with Mylovanov and Zapechelnyuk (2017). Second, I find that limiting the principal's ability to punish agents also limits her ability to

[1] See Proposition 5A in Mylovanov and Zapechelnyuk (2017).

treat agents differently. In particular, when a one-threshold mechanism is optimal, the set of all optimal threshold mechanisms shrinks as punishment becomes more limited. Eventually, the unique optimal threshold mechanism is symmetric. If punishment is sufficiently limited so that a two-threshold mechanism or a shortlisting procedure is optimal, then the principal can once again treat agents differently, although only to the extent that they share the same set of thresholds. The comparison is less clear in this case because the sets of optimal mechanisms are disjoint for different levels of punishments.

Technically, I follow Vohra (2012) and use tools from linear programming, which allows me to analyze Ben-Porath et al. (2014) and Mylovanov and Zapechelnyuk (2017) in a unified framework. It also allows me to obtain results on optimal mechanisms in the asymmetric environment with sufficiently limited punishments, which are unavailable in Mylovanov and Zapechelnyuk (2017).

The rest of the paper is organized as follows. Section 1.1 discusses other related work. Section 2 presents the model. Section 3 characterizes an optimal symmetric mechanism when agents are ex ante identical. Section 4 discusses the properties of this optimal symmetric mechanism. Section 5 studies a general asymmetric environment. Finally, Section 6 concludes the paper.

1.1 Other related literature

This paper is related to the literature on costly state verification. The first contribution in the series is Townsend (1979), who has studied a model of a principal and a single agent. In Townsend (1979), verification is deterministic. Border and Sobel (1987) and Mookherjee and Png (1989) have generalized it by allowing random inspection. Gale and Hellwig (1985) have considered the effects of costly verification in the context of credit markets. These models differ from what I consider here in that in their models there is only one agent and monetary transfers are allowed. Recently, Patel and Urgan (2017) have also studied the problem of a principal who must allocate a good among multiple agents when transfers are not allowed. As in Ben-Porath et al. (2014), in Patel and Urgan (2017), verification is costly and punishment is large enough. But, in addition to costly verification, the principal can deploy another instrument: money burning. They have shown that both instruments are present in the optimal mechanism. Furthermore, the optimal mechanism admits monotonicity in the allocation probability with regards to an agent's value, and takes a threshold form where all the values below a certain threshold are not subject to verification or money burning.

Technically, this paper is related to the literature on reduced form implementation, including Maskin and Riley (1984), Matthews (1984), Border (1991) and Mierendorff (2011).

The most related paper is that of Pai and Vohra (2014), who have also used reduced form implementation and linear programming to derive optimal mechanisms for financially constrained agents.

2 Model

The set of agents is $\mathcal{I} := \{1, \ldots, n\}$. There is a single indivisible object to be allocated among them. The value of the principal of assigning the object to agent i is v_i, where v_i is agent i's private information. I assume $\{v_i\}$ to be independently distributed and that their density f_i is strictly positive on $V_i := [\underline{v}_i, \overline{v}_i] \subset \mathbb{R}_+$. The assumption that an agent's value to the principal is always non-negative simplifies some statements, but the results in this paper can easily extend to include negative values. I use F_i to denote the corresponding cumulative distribution function. Let $\mathcal{V} := \prod_i V_i$. Agent i gets a private benefit of $b_i(v_i)$ if he receives the object, and 0 otherwise. The principal can verify agent i's report at a cost $k_i \geq 0$. Verification perfectly reveals an agent's type. The cost to an agent to have his report verified is zero. If agent i receives the object and is inspected, the principal can impose a non-negative penalty $c_i(v_i)$ on him.[2] Note that an agent can be punished only if he receives the object. The interpretation of this assumption is that the principal can only punish an agent by taking the object back, possibly after a number of periods (e.g., rejecting a job applicant or firing him after a certain length of employment).

For tractability, I assume that penalty is linear in private benefit: $c_i(v_i) = c_i b_i(v_i)$ for all v_i for some $0 < c_i \leq 1$. This assumption is natural in some applications. In the job slot example, this assumption is satisfied if an agent receives a private benefit for each period he is employed and the penalty is being fired after a pre-specified number of periods. In the example of venture capital firms or funding agencies, this assumption is satisfied if agents receive funds periodically and the penalty is cutting off funding after certain periods. Furthermore, this assumption allows us to obtain a clear analysis on the interaction between the verification cost (k_i) and the level of punishment (c_i). Lastly, this assumption can be relaxed, and the results in this paper can easily extend if $c_i(v_i)/b_i(v_i)$ is minimized at $\underline{v}_i$.[3]

I invoke the Revelation Principle and focus on direct mechanisms in which truth-telling

[2] I will use the words "verify" and "inspect" interchangeably in this paper.

[3] (IC) can be rewritten as: for each agent i,

$$Q_i(v_i') \geq \frac{b_i(v_i)}{c_i(v_i)} \left(1 - \frac{P_i(v_i)}{P_i(v_i')}\right), \forall v_i, v_i'.$$

Suppose that $c_i(v_i)/b_i(v_i)$ is minimized at $\underline{v}_i$ and $P_i(v_i)$ is non-decreasing. Then, for any given v_i', the left-hand side of the above inequality is maximized at $\underline{v}_i$. If redefining $c_i := c_i(\underline{v}_i)/b_i(\underline{v}_i)$, then (IC) hold if and only if (2) holds.

is a Bayes-Nash equilibrium. Clearly, if an agent is inspected, it is optimal to penalize him if and only if he is found to have lied. Using this result, a direct mechanism can be written as a pair $(\boldsymbol{p}, \boldsymbol{q})$, where $\boldsymbol{p} := (p_1, \dots, p_n) : \mathcal{V} \to [0,1]^n$ and $\boldsymbol{q} := (q_1, \dots, q_n) : \mathcal{V} \to [0,1]^n$. For each i and each profile of reported values, $\boldsymbol{v} \in \mathcal{V}$, $p_i(\boldsymbol{v})$ specifies the probability with which i is assigned the object, and $q_i(\boldsymbol{v})$ specifies the probability of inspecting i conditional on the object being assigned to agent i. The utility of an agent whose true type is v_i and who reports v_i' is $p_i(v_i, v_{-i})b_i(v_i)$ if $v_i' = v_i$, and it is

$$p_i(v_i', v_{-i}) \left(b_i(v_i) - q_i(v_i', v_{-i})c_i(v_i)\right)$$

otherwise. A mechanism is *feasible* if $\sum_i p_i(\boldsymbol{v}) \leq 1$ for all $\boldsymbol{v} \in \mathcal{V}$. A mechanism satisfies the *incentive compatibility* (IC) constraints if, for each agent i,

$$\mathbb{E}_{v_{-i}} \left[p_i(v_i, v_{-i})b_i(v_i)\right] \geq \mathbb{E}_{v_{-i}} \left[p_i(v_i', v_{-i}) \left(b_i(v_i) - q_i(v_i', v_{-i})c_i(v_i)\right)\right], \forall v_i, v_i'.$$

The principal's objective is to maximize her expected gain from allocating the object minus the expected verification cost,

$$\mathbb{E}_{\boldsymbol{v}} \left[\sum_{i=1}^{n} p_i(\boldsymbol{v}) \left(v_i - q_i(\boldsymbol{v})k_i\right)\right], \tag{1}$$

subject to the feasibility and IC constraints.

Note that if $k_i = 0$, then the above principal's problem reduces to that considered in Mylovanov and Zapechelnyuk (2017); and if $c_i(v_i) = b_i(v_i)$ for all v_i, then it reduces to that considered in Ben-Porath et al. (2014).

For each agent i and each $v_i \in V_i$, let $P_i(v_i) := \mathbb{E}_{v_{-i}} \left[p_i(v_i, v_{-i})\right]$ denote the interim probability with which agent i is assigned the object. If $P_i(v_i) \neq 0$, then let $Q_i(v_i) := \mathbb{E}_{v_{-i}} \left[p_i(v_i, v_{-i})q_i(v_i, v_{-i})\right] / P_i(v_i)$; and otherwise let $Q_i(v_i) := 0$. Then $P_i(v_i)Q_i(v_i)$ is the interim probability with which agent i is inspected. Let $\boldsymbol{P} := (P_1, \dots, P_n)$ and $\boldsymbol{Q} := (Q_1, \dots, Q_n)$. The principal's problem can then be written in the following reduced form:

$$\max_{\boldsymbol{P}, \boldsymbol{Q}} \sum_{i=1}^{n} \mathbb{E}_{v_i} \left[P_i(v_i) \left(v_i - Q_i(v_i)k_i\right)\right],$$

subject to

$$P_i(v_i)b_i(v_i) \geq P_i(v_i')\left(b_i(v_i) - Q_i(v_i')c_i(v_i)\right), \forall v_i, v_i', \tag{IC}$$

$$0 \leq Q_i(v_i) \leq 1, \forall v_i, \tag{F1}$$

$$\sum_i \int_{S_i} P_i(v_i)\mathrm{d}F_i(v_i) \leq 1 - \prod_i \left(1 - \int_{S_i} dF_i(v_i)\right), \forall S_i \subset V_i. \tag{AF2}$$

In particular, an allocate rule $\boldsymbol{p}$ is feasible if and only if the corresponding reduced form allocation rule $\boldsymbol{P}$ satisfies (AF2) by Theorem 2 in Mierendorff (2011), which generalizes the well-known Maskin-Riley-Matthews-Border conditions to asymmetric environments.

I begin solving the principal's problem by solving for the optimal $\boldsymbol{Q}$ for a given $\boldsymbol{P}$. Because penalty, $c_i(v_i)$, is linear in private benefit, $b_i(v_i)$, (IC) becomes $P_i(v_i) \geq P_i(v_i')\left(1 - c_i Q_i(v_i')\right)$ for all v_i and v_i'. Then (IC) holds if and only if

$$\varphi_i \geq P_i(v_i')\left(1 - c_i Q_i(v_i')\right), \forall v_i', \tag{2}$$

where $\varphi_i := \inf_{v_i} P_i(v_i)$ is agent i's lowest interim probability of receiving the object. Because $Q_i(v_i') \leq 1$, (2) holds only if

$$(1 - c_i)P_i(v_i') \leq \varphi_i, \forall v_i'. \tag{3}$$

Remark 1 *Note that if $c_i = 1$, as in Ben-Porath et al. (2014), then (3) is satisfied automatically. This explains why there is no pooling at the top of the value distribution in Ben-Porath et al. (2014). In contrast, if $0 < c_i < 1$, then (3) imposes an upper-bound on P_i and, as I demonstrate later, there can be pooling at the top for a sufficiently large number of agents.*

For the rest of the paper, I assume that punishment is sufficiently limited, i.e. $0 < c_i < 1$ for some i. This is to say that the principal cannot reduce agent i's payoff to his outside option by punishing him. If we interpret verification as acquiring information, then punishment can be sufficiently limited because information is imperfect.

If (3) holds, then it is optimal to set $Q_i(v_i) = \left(1 - \varphi_i / P_i(v_i)\right)/c_i$ for all $v_i \in V_i$ since the principal's objective function is strictly decreasing in Q_i. Substituting this into the principal's objective function results in

$$\sum_{i=1}^{n} \mathbb{E}_{v_i}\left[P_i(v_i)\left(v_i - \frac{k_i}{c_i}\right)\right] + \frac{\varphi_i k_i}{c_i}. \tag{4}$$

In Sections 3 and 4, I assume $\{v_i\}$ to be identically distributed and that their density f is strictly positive on $V = [\underline{v}, \overline{v}] \subset \mathbb{R}_+$. I use F to denote the corresponding cumulative distribution function. In addition, I assume $c_i = c$ and $k_i = k$ for all i. In this symmetric setting, there exists an optimal mechanism that is symmetric. Hence, I focus on symmetric mechanisms in Sections 3 and 4. In what follows, I suppress the subscript i whenever the meaning is clear. The results of Section 3 can be extended to environments in which the values (v_i) of different agents can follow different distributions (F_i), and both the punishments (c_i) and the verification costs (k_i) can be different for different agents. I discuss this general asymmetric setting in Section 5.

3 Optimal mechanisms

In this section, I demonstrate that a simple threshold mechanism is optimal. As an overview of the proof idea, I solve the principal's problem in two steps. In the first step, I characterize an optimal mechanism for any given lowest probability with which an agent receives the object (φ). In the second step, I solve for the optimal φ.

3.1 Optimal mechanisms for fixed φ

Fix $\varphi = \inf_v P(v) \leq 1/n$.[4] I first solve the following problem ($OPT - \varphi$):

$$\max_P \mathbb{E}_v \left[P(v) \left(v - \frac{k}{c} \right) \right] + \frac{\varphi k}{c},$$

subject to

$$\varphi \leq P(v) \leq \frac{\varphi}{1-c}, \forall v, \tag{IC$'$}$$

$$n \int_S P(v) \mathrm{d}F(v) \leq 1 - \left(1 - \int_S \mathrm{d}F(v) \right)^n, \forall S \subset V. \tag{F2}$$

In this symmetric setting, when mechanisms are symmetric, (AF2) can be simplifed to (F2). Recall that Q exists such that (F1) and (IC) hold if and only if P satisfies (IC$'$). To solve ($OPT - \varphi$), I approximate the continuum-type space with a finite partition, characterize an optimal mechanism in the finite model and take limits. Later, I show that the limiting mechanism is optimal in the original model.

[4]Note that the problem ($OPT - \varphi$) is feasible only if $\varphi \leq 1/n$.

3.1.1 Finite case

Fix an integer $m \geq 2$. For $t = 1, \ldots m$, let

$$v^t := \underline{v} + \frac{(2t-1)(\overline{v} - \underline{v})}{2m},$$
$$f^t := F\left(\underline{v} + \frac{t(\overline{v} - \underline{v})}{m}\right) - F\left(\underline{v} + \frac{(t-1)(\overline{v} - \underline{v})}{m}\right).$$

Consider the finite model in which v_i can take only m possible different values (i.e. $v_i \in \{v^1, \ldots, v^m\}$) and the probability mass function satisfies $f(v^t) = f^t$ for $t = 1, \ldots, m$. I slightly abuse notation and let $P := (P^1, \ldots, P^m)$, where P^t is the interim probability with which a type v^t agent is assigned the good. Then, the corresponding problem of $(OPT - \varphi)$ in the finite model, denoted by $(OPTm - \varphi)$, is given by

$$\max_P \sum_{t=1}^{m} f^t P^t \left(v^t - \frac{k}{c}\right) + \frac{\varphi k}{c},$$

subject to

$$\varphi \leq P^t \leq \frac{\varphi}{1-c}, \forall t, \tag{IC$'m$}$$

$$n \sum_{t \in S} f^t P^t \leq 1 - \left(\sum_{t \notin S} f^t\right)^n, \forall S \subset \{1, \ldots, m\}. \tag{F2m}$$

To solve $(OPTm - \varphi)$, I first rewrite it as a polymatroid optimization problem. Define $G(S) := 1 - \left(\sum_{t \notin S} f^t\right)^n$ and $H(S) := G(S) - n\varphi \sum_{t \in S} f^t$ for all $S \subset \{1, \ldots, m\}$. Define $z^t := f^t(P^t - \varphi)$ for all $t = 1, \ldots, m$ and $z := (z^1, \ldots, z^m)$. Clearly, $P^t \geq \varphi$ if and only if $z^t \geq 0$ for all $t = 1, \ldots, m$. Using these notations, (F2m) can be rewritten as

$$n \sum_{t \in S} z^t \leq H(S), \forall S \subset \{1, \ldots, m\}.$$

It is easy to verify that $H(\emptyset) = 0$ and H is submodular. However, H is not monotonic. Define $\overline{H}(S) := \min_{S' \supset S} H(S)$. Then $\overline{H}(\emptyset) = 0$, and $\overline{H}$ is non-decreasing and submodular. Furthermore, by Lemma 2 in Appendix A,

$$\left\{ z \,\middle|\, z \geq 0, n \sum_{t \in S} z^t \leq H(S), \forall S \right\} = \left\{ z \,\middle|\, z \geq 0, n \sum_{t \in S} z^t \leq \overline{H}(S), \forall S \right\}.$$

Thus, $(OPTm - \varphi)$ can be rewritten as $(OPTm1 - \varphi)$

$$\max_{z} \sum_{t=1}^{m} z^t \left(v^t - \frac{k}{c} \right) + \varphi \sum_{t=1}^{m} f^t v^t,$$

subject to

$$0 \le z^t \le \frac{c\varphi f^t}{1-c}, \forall t, \tag{IC$'m1$}$$

$$n \sum_{t \in S} z^t \le \overline{H}(S), \forall S \subset \{1, \ldots, m\}. \tag{F2$m1$}$$

Without the upper-bound on z^t in (IC$'m1$), this is a standard polymatroid optimization problem and can be solved using the greedy algorithm. With the upper-bound, this is a weighted polymatroid intersection problem and efficient algorithms exist that solve the optima if the weights $(v^t - k/c)$ are rational.[5] In this paper, I solve the problem using a "guess-and-verify" approach. Although we cannot directly apply the greedy algorithm to $(OPTm1 - \varphi)$, it is not difficult to conjecture the optimal solution. Intuitively, $z^t = 0$ if $v^t < k/c$. Consider $v^t \ge k/c$. Because $\overline{H}$ is non-decreasing and submodular, and the upper-bound on z^t is linear in f^t, the solution found by the greedy algorithm potentially violates the upper-bound for large t. Hence, it is natural to conjecture that a cutoff $\bar{t}$ exists such that the upper-bounds in (IC$'m1$) bind if and only if $t > \bar{t}$.

Formally, let $S^t := \{t, \ldots, m\}$ for all $t = 1, \ldots, m$, and $S^{m+1} := \emptyset$. If $\varphi \le (1-c)/n$, let $\bar{t} := 0$; otherwise, I show in the proof of Lemma 1 that a unique $\bar{t} \in \{1, \ldots, m+1\}$ exists such that

$$\overline{H}(S^{\bar{t}}) \le n \sum_{\tau=\bar{t}}^{m} \frac{c\varphi f^\tau}{1-c} \text{ and } \overline{H}(S^{\bar{t}+1}) > n \sum_{\tau=\bar{t}+1}^{m} \frac{c\varphi f^\tau}{1-c}.$$

Note that, by definition, if we assign the highest possible value allowed by (F2$m1$) to $\sum_{\tau=\bar{t}+1}^{m} z^\tau$, then (IC$'m1$) must be violated for some $t \ge \bar{t}+1$. However, it is possible to assign the highest possible value allowed by (F2$m1$) to $\sum_{\tau=\bar{t}}^{m} z^\tau$ while respecting (IC$'m1$) for all $t \ge \bar{t}$. Hence, it is natural to conjecture that $\bar{t}$ defined above is the cutoff above which the upper-bounds in (IC$'m1$) bind. I can now construct my candidate optimal solution of $(OPTm1 - \varphi)$ as follows

$$\hat{z}^t := \begin{cases} \overline{z}^t \text{ if } v^t \ge \frac{k}{c} \\ 0 \text{ if } v^t < \frac{k}{c} \end{cases}, \tag{5}$$

[5] See, for example, Cook et al. (2011) and Frank (2011).

where

$$\overline{z}^t := \begin{cases} \frac{c\varphi f^t}{1-c} & \text{if } t > \bar{t} \\ \frac{1}{n}\overline{H}(S^{\bar{t}}) - \sum_{\tau=\bar{t}+1}^{m} \frac{c\varphi f^t}{1-c} & \text{if } t = \bar{t} \\ \frac{1}{n}\left[\overline{H}(S^t) - \overline{H}(S^{t+1})\right] & \text{if } t < \bar{t} \end{cases},$$

As previously discussed, if $t > \bar{t}$ and $v^t - k/c > 0$, then I conjecture that the upper-bound in (IC′m1) binds and let $\hat{z}^t = c\varphi f^t/(1-c)$. If $t \leq \bar{t}$ and $v^t - k/c > 0$, then, in the spirit of greedy algorithms, I start by assigning the highest possible value allowed by (F2m1) to $\hat{z}^{\bar{t}}$ and continue to assign values to $\hat{z}^{\bar{t}-1}, \hat{z}^{\bar{t}-2}, \ldots$ in the same fashion. Finally, it is clear that if $v^t - k/c < 0$, then it is optimal to set $\hat{z}^t = 0$. $\hat{z}$ is feasible following from the fact that $\overline{H}(\emptyset) = 0$, and $\overline{H}$ is non-decreasing and submodular. Furthermore, we can verify the optimality of $\hat{z}$ by the duality theorem:

Lemma 1 *$\hat{z}$ defined in (5) is an optimal solution to ($OPTm1 - \varphi$).*

For each $t = 1, \ldots, m$, let

$$P_m^t := \frac{\hat{z}^t}{f^t} + \varphi \tag{6}$$

The following corollary directly follows from Lemma 1:

Corollary 1 *P_m defined in (6) is an optimal solution to ($OPTm - \varphi$).*

3.1.2 Continuum case

I characterize an optimal solution of ($OPT - \varphi$) by taking m to infinity. Let v^l be such that $F(v^l)^{n-1} = n\varphi$ and

$$v^u := \inf\left\{ v \,\middle|\, 1 - F(v)^n - \frac{n\varphi}{1-c}\left[1 - F(v)\right] \geq 0 \right\}. \tag{7}$$

v^l is chosen so that if all agents whose values are below v^l are pooled together and ranked below any other agents with higher values, their interim probability of receiving the object $F(v^l)^{n-1}/n$ is equal to the lower-bound in (IC′), φ. The definition of v^u mirrors that of $\bar{t}$. Informally, v^u is chosen so that if all agents whose values are above v^u are pooled together and ranked above any other agents with lower values, then their interim probability of receiving the object $[1-F(v)^n]/n[1-F(v)]$ is equal to the upper-bound in (IC′), $\varphi/(1-c)$. Note that if $\varphi \leq (1-c)/n$, then $v^u = \underline{v}$. Let $\overline{P}_\varphi$ be defined as follows: If $v^l < v^u$, let

$$\overline{P}_\varphi(v) := \begin{cases} \frac{\varphi}{1-c} & \text{if } v \geq v^u \\ F(v)^{n-1} & \text{if } v^l < v < v^u \\ \varphi & \text{if } v \leq v^l \end{cases}.$$

If $v^l \geq v^u$, let

$$\hat{v} := \inf \left\{ v \,\middle|\, 1 - n\varphi F(v) - \frac{n\varphi}{1-c}\left[1 - F(v)\right] \geq 0 \right\} \in [v^u, v^l], \tag{8}$$

and

$$\overline{P}_{\varphi}(v) := \begin{cases} \frac{\varphi}{1-c} & \text{if } v \geq \hat{v} \\ \varphi & \text{if } v < \hat{v} \end{cases}.$$

Finally, let

$$P^*_{\varphi}(v) := \begin{cases} \overline{P}(v) & \text{if } v \geq \frac{k}{c} \\ \varphi & \text{if } v < \frac{k}{c} \end{cases}. \tag{9}$$

I show in Appendix A that P^*_{φ} is the "pointwise limit" of P_m as $m \to \infty$. Moreover, P^*_{φ} is an optimal solution to $(OPT - \varphi)$.

Theorem 1 *P^*_{φ} defined in (9) is an optimal solution to $(OPT - \varphi)$.*

3.2 Optimal φ

I complete the characterization of an optimal mechanism by solving for the optimal φ. Firstly, if verification is sufficiently costly or the principal's ability to punish an agent is sufficiently limited, then pure randomization is optimal.

Theorem 2 *If $\overline{v} - k/c \leq \mathbb{E}_v[v]$, then pure randomization is optimal: $P^* = 1/n$ and $Q^* = 0$.*

To make the problem more interesting, in what follows I assume that

Assumption 1 $\overline{v} - k/c > \mathbb{E}_v[v]$.

Recall that given φ, v^l is uniquely pinned down by $F(v^l)^{n-1} = n\varphi$ and v^u is uniquely pinned down by (7). Define v^* and v^{**} by equations (10) and (11), respectively:

$$\mathbb{E}_v[v] - \mathbb{E}_v[\max\{v, v^*\}] + \frac{k}{c} = 0, \tag{10}$$

$$\mathbb{E}_v[v] - \mathbb{E}_v[\min\{v, v^{**}\}] + (1-c)\left[\mathbb{E}_v[v] - \mathbb{E}_v[\max\{v, v^{**}\}] + \frac{k}{c}\right] = 0. \tag{11}$$

They are well defined under Assumption 1. Furthermore, $v^{**} > v^* \geq k/c$. Finally, let

$$v^{\natural} := \sup \left\{ v \,\middle|\, \frac{F(v)^{n-1}(1 - F(v))}{1-c} - 1 + F(v)^n \leq 0 \right\}. \tag{12}$$

An optimal mechanism is characterized by the following theorem:

Theorem 3 *Supposing that Assumption 1 holds, there are three cases.*

1. *If $F(v^*)^{n-1} \geq n(1-c)$, then the optimal $\varphi^* = F(v^*)^{n-1}/n$, the optimal inspection rule satisfies $Q^* = (1 - \varphi^*/P^*)/c$ and the following allocation rule is optimal:*

$$P^*(v) := \begin{cases} F(v)^{n-1} & \text{if } v \geq v^* \\ \varphi^* & \text{if } v < v^* \end{cases}.$$

2. *If $F(v^*)^{n-1} < n(1-c)$ and $v^{**} \leq v^\natural$, then the optimal $\varphi^* = (1-c)/n(1-cF(v^{**}))$, the optimal inspection rule satisfies $Q^* = (1 - \varphi^*/P^*)/c$ and the following allocation rule is optimal:*

$$P^*(v) := \begin{cases} \frac{\varphi^*}{1-c} & \text{if } v \geq v^{**} \\ \varphi^* & \text{if } v < v^{**} \end{cases}.$$

3. *If $F(v^*)^{n-1} < n(1-c)$ and $v^{**} > v^\natural$, then the optimal φ^* is defined by*

$$\mathbb{E}_v[v] - \mathbb{E}_v[\min\{v, v^u(\varphi^*)\}] + (1-c)\left[\mathbb{E}_v[v] - \mathbb{E}_v[\max\{v, v^l(\varphi^*)\}] + \frac{k}{c}\right] = 0, \tag{13}$$

the optimal inspection rule satisfies $Q^* = (1 - \varphi^*/P^*)/c$ and the following allocation rule is optimal:

$$P^*(v) := \begin{cases} \frac{\varphi^*}{1-c} & \text{if } v \geq v^u(\varphi^*) \\ F(v)^{n-1} & \text{if } v^l(\varphi^*) < v < v^u(\varphi^*) \\ \varphi^* & \text{if } v \leq v^l(\varphi^*) \end{cases}.$$

To understand the result, consider the following implementation of the optimal mechanism in Theorem 3. There are two thresholds. I abuse notation here and denote them by v^l and v^u with $\underline{v} \leq v^l \leq v^u \leq \overline{v}$. If every agent reports a value below v^l, then an agent is selected uniformly at random and receives the good, and no one is inspected. If any agent reports a value above v^l but all reports are below v^u, then the agent with the highest reported value receives the good, is inspected with some probability (proportional to $1/c$) and is penalized if he is found to have lied. If any agent reports a value above v^u, then an agent is selected uniformly at random among all the agents whose reported values are above v^u, receives the good, is inspected with a probability of 1 and is penalized if he is found to have lied. I call a mechanism a one-threshold mechanism if $v^u = \overline{v}$, a two-threshold mechanism if $v^l < v^u < \overline{v}$, and a shortlisting mechanism if $v^l = v^u < \overline{v}$.

To understand conditions (10), (11) and (13), consider the impact of a reduction in v^l. Intuitively, this improves allocation efficiency at the bottom of the value distribution.

After some algebra, one can verify that the increase in allocation efficiency is proportional to $\mathbb{E}_v[\max\{v, v^l\}] - \mathbb{E}_v[v]$. However, as v^l decreases, agents with low v's become worse off and have stronger incentives to exaggerate their types. To restore IC, the principal must now inspect agents more frequently, which raises the total verification cost by an amount proportional to k/c. Furthermore, because the principal's ability to penalize an agent is sufficiently limited, more pooling at the top (i.e. a lower v^u) may also be required to restore IC. This reduces the allocation efficiency at the top by an amount proportional to $[\mathbb{E}_v[v] - \mathbb{E}_v[\min\{v, v^u\}]]/(1-c)$. In an optimal mechanism, the marginal gain from a reduction in v^l (proportional to the left-hand side of (14)) must equal the marginal cost (proportional to the right-hand side of (14)):

$$\mathbb{E}_v[\max\{v, v^l\}] - \mathbb{E}_v[v] = \frac{\mathbb{E}_v[v] - \mathbb{E}_v[\min\{v, v^u\}]}{1-c} + \frac{k}{c}. \tag{14}$$

This is precisely the case captured by the third part of Theorem 3 (compare (14) with (13)). If the limited punishment constraint does not bind (i.e. $v^u = \overline{v}$), there is no efficiency loss at the top and $[\mathbb{E}_v[v] - \mathbb{E}_v[\min\{v, v^u\}]]/(1-c) = 0$. In this case, (14) becomes (10) ($v^l = v^*$) and an optimal mechanism is characterized by the first part of Theorem 3. If the principal's ability to punish an agent is sufficiently limited so that $v^u = v^l (= v^{**})$, then (14) becomes (11) and an optimal mechanism is characterized by the second part of Theorem 3.

Remark 2 *If $k = 0$, then $v^* = \underline{v}$ and $F(v^*)^{n-1} = 0 < n(1-c)$ for any $0 < c < 1$. That is, when verification is free, there is always pooling at the top (Mylovanov and Zapechelnyuk (2017)).*

4 Properties of optimal mechanisms

Theorem 3 in the previous section shows that either one-threshold mechanisms, two-threshold mechanisms or shortlisting mechanisms are optimal. In this section, I show that which of the above three kinds of mechanisms are optimal crucially depends on the number of agents (n). Specifically, I show that there exist $n^*(\rho, c)$ and $n^{**}(\rho, c)$ with $n^*(\rho, c) < n^{**}(\rho, c)$ such that if $n \leq n^*(\rho, c)$, then one-threshold mechanisms are optimal; if $n^*(\rho, c) < n < n^{**}(\rho, c)$, then two-thresholds mechanisms are optimal; and if $n \geq n^{**}(\rho, c)$, then shortlisting mechanisms are optimal. Here $\rho := k/c \geq 0$ is referred as the *effective verification cost.* The effective verification cost, ρ, is strictly decreasing in c. This is because a smaller c implies a lower level of punishment, which essentially makes verification costlier as the principal must inspect agents more frequently to maintain IC.

Formally, let $n^*(\rho, c) < 1/(1-c)$ be defined by

$$F(v^*)^{n^*(\rho,c)-1} = n^*(\rho, c)(1-c), \tag{15}$$

where v^* is defined by

$$\mathbb{E}_v[v] - \mathbb{E}_v[\max\{v, v^*\}] + \rho = 0. \tag{10}$$

Because v^* is independent of n, by Theorem 3, one-threshold mechanisms are optimal if and only if $n \leq n^*(\rho, c)$. Intuitively, for fixed v^*, an agent whose type below v^* gets the object with probability

$$\varphi^* = \frac{1}{n} F(v^*)^{n-1},$$

which is strictly decreasing in n. In particular, an agent with the lowest type becomes worse off and has stronger incentives to exaggerate his type when the number of agents, n, increases. For an n sufficiently large, IC cannot be sustained without pooling at the top of the value distribution.

Because v^* is strictly increasing in ρ, the left-hand side of (15) is strictly decreasing in n and the right-hand side of (15) is strictly increasing in n, n^* is strictly increasing in ρ. Intuitively, as the effective verification cost (ρ) increases, the principal optimally reduces the use of verification and instead enlarges the pooling area at the bottom of the value distribution (v^* increases) to maintain IC. As a result, an agent with the lowest type becomes better off (φ increases), and IC can therefore be sustained without pooling at the top for a larger number of agents. For a fixed ρ, v^* is independent of c but the right-hand side of (15) is strictly decreasing in c. Hence, n^* is strictly increasing in c. Intuitively, the upper-bound on P in (IC′) becomes larger as c increases, and IC can therefore be sustained without pooling at the top for a larger number of agents.

Next, let $n^{**}(\rho, c) < 1/(1-c)$ be defined by

$$\frac{1 - F(v^{**})^{n^{**}(\rho,c)}}{1 - F(v^{**})} = \frac{F(v^{**})^{n^{**}(\rho,c)-1}}{1-c}, \tag{16}$$

where v^{**} is given by

$$\mathbb{E}_v[v] - \mathbb{E}_v[\min\{v, v^{**}\}] + (1-c)\left[\mathbb{E}_v[v] - \mathbb{E}_v[\max\{v, v^{**}\}] + \rho\right] = 0. \tag{11}$$

When comparing (16) with (12), it is easy to see that $v^{**} \leq v^\natural$ if and only if $n \geq n^{**}(\rho, c)$. Per Theorem 3, shortlisting mechanisms are optimal if and only if $n \geq n^{**}(\rho, c)$. As previously discussed, an agent with the lowest type becomes worse off and has stronger incentives to exaggerate his type when the number of agents, n, increases. As a result, pooling areas at both the bottom and the top of the value distribution must be enlarged to ensure that the mechanism is incentive compatible and to save verification cost. Formally, I show in Appendix 4 that $v^l(n, \rho, c)$ is strictly increasing in n and $v^u(n, \rho, c)$ is strictly decreasing in n. Eventually, for a sufficiently large number of agents, the two pooling areas meet and there is a unique threshold such that all agents whose values are above the threshold and all agents whose values are below the threshold are pooled, respectively.

Recall that $v^{**} > v^*$. Hence,

$$\frac{F(v^{**})^{n^*(\rho,c)-1}}{1-c} > \frac{F(v^*)^{n^*(\rho,c)-1}}{1-c} = n^*(\rho, c) \geq \frac{1-F(v^{**})^{n^*(\rho,c)}}{1-F(v^{**})}.$$

Because the left-hand side of (16) is strictly increasing in n, and the right-hand side of (16) is strictly decreasing in n, we have $n^{**}(\rho, c) > n^*(\rho, c)$. It is easy to see that $v^{**}(\rho, c)$ is strictly increasing in both ρ and c, and independent of n. Recall that $v^\natural$ is independent of ρ. I show in Lemma 4 in Appendix A that if $n(1-c) < 1$, then $v^\natural$ is strictly increasing in n and strictly decreasing in c. Hence, $n^{**}(\rho, c)$ is strictly increasing in both ρ and c.

Fixed ρ, an increase in c has two opposite impacts on the size of the pooling areas. On the one hand, the upper-bound on P in (IC′) becomes larger as c increases, which reduces the pooling area at the top (v^u increases) needed to sustain IC. On the other hand, it follows from the analysis in Section 3 that the marginal cost from a reduction in v^l increases as c increases.[6] Hence, it is optimal for the principal to enlarge the pooling area at the bottom (v^l increases). Formally, I show in Appendix B that both $v^l(n, \rho, c)$ and $v^u(n, \rho, c)$ are strictly increasing in c. The analysis above on n^{**} shows that the first effect dominates, and two-thresholds mechanisms are optimal for a larger number of agents as c increases.

Fixed c, an increase in ρ also has two opposite impacts on the size of the pooling areas. On the one hand, as previously discussed, as the effective verification cost (ρ) increases, the principal optimally reduces the use of verification and instead enlarges the pooling area at the bottom of the value distribution to maintain IC. On the other hand, as the pooling area at the bottom increases, an agent with the lowest type becomes better off, and IC can be sustained with less pooling at the top (v^u increases). Formally, I show in Appendix B that both $v^l(n, \rho, c)$ and $v^u(n, \rho, c)$ are strictly increasing in ρ. The analysis above on n^{**} shows that the second effect dominates, and two-thresholds mechanisms are optimal for a larger number of agents as ρ increases.

These results are summarized by the following corollary:

Corollary 2 *Suppose that Assumption 1 holds. Given $k > 0$, $c \in (0, 1)$ and $\rho = k/c$, there*

[6] Fixed $\rho = k/c$, the right-hand side of (14) is strictly increasing in c.

exists $0 < n^*(\rho, c) < n^{**}(\rho, c) < 1/(1-c)$ *such that the following statements are true:*

1. *If* $n \leq n^*(\rho, c)$*, then one-threshold mechanisms are optimal; if* $n^*(\rho, c) < n < n^{**}(\rho, c)$*, then two-thresholds mechanisms are optimal; if* $n \geq n^{**}(\rho, c)$*, then shortlisting mechanisms are optimal.*

2. $n^*(\rho, c)$ *and* $n^{**}(\rho, c)$ *are strictly increasing in* ρ *and* c*.*

3. $v^*(n, \rho, c)$ *is strictly increasing in* ρ*, and independent of* n *and* c*.* v^{**} *is strictly increasing in* ρ *and* c*, and independent of* n*. If* $n^*(\rho, c) < n < n^{**}(\rho, c)$*, then* $v^l(n, \rho, c)$ *is strictly increasing in* n*,* ρ *and* c*, and* $v^u(n, \rho, c)$ *is strictly decreasing in* n*, and strictly increasing in* ρ *and* c*.*

Corollary 2 gives comparative statics results in terms of (ρ, c). It is also interesting to see the comparative statics results with respect to the model primitives (k, c). The impact of k is straightforward. As k increases, verification becomes costlier. The optimal mechanism given in Theorem 3 sees more pooling at the bottom (measured by φ) in order to save verification cost. An increase in φ relaxes the upper-bound on P, which leads to less or no pooling at the top. The impact of c is ambiguous. On the one hand, given the amount of pooling at the bottom (measured by φ), a reduction in c lowers the upper-bound on P in (IC$'$), which implies more pooling at the top. On the other hand, a reduction in c makes verification costlier. Similar to the case of an increase in k, this change increases the amount of pooling at the bottom (φ increases) and relaxes the upper-bound on P. As a result, there may be less or no pooling at the top. This result is in contrast to the findings of Mylovanov and Zapechelnyuk (2017). In Mylovanov and Zapechelnyuk (2017), the top pooling area increases as the penalty decreases.[7] This is because the second channel is absent if verification is free ($k = 0$). The non-monotonicity of the pooling area at the top is further illustrated by the following numerical example.

Example 1 *Consider a numerical example in which* $\{v_i\}$ *are uniformly distributed on* $[0, 1]$*. There are* $n = 8$ *agents. The verification cost is* $k = 0.4$*. I slightly abuse notation and redefine* $v^l = v^u = v^{**}$ *if* $v^l > v^u$*. Figure 1 plots* v^l *and* v^u *as functions of* c*. Observe that the change of* v^u *is not monotonic. As* c *increases, the pooling area at the top first expands and then shrinks.*

Finally, a careful examination of (15) and (10) proves the following corollary:

Corollary 3 $\lim_{c \to 1} n^*(k/c, c) = \infty$ *and* $\lim_{k \to 0} n^*(k/c, c) = 0$*.*

[7] See Proposition 5A in Mylovanov and Zapechelnyuk (2017).

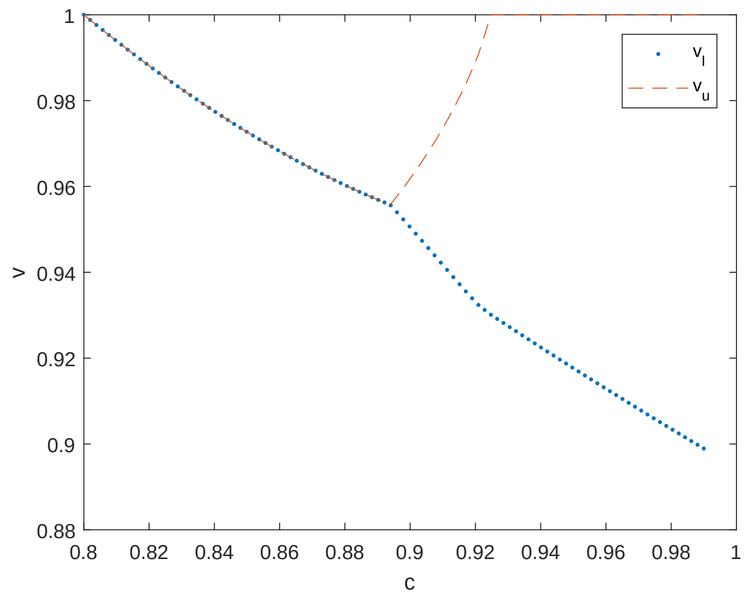


Figure 1: The impact of level of punishment (c)

Corollary 3 shows that as the principal's ability to punish an agent becomes large enough, the model collapses to Ben-Porath et al. (2014); and as the verification cost diminishes, the model collapses to Mylovanov and Zapechelnyuk (2017).

5 Asymmetric environment

In this section, I consider the general model with asymmetric agents and show that a generalized threshold mechanism is optimal in this case. The analysis, however, is much more complex. Although there is still a unique lower threshold for all agents, different agents may face different upper thresholds. Because of the complication of the pooling areas at the top, it is generally extremely difficult to fully characterize the set of optimal mechanisms. Thus, I provide results only for some important special cases. Section 5.1.1 studies the case in which one group agents' value distribution is "better" than the other's in the sense of first-order stochastic dominance. Section 5.1.2 studies the case in which one group agents's value distribution is "more risky" than the other's in the sense of mean-preserving spread. Section 5.2 revisits the symmetric environment. Finally, in Appendix C.3, I characterize the set of optimal one-threshold mechanisms. If $c_i = 1$ for all i, then these are the set of optimal mechanisms found in Ben-Porath et al. (2014).

Similar to that in Section 3, I first characterize an optimal mechanism given the lowest probabilities with which each agent receives the object ($\boldsymbol{\varphi} := (\varphi_1, \ldots, \varphi_n)$). Formally, fix

$\varphi_i = \inf_{v_i} P_i(v_i)$ for all i and consider the following problem $(OPTA-\varphi)$:

$$\max_{\boldsymbol{P},\boldsymbol{Q}} \sum_{i=1}^{n} \mathbb{E}_{v_i}\left[P_i(v_i)\left(v_i - \frac{k_i}{c_i}\right)\right] + \frac{\varphi_i k_i}{c_i},$$

subject to

$$\varphi_i \leq P_i(v_i) \leq \frac{\varphi_i}{1-c_i}, \forall v_i, \tag{AIC$'$}$$

$$0 \leq Q_i(v_i) \leq 1, \forall v_i, \tag{F1}$$

$$\sum_i \int_{S_i} P_i(v_i)\mathrm{d}F_i(v_i) \leq 1 - \prod_i \left(1 - \int_{S_i} dF_i(v_i)\right), \forall S_i \subset V_i. \tag{AF2}$$

Clearly, $(OPTA-\varphi)$ is feasible only if $\sum_i \varphi_i \leq 1$. As in the symmetric case, I approximate the continuum type space with a finite partition, solve an optimal mechanism in the finite model and take limits. The following theorem gives an optimal solution to $(OPTA-\varphi)$:

Theorem 4 *There exist d^l and d_i^u for $i = 1, \ldots, n$ such that $\boldsymbol{P}^*$ defined by*

$$P_i^*(v_i) := \begin{cases} \overline{P}_i(v_i) & \text{if } v_i > \frac{k_i}{c_i} \\ \varphi_i & \text{if } v_i < \frac{k_i}{c_i}. \end{cases}, \tag{17}$$

where

$$\overline{P}_i(v_i) := \begin{cases} \frac{\varphi_i}{1-c_i} & \text{if } v_i > d_i^u + \frac{k_i}{c_i} \\ \prod_{j \neq i, d_j^u \geq v_i - \frac{k_i}{c_i}} F_j\left(v_i - \frac{k_i}{c_i} + \frac{k_j}{c_j}\right) & \text{if } d^l + \frac{k_i}{c_i} < v_i < d_i^u + \frac{k_i}{c_i} \\ \varphi_i & \text{if } v_i < d^l + \frac{k_i}{c_i} \end{cases}. \tag{18}$$

is an optimal solution to $(OPTA-\varphi)$.

Unsurprisingly, agents are now ordered by their "net" values $v_i - k_i/c_i$, which is equal to their values to the principal minus the effective verification cost borne by the principal.[8]. As before, there is a unique lower threshold d^l such that all agents whose net values $v_i - k_i/c_i$ below the threshold are pooled. However, there can be up to n distinct upper thresholds d_i^u $(i = 1, \ldots, n)$.

To illustrate how an optimal mechanism in Theorem 4 can be implemented, assume that there are two distinct upper thresholds: $d_1^u = \cdots = d_j^u > d_{j+1}^u = \ldots d_n^u$. Then the first j agents whose net values are above d_1^u are pooled together, while the other $n-j$ agents whose

[8] This is consistent with the result in Section 3 because when $k_i = k$ and $c_i = c$ for all i, ordering agents by net values produces the same result as ordering them by values.

net values are above d^u_{j+1} are pooled together and ranked below any of the first j agents whose net value is above d^u_{j+1}. Specifically, the following procedure implements the truth-telling equilibrium in a threshold mechanism: If there exists some agent i ($1 \leq i \leq j$) whose net value $v_i - k_i/c_i$ is above d^u_1, then one such agent is selected at random, receives the good and is inspected with probability one. If $v_i - k_i/c_i < d^u_1$ for all $1 \leq i \leq j$ but $v_i - k_i/c_i \geq d^u_{j+1}$ for some $1 \leq i \leq j$, then the agent with the highest reported net value among the first j agents receives the good and is inspected with some probability. If $v_i - k_i/c_i < d^u_{j+1}$ for all $1 \leq i \leq j$ and $v_i - k_i/c_i \geq d^u_{j+1}$ for some $j+1 \leq i \leq n$, then one agent is selected at random among the last $n-j$ agents whose reported net values are above d^u_{j+1}, receives the good and is inspected with probability one. If $v_i - k_i/c_i < d^u_{j+1}$ for all i but $v_i - k_i/c_i \geq d^l$ for some i, then the agent with the highest reported net value receives the good and is inspected with some probability. If $v_i - k_i/c_i < d^l$ for all i, then one agent is selected at random and receives the good and no one is inspected. Finally, an agent is punished if and only if he is found to have lied.

Because of the complication of the pooling areas at the top, it is much harder to find an optimal solution to $(OPTA - \varphi)$. Specifically, d^u_i's are solved recursively from the largest to the smallest. Furthermore, to characterize the set of optimal $\boldsymbol{\varphi}$'s, without prior knowledge of which set of agents share the same upper threshold, one must consider 2^n different cases.[9] Thus, I leave the full characterization of optimal mechanisms to future research. In what follows, I provide results only for some important special cases.

Before proceeding, I introduce the following assumption:

Assumption 2 $\overline{v}_i - k_i/c_i > \mathbb{E}_{v_i}[v_i]$ *for some* i.

By a similar argument to that in the proof of Theorem 2, we can show that pure randomization is optimal if verification is sufficiently costly or the principal's ability to punish an agent is sufficiently limited, i.e., $\overline{v}_i - k_i/c_i \leq \mathbb{E}_{v_i}[v_i]$ for all i. To make the problem interesting, in the rest of this section I assume that Assumption 2 holds.

5.1 First-order stochastic dominance and mean-preserving spread

For simplicity, suppose that there are two groups of agents $g \in \{H, L\}$. The number of agents in group g is n_g. The value (v) of an agent in group g follows the cumulative distribution F_g, with density f_g and support $[\underline{v}, \overline{v}] \subset \mathbb{R}_+$. The two group of agents are otherwise identical. An agent gets a private benefit $b(v)$ if he receives the object and 0

[9] Assume, without loss of generality, that $d^u_1 \geq \cdots \geq d^u_n$. If there are ν distinct upper thresholds, then there are C^ν_n possibilities to consider. In total, there are $\sum_{\nu=1}^{n} C^\nu_n = 2^n$ possibilities to consider.

otherwise. The principal can verify an agent's report at a cost $k \geq 0$ and impose a penalty $cb(v)$ on him, where $0 < c \leq 1$.

In this setting, there exists an optimal mechanism that is group symmetric. Hence, I focus on group symmetric mechanisms in the rest of this subsection. Let φ_g denote the lowest probability with which a group g agent receives the object. Let P_g denote the interim probability with which a group g agent receives the object, and Q_g be such that $P_g Q_g$ is the interim probability with which a group g agent is inspected. I slightly abuse notation, and let $\boldsymbol{\varphi} := (\varphi_H, \varphi_L)$, $\boldsymbol{P} := (P_H, P_L)$ and $\boldsymbol{Q} := (Q_H, Q_L)$.

Recall that $\rho := k/c$ denotes the effective verification cost. Define d_g^* and d_g^{**}, $g \in \{H, L\}$ by equations (19) and (20), respectively: For $g \in \{H, L\}$,

$$\int_{\underline{v}_g}^{\overline{v}_g} \left(v - \max\{v, d_g^* + \rho\}\right) \mathrm{d}F_g(v) + \rho = 0, \tag{19}$$

$$(1-c)\left[\int_{\underline{v}_g}^{\overline{v}_g} \left(v - \max\{v, d_g^{**} + \rho\}\right) \mathrm{d}F_g(v) + \rho\right] + \int_{\underline{v}_g}^{\overline{v}_g} \left(v - \min\{v, d_g^{**} + \rho\}\right) \mathrm{d}F_g(v) = 0. \tag{20}$$

5.1.1 First-order stochastic dominance

Suppose that agents in group H are "better" than those in group L in the sense that F_H *first-order stochastically dominates* F_L.

Recall that d_g^* and d_g^{**}, $g \in \{H, L\}$ are defined by equations (19) and (20), respectively. I argue that $d_H^* \geq d_L^*$ and $d_H^{**} \geq d_L^{**}$. Because $v - \max\{v, d^l + \rho\}$ is strictly increasing in v and F_H first-order stochastically dominates F_L, we have $\int_{\underline{v}}^{\overline{v}} \left(v - \max\{v, d^l + \rho\}\right) \mathrm{d}F_H(v) \geq \int_{\underline{v}}^{\overline{v}} \left(v - \max\{v, d^l + \rho\}\right) \mathrm{d}F_L(v)$ for all d^l. Furthermore, the left-hand side of equation (19) is strictly decreasing in d_g^*. Hence, $d_H^* \geq d_L^*$. By a similar argument, $d_H^{**} \geq d_L^{**}$.

To simplify the analysis, assume further that $d_H^* \geq d_L^{**}$. In this case, an optimal mechanism is characterized by the following theorem:

Theorem 5 *Suppose that Assumption 2 holds and F_H first-order stochastically dominates F_L so that $d_H^* \geq d_L^{**}$. There are two cases.*

1. *If $n_H(1-c)F_H(d_H^* + \rho) + n_L(1-c)F_L(d_H^* + \rho) \leq F_H(d_H^* + \rho)^{n_H} F_L(d_H^* + \rho)^{n_L}$, then the optimal*

$$\boldsymbol{\varphi}^* = \left(\frac{F_H(d_H^* + \rho)^{n_H} F_L(d_H^* + \rho)^{n_L} - n_L(1-c)F_L(d_H^* + \rho)}{n_H F_H(d_H^* + \rho)}, 1 - c\right),$$

the optimal inspection rule satisfies $\boldsymbol{Q}^* = ((1-\varphi_H^*/P_H^*)/c, (1-\varphi_L^*/P_L^*)/c)$ *and the following allocation rule is optimal:*

$$P_g^*(v) = \begin{cases} F_g(v)^{n_g-1}F_{g'}(v)^{n_{g'}} & \text{if } v-\rho \geq d_H^* \\ \varphi_g^* & \text{if } v-\rho < d_H^* \end{cases}, \quad \text{for } g,g' \in \{H,L\} \text{ and } g \neq g'.$$

2. *If* $n_H(1-c)F_H(d_H^*+\rho) + n_L(1-c)F_L(d_H^*+\rho) > F_H(d_H^*+\rho)^{n_H}F_L(d_H^*+\rho)^{n_L}$*, then in any optimal mechanism a group* L *agent never receives the object* ($P_L^* = 0$) *and is never inspected* ($Q_L^* = 0$)*. The mechanism given in Theorem 3, when applied to group* H *agents, is optimal.*

The inequality in part 1 of Theorem 5 is satisfied if n_H and n_L are small. If the number of agents is small, then part 1 of Theorem 5 proves that a one-threshold mechanism is optimal. Specifically, if there exists an agent whose net value $(v-\rho)$ is above the threshold d^l, then the agent with the highest net value receives the object and is inspected with some probability. If all agents' net values are below d^l, then one agent is selected randomly and no one is inspected. Since agents in different groups have different value distributions, the principal may benefit by selecting agents from one group more frequently than the other when all agents' net values are below d^l. We say group g is *favored* if for $g, g' \in \{H, L\}$ and $g \neq g'$,

$$\varphi_g = \frac{F_H(d^l+\rho)^{n_H}F_L(d^l+\rho)^{n_L} - n_{g'}(1-c)F_{g'}(d^l+\rho)}{n_gF_g(d^l+\rho)}, \varphi_{g'} = 1-c.$$

Note that if all agents' net values are below d^l, group g' agents are selected just frequently enough to satisfy their IC, and group g agents are favored in the sense that they are selected more frequently. To understand the definition of d_g^*, consider a marginal reduction in d^l. Clearly, this change has no first-order impact on the allocation efficiency and IC of group g'. We can apply the analysis in Section 3 to understand the impact of this change on group g. (Compare (19) with (10).) Hence, it is optimal to choose $d^l = d_g^*$ when group g is favored.

An alternative interpretation of d_g^* is as follows.[10] Consider a principal who has two choices: choosing a group g agent without checking, and checking the group g agent and then choosing between him and an outside option v. If the value of the outside option is $v = d_g^* + \rho$, then the principal is indifferent between her two choices. Intuitively, the higher d_g^* is, the principal is more inclined to select a group g agent without checking. Naturally, the group with the highest d_g^* is the principal's choice for the favored group.

Since F_H first-order stochastically dominates F_L, $d_H^* \geq d_L^*$. This implies that the principal's optimal choice of favored group is group H. This result is consistent with Ben-Porath

[10] See Ben-Porath et al. (2014) and Doval (2014).

et al. (2014). The intuition for this result is also clear. Since agents from group H are more likely to have high values, group H is a better choice to be the favored group.

More interestingly, when the number of agents is large, part 2 of Theorem 5 proves that in any optimal mechanism agents in group L never receive the object and are never inspected. It is optimal for the principle to choose a mechanism as if she faces only group H agents. The intuition of this result is as follows. As the number of agents increases, an agent with the lowest type becomes worse off and has stronger incentives to exaggerate his type. In order to maintain IC, the principal must introduce pooling at the top of value distributions. Because group H is "better" than group L in the sense of first-order stochastic dominance, it is less costly for the principal to distort group L's allocations. If $d_H^* \geq d_L^{**}$, then group H is so much "better" than group L that the principal finds it optimal to ignore group L agents all together.

To better understand this result, assume that $F_H(d_H^* + \rho)^{n_H - 1} \geq n_H(1-c)$. In this case, there is a threshold d^l such that the following mechanism is optimal: If there is a group H agent whose net value is above d^l, then select the group H agent with the highest net value and inspect him with some probability. If all group H agents' net values are below d^l but some group L agents' net values are above d^l, then randomly select one agent among all group L agents whose net values are above d^l and inspects him with some probability. If all agents' net values are below d^l, then select one agent randomly and inspect no one.

We say group H is favored if

$$\varphi_H = \frac{F_H(d^l + \rho)^{n_H - 1}}{n_H}, \varphi_L = 0,$$

and group L is favored if

$$\varphi_H = 1 - c, \varphi_L = \frac{1-c}{n_L} \frac{F_H(d^l + \rho)^{n_H} - n_H(1-c)F_H(d^l + \rho)}{1 - cF_L(d^l + \rho)}.$$

As before, if group g is favored and all agents' net values are below d^l, group g' agents are selected just frequently enough to satisfy their IC, and group g agents are favored in the sense that they are selected more frequently. By a similar analysis to that above, it is optimal to set $d^l = d_H^*$ when group H is favored, and it is optimal to set $d^l = d_L^{**}$ when group L is favored (compare (20) with (11)).[11] If $d_H^* \geq d_L^{**}$, then group H is the principal choice's for favored group. In other words, group H is so much "better" than group L that the principal finds it optimal to ignore group L agents all together.

Next, we turn to the impact of an increase in the number of agents on agents' payoffs.

[11] d_g^{**} does not have a straightforward interpretation in terms of search.

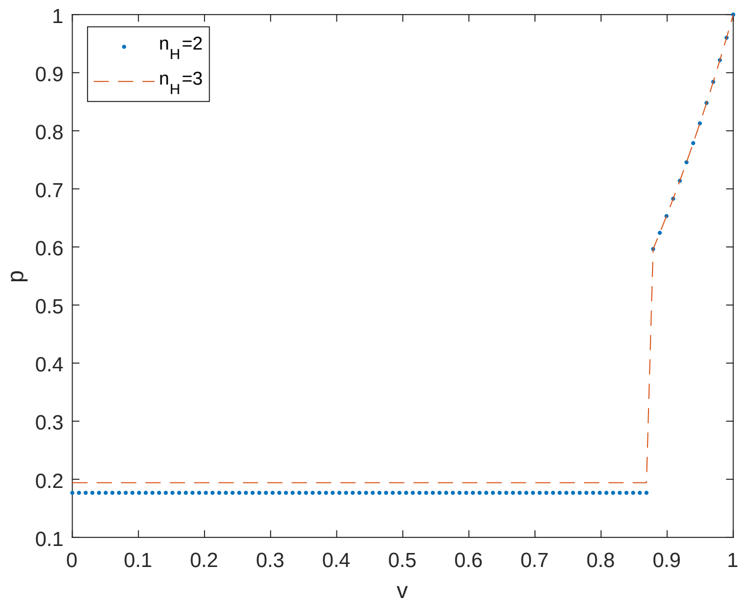


Figure 2: A group H agent's probability of receiving the good can increase as the number of agents increases.

The result is somewhat surprising. The common sense is that an agent's expected payoff is lower if he faces more fierce competition. This is true for group L agents. However, in the environment considered here, a group H agent can actually get better off when he has more competitors. This is possible even if the additional competitors all come from group H. This is because when the number of agents is small, group H agents need to compete with both agents in their own group and group L agents; but when the number of agents is large, they need to compete with only people from their own group. As a result, they essentially face less competition and their chances of receiving the object can actually increase. This point is further illustrated by the following numerical example.

Example 2 *Consider a numerical example in which the verification cost is* $k = 0.2$ *and the punishment is* $c = 0.9$. *Assume that the value of a group* L *agent is uniformly distributed on* $[0, 1]$ *(i.e.* $F_L(v) = v$ *for* $v \in [0, 1]$*) and the value of a group* H *agent follows the cumulative distribution* $F_H(v) = v^2$ *for* $v \in [0, 1]$. *Clearly,* F_H *first-order stochastically dominates* F_L. *Furthermore,* $d_H^* \approx 0.65 > d_L^{**} \approx 0.63$. *Hence, Theorem 5 applies.*

Assume that there are $n_L = 2$ *agents in group* L. *Figure 2 plots a group* H *agent's interim allocation rules (*P_H^**) when* $n_H = 2$ *and* $n_H = 3$*, respectively. Observe that when the number of group* H *agents increases from* $n_H = 2$ *to* $n_H = 3$*, the interim probability with which a group* H *agent (*$P_H^*(v)$*) receives the object weakly increases for all* $v \in [0, 1]$ *and strictly increases for* $v \leq 0.87$.

5.1.2 Mean-preserving spread

Suppose that agents in group H are "less risky" than those in group L in the sense that F_L is a *mean-preserving spread* of F_H. Then $[\underline{v}_H, \overline{v}_H] \subseteq [\underline{v}_L, \overline{v}_L]$.

Recall that d_g^* and d_g^{**}, $g \in \{H, L\}$ are defined by equations (19) and (20), respectively. I argue that $d_H^* \geq d_L^*$ and $d_H^{**} \leq d_L^{**}$. Firstly, because $v - \max\{v, d^l + \rho\}$ is strictly increasing and concave in v, and F_L is a mean-preserving spread of F_H, $\int_{\underline{v}_H}^{\overline{v}_H} \left(v - \max\{v, d^l + \rho\}\right) \mathrm{d}F_H(v) \geq \int_{\underline{v}_L}^{\overline{v}_L} \left(v - \max\{v, d^l + \rho\}\right) \mathrm{d}F_L(v)$ for all d^l. Furthermore, the left-hand side of equation (19) is strictly decreasing in d_g^*. Hence, $d_H^* \geq d_L^*$. Secondly, since F_L is a mean-preserving spread of F_H, for all d^l,

$$
\begin{aligned}
&\int_{\underline{v}_H}^{\overline{v}_H} \left[(1-c)\left(v - \max\{v, d^l + \rho\}\right) + \left(v - \min\{v, d^l + \rho\}\right)\right] \mathrm{d}F_H(v) \\
=&\int_{\underline{v}_H}^{\overline{v}_H} \left[v - \left(d^l + \rho\right)\right] \mathrm{d}F_H(v) - c\int_{\underline{v}_H}^{\overline{v}_H} \left(v - \max\{v, d^l + \rho\}\right) \mathrm{d}F_H(v) \\
\leq&\int_{\underline{v}_L}^{\overline{v}_L} \left[v - \left(d^l + \rho\right)\right] \mathrm{d}F_L(v) - c\int_{\underline{v}_L}^{\overline{v}_L} \left(v - \max\{v, d^l + \rho\}\right) \mathrm{d}F_L(v) \\
=&\int_{\underline{v}_L}^{\overline{v}_L} \left[(1-c)\left(v - \max\{v, d^l + \rho\}\right) + \left(v - \min\{v, d^l + \rho\}\right)\right] \mathrm{d}F_L(v).
\end{aligned}
$$

Furthermore, the left-hand side of equation (20) is strictly decreasing in d_g^{**}. Hence, $d_H^{**} \leq d_L^{**}$.

Let $n_L^{**}(\rho, c)$ be defined by

$$
\frac{1 - F_L(d_L^{**} + \rho)^{n_L^{**}(\rho,c)}}{1 - F_L(d_L^{**} + \rho)} = \frac{F_L(d_L^{**} + \rho)^{n_L^{**}(\rho,c)-1}}{1-c}, \tag{21}
$$

Theorem 6 gives an optimal mechanism when the number of agents is sufficiently small and sufficiently large, respectively:

Theorem 6 *Suppose that Assumption 2 holds and F_L is a* mean-preserving spread *of F_H.*

1. *If $n_H(1-c)F_H(d_H^* + \rho)F_L(\overline{v}_H)^{n_L} + n_L(1-c)F_L(d_H^* + \rho) \leq F_H(d_H^* + \rho)^{n_H} F_L(d_H^* + \rho)^{n_L}$, then the optimal*

$$
\boldsymbol{\varphi}^* = \left(\frac{F_H(d_H^* + \rho)^{n_H} F_L(d_H^* + \rho)^{n_L} - n_L \varphi_L^* F_L(d_L^* + \rho)}{n_H F_H(d_H^* + \rho)}, 1 - c\right),
$$

the optimal inspection rule satisfies $\boldsymbol{Q}^* = \left(\left(1 - \varphi_H^*/P_H^*\right)/c, \left(1 - \varphi_L^*/P_L^*\right)/c\right)$ and the *following allocation rule is optimal:*

$$P_g^*(v) = \begin{cases} F_g(v)^{n_g-1}F_{g'}(v-\rho+\rho_{g'})^{n_g'} & \text{if } v-\rho \geq d_H^* \\ \varphi_g & \text{if } v-\rho < d_H^* \end{cases}, \quad \text{for } g,g' \in \{H,L\} \text{ and } g \neq g'.$$

2. *If* $n_H(1-c)F_H(d_H^*+\rho)F_L(\overline{v}_H)^{n_L} + n_L(1-c)F_L(d_H^*+\rho) > F_H(d_H^*+\rho)^{n_H}F_L(d_H^*+\rho)^{n_L}$, $n_{g'}(1-c)F_{g'}(d_g^{**}+\rho) + n_g\left[1-cF_g(d_g^{**}+\rho)\right] \geq F_{g'}(d_g^{**}+\rho)^{n_{g'}}$ *for* $g,g' \in \{H,L\}$ *and* $g \neq g'$, *and* $n_L \geq n_L^{**}(\rho,c)$, *then the optimal*

$$\boldsymbol{\varphi}^* = \left(0, \frac{1-c}{n_L\left[1-cF_L(d_L^{**}+\rho)\right]}\right),$$

the optimal inspection rule satisfies $\boldsymbol{Q}^* = (0, (1-\varphi_L^*/P_L^*)/c)$ *and the following allocation rule is optimal:* $P_H^* = 0$ *and*

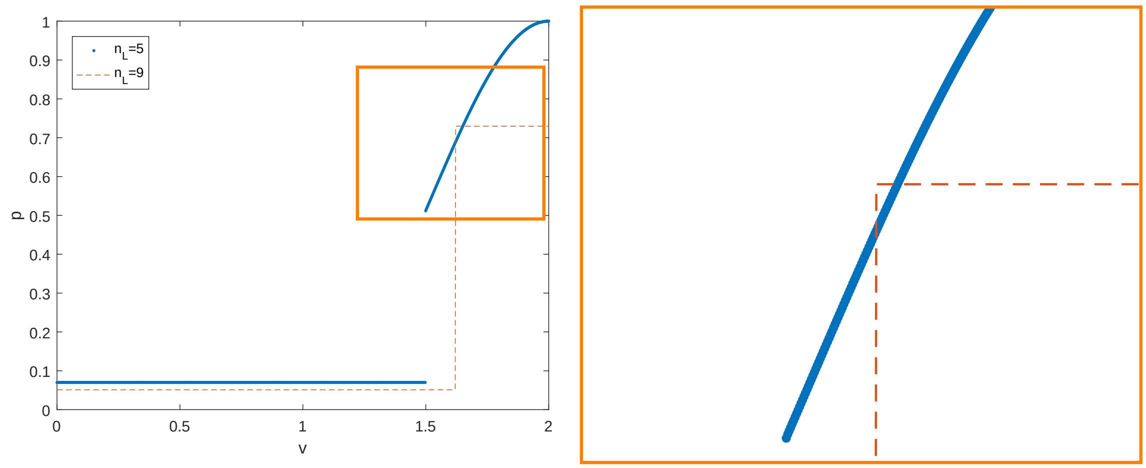
$$P_L^*(v) = \begin{cases} \frac{\varphi_L^*}{1-c} & \text{if } v-\rho \geq d_L^{**} \\ \varphi_L^* & \text{if } v-\rho < d_L^{**} \end{cases}.$$

Note that the result in part 1 of Theorem 6 is identical to that in part 1 of Theorem 5. The inequality in part 1 of Theorem 6 is satisfied if n_H and n_L are small. If the number of agents is small, then part 1 of Theorem 6 proves that a one-threshold mechanism is optimal. The analysis in Section 5.1.1 shows that in an one-threshold mechanism it is optimal to choose $d^l = d_g^*$ when group g is favored. Furthermore, the group with the highest d_g^* is the principal's choice for the favored group. Since F_L is a mean-preserving spread of F_H, $d_H^* \geq d_L^*$. This implies that the principal's optimal choice of favored group is group H. This result is consistent with Ben-Porath et al. (2014).[12] They argue that the reason "why less risky agents are favored is that there is less benefit from checking an agent if there is less uncertainty about his type".

However, part 2 of Theorem 6 proves that if the number of agents is sufficiently large, a shortlisting procedure is optimal and *group L is favored*.

Specifically, there exists $\hat{d}$ such that the following shortlisting procedure is optimal: Agents whose net values are above $\hat{d}$ are shortlisted with probability one, and agents whose net values are below $\hat{d}$ are shortlisted with some probability. The principal then randomly selects an agent from the shortlist. The selected agent is inspected if and only if his net value is above $\hat{d}$. Among all the shortlisted agents, the principal may benefit from selecting one group more frequently than another. We say group g is favored if for $g,g' \in \{H,L\}$ and

[12] Note that mean-preserving spread is a special case of second-order stochastic dominance.



(a) (b)

Figure 3: A group L agent's probability of receiving the good can increase as the number of agents increases.

$g \neq g'$,

$$\varphi_g = \frac{1-c}{n_g\left[1 - cF_g(\hat{d}+\rho)\right]}, \varphi_{g'} = 0.$$

Note that group g is favored in the sense that the principal select only group g agents who are shortlisted. The analysis in Section 3 and Section 5.1.1 implies that it is optimal to choose $\hat{d} = d_g^{**}$ when group g is favored. A simple comparison tells us that the group with the highest d_g^{**} is the principal's choice for the favored group.

Since F_L is a mean-preserving spread of F_H, $d_H^{**} \leq d_L^{**}$. This implies the the principal's optimal choice of favored group is group L. This is in contrast to Ben-Porath et al. (2014). Why are more risky agents favored now? The intuition is that when the number of agents is sufficiently large, randomization at the top is introduced to sustain IC. In this case, if a group is favored, the agents in that group are more likely to be selected both when their net values are above and below the threshold $\hat{d}$. In contrast, when the number of agents is small and one-threshold mechanisms are optimal, a favored group is selected more frequently only when all agents' net values are below the threshold. Since it is more likely to select an agent with high type from the right tail if his value distribution is more spread, the more risky group becomes the principal's favorite.

If we turn to the payoffs of group L agents, they may get better off as the number of their competitors increases and group L becomes the principal's favored group. This possibility is illustrated by the following numerical example.

Example 3 *Consider a numerical example in which the verification cost is $k = 0.464$ and the punishment is $c = 0.93$. Assume that the value of a group H agent is uniformly distributed* *on $[1/2, 3/2]$ (i.e. $F_H(v) = v - (1/2)$ for $v \in [1/2, 3/2]$) and the value of a group L agent follows the cumulative distribution $F_L(v) = v^2/2$ for $v \in [0, 1]$ and $F_L(v) = -v^2/2 + 2v - 1$ for $v \in [1, 2]$. Note that F_L has the same distribution as the sum of F_H and a uniform random variable on $[-1/2, 1/2]$. Clearly, F_L is a mean-preserving spread of F_H.*

Assume that there is $n_H = 1$ agent in group H. If $n_L = 5$, part 1 of Theorem 6 applies. If $n_L = 9$, part 2 of Theorem 6 applies. Figure 2 plots a group L agent's interim allocation rules (P_L^*) when $n_L = 5$ and $n_L = 9$, respectively. Observe that when the number of group L agents increases from $n_L = 5$ to $n_L = 9$, the interim probability with which a group L agent ($P_L^*(v)$) receives the object increases for $v \in [1.62, 1.65]$.

5.2 Symmetric environment revisited

In this subsection, I revisit the symmetric environment. Firstly, I argue that, in the symmetric environment, an optimal mechanism must satisfy: $d_1^u = \cdots = d_n^u$. To understand the intuition behind this result, note first that in the symmetric environment $d_i^u \geq d_j^u$ only if $\varphi_i \geq \varphi_j$. Assume, without loss of generality, that $d_1^u \geq \cdots \geq d_n^u$. Consider, for simplicity, a mechanism in which $\max_j\{\bar{v}_j - k_j/c_j\} > d_1^u > d_2^u > d_3^u$, which implies that $\varphi_1 > \varphi_2$. A new mechanism can then be constructed in which $\varphi_1^* = \varphi_2^* = \sum_{i=1}^2 \varphi_i/2$ and $\varphi_i = \varphi_i^*$ for all $i \geq 3$. In this new mechanism, agents 1 and 2 share the same upper threshold $d^{u*} \in (d_1^u, d_2^u)$ and the upper thresholds of the other agents remain the same. If agents 1 and 2 are ex ante identical, then this new mechanism improves the principal's payoff by allocating the good between agents 1 and 2 more efficiently when their net values, $v_i - k_i/c_i$, lie between (d_1^u, d_2^u).

This property of optimal mechanisms facilitates our analysis of optimal $\boldsymbol{\varphi}$. Theorem 7 below characterizes the set of all optimal $\boldsymbol{\varphi}$. Let v^*, v^{**} and $v^\natural$ be defined by (10), (11) and (12), respectively.

Theorem 7 *Suppose that Assumption 1 holds. There are three cases.*

1. *If $F(v^*)^{n-1} \geq n(1-c)$, then the set of optimal $\boldsymbol{\varphi}$ is the convex hull of*

$$\left\{\boldsymbol{\varphi} \,\middle|\, \varphi_{i^*} = F(v^*)^{n-1} - (n-1)(1-c), \varphi_j = 1 - c \;\forall j \neq i^*, i^* \in \mathcal{I}\right\}.$$

   For each optimal $\boldsymbol{\varphi}^*$ and each agent i, the optimal inspection rule satisfies $Q_i^* = \left(1 - \varphi_i^*/P_i^*\right)/c_i$ and the following allocation rule is optimal:

$$P_i^*(v_i) := \begin{cases} F(v_i)^{n-1} & \text{if } v_i \geq v^* \\ \varphi_i^* & \text{if } v_i < v^* \end{cases}.$$

2. If $F(v^*)^{n-1} < n(1-c)$ and $v^{**} \leq v^\natural$, then the set of optimal $\boldsymbol{\varphi}$ is the convex hull of

$$\left\{ \boldsymbol{\varphi} \;\middle|\; \begin{array}{l} \varphi_{i_j} = (1-c)F(v^{**})^{j-1} \text{ if } j \leq h-1, \varphi_{i_h} = \frac{1-c}{1-cF(v^{**})} - \sum_{j=1}^{h-1}(1-c)F(v^{**})^{j-1}, \\ \varphi_{i_j} = 0 \text{ if } j \geq h+1 \text{ and } (i_1, \ldots, i_n) \text{ is a permutation of } (1, \ldots, n) \end{array} \right\},$$

where $1 \leq h \leq n$ is such that

$$\frac{1-F(v^{**})^{h-1}}{1-F(v^{**})} \leq \frac{1}{1-cF(v^{**})} < \frac{1-F(v^{**})^h}{1-F(v^{**})}.$$

For each optimal $\boldsymbol{\varphi}^*$ and each agent i, the optimal inspection rule satisfies $Q_i^* = (1-\varphi_i^*/P_i^*)/c_i$ and the following allocation rule is optimal:

$$P_i^*(v_i) := \begin{cases} \frac{\varphi_i^*}{1-c} & \text{if } v_i \geq v^{**} \\ \varphi_i^* & \text{if } v_i < v^{**} \end{cases}.$$

3. If $F(v^*)^{n-1} < n(1-c)$ and $v^{**} > v^\natural$, then the the set of optimal $\boldsymbol{\varphi}$ is the convex hull of

$$\left\{ \boldsymbol{\varphi} \,\middle|\, \varphi_{i_j} = (1-c)F(v^u(\varphi^*))^{j-1} \; \forall j \text{ and } (i_1, \ldots, i_n) \text{ is a permutation of } (1, \ldots, n) \right\},$$

where φ^* is defined by (13) and, for each φ, v^l is such that $F(v^l)^{n-1} = \varphi$ and v^u is defined by (7). For each optimal $\boldsymbol{\varphi}^*$ and each agent i, the optimal inspection rule satisfies $Q_i^* = (1-\varphi_i^*/P_i^*)/c_i$ and the following allocation rule is optimal:

$$P_i^*(v_i) := \begin{cases} \frac{\varphi_i^*}{1-c} & \text{if } v_i \geq v^u(\varphi^*) \\ F(v_i)^{n-1} & \text{if } v^l(\varphi^*) < v_i < v^u(\varphi^*) \\ \varphi_i^* & \text{if } v_i \leq v^l(\varphi^*) \end{cases}.$$

Theorem 7 illustrates how limiting the principal's ability to punish agents restricts her ability to treat agents differently. Supposing that $F(v^*)^{n-1} \geq n(1-c)$, the upper-bounds on P_i do not bind in an optimal mechanism. This inequality is trivially satisfied if $c = 1$ as in Ben-Porath et al. (2014). In their study, there is a class of optimal mechanisms called *favored-agent mechanisms*. In a favored-agent mechanism, there exists a favored-agent i^* whose $\varphi_{i^*} = F(v^*)^{n-1}$ while $\varphi_i = 0$ for any other agent $i \neq i^*$. However, if $c < 1$, then in an optimal mechanism it must be that $\varphi_i \geq 1-c$ for all i because otherwise some upper-bounds on P_i would be violated. Intuitively, the worse an agent is treated when he reports a low type, the stronger incentive he has to exaggerate his type. As a result, as the level of punishment declines, the extent to which the principal can favor one agent at the cost of others without violating IC also declines. Fix the ratio of $\rho = k/c$ so that v^* remains the same. The optimal set of $\boldsymbol{\varphi}$ shrinks as c becomes smaller. When c is such that $F(v^*)^{n-1} = n(1-c)$, the unique optimal $\boldsymbol{\varphi}^*$ is such that $\varphi_1^* = \cdots = \varphi_n^*$. These results are summarized in Corollary 4.

Corollary 4 *Suppose that Assumption 1 holds. Let $\Phi(\rho, c)$ denote the set of optimal $\boldsymbol{\varphi}^*$. If $c \geq 1 - F(v^*)^{n-1}/n$, then $c < c'$ implies that $\Phi(\rho, c) \subsetneq \Phi(\rho, c')$ and*

$$\lim_{c \searrow 1-F(v^*)^{n-1}/n} \Phi(\rho, c) = \left\{ \left(\frac{F(v^*)^{n-1}}{n}, \ldots, \frac{F(v^*)^{n-1}}{n} \right) \right\},$$

where v^* is given by (10).

If c is small enough so that $F(v^*)^{n-1} < n(1-c)$, then the comparison is less clear because the sets of optimal mechanisms are disjoint for different levels of punishment. In this case, the principal can again treat agents differently but only to the extent that they share the same upper threshold. Assume, without loss of generality, that an agent with a smaller index is more favored by the principal in terms of a larger φ_i. Then, in an optimal mechanism, the first h agents cannot be favored too much in the sense that $\sum_{i=1}^{h} \varphi_i \leq (1-c) \sum_{i=1}^{h} F(v^u)^{i-1}$ for all $h = 1, \ldots, n$.

6 Concluding remarks

I conclude by comparing the results in this paper with those in previous papers regarding the role played by the number of agents. In this paper, I study the problem of a principal who has a single indivisible object to allocate among a finite number of agents. Each agent has private information regarding the principal's payoff of allocating the object to him. There are no monetary transfers. The principal can inspect agents' reports at a cost and punish them, but punishments are sufficiently limited. As the number of agents increases, the form of optimal mechanisms changes and the comparative statics results can be reversed. Specifically, if the number of agents is small, the optimal mechanism only involves a pooling area at the bottom of value distribution as in Ben-Porath et al. (2014). As the number of agents increases, pooling at the top is required to guarantee incentive compatibility as in Mylovanov and Zapechelnyuk (2017). One implication of this result is that the principal favors "less risky" agents when the number of agents is small, but "more risky" agents when the number of agents is large.

Earlier mechanism design papers studying an allocation problem have often focused on mechanisms with monetary transfers and ignored the possibility of the principal verifying agents' information. In these papers, a robust feature of optimal mechanisms is that they are independent of the number of agents. For example, in the seminal work of Myerson (1981), under some regularity conditions, the revenue-maximizing mechanism can be implemented by a first-price or second-price auction with a reserve price. In particular, this optimal reserve price is independent of the number of agents. This difference can be explained by the difference in binding IC constraints in the two settings. In Myerson (1981), the binding IC constraints are between adjacent types, and the difference between two adjacent types' allocation rules is insensitive to a change in the number of agents. However, in this paper, the binding IC constraints correspond to those of the lowest possible type misreports as higher types. Note that, as the number of agents increases, the lowest possible type's probability of receiving the object declines much faster than that of a much higher type.

In Ben-Porath et al. (2014), the optimal mechanisms are also independent of the number of agent.[13] What makes the difference? The analysis in this paper implies that, although verification is costly, given the rationing area at the bottom of the value distribution, the principal prefers to guarantee IC by verifying an agent's information and punishing him rather than by introducing rationing area to the top. If the level of punishment is sufficiently limited as in this paper, rationing at the top becomes indispensable as the number of agents increases. Furthermore, the size of the rationing area at the top required to sustain IC increases as the number of agents increases. This is why the optimal mechanisms depend on the number of agents. In contrast, if the level of punishment is high enough so that the principal can always guarantee IC by verification as in Ben-Porath et al. (2014), the optimal mechanisms are independent of the number of agents. In this sense, introducing limited punishment is important for us to understand the role played by the number of agents in shaping the optimal mechanisms. This channel is also present in Mylovanov and Zapechelnyuk (2017) in which verification is free. The difference between Mylovanov and Zapechelnyuk (2017) and this paper is that, since verification is free in their paper, an optimal mechanism contains a smaller rationing area at the bottom and rationing at the top is required to sustain IC regardless of the number of agents.

Finally, the change in the form of optimal mechanism also affects the principal's choice of favored agents. Recall that whenever the principal chooses an agent randomly, she can favor one agent by selecting him more frequently than the others. When the number of agents is small, the principal favors "less risky" agents because she benefits less from checking an agent if there is less uncertainty about his type. When the number of agents is large, the principal favors "more risky" agents because now there is also rationing at the top and it is more likely to select an agent with high type if there is more uncertainty about his type.

[13] Recall that when punishment is large enough, a one-threshold mechanism is optimal and the threshold is independent of the number of agents by the third part of Corollary 2.

A Omitted proofs in Sections 3

A polymatroid is a polytope of the following type

$$P(g) := \left\{ x \in \mathbb{R}^E \,\middle|\, x \geq 0, \sum_{e \in A} x_e \leq g(A) \text{ for all } A \subset E \right\}, \tag{22}$$

where E is a finite set and $g : 2^E \to \mathbb{R}_+$ is a submodular function.

Lemma 2 *There exists a monotone and submodular function $\overline{g} : 2^E \to \mathbb{R}_+$ with $\overline{g}(\emptyset) = 0$ and $P(g) = P(\overline{g})$.*

Proof. Let $\overline{g}(\emptyset) := 0$ and $\overline{g}(X) := \min_{A \supset X} g(A)$ for $X \neq \emptyset$. Let $X \subset Y \subset E$. If $X = \emptyset$, then $\overline{g}(X) = 0 \leq \overline{g}(Y)$. If $X \neq \emptyset$, then $A \supset Y$ implies that $A \supset X$, and therefore we have

$$\overline{g}(X) = \min_{A \supset X} g(A) \leq \min_{A \supset Y} g(A) = \overline{g}(Y).$$

Hence, $\overline{g}$ is monotone. Let $e \in E \backslash Y$. To show that $\overline{g}$ is submodular, it suffices to show that

$$\overline{g}(Y \cup \{e\}) - \overline{g}(Y) \leq \overline{g}(X \cup \{e\}) - \overline{g}(X).$$

Because $\overline{g}(\emptyset) = 0 \leq \min_A g(A)$, it suffices to show that

$$\min_{C \supset Y \cup \{e\}} g(C) + \min_{D \supset X} g(D) \leq \min_{A \supset X \cup \{e\}} g(A) + \min_{B \supset Y} g(B).$$

Let $A^* \in \arg\min_{A \supset X \cup \{e\}} g(A)$ and $B^* \in \arg\min_{B \supset Y} g(B)$. Then $A^* \cup B^* \supset Y \cup \{e\}$ and $A^* \cap B^* \supset X$. Hence,

$$\begin{aligned}
\min_{A \supset X \cup \{e\}} g(A) + \min_{B \supset Y} g(B) &= g(A^*) + g(B^*) \\
&\geq g(A^* \cup B^*) + g(A^* \cap B^*) \\
&\geq \min_{C \supset Y \cup \{e\}} g(C) + \min_{D \supset X} g(D),
\end{aligned}$$

where the first inequality holds because g is submodular. Hence, $\overline{g}$ is submodular. Finally, I want to show that $P(g) = P(\overline{g})$. Because $g(A) \geq \overline{g}(A)$ for all $A \subset E$, we have $P(\overline{g}) \subset P(g)$. Suppose that there exists $x \in \mathbb{R}^E$ such that $x \in P(g)$ and $x \notin P(\overline{g})$. Then there exists $A \neq \emptyset$ such that $\sum_{e \in A} x_e > \overline{g}(A)$. By construction, there exists $B \supset A$ such that $\overline{g}(A) = g(B)$. However, then we have $\sum_{e \in B} x_e \geq \sum_{e \in A} x_e > \overline{g}(A) = g(B)$, which is a contradiction to that $x \in P(g)$. Hence, $P(g) = P(\overline{g})$. ■

Proof of Lemma 1. First, because $\overline{H}(\emptyset)=0$, and $\overline{H}$ is non-decreasing and submodular, $\hat{z}^t$ is feasible. Next, I show that $\hat{z}^t$ is optimal.

I begin the analysis by characterizing $\overline{H}$. Clearly, there exists a unique $\underline{t} \in \{1,\ldots,m\}$ such that

$$\frac{1}{n}\left(\sum_{\tau=1}^{\underline{t}-1} f^\tau\right)^{n-1} < \varphi \leq \frac{1}{n}\left(\sum_{\tau=1}^{\underline{t}} f^\tau\right)^{n-1}.$$

Here, $\underline{t}$ is the minimum t such that if all agents whose values are weakly less than v^t are pooled together and ranked below any other agents with higher values, then they receive the object with probability of at least φ. It is easy to verify that[14]

$$\overline{H}(S^t) = \begin{cases} 1-\left(\sum_{\tau=1}^{t-1} f^\tau\right)^n - n\varphi\sum_{\tau=t}^m f^\tau & \text{if } t > \underline{t} \\ 1-n\varphi & \text{if } t \leq \underline{t} \end{cases}. \tag{23}$$

Let $\Delta(t) := \overline{H}(S^t) - n\sum_{\tau=t}^m \frac{c\varphi f^\tau}{1-c}$ for $t=1,\ldots,m+1$. Then $\Delta(m+1)=0$ and $\Delta(t) = \tilde{\Delta}\left(\sum_{\tau=1}^{t-1} f^\tau\right)$, where $\tilde{\Delta}(x) := 1-\frac{n\varphi}{1-c} - x^n - \frac{n\varphi x}{1-c}$ is concave in x. If $\Delta(1) = 1-n\varphi/(1-c) \geq 0$, then let $\bar{t} := 0$; otherwise, there exists a unique $\bar{t} \in \{1,\ldots,m+1\}$ such that

$$\overline{H}(S^{\bar{t}}) \leq n\sum_{\tau=\bar{t}}^m \frac{c\varphi f^\tau}{1-c} \text{ and } \overline{H}(S^{\bar{t}+1}) > n\sum_{\tau=\bar{t}+1}^m \frac{c\varphi f^\tau}{1-c}.$$

Let $\lambda := (\lambda^1,\ldots,\lambda^m)$ and $\mu := (\mu^1,\ldots,\mu^m)$ denote the dual variables corresponding to the upper-bounds and lower-bounds in (IC′m1), and $\beta := (\beta(S))_S$ denote the dual variables corresponding to (F2m1) in problem $(OPTm1-\varphi)$. Consider the dual to problem $(OPTm1-\varphi)$, denoted by $(DOPTm1-\varphi)$,

$$\min_{\lambda,\beta,\mu} \sum_{t=1}^m \frac{c\varphi f^t\lambda^t}{1-c} + \sum_S \beta(S)\overline{H}(S) + \varphi\sum_{t=1}^m f^t v^t,$$

subject to

$$v^t - \frac{k}{c} - \lambda^t + \mu^t - n\sum_{S\ni t}\beta(S) \geq 0, \forall t,$$
$$\lambda \geq 0, \beta \geq 0, \mu \geq 0.$$

Let $\hat{z}$ be define in (5), and $(\hat{\lambda},\hat{\beta},\hat{\mu})$ be the corresponding dual variables. Let t^0 be such that

[14] This result can be seen as a corollary of Lemmas 8 and 9 in Appendix C.

$v^t \geq k/c$ if and only if $t \geq t^0$.

Case 1: $v^{\bar{t}} < \frac{k}{c}$ or $\bar{t} < t^0$. In this case, we have

$$\hat{z}^t := \begin{cases} \frac{c\varphi f^t}{1-c} & \text{if } t > \bar{t} \\ 0 & \text{if } t \leq \bar{t} \end{cases}.$$

Let $\hat{\beta}(S) = 0$ for all S. If $v^t < k/c$, then let $\hat{\lambda}^t = 0$ and $\hat{\mu}^t = k/c - v^t > 0$; if $v^t \geq k/c$, then let $\hat{\mu}^t = 0$ and $\hat{\lambda}^t = v^t - k/c \geq 0$. It is easy to verify that this is a feasible solution to $(DOPTm1 - \varphi)$, and the complementary slackness conditions are satisfied. Finally, the dual objective is equal to the primal objective:

$$\sum_{t=t^0}^{m} \frac{c\varphi f^t}{1-c}\left(v^t - \frac{k}{c}\right) + \varphi \sum_{t=1}^{m} f^t v^t.$$

By the duality theorem, $\hat{z}$ is an optimal solution to $(OPTm1 - \varphi)$.

Case 2: $v^{\bar{t}} \geq \frac{k}{c}$ or $\bar{t} \geq t^0$. In this case, we have

$$\hat{z}^t := \begin{cases} \frac{c\varphi f^t}{1-c} & \text{if } t > \bar{t} \\ \frac{1}{n}\overline{H}(S^{\bar{t}}) - \sum_{\tau=\bar{t}+1}^{m} \frac{c\varphi f^t}{1-c} & \text{if } t = \bar{t} \\ \frac{1}{n}\left[\overline{H}(S^t) - \overline{H}(S^{t+1})\right] & \text{if } t^0 \leq t < \bar{t} \\ 0 \text{ if } t < t^0 & \end{cases},$$

Let $\hat{\beta}(S) > 0$ if $S = S^t$ for $t^0 \leq t \leq \bar{t}$; and $\hat{\beta}(S) = 0$ otherwise. If $t < t^0$, then let $\hat{\lambda}^t = 0$ and $\hat{\mu}^t = k/c - v^t \geq 0$. If $t^0 \leq t \leq \bar{t}$, then let $\hat{\lambda}^t = \hat{\mu}^t = 0$, $\hat{\beta}(S^t) = (v^t - v^{t-1})/n$ for $t > t^0$ and $\hat{\beta}(S^{t^0}) = (v^{t^0} - k/c)/n$. If $t > \bar{t}$, then let $\hat{\lambda}^t = v^t - v^{\bar{t}}$ and $\hat{\mu}^t = 0$. It is easy to verify that this is a feasible solution to $(DOPTm1 - \varphi)$, and the complementary slackness conditions are satisfied. Finally, the dual objective is equal to the primal objective:

$$\frac{1}{n}\overline{H}(S^{t^0})\left(v^{t^0} - \frac{k}{c}\right) + \sum_{t=t^0+1}^{\bar{t}} \frac{1}{n}\overline{H}(S^t)\left(v^t - v^{t-1}\right) + \sum_{t=\bar{t}+1}^{m} \frac{c\varphi f^t}{1-c}\left(v^t - \frac{k}{c}\right) + \varphi \sum_{t=1}^{m} f^t v^t.$$

By the duality theorem, $\hat{z}$ is an optimal solution to $(OPTm1 - \varphi)$. ■

Lemma 3 *An optimal solution to $(OPT - \varphi)$ exists.*

Proof. Let $\mathscr{D}$ denote the set of feasible solutions, i.e., solutions satisfying (IC′) and (F2). Consider $\mathscr{D}$ as a subset of L_2, the set of square integrable functions with respect to the probability measure corresponding to F. Topologize L_2 with its weak*, or $\sigma(L_2, L_2)$, topology. It is straightforward to verify that $\mathscr{D}$ is $\sigma(L_2, L_2)$ compact. See, for example, Border (1991).

Let $V(\varphi) := \sup_{P\in\mathscr{D}} \mathbb{E}_v \left[P(v)\left(v - \frac{k}{c}\right)\right] + \frac{\varphi k}{c}$. Let $\{P_\nu\}$ be a sequence of feasible solutions to $(OPT - \varphi)$ such that

$$\int P_\nu(v)\left(v - \frac{k}{c}\right) \mathrm{d}F(v) + \frac{\varphi k}{c} \to V(\varphi).$$

By Helly's selection theorem, after taking subsequences, I can assume that there exists P such that $\{P_\nu\}$ converges pointwise to P. Because $\mathscr{D}$ is $\sigma(L_2, L_2)$ compact, after taking subsequences again, I can assume that there exists $P \in \mathscr{D}$ such that $\{P_\nu\}$ converges to P in $\sigma(L_2, L_2)$ topology. Because $v - k/c \in L_2$, the weak convergence of $\{P_\nu\}$ implies that

$$\int P(v)\left(v - \frac{k}{c}\right) \mathrm{d}F(v) + \frac{\varphi k}{c} = V(\varphi).$$

■

Proof of Theorem 1. Let $\{P_m\}$ be the sequence of optimal solutions to $(OPTm - \varphi)$ defined in Corollary 1. Let $\overline{P}_m^t := \overline{z}^t/f^t + \varphi$ for all t. Then

$$P_m^t := \begin{cases} \overline{P}_m^t & \text{if } v^t > \frac{k}{c} \\ \varphi & \text{if } v^t < \frac{k}{c} \end{cases}.$$

Recall that $\overline{H}$ is given by (23). Thus, there are three cases. If $\overline{t} > \underline{t}$, then

$$\overline{P}_m^t = \begin{cases} \frac{\varphi}{1-c} & \text{if } t > \overline{t} \\ \frac{\frac{1}{n} - \frac{1}{n}\left(\sum_{\tau=1}^{\overline{t}} f^\tau\right)^n - \sum_{\tau=\overline{t}+1}^{m} \frac{\varphi f^t}{1-c}}{f^{\overline{t}}} & \text{if } t = \overline{t} \\ \frac{\frac{1}{n}\left(\sum_{\tau=1}^{t} f^\tau\right)^n - \frac{1}{n}\left(\sum_{\tau=1}^{t-1} f^\tau\right)^n}{f^t} & \text{if } \underline{t} < t < \overline{t} \\ \frac{\frac{1}{n}\left(\sum_{\tau=1}^{t} f^\tau\right)^n - \varphi \sum_{\tau=1}^{t-1} f^\tau}{f^t} & \text{if } t = \underline{t} \\ \varphi & \text{if } t < \underline{t} \end{cases}.$$

If $\overline{t} = \underline{t}$, then

$$\overline{P}_m^t = \begin{cases} \frac{\varphi}{1-c} & \text{if } t > \overline{t} \\ \frac{\frac{1}{n} - \varphi \sum_{t=1}^{t-1} f^t - \sum_{\tau=t+1}^{m} \frac{\varphi f^t}{1-c}}{f^{\overline{t}}} & \text{if } t = \overline{t} \\ \varphi & \text{if } t < \overline{t} \end{cases}.$$

If $\overline{t} < \underline{t}$, then

$$\overline{P}_m^t = \begin{cases} \frac{\varphi}{1-c} & \text{if } t > \overline{t} \\ \varphi & \text{if } t < \overline{t} \end{cases}.$$

I can extend P_m to V by setting

$$P_m(v) := P_m^t \text{ for } v \in \left[\underline{v} + \frac{(t-1)(\overline{v}-\underline{v})}{m}, \underline{v} + \frac{t(\overline{v}-\underline{v})}{m}\right], t = 1, \ldots, m.$$

Extend $\overline{P}_m$ to V in a similar fashion. Compare $\overline{P}_m$ and $\overline{P}_\varphi$. It is easy to see that $\{\overline{P}_m\}$ converges pointwise to $\overline{P}_\varphi$. Hence, $\{P_m\}$ converges pointwise to P_φ^*, which is a feasible solution to $(OPT - \varphi)$.

To show the optimality of P_φ^*, let $\hat{P}$ be an optimal solution to $(OPT - \varphi)$, which exists by Lemma 3 in the appendix. Define $\hat{P}_m$ be such that

$$\hat{P}_m^t := \frac{1}{f^t} \int_{\underline{v} + \frac{(t-1)(\overline{v}-\underline{v})}{m}}^{\underline{v} + \frac{t(\overline{v}-\underline{v})}{m}} \hat{P}(v) \mathrm{d}F(v) \text{ for } t = 1, \ldots, m,$$

and it can be extended to V by setting

$$\hat{P}_m(v) := \hat{P}_m^t \text{ for } v \in \left[\underline{v} + \frac{(t-1)(\overline{v}-\underline{v})}{m}, \underline{v} + \frac{t(\overline{v}-\underline{v})}{m}\right], t = 1, \ldots, m.$$

By the Lebesgue differentiation theorem, $\{\hat{P}_m\}$ converges pointwise to $\hat{P}$. It is easy to verify that $\hat{P}_m$ defined on $\{v^1, \ldots, v^m\}$ is a feasible solution to $(OPT - \varphi)$. Hence

$$\sum_{t=1}^{m} f^t \hat{P}_m^t \left(v^t - \frac{k}{c}\right) + \frac{\varphi k}{c} \leq \sum_{t=1}^{m} f^t P_m^t \left(v^t - \frac{k}{c}\right) + \frac{\varphi k}{c}$$

By the dominated convergence theorem,

$$\sum_{t=1}^{m} f^t \hat{P}_m^t \left(v^t - \frac{k}{c}\right) = \int_V \hat{P}_m(v) \left(v - \frac{k}{c}\right) \mathrm{d}F(v) \to \int_V \hat{P}(v) \left(v - \frac{k}{c}\right) \mathrm{d}F(v),$$

and

$$\sum_{t=1}^{m} f^t P_m^t \left(v^t - \frac{k}{c}\right) = \int_V P_m(v) \left(v - \frac{k}{c}\right) \mathrm{d}F(v) \to \int_V P_\varphi^*(v) \left(v - \frac{k}{c}\right) \mathrm{d}F(v).$$

Hence,

$$\int_V P_\varphi^*(v) \left(v - \frac{k}{c}\right) \mathrm{d}F(v) = \int_V \hat{P}(v) \left(v - \frac{k}{c}\right) \mathrm{d}F(v),$$

which implies that P_φ^* is optimal. ■

Lemma 4 *Suppose that $(1-c)/n \leq \varphi \leq \min\{1/n, 1-c\}$. Then $v^l \geq v^u$ if and only if $v^l \leq v^\natural$, where $v^\natural$ is defined by (12). Furthermore, if $n(1-c) < 1$, then $v^\natural$ is strictly increasing in n*

and strictly decreasing in c.

Proof. Because $(1-c)/n \leq \varphi \leq \min\{1/n, 1-c\}$, v^l and v^u satisfies:

$$\frac{1-F(v^u)^n}{1-F(v^u)} = \frac{F(v^l)^{n-1}}{1-c}. \tag{24}$$

Define

$$\Delta(v) := \frac{F(v)^{n-1}(1-F(v))}{1-c} - 1 + F(v)^n.$$

Then $\Delta(\underline{v}) = -1 < 0$ and $\Delta(\overline{v}) = 0$. Then

$$\Delta'(v) = \frac{F(v)^{n-2}f(v)}{1-c}\left[-cnF(v) + n - 1\right].$$

Clearly, the term in the brackets is strictly decreasing in v. Moreover, $\Delta'(\underline{v}) = n-1 > 0$ and $\Delta'(\overline{v}) = n(1-c) - 1$.

If $n(1-c) \geq 1$, then $\Delta'(v) \geq 0$ for all v. Hence, $\Delta(v)$ is non-decreasing, and therefore $\Delta(v) \leq 0$ for all v. Hence,

$$\frac{1-F(v^u)^n}{1-F(v^u)} = \frac{F(v^l)^{n-1}}{1-c} \leq \frac{1-F(v^l)^n}{1-F(v^l)},$$

which implies $v^l \geq v^u$.

If $n(1-c) < 1$, then there exists $v^\flat$ such that $\Delta'(v) > 0$ for $v \in [\underline{v}, v^\flat]$ and $\Delta'(v) < 0$ for $v \in [v^\flat, \overline{v}]$. Hence, $\Delta(v)$ in strictly increasing in $[\underline{v}, v^\flat]$, and strictly decreasing in $[v^\flat, \overline{v}]$. Hence, there exists a unique $v^\natural \in (\underline{v}, \overline{v})$ such that $\Delta(v) \leq 0$ if and only if $v \leq v^\natural$. By (24), this implies that $v^l \geq v^u$ if and only if $v^l \leq v^\natural$. Finally, for any v, $\Delta(v)$ is strictly decreasing in n and strictly increasing in c. Hence, $v^\natural$ is strictly increasing in n and strictly decreasing in c. ■

Proof of Theorem 3. First, if $\varphi \leq (1-c)/n$, then $v^u = \hat{v} = \underline{v}$, and

$$P^*_\varphi(v) := \begin{cases} \frac{\varphi}{1-c} & \text{if } v \geq \frac{k}{c} \\ \varphi & \text{if } v < \frac{k}{c} \end{cases}.$$

The principal's objective becomes

$$\frac{c\varphi}{1-c}\int_{\frac{k}{c}}^{\overline{v}}\left(v - \frac{k}{c}\right)\mathrm{d}F(v) + \varphi\int_{\underline{v}}^{\overline{v}} v\mathrm{d}F(v),$$

which is strictly increasing in φ. Hence, in optimum, $\varphi \geq (1-c)/n$.

Given φ, let $Z(\varphi)$ denote the principal's optimal payoff. Suppose that $\varphi \geq 1-c$ or equivalently $F(v^l)^{n-1} \geq n(1-c)$. Then $v^u = \overline{v}$ and the principal's payoff is $Z(\varphi) = Z_1(v^l(\varphi))$, where

$$Z_1(v^l) := \int_{\max\left\{v^l, \frac{k}{c}\right\}}^{\overline{v}} \left(v - \frac{k}{c}\right) F(v)^{n-1} \mathrm{d}F(v) + \frac{1}{n} F(v^l)^{n-1} \int_{\underline{v}}^{\max\left\{v^l, \frac{k}{c}\right\}} \left(v - \frac{k}{c}\right) \mathrm{d}F(v) + \frac{1}{n} F(v^l)^{n-1} \frac{k}{c}.$$

If $v^l < k/c$, then $Z_1(v^l)$ is strictly increasing in v^l. If $v^l \geq k/c$, then

$$Z_1'(v^l) = \frac{n-1}{n} F(v^l)^{n-2} f(v^l) \left\{ \mathbb{E}_v[v] - \mathbb{E}_v[\max\{v, v^l\}] + \frac{k}{c} \right\}.$$

Clearly, the term inside the braces is strictly decreasing in v^l. Recall that $v^* \geq k/c$ is defined by (10). Hence, $Z_1'(v^l) \geq 0$ if and only if $v^l \leq v^*$, and Z_1 achieves its maximum at $v^l = v^*$. I show in Lemma 5 below that, for any φ and the corresponding v^l, we have $Z(\varphi) \leq Z_1(v^l(\varphi))$. Hence,

$$Z(\varphi) \leq Z_1(v^l(\varphi)) \leq Z_1(v^*).$$

Thus, if $F(v^*)^{n-1} \geq n(1-c)$, then it is optimal to set $\varphi^* = F(v^*)^{n-1}/n$ and $v^l = v^*$. This proves the first part of Theorem 3.

Suppose that $F(v^*)^{n-1} < n(1-c)$. Then in optimum $\varphi \leq 1-c$. Because $(1-c)/n \leq \varphi \leq 1/n$, there is a one-to-one correspondence between $\hat{v}$ and φ. Given φ, $\hat{v}(\varphi)$ is uniquely pinned down by

$$1 - n\varphi F(\hat{v}) - \frac{n\varphi}{1-c} [1 - F(\hat{v})] = 0.$$

If φ is such that $v^l \geq v^u$, then $Z(\varphi) = Z_2(\hat{v}(\varphi))$, where

$$Z_2(\hat{v}) := \frac{1-c}{n(1-cF(\hat{v}))} \int_{\underline{v}}^{\max\left\{\hat{v}, \frac{k}{c}\right\}} \left(v - \frac{k}{c}\right) \mathrm{d}F(v) + \frac{1}{n(1-cF(\hat{v}))} \int_{\max\left\{\hat{v}, \frac{k}{c}\right\}}^{\overline{v}} \left(v - \frac{k}{c}\right) \mathrm{d}F(v) + \frac{1-c}{n(1-cF(\hat{v}))} \frac{k}{c}.$$

If $\hat{v} < k/c$, then $Z_2(\hat{v})$ is strictly increasing in $\hat{v}$. If $\hat{v} \geq k/c$, then

$$Z_2'(\hat{v}) = \frac{cf(\hat{v})}{n(1-cF(\hat{v}))^2}\left\{\mathbb{E}_v[v] - \mathbb{E}_v[\min\{v,\hat{v}\}] + (1-c)\left[\mathbb{E}_v[v] - \mathbb{E}_v[\max\{v,\hat{v}\}] + \frac{k}{c}\right]\right\}.$$

Clearly, the term inside the braces is strictly decreasing in $\hat{v}$. Recall that $v^{**} > v^* \geq k/c$ is defined by (11). Hence, $Z_2'(\hat{v}) \geq 0$ if and only if $\hat{v} \leq v^{**}$, and Z_2 achieves its maximum at $\hat{v} = v^{**}$. I show in Lemma 6 below that, for any $\varphi \leq 1-c$ and the corresponding $\hat{v}$, we have $Z(\varphi) \leq Z_2(\hat{v}(\varphi))$. Hence,

$$Z(\varphi) \leq Z_2(\hat{v}(\varphi)) \leq Z_2(v^{**}).$$

Finally, by Lemma 4, $v^l \geq v^u$ if and only if $v^l \leq v^\natural$. Thus, if $v^{**} \leq v^\natural$, then it is optimal to set $\varphi^* = (1-c)/n(1-cF(v^{**}))$ and $\hat{v} = v^{**}$. This proves the second part of Theorem 3.

Suppose that $F(v^*)^{n-1} < n(1-c)$ and $v^{**} > v^\natural$. Then

$$\begin{aligned} Z(\varphi) =& \varphi\int_{\underline{v}}^{\max\left\{v^l,\frac{k}{c}\right\}}\left(v-\frac{k}{c}\right)\mathrm{d}F(v) + \int_{\max\left\{v^l,\frac{k}{c}\right\}}^{\max\left\{v^u,\frac{k}{c}\right\}}\left(v-\frac{k}{c}\right)F(v)^{n-1}\mathrm{d}F(v) \\ &+ \frac{\varphi}{1-c}\int_{\max\left\{v^u,\frac{k}{c}\right\}}^{\overline{v}}\left(v-\frac{k}{c}\right)\mathrm{d}F(v) + \frac{\varphi k}{c}. \end{aligned}$$

If φ is such that $v^l < k/c$, then $Z(\varphi)$ is strictly increasing in φ. If $v^l \geq k/c$, then

$$Z'(\varphi) = \frac{1}{1-c}\left[\mathbb{E}_v[v] - \mathbb{E}_v[\min\{v,v^u\}]\right] + \mathbb{E}_v[v] - \mathbb{E}_v[\max\{v,v^l\}] + \frac{k}{c}.$$

Because both v^l and v^u are strictly increasing in φ, $Z'(\varphi)$ is strictly decreasing in φ. Let φ^* be such that

$$\mathbb{E}_v[v] - \mathbb{E}_v[\min\{v,v^u(\varphi^*)\}] + (1-c)\left[\mathbb{E}_v[v] - \mathbb{E}_v[\max\{v,v^l(\varphi^*)\}] + \frac{k}{c}\right] = 0. \qquad (13)$$

Compare (13) with (11) and (10). It is easy to see that $v^u(\varphi^*) > v^{**} > v^l(\varphi^*) > v^*$. Hence, $Z'(\varphi) \geq 0$ if and only if $\varphi \leq \varphi^*$, and Z achieves its maximum at $\varphi = \varphi^*$. This proves the third part of Theorem 3. ■

Definition 1 **(Karamardian and Schaible (1990))** *A function $g(v)$ is* quasi-monotone *if $v' > v$ and $g(v) > 0$ imply $g(v') \geq 0$.*

Lemma 5 *Let Z and Z_1 be defined as in the proof of Theorem 3. Then $Z(\varphi) \leq Z_1(v^l(\varphi))$.*

Proof. Fix φ and the corresponding v^l. Note that $Z_1(v^l)$ is attained by the following allocation rule

$$P_1(v) := \begin{cases} F(v)^{n-1} & \text{if } v \geq \max\left\{v^l, \frac{k}{c}\right\} \\ \varphi & \text{if } v < \max\left\{v^l, \frac{k}{c}\right\} \end{cases}.$$

It is easy to see that $P_1 - P^*_\varphi$ is quasi-monotone and

$$\int_{\underline{v}}^{\overline{v}} P_1(v)\mathrm{d}F(v) = \int_{\underline{v}}^{\overline{v}} P^*_\varphi(v)\mathrm{d}F(v) = \frac{1}{n}.$$

Moreover, $v - k/c$ is non-decreasing in v. Hence, by Lemma 1 in Persico (2000), $Z(\varphi) \leq Z_1(v^l(\varphi))$. ■

Lemma 6 *Let Z and Z_2 be defined as in the proof of Theorem 3. If $\varphi \leq 1 - c$, then $Z(\varphi) \leq Z_2(\hat{v}(\varphi))$.*

Proof. Fix φ and the corresponding $\hat{v}$. Note that $Z_2(\hat{v})$ is attained by the following allocation rule

$$P_2(v) := \begin{cases} \frac{\varphi}{1-c} & \text{if } v \geq \max\left\{\hat{v}, \frac{k}{c}\right\} \\ \varphi & \text{if } v < \max\left\{\hat{v}, \frac{k}{c}\right\} \end{cases}.$$

It is easy to see that $P_2 - P^*_\varphi$ is quasi-monotone and

$$\int_{\underline{v}}^{\overline{v}} P_2(v)\mathrm{d}F(v) = \int_{\underline{v}}^{\overline{v}} P^*_\varphi(v)\mathrm{d}F(v) = \frac{1}{n},$$

Moreover, $v - k/c$ is non-decreasing in v. Hence, by Lemma 1 in Persico (2000), $Z(\varphi) \leq Z_1(v_2(\hat{\varphi}))$. ■

Proof of Theorem 2. Let Z and Z_1 be defined as in the proof of Theorem 3. If $\overline{v} - k/c \leq \mathbb{E}_v[v]$, then $Z_1(v)$ is strictly increasing in v^l and achieves its maximum when $v^l = \overline{v}$. By Lemma 5, $Z(\varphi) \leq Z_1(v^l(\varphi)) \leq Z_1(\overline{v})$. Note that $Z_1(\overline{v})$ can be achieved via pure randomization. This completes the proof. ■

B Omitted proofs in Section 4

Proof of Corollary 2. The analysis in Section 4 has proved most results of Corollary 2. What is left to prove is that if $n^*(\rho, c) < n < n^{**}(\rho, c)$, then $v^l(n, \rho, c)$ is strictly increasing in n, ρ and c and $v^u(n, \rho, c)$ is strictly decreasing in n and strictly increasing in ρ and c. If $n^*(\rho, c) < n < n^{**}(\rho, c)$, then v^l and v^u satisfy (24). By (24), v^u is strictly increasing in v^l and vice versa.

To prove the properties of v^l, let

$$\Delta_l(v^l, n, \rho, c) := \mathbb{E}_v[v] - \mathbb{E}_v[\min\{v, v^u\}] + (1-c)\left[\mathbb{E}_v[v] - \mathbb{E}_v[\max\{v, v^l\}] + \rho\right],$$

where v^u is a function of v^l, n and c defined by (24). Then $\Delta_l(v^l, n, \rho, c) \equiv 0$ by (13). Furthermore, we have

$$\begin{aligned}
\frac{\partial \Delta_l}{\partial v^l} &= -[1 - F(v^u)]\frac{\partial v^u}{\partial v^l} - (1-c)F(v^l) < 0, \\
\frac{\partial \Delta_l}{\partial n} &= -[1 - F(v^u)]\frac{\partial v^u}{\partial n} > 0, \\
\frac{\partial \Delta_l}{\partial \rho} &= 1 - c > 0, \\
\frac{\partial \Delta_l}{\partial c} &= -\left[\mathbb{E}_v[v] - \mathbb{E}_v[\max\{v, v^l\}] + \rho\right] > 0.
\end{aligned} \tag{25}$$

Hence, by the implicit function theorem, we have $\partial v^l/\partial n > 0$, $\partial v^l/\partial \rho > 0$ and $\partial v^l/\partial c > 0$. To see that $\partial v^u/\partial n < 0$ in the second line in (25), let

$$\Delta(v^u, v^l, n) := \frac{F(v^l)^{n-1}(1 - F(v^u))}{1-c} - 1 + F(v^u)^n.$$

Then $\Delta(v^u, v^l, n) \equiv 0$ by (24). Furthermore, we have

$$\begin{aligned}
\frac{\partial \Delta}{\partial v^u} &= \left[-\frac{F(v^l)^{n-1}}{1-c} + nF(v^u)^{n-1}\right] f(v^u) = \left[-\frac{1 - F(v^u)^n}{1 - F(v^u)} + nF(v^u)^{n-1}\right] f(v^u) < 0, \\
\frac{\partial \Delta}{\partial n} &= \frac{F(v^l)^{n-1}[1 - F(v^u)]\log F(v^l)}{1-c} + F(v^u)^n \log F(v^u) < 0.
\end{aligned}$$

Hence, by the implicit function theorem, $\partial v^u/\partial n = -(\partial\Delta/\partial n)/(\partial\Delta/\partial v^u) < 0$.

To prove the properties of v^u, let

$$\Delta_u(v^u, n, \rho, c) := \mathbb{E}_v[v] - \mathbb{E}_v[\min\{v, v^u\}] + (1-c)\left[\mathbb{E}_v[v] - \mathbb{E}_v[\max\{v, v^l\}] + \rho\right],$$

where v^l is a function of v^u, n and c defined by (24). Then $\Delta_u(v^u, n, \rho, c) \equiv 0$ by (24).

Furthermore, we have

$$\begin{aligned}
\frac{\partial \Delta_u}{\partial v^u} &= -[1 - F(v^u)] - (1-c)F(v^l)\frac{\partial v^l}{\partial v^u} < 0, \\
\frac{\partial \Delta_u}{\partial n} &= -(1-c)F(v^l)\frac{\partial v^l}{\partial n} < 0, \\
\frac{\partial \Delta_u}{\partial \rho} &= 1 - c > 0, \\
\frac{\partial \Delta_u}{\partial c} &= -\left[\mathbb{E}_v[v] - \mathbb{E}_v[\max\{v, v^l\}] + \rho\right] > 0.
\end{aligned} \tag{26}$$

Hence, by the implicit function theorem, we have $\partial v^u/\partial n < 0$, $\partial v^u/\partial \rho > 0$ and $\partial v^u/\partial c > 0$. To see that $\partial v^l/\partial n > 0$ in the second line in (26), note that

$$\frac{\partial \Delta}{\partial v^l} = \frac{(n-1)F(v^l)^{n-2}f(v^l)[1-F(v^u)]}{1-c} > 0.$$

Hence, by the implicit function theorem, $\partial v^l/\partial n = -(\partial \Delta/\partial n)/(\partial \Delta/\partial v^l) > 0$. ■

C Asymmetric environment

C.1 Finite case

Let $\mathcal{D} := \cup_i[\underline{v}_i - k_i/c_i, \overline{v}_i - k_i/c_i]$. Let $\underline{d} := \inf \mathcal{D}$ and $\overline{d} := \sup \mathcal{D}$. Fix an integer $m \geq 2$. For $t = 1, \ldots m$, let

$$\begin{aligned}
d^t :=& \underline{d} + \frac{(2t-1)(\overline{d} - \underline{d})}{2m}, \\
f_i^t :=& F_i\left(\underline{d} + \frac{t(\overline{d} - \underline{d})}{m} + \frac{k_i}{c_i}\right) - F_i\left(\underline{d} + \frac{(t-1)(\overline{d} - \underline{d})}{m} + \frac{k_i}{c_i}\right), i = 1, \ldots, n.
\end{aligned}$$

Consider the finite model in which, for each agent i, $v_i - k_i/c_i$ can take only m possible different values (i.e. $v_i - k_i/c_i \in \{d^1, \ldots, d^m\}$) and the probability mass function satisfies $f_i(d^t) = f_i^t$ for $t = 1, \ldots, m$. It is possible that $f_i^t = 0$ for some t. The corresponding problem of $(OPTA - \varphi)$ in the finite model, denoted by $(OPTAm - \varphi)$, is given by:

$$\max_P \sum_{i=1}^n \left[\sum_{t=1}^m f_i^t P_i^t d^t + \frac{\varphi_i k_i}{c_i}\right],$$

subject to

$$\varphi_i \le P_i^t \le \frac{\varphi_i}{1-c_i}, \forall t, \tag{AIC$'m$}$$

$$\sum_{i=1}^{n}\sum_{t\in S_i} f_i^t P_i^t \le 1 - \prod_{i=1}^{n}\sum_{t\notin S_i} f_i^t, \forall S_i \subset \{1,\dots,m\}. \tag{AF2m}$$

Define $H(\mathcal{S}) := 1 - \prod_{i=1}^{n}\sum_{t\notin S_i} f_i^t - \sum_{i=1}^{n}\sum_{t\in S_i}\varphi_i f_i^t$ for all $\mathcal{S} := (S_1,\dots,S_n)$ and $S_i \subset \{1,\dots,m\}$ for all i. Define $\overline{H}(\mathcal{S}) := \min_{\mathcal{S}'\supset\mathcal{S}} H(\mathcal{S}')$ for all $\mathcal{S}$. Let $z_i^t := f_i^t\left(P_i^t - \varphi_i\right)$ for all i and t. By Lemma 2, $(OPTAm-\varphi)$ can be rewritten as $(OPTAm1-\varphi)$

$$\max_z \sum_{i=1}^{n}\sum_{t=1}^{m} z_i^t d^t + \sum_{i=1}^{n}\varphi_i\left(\sum_{t=1}^{m} f_i^t d^t + \frac{k_i}{c_i}\right),$$

subject to

$$0 \le z_i^t \le \frac{c_i\varphi_i f_i^t}{1-c_i}, \forall i, \forall t, \tag{AIC$'m1$}$$

$$\sum_{i=1}^{n}\sum_{t\in S_i} z_i^t \le \overline{H}(\mathcal{S}), \forall \mathcal{S}\subset\{1,\dots,m\}^n. \tag{AF2$m1$}$$

Note that if $f_i^t = 0$, then $z_i^t = 0$ by definition and therefore satisfies (AIC$'m1$) automatically.

Algorithm 1 below describes an algorithm that finds a feasible solution to $(OPTAm-\varphi)$. I start by giving a verbal overview of the algorithm. It is in the spirit of greedy algorithms. It begins by assigning values to $\{z_i^m\}_i$ who have the largest weight d^m in the objective function. Let the set $\mathcal{I}_0^m$ collect all the agents whose highest net values are below d^m. If $i \in \mathcal{I}_0^m$, then $f_i^m = 0$ by definition and $z_i^m = 0$ by (AIC$'m1$). Next check whether there exists some agent $i \notin \mathcal{I}_0^m$ such that if z_i^m is assigned the highest value allowed by (AF2$m1$), the upper-bound on z_i^m in (AIC$'m1$) is respected. If so, assign z_i^m this highest value. Continue until no such agent can be found. Then, among all the agents whose z_i^m have not been assigned values yet, check whether there exists a pair of agents, a triple of agents and etc. until there does not exist a group of agents $\mathcal{I}'$ such that we can assign $\sum_{i\in\mathcal{I}'} z_i^m$ the highest value allowed by (AF2$m1$) while respecting the upper-bounds in (AIC$'m1$). If now there still exists an agent i whose z_i^m has not been assigned a value yet, then let $z_i^m = c_i\varphi_i f_i^t/(1-c_i)$ (the upper-bound on z_i^m in (AIC$'m1$)). Let the set $\mathcal{I}_1^m$ collect all the agents not in $\mathcal{I}_0^m$ and for whom the upper-bounds on z_i^m in (AIC$'m1$) do not bind. Continue to assign values to $\{z_i^{m-1}\}_i, \{z_i^{m-2}\}_i, \dots \{z_i^1\}_i$ in the same fashion.

In order to define the algorithm formally, I introduce some notations. Let $S_i^t := \{t,\dots,m\}$

and $S_i^{m+1} := \emptyset$ for all i and t, $\mathcal{S}^t := \{t, \ldots, m\}^n$ for all t and $\mathcal{S}^{m+1} := \emptyset$. Define $\mathcal{S} + (t, i) := (S_1, \ldots, S_{i-1}, S_i \cup \{t\}, S_{i+1} \ldots, S_n)$ and $\mathcal{S} - (t, i) := (S_1, \ldots, S_{i-1}, S_i \backslash \{t\}, S_{i+1} \ldots, S_n)$.

Algorithm 1 *Let $\mathcal{I}_0^m := \{i \,|\, f_i^m = 0\}$ and $\overline{z}_i^m := 0$ for all $i \in \mathcal{I}_0^m$. Define $\mathcal{I}_1^m \subset \mathcal{I} \backslash \mathcal{I}_0^m$, n^m, $\{\pi^{m,1}, \ldots, \pi^{m,n^m}\}$, $\{\mathcal{S}^{m,1}, \ldots, \mathcal{S}^{m,n^m}\}$ and $\overline{z}_i^m$ for all $i \notin \mathcal{I}_0^m$ recursively as follows.*

1. *Let $\mathcal{I}_1^m = \emptyset$ and $\nu = 1$.*

2. *If $\mathcal{I}_1^m = \mathcal{I} \backslash \mathcal{I}_0^m$, then go to step 5. Otherwise, let $\iota = 1$ and go to step 2.*

3. *If there exists $\mathcal{I}' \neq \emptyset$ such that $|\mathcal{I}'| = \iota$, $\mathcal{I}' \cap (\mathcal{I}_0^m \cup \mathcal{I}_1^m) = \emptyset$ and*

$$\overline{H}\left(\mathcal{S} + \sum_{i \in \mathcal{I}'} (m, i)\right) - \overline{H}(\mathcal{S}) \leq \sum_{i \in \mathcal{I}'} \frac{c_i \varphi_i f_i^m}{1 - c_i},$$

where $S_j = S_j^m$ if $j \in \mathcal{I}_1^m$ and $S_j = S_j^{m+1}$ otherwise, then let $\overline{z}_i^m \leq c_i \varphi_i f_i^m / (1 - c_i)$ for $i \in \mathcal{I}'$ be such that

$$\sum_{i \in \mathcal{I}'} \overline{z}_i^m = \overline{H}\left(\mathcal{S} + \sum_{i \in \mathcal{I}'} (m, i)\right) - \overline{H}(\mathcal{S}).$$

Let $\pi^{m,\nu} := \mathcal{I}'$ and $\mathcal{S}^{m,\nu} := \mathcal{S}$. Redefine ν as $\nu + 1$ and $\mathcal{I}_1^m$ as $\mathcal{I}' \cup \mathcal{I}_1^m$, and go to step 2. If there does not exist such an $\mathcal{I}'$, go to step 4.

4. *If $\iota < n - |\mathcal{I}_0^m \cup \mathcal{I}_1^m|$, then redefine ι as $\iota + 1$ and go to step 3. If $\iota = n - |\mathcal{I}_0^m \cup \mathcal{I}_1^m|$, then go to step 5.*

5. *Let $n^m := \nu - 1$ and $\overline{z}_i^m := c_i \varphi_i f_i^m / (1 - c_i)$ for all $i \in \mathcal{I} \backslash (\mathcal{I}_0^m \cup \mathcal{I}_1^m)$.*

Let $1 \leq t \leq m - 1$. Suppose that we have defined $\mathcal{I}_0^\tau$, $\mathcal{I}_1^\tau$, n^τ, $\{\pi^{\tau,1}, \ldots, \pi^{\tau,n^\tau}\}$, $\{\mathcal{S}^{\tau,1}, \ldots, \mathcal{S}^{\tau,n^\tau}\}$ and $\{\overline{z}_i^\tau\}_i$ for all $\tau \geq t + 1$. Let $\mathcal{I}_0^t := \{i | f_i^t = 0\}$ and $\overline{z}_i^t := 0$ for all $i \in \mathcal{I}_0^t$. Define $\mathcal{I}_1^t \subset \mathcal{I} \backslash \mathcal{I}_0^t$, $\{\pi^{t,1}, \ldots, \pi^{t,n^t}\}$, $\{\mathcal{S}^{t,1}, \ldots, \mathcal{S}^{t,n^t}\}$ and $\overline{z}_i^t$ for all $i \notin \mathcal{I}_0^t$ recursively as follows.

1. *Let $\mathcal{I}_1^t := \emptyset$ and $\nu = 1$.*

2. *If $\mathcal{I}_1^t = \mathcal{I} \backslash \mathcal{I}_0^t$, then go to step 5. Otherwise, let $\iota = 1$ and go to step 2.*

3. *If there exists $\mathcal{I}' \neq \emptyset$ such that $|\mathcal{I}'| = \iota$, $\mathcal{I}' \cap (\mathcal{I}_0^t \cup \mathcal{I}_1^t) = \emptyset$ and*

$$\min_{\mathcal{S}} \overline{H}\left(\mathcal{S} + \sum_{i \in \mathcal{I}'} (t, i)\right) - \sum_{j=1}^{n} \sum_{\tau \in S_j} \overline{z}_j^\tau \leq \sum_{i \in \mathcal{I}'} \frac{c_i \varphi_i f_i^t}{1 - c_i},$$

where $\mathcal{S} = (S_1^{t_1}, \ldots, S_n^{t_n})$ *with* $t_j \geq t$ *if* $j \in \pi^{t,1} \cup \cdots \cup \pi^{t,\nu-1}$, $t_j = t+1$ *if* $j \in \mathcal{I}'$ *and* $t_j \geq t+1$ *otherwise, then let* $\overline{z}_i^t \leq c_i \varphi_i f_i^t/(1-c_i)$ *for* $i \in \mathcal{I}'$ *be such that*

$$\sum_{i \in \mathcal{I}'} \overline{z}_i^t = \min_{\mathcal{S}} \overline{H}\left(\mathcal{S} + \sum_{i \in \mathcal{I}'} (m,i)\right) - \sum_{i=1}^n \sum_{\tau \in S_i} \overline{z}_i^\tau.$$

Let $\pi^{t,\nu} := \mathcal{I}'$ *and* $\mathcal{S}^{t,\nu}$ *as a minimizer of the right-hand side of the above equation such that there is no* $\mathcal{S} \supsetneq \mathcal{S}^{t,\nu}$ *which is also a minimizer. Redefine* ν *as* $\nu+1$ *and* $\mathcal{I}_1^t$ *as* $\mathcal{I}' \cup \mathcal{I}_1^t$, *and go to step 2. If there does not exist such an* $\mathcal{I}'$, *then go to step 4.*

4. *If* $\iota < n - |\mathcal{I}_0^t \cup \mathcal{I}_1^t|$, *then redefine* ι *as* $\iota + 1$ *and go to step 3. If* $\iota = n - |\mathcal{I}_0^t \cup \mathcal{I}_1^t|$, *then go to step 5.*

5. *Let* $n^t := \nu - 1$ *and* $\overline{z}_i^t := c_i \varphi_i f_i^t/(1-c_i)$ *for all* $i \in \mathcal{I} \backslash (\mathcal{I}_0^t \cup \mathcal{I}_1^t)$.

Note that $\{S^{t,\nu} + \sum_{i \in \pi^{t,\nu}} (t,i)\}$ is the collection of sets for which (AF2m1) bind.

Let $\overline{z}$ be a solution found by Algorithm 1. I first prove that $\overline{z}$ is a feasible solution to ($OPTAm1-\varphi$). For each i and t, let $\overline{P}_i^t := \overline{z}_i^t/f_i^t + \varphi_i$ if $f_i^t > 0$ and $\overline{P}_i^t := 0$ otherwise. Then $\overline{z}$ is a feasible solution to ($OPTAm1-\varphi$) if and only if $\overline{P}$ is a feasible solution to ($OPTAm-\varphi$). Lemma 11 below proves that $\overline{P}$ is non-decreasing. By Theorem 2 in Mierendorff (2011), $\overline{P}$ is a feasible solution to ($OPTAm-\varphi$) if and only if for all $t_1, \ldots, t_n \in \{1, \ldots, m\}$

$$\sum_{i=1}^n \sum_{t \in S_i} \overline{P}_i^t \leq 1 - \prod_{i=1}^n \sum_{t \notin S_i^{t_i}} f_i^t.$$

By construction, this is true if and only if for all $t_1, \ldots, t_n \in \{1, \ldots, m\}$,

$$\sum_{i=1}^n \sum_{t \in S_i} \overline{z}_i^t \leq \overline{H}(\mathcal{S}), \tag{27}$$

where $\mathcal{S} = (S_1^{t_1}, \ldots, S_n^{t_n})$. Lemma 10 below proves that $\overline{z}$ satisfies (27).

Hence, $\overline{P}$ is a feasible solution to ($OPTAm-\varphi$), or equivalently, $\overline{z}$ is a feasible solution to ($OPTAm1-\varphi$). For each i and t, let

$$\hat{z}_i^t := \begin{cases} \overline{z}_i^t & \text{if } d^t \geq 0 \\ 0 & \text{if } d^t < 0 \end{cases}. \tag{28}$$

Clearly, $\hat{z}$ is also a feasible solution to ($OPTAm1-\varphi$). Furthermore, one can verify that $\hat{z}$ is an optimal solution to ($OPTAm1-\varphi$) by the duality theorem:

Lemma 7 *$\hat{z}$ define in (28) is an optimal solution to $(OPTAm1 - \varphi)$.*

The formal proof of Lemma 7 can be found in Appendix C.1.3. Finally, let $\boldsymbol{P}^m := (P_i^{m,t})_{i,t}$, where

$$P_i^{m,t} := \begin{cases} \overline{P}_i^{m,t} & \text{if } d^t \geq 0 \\ \varphi_i & \text{if } d^t < 0 \end{cases}. \tag{29}$$

The following corollary directly follows from Lemma 7:

Corollary 5 *$\boldsymbol{P}^m$ defined in (29) is an optimal solution to $(OPTAm - \varphi)$.*

The rest of this subsection is organized as follows. In Appendix C.1.1, I prove two technical lemmas on H and $\overline{H}$, which are useful for later proofs. In Appendix C.1.2, I prove that $\overline{z}$ is a feasible solution to $(OPTAm - \varphi)$ by proving Lemmas 10 and 11. In Appendix C.1.3, I prove that $\hat{z}$ is an optimal solution to $(OPTAm - \varphi)$. In Appendix C.1.4, I prove two technical lemmas that are useful in characterizing the limit of $\{P^m\}$.

C.1.1 Properties of H and $\overline{H}$

Here, I introduce two technical lemmas on H and $\overline{H}$. Lemma 8 proves a useful property of H. Lemma 9 characterizes $\overline{H}$.

Lemma 8 *If $H(\mathcal{S}) < 1 - \sum_{i=1}^n \varphi_i$ and $\mathcal{S}' \subset \mathcal{S}$, then $H(\mathcal{S}') \leq H(\mathcal{S})$.*

Proof. Consider $\mathcal{S} = (S_1, \ldots, S_n)$. We have

$$H(\mathcal{S}) - 1 + \sum_{i=1}^n \varphi_i = \sum_{i=1}^n \varphi_i \sum_{\tau \notin S_i} f_i^\tau - \prod_{i=1}^n \sum_{\tau \notin S_i} f_i^\tau.$$

Let $\mathcal{S}_i^{supp} := \{t | f_i^t > 0\}$. If $S_i = S_i^{supp}$ for some i, then $\sum_{\tau \notin S_i} f_i^\tau = 0$ and therefore $H(\mathcal{S}) \geq 1 - \sum_{i=1}^n \varphi_i$. Hence, $H(\mathcal{S}) < 1 - \sum_{i=1}^n \varphi_i$ implies that $S_i \neq S_i^{supp}$ or $\sum_{\tau \notin S_i} f_i^\tau > 0$ for all i. Thus, $\varphi_i \leq \prod_{j \neq i} \sum_{\tau \notin S_j} f_j^\tau$ for all i. Let $\mathcal{S}' := (S_1, \ldots, S_{i-1}, S_i \backslash \{t\}, S_{i+1} \ldots, S_n)$. Then

$$H(\mathcal{S}) - H(\mathcal{S}') = f_i^t \left(\prod_{j \neq i} \sum_{\tau \notin S_j} f_j^\tau - \varphi_i \right) \geq 0.$$

Hence, $H(\mathcal{S}') \leq H(\mathcal{S})$. By induction, $H(\mathcal{S}') \leq H(\mathcal{S})$ for all $\mathcal{S}' \subset \mathcal{S}$. ■

Lemma 9 $\overline{H}(\mathcal{S}) = \min \{H(\mathcal{S}), 1 - \sum_{i=1}^n \varphi_i\}$.

Proof. Recall that $\overline{H}(\mathcal{S}) = \min_{\mathcal{S}'' \supset \mathcal{S}} H(S)$. Recall that $\mathcal{S}^1 := \{1, \ldots, m\}^n$. Because $\mathcal{S}^1 \supset \mathcal{S}$ and $H(\mathcal{S}^1) = 1 - \sum_{i=1}^n \varphi_i$, we have $\overline{H}(\mathcal{S}) \leq 1 - \sum_{i=1}^n \varphi_i$.

Suppose that $H(\mathcal{S}) \leq 1 - \sum_{i=1}^n \varphi_i$. Let $\mathcal{S}'' \supset \mathcal{S}$. If $H(\mathcal{S}'') \geq 1 - \sum_{i=1}^n \varphi_i$, then $H(\mathcal{S}) \leq 1 - \sum_{i=1}^n \varphi_i \leq H(\mathcal{S}'')$. If $H(\mathcal{S}'') < 1 - \sum_{i=1}^n \varphi_i$, then $H(\mathcal{S}) \leq H(\mathcal{S}'')$ by Lemma 8. Hence, $\overline{H}(\mathcal{S}) = H(\mathcal{S})$.

Suppose that $H(\mathcal{S}) > 1 - \sum_{i=1}^n \varphi_i$. I claim that $\overline{H}(\mathcal{S}) = 1 - \sum_{i=1}^n \varphi_i$. Suppose not, then there exists $\mathcal{S}'' \supset \mathcal{S}$ such that $H(\mathcal{S}'') < 1 - \sum_{i=1}^n \varphi_i$. Then, by Lemma 8, $H(\mathcal{S}) \leq H(\mathcal{S}'') < 1 - \sum_{i=1}^n \varphi_i$, which is a contradiction to the fact that $H(\mathcal{S}) > 1 - \sum_{i=1}^n \varphi_i$. Hence, $\overline{H}(\mathcal{S}) = 1 - \sum_{i=1}^n \varphi_i$. ■

C.1.2 Proofs of feasibility

Lemma 10 *For all* $t_1, \ldots, t_n \in \{1, \ldots, m\}$,

$$\sum_{i=1}^n \sum_{t \in S_i} \overline{z}_i^t \leq \overline{H}(\mathcal{S}), \tag{27}$$

where $\mathcal{S} = (S_1^{t_1}, \ldots, S_n^{t_n})$.

Proof. For each t, let $\pi^{t,0} := \emptyset$ and $\pi^{t,n^t+1} := \mathcal{I} \backslash (\mathcal{I}_0^t \cup \mathcal{I}_1^t)$. Suppose that $\mathcal{S} \subset \mathcal{S}^m$, i.e., $t_i \geq m$ for all i. By Algorithm 1, we have

$$\sum_{i \in \pi^{m,1}} \overline{z}_i^m = \overline{H}\left(\emptyset + \sum_{i \in \pi^{m,1}} (m, i)\right),$$

$$\sum_{i \in \mathcal{I}''} \overline{z}_i^m \leq \sum_{i \in \mathcal{I}''} \frac{c_i \varphi_i f_i^m}{1 - c_i} \leq \overline{H}\left(\emptyset + \sum_{i \in \mathcal{I}''} (m, i)\right), \forall \emptyset \neq \mathcal{I}'' \subsetneq \pi^{m,1},$$

where the second inequality in the second line holds because otherwise $|\pi^{m,1}| \leq |\mathcal{I}''|$ by Algorithm 1, which is a contradiction to $\mathcal{I}'' \subsetneq \pi^{m,1}$. Thus, (27) holds if $t_i = m+1$ for all $i \notin \pi^{m,1}$. Suppose that we have shown that (27) holds if $t_i = m+1$ for all $i \notin \pi^{m,1} \cup \cdots \cup \pi^{m,\nu-1}$ and $\nu \geq 2$. Suppose that $t_i = m+1$ for all $i \notin \pi^{m,1} \cup \cdots \cup \pi^{m,\nu}$. Let $\mathcal{S}' := \emptyset + \sum_{i \in \pi^{m,1} \cup \cdots \cup \pi^{m,\nu-1}} (m, i)$ and $\mathcal{I}'' := \{i \in \pi^{m,\nu} | t_i = m\} \subset \pi^{m,\nu}$. By Algorithm 1, we have

$$\sum_{i \in \mathcal{I}''} \overline{z}_i^m = \overline{H}\left(\mathcal{S}' + \sum_{i \in \mathcal{I}'} (m, i)\right) - \overline{H}(\mathcal{S}') \text{ if } \mathcal{I}'' = \pi^{m,\nu} \text{ and } \nu \leq n^m,$$

$$\sum_{i \in \mathcal{I}''} \overline{z}_i^m \leq \sum_{i \in \mathcal{I}''} \frac{c_i \varphi_i f_i^m}{1 - c_i} \leq \overline{H}\left(\mathcal{S}' + \sum_{i \in \mathcal{I}''} (m, i)\right) - \overline{H}(\mathcal{S}') \text{ if } \mathcal{I}'' \subsetneq \pi^{m,\nu} \text{ or } \nu = n^m + 1.$$

Because $\mathcal{S} - \sum_{i\in\pi^{m,\nu}}(m,i) \subset \mathcal{S}'$, we have

$$\begin{aligned}\sum_{i=1}^{n}\sum_{t\in S_i'\setminus S_i}\overline{z}_i^t &= \sum_{i=1}^{n}\sum_{t\in S_i'}\overline{z}_i^t - \sum_{i\notin\pi^{m,\nu}}\sum_{t\in S_i}\overline{z}_i^t\\ &\geq \overline{H}(\mathcal{S}') - \overline{H}\left(\mathcal{S} - \sum_{i\in\pi^{m,\nu}}(m,i)\right)\\ &\geq \overline{H}\left(S' + \sum_{i\in\mathcal{I}'}(m,i)\right) - \overline{H}(\mathcal{S}),\end{aligned}$$

where the last inequality holds because $\overline{H}$ is submodular. Hence,

$$\begin{aligned}\sum_{i=1}^{n}\sum_{t\in S_i}\overline{z}_i^t &\leq \sum_{i=1}^{n}\sum_{t\in S_i'}\overline{z}_i^t - \sum_{i=1}^{n}\sum_{t\in S_i'\setminus S_i}\overline{z}_i^t + \overline{H}\left(\mathcal{S}' + \sum_{i\in\mathcal{I}'}(m,i)\right) - \overline{H}(\mathcal{S}')\\ &\leq \overline{H}(\mathcal{S}') - \overline{H}\left(\mathcal{S}' + \sum_{i\in\mathcal{I}'}(m,i)\right) + \overline{H}(\mathcal{S}) + \overline{H}\left(\mathcal{S}' + \sum_{i\in\mathcal{I}'}(m,i)\right) - \overline{H}(\mathcal{S}')\\ &= \overline{H}(\mathcal{S}).\end{aligned}$$

By induction, (27) holds for all $\mathcal{S} \subset \mathcal{S}^m$.

Suppose that $\mathcal{S} \subset \mathcal{S}^{t+1} + \sum_{i\in\pi^{t,1}\cup\cdots\cup\pi^{t,\nu}}(t,i)$ for $t \leq m-1$ and $1 \leq \nu \leq n^t + 1$. Let $\mathcal{I}' := \{i \in \pi^{t,\nu} | t_i = t\}$ and $\mathcal{S}' := \mathcal{S} - \sum_{i\in\mathcal{I}'}(t,i)$. Suppose, w.l.o.g., that $\mathcal{I}' \neq \emptyset$. If $\mathcal{I}' = \pi^{t,\nu}$, then, by Algorithm 1, we have

$$\sum_{i\in\mathcal{I}'}\overline{z}_i^\tau \leq \overline{H}\left(\mathcal{S}' + \sum_{i\in\mathcal{I}'}(t,i)\right) - \sum_{i=1}^{n}\sum_{\tau\in S_i'}\overline{z}_i^\tau = \overline{H}\left(\mathcal{S}\right) - \sum_{i=1}^{n}\sum_{\tau\in S_i'}\overline{z}_i^\tau.$$

If $\mathcal{I}' \subsetneq \pi^{t,\nu}$, then, by Algorithm 1, we have

$$\sum_{i\in\mathcal{I}'}\overline{z}_i^\tau \leq \sum_{i\in\mathcal{I}'}\frac{c_i\varphi_i f_i^t}{1-c_i} \leq \overline{H}\left(\mathcal{S}' + \sum_{i\in\mathcal{I}'}(t,i)\right) - \sum_{i=1}^{n}\sum_{\tau\in S_i'}\overline{z}_i^\tau = \overline{H}\left(\mathcal{S}\right) - \sum_{i=1}^{n}\sum_{\tau\in S_i'}\overline{z}_i^\tau,$$

where the second inequality holds because otherwise $|\pi^{t,\nu}| \leq |\mathcal{I}'|$ by Algorithm 1, which is a contradiction to $\mathcal{I}' \subsetneq \pi^{t,\nu}$. Hence, (27) holds for $\mathcal{S}$. ■

Lemma 11 *$\overline{P}_i^t$ is non-decreasing in t on $\{t\,|\,f_i^t > 0\}$.*

To prove Lemma 11, I first prove the following lemma which says that if the upper-bound in (AIC$'m$1) does not bind for z_i^{t+1}, then it does not bind for z_i^t.

Lemma 12 *Suppose that $f_i^t, f_i^{t+1} > 0$. Then $\overline{z}_i^{t+1} \in \mathcal{I}_1^{t+1}$ implies that $\overline{z}_i^t \in \mathcal{I}_1^t$.*

Proof. Suppose that $f_i^t, f_i^{t+1} > 0$ and $\overline{z}_i^{t+1} \in \mathcal{I}_1^{t+1}$. Then, by Algorithm 1, there exists $\mathcal{S}$ with $S_j = S_j^{t_j} \subset S_j^{t+1}$ for all $j \neq i$ and $S_i = S_i^{t+1}$ such that

$$\sum_{j=1}^{n} \sum_{\tau \in S_j} \overline{z}_j^\tau = \overline{H}(\mathcal{S}).$$

Suppose that $H(\mathcal{S}) < 1 - \sum_{j=1}^n \varphi_j$. Because, by Lemma 10,

$$\sum_{j \neq i} \sum_{\tau \in S_j} \overline{z}_j^\tau + \sum_{\tau \in S_i \setminus \{t+1\}} \overline{z}_i^\tau \leq \overline{H}(\mathcal{S} - (t+1, i)),$$

we have

$$\frac{c_i \varphi_i f_i^{t+1}}{1 - c_i} \geq \overline{z}_i^{t+1} \geq \overline{H}(\mathcal{S}) - \overline{H}(\mathcal{S} - (t+1, i)) = f_i^{t+1} \left(\prod_{j \neq i} \sum_{\tau \in S_j} f_j^\tau - \varphi_i \right),$$

where the last equality holds by Lemmas 8 and 9. This implies that $\prod_{j \neq i} \sum_{\tau \in S_j} f_j^\tau \leq \frac{\varphi_i}{1 - c_i}$. Hence,

$$\begin{aligned}
\overline{z}_i^t \leq & \overline{H}(\mathcal{S} + (t, i)) - \sum_{j=1}^{n} \sum_{\tau \in S_j} \overline{z}_j^\tau \\
\leq & H(\mathcal{S} + (t, i)) - \overline{H}(\mathcal{S}) \\
= & f_i^t \left(\prod_{j \neq i} \sum_{\tau \in S_j} f_j^\tau - \varphi_i \right) \leq \frac{c_i \varphi_i f_i^t}{1 - c_i},
\end{aligned}$$

where the equality holds by Lemmas 8 and 9.

Suppose that $H(\mathcal{S}) \geq 1 - \sum_{j=1}^n \varphi_j$, then by Lemmas 8 and 9, $\overline{H}(\mathcal{S}) = \overline{H}(\mathcal{S} + (t, i)) = 1 - \sum_{j=1}^n \varphi_j$. Hence,

$$\begin{aligned}
\overline{z}_i^t \leq & \overline{H}(\mathcal{S} + (t, i)) - \sum_{j=1}^{n} \sum_{\tau \in S_j} \overline{z}_j^\tau \\
\leq & \overline{H}(\mathcal{S} + (t, i)) - \overline{H}(\mathcal{S}) \\
= & 0 \leq \frac{c_i \varphi_i f_i^t}{1 - c_i}.
\end{aligned}$$

Hence, $z_i^t \in \mathcal{I}_1^t$. ∎

Proof of Lemma 11. Suppose that $f_i^t, f_i^{t+1} > 0$. Recall that $\overline{P}_i^t = \overline{z}_i^t / f_i^t + \varphi_i$ if $f_i^t > 0$. Suppose that $\overline{z}_i^{t+1} \notin \mathcal{I}_1^{t+1}$, then $\overline{P}_i^t \leq \frac{\varphi_i}{1-c_i} = \overline{P}_i^{t+1}$. Suppose that $\overline{z}_i^{t+1} \in \mathcal{I}_1^{t+1}$. Then there exists $\mathcal{S}$ with $S_j = S_j^{t_j} \subset S_j^{t+1}$ for all $j \neq i$ and $S_i = S_i^{t+1}$ such that

$$\sum_{j=1}^{n} \sum_{\tau \in S_j} \overline{z}_j^{\tau} = \overline{H}(\mathcal{S}).$$

Suppose that $H(\mathcal{S}) < 1 - \sum_{j=1}^n \varphi_j$. In the proof of Lemma 12, we have shown that $\overline{z}_i^{t+1} \geq f_i^{t+1} \left(\prod_{j \neq i} \sum_{\tau \in S_j} f_j^{\tau} - \varphi_i \right)$ and $\overline{z}_i^t \leq f_i^t \left(\prod_{j \neq i} \sum_{\tau \in S_j} f_j^{\tau} - \varphi_i \right)$. Hence,

$$\overline{P}_i^t \leq \prod_{j \neq i} \sum_{\tau \in S_j} f_j^{\tau} \leq \overline{P}_i^{t+1}.$$

Suppose that $H(\mathcal{S}) \geq 1 - \sum_{j=1}^n \varphi_j$. By the proof of Lemma 12, we have $\overline{P}_i^t = \varphi_i \leq \overline{P}_i^{t+1}$.
■

C.1.3 Proofs of optimality

Before proving Lemma 7, I first prove some useful properties of $\mathcal{S}^{t,\nu}$ and $\overline{z}$. Recall that $\{S^{t,\nu} + \sum_{i \in \pi^{t,\nu}} (t,i)\}$ is the collection of sets for which (AF2m1) bind. The result in Lemma 13 implies that this collection is a *nested sequence of sets*. In fact, Lemma 13 proves a stronger statement.

Lemma 13 *$\mathcal{S}^{t,1} \supset \mathcal{S}^{t+1,n^{t+1}} + \sum_{i \in \pi^{t+1,n^{t+1}}} (t+1, i)$ for $1 \leq t \leq m-1$; and $\mathcal{S}^{t,\nu+1} \supset \mathcal{S}^{t,\nu} + \sum_{i \in \pi^{t,\nu}} (t,i)$ for $1 \leq t \leq m$.*

Proof. By Algorithm 1, $\mathcal{S}^{m,\nu+1} \supset \mathcal{S}^{m,\nu} + \sum_{i \in \pi^{m,\nu}} (m,i)$. Let $t \leq m-1$ and $\mathcal{I}' = \pi^{t,1}$. Let $\mathcal{S} := \mathcal{S}^{t+1,n^{t+1}} + \sum_{i \in \pi^{t+1,n^{t+1}}} (t+1,i)$. Then $\sum_{j=1}^n \sum_{\tau \in S_j} \overline{z}_j^{\tau} = \overline{H}(\mathcal{S})$. Suppose $\mathcal{S}^{t,1} \not\supset \mathcal{S}$. Let $\mathcal{S}' := \mathcal{S} \cup \mathcal{S}^{t,1}$. Then $S_j' = S_j^{t_j}$ for some $t_j \geq t+1$ for all j. By Lemma 10, we have

$$\begin{aligned}
&\sum_{j=1}^{n} \sum_{\tau \in S_j' \setminus S_j^{t,1}} \overline{z}_j^{\tau} \\
=&\sum_{j=1}^{n} \sum_{\tau \in S_j} \overline{z}_j^{\tau} - \sum_{j=1}^{n} \sum_{\tau \in S_j^{t,1} \cap S_j} \overline{z}_j^{\tau} \\
\geq& \overline{H}(\mathcal{S}) - \overline{H}(\mathcal{S} \cap \mathcal{S}^{t,1}).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\overline{H}\left(\mathcal{S}'+\sum_{i\in\mathcal{I}'}(t,i)\right)-\sum_{j=1}^{n}\sum_{\tau\in S_j'}\overline{z}_j^\tau-\overline{H}\left(\mathcal{S}^{t,1}+\sum_{i\in\mathcal{I}'}(t,i)\right)+\sum_{j=1}^{n}\sum_{\tau\in S_j^{t,1}}\overline{z}_j^\tau\\
=&\overline{H}\left(\mathcal{S}'+\sum_{i\in\mathcal{I}'}(t,i)\right)-\overline{H}\left(\mathcal{S}^{t,1}+\sum_{i\in\mathcal{I}'}(t,i)\right)-\sum_{j=1}^{n}\sum_{\tau\in S_j'\setminus S_j^{t,1}}\overline{z}_j^\tau\\
\leq&\left[\overline{H}\left(\mathcal{S}'+\sum_{i\in\mathcal{I}'}(t,i)\right)-\overline{H}(\mathcal{S})\right]-\left[\overline{H}\left(\mathcal{S}^{t,1}+\sum_{i\in\mathcal{I}'}(t,i)\right)-\overline{H}(\mathcal{S}\cap\mathcal{S}^{t,1})\right]\\
\leq&0,
\end{aligned}$$

where the last inequality holds because $\overline{H}$ is submodular, which is a contradiction to the definition of $\mathcal{S}^{t,\nu}$. Hence, $\mathcal{S}^{t,1}\supset\mathcal{S}^{t+1,n^{t+1}}+\sum_{i\in\pi^{t+1,n^{t+1}}}(t+1,i)$. By a similar argument, one can show that $\mathcal{S}^{t,\nu+1}\supset\mathcal{S}^{t,\nu}+\sum_{i\in\pi^{t,\nu}}(t,i)$ for all $t\leq m-1$. ■

By Lemmas 8, 9 and 13, there exists $\underline{t}$ and $\overline{\nu}$ such that

$$\overline{H}\left(\mathcal{S}^{t,\nu}+\sum_{i\in\pi^{t,\nu}}(t,i)\right)=\begin{cases}1-\sum_i\varphi_i & \text{if } t<\underline{t} \text{ or } \nu\geq\overline{\nu}, t=\underline{t}\\ H\left(\mathcal{S}^{t,\nu}+\sum_{i\in\pi^{t,\nu}}(t,i)\right)<1-\sum_i\varphi_i & \text{otherwise}\end{cases}. \tag{30}$$

The definition of $\underline{t}$ is analogous to that in the symmetric case. By a similar argument to that in Lemma 12, we have

Lemma 14 *If $t<\underline{t}$, or $t=\underline{t}$ and $i\notin\pi^{t,1}\cup\cdots\cup\pi^{t,\overline{\nu}}$, then $\overline{z}_i^t=0$.*

Proof of Lemma 7. Consider the dual to problem $(OPTAm1-\varphi)$, which is denoted by $(DOPTAm1-\varphi)$,

$$\min_{\lambda,\beta,\mu}\sum_{i=1}^{n}\sum_{t=1}^{m}\frac{\lambda_i^t c_i\varphi_i f_i^t}{1-c_i}+\sum_{\mathcal{S}}\beta(\mathcal{S})\overline{H}(\mathcal{S})+\sum_{i=1}^{n}\varphi_i\left(\sum_{t=1}^{m}f_i^t d^t+\frac{k_i}{c_i}\right),$$

subject to

$$\begin{aligned}
d^t-\lambda_i^t+\mu_i^t-\sum_{S_i\ni t}\beta(\mathcal{S})\geq 0 \text{ if } f_i^t>0,\forall i,\forall t,\\
\lambda\geq 0,\mu\geq 0,\beta\geq 0.
\end{aligned}$$

Let $\hat{z}$ be defined by (28) and $(\hat{\beta},\hat{\lambda},\hat{\mu})$ be the corresponding dual variables. Let t^0 be such that $d^{t^0}\geq 0$ if and only if $t\geq t^0$.

Let $\hat{\beta}\left(\mathcal{S}^{t,n^t}+\sum_{i\in\pi^{t,n^t}}(t,i)\right)\geq 0$ for $t\geq\max\{t^0,\underline{t}\}$ and $\hat{\beta}(\mathcal{S})=0$ otherwise. (i) If $t<\max\{t^0,\underline{t}\}$, then let $\hat{\lambda}_i^t=0$ and $\hat{\mu}_i^t=-d^t\geq 0$. (ii) If $t=\max\{t^0,\underline{t}\}$, then let $\hat{\beta}\left(\mathcal{S}^{t,n^t}+\sum_{j\in\pi^{t,n^t}}(t,j)\right)=d^t\geq 0$ and $\hat{\mu}_i^t=0$. If $i\in\mathcal{I}_1^t$, then let $\hat{\lambda}_i^t=0$. If $i\notin\mathcal{I}_0^t\cup\mathcal{I}_1^t$, let $\hat{\lambda}_i^t=d^t\geq 0$. (iii) If $t>\max\{t^0,\underline{t}\}$, let $\hat{\beta}\left(\mathcal{S}^{t,n^t}+\sum_{j\in\pi^{t,n^t}}(t,j)\right)=d^t-d^{t-1}\geq 0$ and $\hat{\mu}_i^t=0$. If $i\in\mathcal{I}_1^t$, then let $\hat{\lambda}_i^t=0$. If $i\notin\mathcal{I}_0^t\cup\mathcal{I}_1^t$ and $i\in\mathcal{I}_1^{\max\{t^0,\underline{t}\}}$, then let $\hat{\lambda}_i^t=d^t-d^{t^*}\geq 0$ where $t^*=\min\{t'\geq\max\{t^0,\underline{t}\}|\mathcal{I}_1^{t'}\ni i\}$. If $i\notin\mathcal{I}_1^{\max\{t^0,\underline{t}\}}$, then let $\hat{\lambda}_i^t=0$. Hence, $(\hat{\lambda},\hat{\mu},\hat{\beta})$ is a feasible solution to $(DOPTAm1-\varphi)$ and the complementary slackness conditions are satisfied. Finally, it is easy to verify that the dual objective is equal to the primal objective. By the duality theorem, $\hat{z}$ is an optimal solution to $(OPTAm1-\varphi)$. ■

C.1.4 Properties of $S^{t,\nu}$

Before moving on to the continuum case, I prove the following two lemmas which are useful in characterizing the limit of $\{P^m\}$.

Lemma 15 *Suppose $\mathcal{S}^{t,\nu}=(S_1^{t_1^*},\dots,S_n^{t_n^*})$. Then $t_i^*=t$ if $i\in\pi^{t,1}\cup\dots\cup\pi^{t,\nu-1}$, $t_i^*=t+1$ if $i\in\mathcal{I}_1^{t+1}\cup\pi^{t,\nu}\backslash(\pi^{t,1}\cup\dots\cup\pi^{t,\nu-1})$, and $t_i^*\in\{t+1,m+1\}$ otherwise. Furthermore, for $h\notin\mathcal{I}_1^{t+1}\cup\pi^{t,1}\cup\dots\cup\pi^{t,\nu}$, we have*

1. *If $\frac{\varphi_h}{1-c_h}-\prod_{i\neq h}\sum_{\tau=1}^{t_i^*-1}f_i^\tau\geq 0$, then $t_h^*=t+1$.*
2. *If $\frac{\varphi_h}{1-c_h}-\prod_{i\neq h}\sum_{\tau=1}^{t_i^*-1}f_i^\tau<0$ and $\overline{H}\left(\mathcal{S}^{t,\nu}+\sum_{i\in\pi^{t,\nu}}(t,i)\right)<1-\sum_{i=1}^n\varphi_i$, then $t_h^*=m+1$.*

Proof. By Algorithm 1, $t_i^*=t+1$ if $i\in\pi^{t,\nu}$. By Lemma 13, $t_i^*=t$ if $i\in\pi^{t,1}\cup\dots\cup\pi^{t,\nu-1}$ and $t_i^*=t+1$ if $i\in\mathcal{I}_1^{t+1}\backslash(\pi^{t,1}\cup\dots\cup\pi^{t,\nu-1})$. If $t=m$, then, by Algorithm 1, $t_i^*=m+1$ for $i\notin\pi^{t,1}\cup\dots\cup\pi^{t,\nu-1}$.

Let $t\leq m-1$. For the ease of notation, let $\mathcal{I}'=\pi^{t,\nu}$ and $\mathcal{S}=(S_1^{t_1},\dots,S_n^{t_n})$ be such that $t_i=t$ if $i\in\pi^{t,1}\cup\dots\cup\pi^{t,\nu-1}$, $t_i=t+1$ if $i\in\mathcal{I}_1^{t+1}\cup\pi^{t,\nu}\backslash(\pi^{t,1}\cup\dots\cup\pi^{t,\nu-1})$ and $t_i\geq t+1$ otherwise. Fix $h\notin\mathcal{I}_1^{t+1}\cup\pi^{t,1}\cup\dots\cup\pi^{t,\nu}$ and t_i for all $i\neq h$. Define

$$\Delta(t_h):=\overline{H}\left(\mathcal{S}+\sum_{i\in\mathcal{I}'}(t,i)\right)-\sum_{i=1}^n\sum_{\tau\in S_i}\overline{z}_i^\tau.$$

By Lemma 14 and the fact that $h\notin\mathcal{I}_1^{t+1}$, there exists $t\leq t^*\leq m+1$ such that if $t+1\leq t_h\leq t^*$, then $\Delta(t_h)=1-\sum_{i=1}^n\varphi_i-\sum_{i=1}^n\sum_{\tau\in S_i}\overline{z}_i^\tau$; and if $t^*<t_h\leq m+1$, then

$$\Delta(t_h)=1-\left(\prod_{i\notin\mathcal{I}'}\sum_{\tau=1}^{t_i-1}f_i^\tau\right)\left(\prod_{i\in\mathcal{I}'}\sum_{\tau=1}^{t-1}f_i^\tau\right)-\sum_{i\notin\mathcal{I}'}\sum_{\tau=t_i}^m f_i^\tau\varphi_i-\sum_{i\in\mathcal{I}'}\sum_{\tau=t}^m f_i^\tau\varphi_i-\sum_{i=1}^n\sum_{\tau=t_i}^m\overline{z}_i^\tau.$$

Because $h \notin \mathcal{I}_1^{t+1}$, we have $\overline{z}_h^{t_h} = c_h\varphi_h f_h^{t_h}/(1-c_h)$ for all $t_h \geq t+1$. If $t_h < t^*$, then we have $\Delta(t_h+1) - \Delta(t_h) = c_h\varphi_h f_h^{t_h}/(1-c_h) \geq 0$. Hence, $\Delta(t+1) \leq \Delta(t_h)$ for all $t_h \leq t^*$. If $t_h > t^*$, we have

$$\begin{aligned}&\Delta(t_h+1) - \Delta(t_h)\\ =&f_h^{t_h}\left(\frac{\varphi_h}{1-c_h} - \left(\prod_{i\notin\mathcal{I}',i\neq h}\sum_{\tau=1}^{t_i-1} f_i^\tau\right)\left(\prod_{i\in\mathcal{I}'}\sum_{\tau=1}^{t-1} f_i^\tau\right)\right).\end{aligned}$$

If $t_h = t^*$, we have

$$\begin{aligned}&\Delta(t_h+1) - \Delta(t_h)\\ \geq&f_h^{t_h}\left(\frac{\varphi_h}{1-c_h} - \left(\prod_{i\notin\mathcal{I}',i\neq h}\sum_{\tau=1}^{t_i-1} f_i^\tau\right)\left(\prod_{i\in\mathcal{I}'}\sum_{\tau=1}^{t-1} f_i^\tau\right)\right).\end{aligned}$$

Hence, if $\frac{\varphi_h}{1-c_h} - \left(\prod_{i\notin\mathcal{I}',i\neq h}\sum_{\tau=1}^{t_i-1} f_i^\tau\right)\left(\prod_{i\in\mathcal{I}'}\sum_{\tau=1}^{t-1} f_i^\tau\right) \geq 0$, then $\Delta(t_h+1) \geq \Delta(t_h)$ for all $t_h \geq t^*$. Furthermore, because $\Delta(t+1) \leq \Delta(t_h)$ for all $t_h \leq t^*$, we have $\Delta(t+1) \leq \Delta(t_h)$ for all $t_h \geq t+1$, hence $t_h^* = t+1$.

If $\frac{\varphi_h}{1-c_h} - \left(\prod_{i\notin\mathcal{I}',i\neq h}\sum_{\tau=1}^{t_i-1} f_i^\tau\right)\left(\prod_{i\in\mathcal{I}'}\sum_{\tau=1}^{t-1} f_i^\tau\right) < 0$, then $\Delta(t_h+1) \leq \Delta(t_h)$ for all $t_h > t^*$. Hence, $\Delta(m+1) \leq \Delta(t_h)$ for all $t_h > t^*$. Recall that $\Delta(t+1) \leq \Delta(t_h)$ for all $t_h \leq t^*$. Hence, $t_h^* \in \arg\min\{\Delta(t+1), \Delta(m+1)\}$. If $\overline{H}\left(\mathcal{S}^{t,\nu} + \sum_{i\in\pi^{t,\nu}}(t,i)\right) < 1 - \sum_{i=1}^n \varphi_i$, then $t^* = t$ by definition, which implies that $t_h^* = m+1$. ■

Lemma 16 *Suppose $\mathcal{S}^{t,\nu} = (S_1^{t_1^*}, \ldots, S_n^{t_n^*})$ and $h \notin \mathcal{I}_1^{t+1} \cup \pi^{t,1} \cup \cdots \cup \pi^{t,\nu}$, then $t_h^* = t+1$ implies that $h \in \mathcal{I}_1^t$.*

Proof. Suppose $\overline{H}\left(\mathcal{S}^{t,\nu} + \sum_{i\in\pi^{t,\nu}}(t,i)\right) = 1 - \sum_{i=1}^n \varphi_i$, then by Lemma 14, $h \in \mathcal{I}_1^t$. Suppose $\overline{H}\left(\mathcal{S}^{t,\nu} + \sum_{i\in\pi^{t,\nu}}(t,i)\right) < 1 - \sum_{i=1}^n \varphi_i$. By Lemma 15, $\frac{\varphi_h}{1-c_h} \geq \prod_{i\neq h}\sum_{\tau=1}^{t_i^*-1} f_i^\tau$. Hence,

$$\begin{aligned}&\overline{H}\left(\mathcal{S}^{t,\nu} + \sum_{i\in\pi^{t,\nu}}(t,i) + (t,h)\right) - \overline{H}\left(\mathcal{S}^{t,\nu} + \sum_{i\in\pi^{t,\nu}}(t,i)\right)\\ \leq&f_h^t\left(\prod_{i\neq h}\sum_{\tau=1}^{t_i^*-1} f_i^\tau - \varphi_h\right)\\ \leq&f_h^t\left(\frac{\varphi_h}{1-c_h} - \varphi_h\right) = \frac{c_h\varphi_h f_h^t}{1-c_h}.\end{aligned}$$

By Algorithm 1, $h \in \mathcal{I}_1^t$. ■

C.2 Continuum case

I characterize an optimal solution in the continuum case by taking m to infinity. Let $\mathcal{I}_1^{m,t}$ denote $\mathcal{I}_1^t$, and $\underline{t}^m$ be defined by (30) when $\mathcal{D}$ is discretized by m grid points. Clearly, if $i \in \mathcal{I}_1^{m,t}$ then $i \in \mathcal{I}_1^{2m,2t-1}$. For each m and i, let $\overline{t}_i^m := \max\left\{t | i \in \mathcal{I}_1^{m,t}\right\}$ and $\overline{d}_i^m := \underline{d} + \frac{(\overline{t}_i^m - 1)(\overline{d} - \underline{d})}{m}$. Then the sequence of $\left\{\overline{d}_i^{2^\kappa}\right\}_\kappa$ is non-decreasing and bounded from above by $\overline{d}$. Hence, the sequence converges and let $d_i^u := \lim_{\kappa\to\infty} \overline{d}_i^{2^\kappa}$ denote its limit. For each κ, let $\underline{d}^{2^\kappa} := \underline{d} + \frac{(\underline{t}^{2^\kappa} - 1)(\overline{d} - \underline{d})}{2^\kappa}$, which is bounded. After taking subsequences, we can assume $\left\{\underline{d}^{2^\kappa}\right\}_\kappa$ converges and let $d^l := \lim_{\kappa\to\infty} \underline{d}^{2^\kappa}$ denote its limit. Let

$$\overline{P}_i(v_i) := \begin{cases} \frac{\varphi_i}{1-c_i} & \text{if } v_i > d_i^u + \frac{k_i}{c_i} \\ \prod_{j\neq i, d_j^u \geq v_i - \frac{k_i}{c_i}} F_j\left(v_i - \frac{k_i}{c_i} + \frac{k_j}{c_j}\right) & \text{if } d^l + \frac{k_i}{c_i} < v_i < d_i^u + \frac{k_i}{c_i} \\ \varphi_i & \text{if } v_i < d^l + \frac{k_i}{c_i} \end{cases}.$$

Finally, let $\boldsymbol{P}^* := (P_i^*)_i$ where

$$P_i^*(v_i) := \begin{cases} \overline{P}_i(v_i) & \text{if } v_i > \frac{k_i}{c_i} \\ \varphi_i & \text{if } v_i < \frac{k_i}{c_i} \end{cases}. \tag{17}$$

We are now ready to prove Theorem 4.

Proof of Theorem 4. We can extend $\overline{P}_i^m$ (P_i^m) to $[\underline{v}_i, \overline{v}_i]$ by setting, for each $t = 1, \ldots, m$,

$$\overline{P}_i^m(v_i) := \overline{P}_i^{m,t} (P_i^m(v_i) := P_i^{m,t}) \text{ for } v_i \in \left[\underline{d} + \frac{(t-1)(\overline{d} - \underline{d})}{m} + \frac{k_i}{c_i}, \underline{d} + \frac{t(\overline{d} - \underline{d})}{m} + \frac{k_i}{c_i}\right].$$

I show that, after taking subsequences, $\overline{P}_i^m$ converges to $\overline{P}_i$ pointwise.

First, by construction and Lemma 14, $\overline{P}_i^{2^\kappa}(v_i) = \varphi_i$ for all $v_i < \underline{d}^{2^\kappa} + \frac{k_i}{c_i}$, we have $\lim_{\kappa\to\infty} \overline{P}_i^{2^\kappa}(v_i) = \overline{P}_i(v_i)$ for all $v_i < d^l + \frac{k_i}{c_i}$. Similarly, by construction, $\overline{P}_i^{2^\kappa}(v_i) = \frac{\varphi_i}{1-c_i}$ for all $v_i > \overline{d}_i^{2^\kappa} + \frac{k_i}{c_i}$, we have $\lim_{\kappa\to\infty} \overline{P}_i^{2^\kappa}(v_i) = \overline{P}_i(v_i)$ for all $v_i > d_i^u + \frac{k_i}{c_i}$.

Suppose $d^l < v_i - \frac{k_i}{c_i} < d_i^u$. Assume without loss of generality that $d_1^u \geq \cdots \geq d_n^u \geq d^l$. If $d_i^u = d^l$, then we are done. Assume for the rest of the proof that $d_i^u > d^l$. Let $d_{n+1}^u := d^l$. Consider v_i such that $d_i^u \geq d_j^u > v_i - \frac{k_i}{c_i} > d_{j+1}^u$ for some $j \geq i$. For m sufficiently large, there exists t such that

$$d_{j+1}^u < \underline{d} + \frac{(t-1)(\overline{d} - \underline{d})}{m} < \underline{d} + \frac{t(\overline{d} - \underline{d})}{m} < v_i - \frac{k_i}{c_i} < \underline{d} + \frac{(t+1)(\overline{d} - \underline{d})}{m} < d_j^u \leq d_i^u.$$

Hence, by construction, we have $\mathcal{I}_1^{m,t} = \mathcal{I}_1^{m,t+1} = \{1, \ldots, j\}$. By Lemmas 15 and 16, there exists $\mathcal{S} = (S_1^{t_1}, \ldots, S_n^{t_n})$ such that $t_i = t+1$, $t_h \in \{t, t+1\}$ if $h \leq j$ and $h \neq i$, $t_h = m+1$

if $h > j$, and

$$f_i^t\left(\overline{P}_i^{m,t} - \varphi_i\right) = \overline{z}_i^{m,t} = \overline{H}(\mathcal{S} + (t,i)) - \overline{H}(\mathcal{S}).$$

Because $\overline{H}$ is submodular, we have

$$\begin{aligned} f_i^t\left(\overline{P}_i^{m,t} - \varphi_i\right) &\leq \overline{H}(\mathcal{S}' + (t,i)) - \overline{H}(\mathcal{S}') \\ &= f_i^t\left(\prod_{h\leq j, h\neq i}\sum_{\tau=1}^{t} f_h^\tau - \varphi_i\right), \end{aligned}$$

where $\mathcal{S}' = (S_1^{t+1}, \ldots, S_j^{t+1}, S_{j+1}^{m+1}, \ldots, S_n^{m+1})$; and

$$\begin{aligned} f_i^t\left(\overline{P}_i^{m,t} - \varphi_i\right) &\geq \overline{H}(\mathcal{S}'' - (t,i)) - \overline{H}(\mathcal{S}') \\ &= f_i^t\left(\prod_{h\leq j, h\neq i}\sum_{\tau=1}^{t-1} f_h^\tau - \varphi_i\right), \end{aligned}$$

where $\mathcal{S}'' = (S_1^{t}, \ldots, S_j^{t}, S_{j+1}^{m+1}, \ldots, S_n^{m+1})$. Hence,

$$\prod_{h\leq j, h\neq i}\sum_{\tau=1}^{t-1} f_h^\tau \leq \overline{P}_i^{m,t} \leq \prod_{h\leq j, h\neq i}\sum_{\tau=1}^{t} f_h^\tau.$$

Take $m = 2^\kappa$ to infinity and we have $\lim_{\kappa\to\infty} \overline{P}_i^{2^\kappa}(v_i) = \overline{P}_i(v_i)$.

It follows that, after taking subsequences, P_i^m converges to P_i^* pointwise. $\boldsymbol{P}^*$ is feasible by a similar argument to that in the proof of Lemma 3, and optimal by a similar argument to that in the proof of Theorem 1. ■

C.3 Optimal one-threshold mechanism

If $d_i^u = \overline{v}_i - \frac{k_i}{c_i}$ for all i, then $d^l \geq \max_j \left\{\underline{v}_j - k_j/c_j\right\}$ satisfies that

$$\sum_{i=1}^n \varphi_i F_i\left(d^l + \frac{k_i}{c_i}\right) = \prod_{i=1}^n F_i\left(d^l + \frac{k_i}{c_i}\right). \tag{31}$$

Lemma 17 below shows that there exists a unique d^l satisfying (31). Note that unless $\varphi_i = 0$ for all i, we have $d^l > \max_j \left\{\underline{v}_j - k_j/c_j\right\}$. Clearly, in optimum, $\varphi_i > 0$ for some i. Hence, $d^l > \max_j \left\{\underline{v}_j - k_j/c_j\right\}$. Let d_i^* $(i = 1, \ldots, n)$ be defined by

$$\mathbb{E}_{v_i}[v_i] - \mathbb{E}_{v_i}\left[\max\left\{v_i, d_i^* + \frac{k_i}{c_i}\right\}\right] + \frac{k_i}{c_i} = 0, \tag{32}$$

and $d^{l*} := \max_i d_i^*$. Now we are ready to state the main result in this subsection which characterizes the set of optimal $\boldsymbol{\varphi}$:

Theorem 8 *Suppose that Assumption 2 holds. If*

$$\sum_{i=1}^{n}(1-c_i)F_i\left(d^{l*}+\frac{k_i}{c_i}\right)\prod_{j\neq i}F_j\left(\overline{v}_i-\frac{k_i}{c_i}+\frac{k_j}{c_j}\right)\leq\prod_{i=1}^{n}F_i\left(d^{l*}+\frac{k_i}{c_i}\right), \tag{33}$$

then the set of optimal $\boldsymbol{\varphi}$ is the convex hull of

$$\left\{\boldsymbol{\varphi}\;\middle|\;\begin{array}{c} i^*\in\arg\max_i d_i^*, \varphi_i=(1-c_i)\prod_{j\neq i}F_j\left(\overline{v}_i-\frac{k_i}{c_i}+\frac{k_j}{c_j}\right)\;\forall i\neq i^*, \\ \varphi_{i^*}=\frac{\prod_{i=1}^{n}F_i\left(d^{l*}+\frac{k_i}{c_i}\right)-\sum_{i\neq i^*}(1-c_i)\prod_{j\neq i}F_j\left(\overline{v}_i-\frac{k_i}{c_i}+\frac{k_j}{c_j}\right)F_i\left(d^*+\frac{k_i}{c_i}\right)}{F_{i^*}\left(d^{l*}+\frac{k_{i^*}}{c_{i^*}}\right)}\end{array}\right\}.$$

For each optimal $\boldsymbol{\varphi}^*$, the following allocation rule is optimal:

$$P_i^{**}(v_i):=\begin{cases}\prod_{j\neq i}F_j\left(v_i-\frac{k_i}{c_i}+\frac{k_j}{c_j}\right) & \textit{if } v_i\geq d^l+\frac{k_i}{c_i}\\ \varphi_i^* & \textit{if } v_i<d^l+\frac{k_i}{c_i}\end{cases}.$$

Proof. Let $\Phi(d^l,d_1^u,\ldots,d_n^u)\subset\{\boldsymbol{\varphi}|\sum\varphi_i\leq 1\}$ denote the feasible set of $\boldsymbol{\varphi}$ given d^l and $d_1^u,\ldots,d_n^u$. I often abuse notation and use Φ to denote the feasible set when its meaning is clear. Fix $d^l>\max_j\left\{\underline{v}_j-k_j/c_j\right\}$ and $d_i^u=\overline{v}_i-\frac{k_i}{c_i}$ for all i. Then $\boldsymbol{\varphi}$ is feasible if and only if

$$\sum_{i=1}^{n}\varphi_iF_i\left(d^l+\frac{k_i}{c_i}\right)=\prod_{i=1}^{n}F_i\left(d^l+\frac{k_i}{c_i}\right),$$
$$\prod_{j\neq i}F_j\left(\overline{v}_i-\frac{k_i}{c_i}+\frac{k_j}{c_j}\right)\leq\frac{\varphi_i}{1-c_i},\forall i.$$

Hence, Φ is non-empty if and only if

$$\sum_{i=1}^{n}(1-c_i)F_i\left(d^l+\frac{k_i}{c_i}\right)\prod_{j\neq i}F_j\left(\overline{v}_i-\frac{k_i}{c_i}+\frac{k_j}{c_j}\right)\leq\prod_{i=1}^{n}F_i\left(d^l+\frac{k_i}{c_i}\right).$$

Suppose that Φ is non-empty. It is not hard to see that Φ is convex. Because the objective function is linear in $\boldsymbol{\varphi}$ and the feasible set is convex, there is an optimal $\boldsymbol{\varphi}$ which is an extreme point.

Clearly, $\boldsymbol{\varphi}$ is an extreme point of Φ if and only if there exists i^* such that

$$\sum_{i=1}^{n} \varphi_i F_i\left(d^l + \frac{k_i}{c_i}\right) = \prod_{i=1}^{n} F_i\left(d^l + \frac{k_i}{c_i}\right),$$
$$\varphi_j = (1-c_j)\prod_{i\neq j} F_i\left(\overline{v}_j - \frac{k_j}{c_j} + \frac{k_i}{c_i}\right), \forall j \neq i^*.$$

In this case, denote the principal's payoff by $Z_{1,i^*}(d^l)$. For ease of notation, let $i^* = 1$. Let $\bar{\varphi}_j := (1-c_j)\prod_{i\neq j} F_i\left(\overline{v}_j - \frac{k_j}{c_j} + \frac{k_i}{c_i}\right)$ for all j. Then the principal's payoff is given as follows:

$$\begin{aligned}
Z_{1,1}(d^l) :=& \sum_{i=1}^{n} \int_{\max\left\{d^l+\frac{k_i}{c_i},\frac{k_i}{c_i}\right\}}^{\overline{v}_i} \left(v_i - \frac{k_i}{c_i}\right) \prod_{j\neq i} F_j\left(v_i - \frac{k_i}{c_i} + \frac{k_j}{c_j}\right) \mathrm{d}F_i(v_i) \\
&+ \sum_{i\neq 1} \int_{\underline{v}_i}^{\max\left\{d^l+\frac{k_i}{c_i},\frac{k_i}{c_i}\right\}} \left(v_i - \frac{k_i}{c_i}\right) \overline{\varphi}_i \mathrm{d}F_i(v_i) \\
&+ \int_{\underline{v}_1}^{\max\left\{d^l+\frac{k_1}{c_1},\frac{k_1}{c_1}\right\}} \left(v_1 - \frac{k_1}{c_1}\right) \frac{\prod_{i=1}^{n} F_i\left(d^l + \frac{k_i}{c_i}\right) - \sum_{i\neq 1} F_i\left(d^l + \frac{k_i}{c_i}\right)\overline{\varphi}_i}{F_1\left(d^l + \frac{k_1}{c_1}\right)} \mathrm{d}F_1(v_1) \\
&+ \sum_{i\neq 1} \frac{\overline{\varphi}_i k_i}{c_i} + \frac{k_1}{c_1} \frac{\prod_{i=1}^{n} F_i\left(d^l + \frac{k_i}{c_i}\right) - \sum_{i\neq 1} F_i\left(d^l + \frac{k_i}{c_i}\right)\overline{\varphi}_i}{F_1\left(d^l + \frac{k_1}{c_1}\right)}.
\end{aligned}$$

If $d^l < 0$, then it is not hard to show that $Z_{1,1}$ is strictly increasing in d^l. If $d^l \geq 0$, then, after some algebra, we have

$$\begin{aligned}
&Z'_{1,1}(d^l) \\
=& \left\{ \sum_{i\neq 1} \left[f_i\left(d^l + \frac{k_i}{c_i}\right) \prod_{j\neq i,1} F_j\left(d^l + \frac{k_j}{c_j}\right) - \overline{\varphi}_i \frac{f_i\left(d^l + \frac{k_i}{c_i}\right) F_1\left(d^l + \frac{k_1}{c_1}\right) - F_i\left(d^l + \frac{k_i}{c_i}\right) f_1\left(d^l + \frac{k_1}{c_1}\right)}{F_1^2\left(d^l + \frac{k_1}{c_1}\right)} \right] \right\} \\
&\cdot \left[\int_{\underline{v}_1}^{d^l + \frac{k_1}{c_1}} \left(v_1 - d^l - \frac{k_1}{c_1}\right) \mathrm{d}F_1(v_1) + \frac{k_1}{c_1} \right].
\end{aligned}$$

Because $\overline{\varphi}_i \leq \prod_{j\neq i} F_j\left(d^l + \frac{k_j}{c_j}\right)$, the first-term in the above equation is strictly positive. The second-term is strictly decreasing in d^l. Let d_1^* be such that

$$\int_{\underline{v}_1}^{d_1^* + \frac{k_1}{c_1}} \left(v_1 - d_1^* - \frac{k_1}{c_1}\right) \mathrm{d}F_1(v_1) + \frac{k_1}{c_1} = 0. \tag{34}$$

Then $Z'_{1,1}(d^l) > 0$ if $d^l < d_1^*$ and $Z'_{1,1}(d^l) < 0$ if $d^l > d_1^*$. Hence, $Z_{1,1}(d^l)$ achieves its maximum at $d^l = d_1^*$.

Define d_i^* for all $i \geq 2$ as in (34). Suppose that $d_1^* \geq d_2^*$. By a similar argument to that in Lemma 17, $\Phi\left(d_2^*, \overline{v}_1 - \frac{k_1}{c_1}, \ldots, \overline{v}_n - \frac{k_n}{c_n}\right) \neq \emptyset$ implies that $\Phi\left(d_1^*, \overline{v}_1 - \frac{k_1}{c_1}, \ldots, \overline{v}_n - \frac{k_n}{c_n}\right) \neq \emptyset$. Suppose that both $\Phi\left(d_2^*, \overline{v}_1 - \frac{k_1}{c_1}, \ldots, \overline{v}_n - \frac{k_n}{c_n}\right)$ and $\Phi\left(d_1^*, \overline{v}_1 - \frac{k_1}{c_1}, \ldots, \overline{v}_n - \frac{k_n}{c_n}\right)$ are non-empty. Then

$$\begin{aligned}
&Z_{1,1}(d^l) - Z_{1,2}(d^l) \\
&= \left[\prod_{i=1}^n F_i\left(d^l + \frac{k_i}{c_i}\right) - \sum_{i=1} F_i\left(d^l + \frac{k_i}{c_i}\right)\overline{\varphi}_i\right] \\
&\quad \cdot \left\{\frac{1}{F_1\left(d^l + \frac{k_1}{c_1}\right)}\left[\int_{\underline{v}_1}^{d^l + \frac{k_1}{c_1}}\left(v_1 - \frac{k_1}{c_1}\right)\mathrm{d}F_1(v_1) + \frac{k_1}{c_1}\right]\right. \\
&\qquad \left. - \frac{1}{F_2\left(d^l + \frac{k_2}{c_2}\right)}\left[\int_{\underline{v}_2}^{d^l + \frac{k_2}{c_2}}\left(v_1 - \frac{k_2}{c_2}\right)\mathrm{d}F_2(v_2) + \frac{k_2}{c_2}\right]\right\}
\end{aligned}$$

If $d^l = d_2^*$, then by definition we have

$$\begin{aligned}
&Z_{1,1}(d_2^*) - Z_{1,2}(d_2^*) \\
&= \left[\prod_{i=1}^n F_i\left(d_2^* + \frac{k_i}{c_i}\right) - \sum_{i=1} F_i\left(d_2^* + \frac{k_i}{c_i}\right)\overline{\varphi}_i\right] \\
&\quad \cdot \left\{\frac{1}{F_1\left(d_2^* + \frac{k_1}{c_1}\right)}\left[\int_{\underline{v}_1}^{d_2^* + \frac{k_1}{c_1}}\left(v_1 - \frac{k_1}{c_1}\right)\mathrm{d}F_1(v_1) + \frac{k_1}{c_1}\right] - d_2^*\right\} \\
&\geq \left[\prod_{i=1}^n F_i\left(d_2^* + \frac{k_i}{c_i}\right) - \sum_{i=1} F_i\left(d_2^* + \frac{k_i}{c_i}\right)\overline{\varphi}_i\right](d_2^* - d_2^*) = 0,
\end{aligned}$$

where the last inequality holds because $d_1^* \geq d_2^*$, and the inequality holds strictly if $d_1^* > d_2^*$. Hence, $Z_{1,1}(d_1^*) \geq Z_{1,2}(d_2^*)$ and the inequality holds strictly if $d_1^* > d_2^*$.

Let $d^{l*} := \max_i d_i^*$. If

$$\sum_{i=1}^n (1 - c_i)\prod_{j \neq i} F_j\left(\overline{v}_j - \frac{k_i}{c_i} + \frac{k_j}{c_j}\right) F_i\left(d^* + \frac{k_i}{c_i}\right) \leq \prod_{i=1}^n F_i\left(d^{l*} + \frac{k_i}{c_i}\right),$$

then $\Phi\left(d^{l*}, \overline{v}_1 - \frac{k_1}{c_1}, \ldots, \overline{v}_n - \frac{k_n}{c_n}\right)$ is feasible. Given $\boldsymbol{\varphi}$, let $Z(\boldsymbol{\varphi})$ denote the principal's optimal payoff. By Lemma 18, $Z(\boldsymbol{\varphi}) \le Z_1(\boldsymbol{\varphi})$, where Z_1 is defined by

$$Z_1(\boldsymbol{\varphi}) := \sum_{i=1}^{n} \int_{\max\left\{d^l+\frac{k_i}{c_i},\frac{k_i}{c_i}\right\}}^{\overline{v}_i} \left(v_i - \frac{k_i}{c_i}\right) \prod_{j\neq i} F_j \left(v_i - \frac{k_i}{c_i} + \frac{k_j}{c_j}\right) \mathrm{d}F_i(v_i)$$
$$+ \sum_{i=1}^{n} \int_{\underline{v}_i}^{\max\left\{d^l+\frac{k_i}{c_i},\frac{k_i}{c_i}\right\}} \left(v_i - \frac{k_i}{c_i}\right) \varphi_i \mathrm{d}F_i(v_i) + \sum_{i=1}^{n} \frac{\varphi_i k_i}{c_i},$$

and $d^l = d^l(\boldsymbol{\varphi})$ is given by (31). By the arguments above, for any $\boldsymbol{\varphi}$,

$$Z(\boldsymbol{\varphi}) \le Z_1(\boldsymbol{\varphi}) \le \max_i Z_{1,i}(d^l(\boldsymbol{\varphi})) \le Z_{1,i^*}(d^{l*}),$$

where $d^*_{i^*} = d^{l^*}$. This completes the proof. ■

Lemma 17 *There exists a unique* $d^l \ge \max_j \left\{\underline{v}_j - k_j/c_j\right\}$ *such that*

$$\sum_{i=1}^{n} \varphi_i F_i \left(d_l + \frac{k_i}{c_i}\right) = \prod_{i=1}^{n} F_i \left(d^l + \frac{k_i}{c_i}\right). \tag{31}$$

Proof. If $\varphi_i = 0$ for all i, then $d^l = \max_j \left\{\underline{v}_j - k_j/c_j\right\}$ is the unique solution to (31). Assume, for the rest of the proof, that $\varphi_i > 0$ for some i. Let

$$\Delta(d^l) := \sum_{i=1}^{n} \varphi_i F_i \left(d^l + \frac{k_i}{c_i}\right) - \prod_{i=1}^{n} F_i \left(d^l + \frac{k_i}{c_i}\right).$$

Then

$$\Delta'(d^l) = \sum_{i=1}^{n} f_i \left(d^l + \frac{k_i}{c_i}\right) \left[\varphi_i - \prod_{j\neq i} F_j \left(d^l + \frac{k_j}{c_j}\right)\right].$$

Because $\Delta_l \left(\max_j \left\{\underline{v}_j - k_j/c_j\right\}\right) > 0$, a solution to (31) must satisfy that $d^l > \max_j \left\{\underline{v}_j - k_j/c_j\right\}$. Assume, for the rest of the proof, that $d^l > \max_j \left\{\underline{v}_j - k_j/c_j\right\}$. Then $F_i \left(d^l + \frac{k_i}{c_i}\right) > 0$ for all i. If $\Delta(d^l) \le 0$, then $\varphi_i \le \prod_{j\neq i} F_j \left(d^l + \frac{k_j}{c_j}\right)$ for all i, and the strict inequality holds for some i, which implies that $\Delta'(d^l) < 0$. Hence, $\Delta(d^l)$ crosses zero at most once, in which case it does so from above. Because $\Delta_l \left(\max_j \left\{\underline{v}_j - k_j/c_j\right\}\right) > 0$ and $\Delta_l \left(\max_j \left\{\overline{v}_j - k_j/c_j\right\}\right) = \sum_i \varphi_i - 1 \le 0$, there exists a unique d^l satisfying (31). ■

Lemma 18 *Let Z and Z_1 be defined as in the proof of Theorem 8. Then $Z(\boldsymbol{\varphi}) \le Z_1(\boldsymbol{\varphi})$.*

Proof. Fix $\boldsymbol{\varphi}$ and the corresponding $d^l = d^l(\boldsymbol{\varphi})$, which is given by (31). Assume, without loss of generality, that $d^l \ge 0$. Note that Z_1 is attained by the following interim allocation rule $\boldsymbol{P}$:

$$P_i(v_i) = \begin{cases} \prod_{j\neq i} F_j\left(v_i - \frac{k_i}{c_j} + \frac{k_j}{c_j}\right) & \text{if } v_i - \frac{k_i}{c_i} > d^l \\ \varphi_i & \text{if } v_i - \frac{k_i}{c_i} \le d^l \end{cases}.$$

$\boldsymbol{P}$ can be implemented by the following ex post allocation rule $\boldsymbol{p}$:

$$p_i(\boldsymbol{v}) = \begin{cases} \frac{1}{\left|\left\{j \middle| v_j - \frac{k_j}{c_j} = v_i - \frac{k_i}{c_i}\right\}\right|} & \text{if } v_i - \frac{k_i}{c_i} > d^l \text{ and } v_i - \frac{k_i}{c_i} = \max_j \left\{v_j - \frac{k_j}{c_j}\right\} \\ \frac{\varphi_i}{\prod_{j\neq i} F_j\left(d^l + \frac{k_j}{c_j}\right)} & \text{if } \max_j \left\{v_j - \frac{k_j}{c_j}\right\} \le d^l \\ 0 & \text{otherwise} \end{cases}.$$

Let $\boldsymbol{P}^*$ denote the optimal mechanism given $\boldsymbol{\varphi}$. By Theorem 4, there exists d^{l*} and d_i^{u*} for $i = 1, \dots, n$ such that $\boldsymbol{P}^*$ is given by (17). Let $\boldsymbol{p}^*$ denote the corresponding ex post allocation rule that implements $\boldsymbol{P}^*$.

First, we show that $d^{l*} \le d^l$. If $\min\{d_i^{u*}\} > d^l$, then $d^{l^*} = d^l$ by construction. Otherwise, d^{l^*} is defined by the following equation

$$\sum_{i=1}^n \varphi_i F_i\left(d^{l^*} + \frac{k_i}{c_i}\right) + \sum_{\left\{i: d_i^{u*} = d^{l*}\right\}} \frac{\varphi_i}{1-c_i}\left[1 - F\left(d^{l^*} + \frac{k_i}{c_i}\right)\right] = \prod_{\left\{i: d_i^{u*} > d^{l*}\right\}} F_i\left(d^{l^*} + \frac{k_i}{c_i}\right).$$

Because (AF2) requires that[15]

$$\sum_{\left\{i: d_i^{u*} = d^{l*}\right\}} \frac{\varphi_i}{1-c_i}\left[1 - F\left(d^{l^*} + \frac{k_i}{c_i}\right)\right] \le \prod_{\left\{i: d_i^{u*} > d^{l*}\right\}} F_i\left(d^{l^*} + \frac{k_i}{c_i}\right) - \prod_{i=1}^n F_i\left(d^{l^*} + \frac{k_i}{c_i}\right),$$

we have

$$\sum_{i=1}^n \varphi_i F_i\left(d^{l^*} + \frac{k_i}{c_i}\right) \ge \prod_{i=1}^n F_i\left(d^{l^*} + \frac{k_i}{c_i}\right).$$

Compare the inequality with (31) and it is easy to see that $d^{l*} \le d^l$.

[15]Let $S_i = \left[d^l + \frac{k_i}{c_i}, \overline{v}_i\right]$ for all $i = 1, \dots, n$.

By construction, $P_i(v_i) = \varphi_i \leq P_i^*(v_i)$ if $v_i \leq d^l + \frac{k_i}{c_i}$ for all $i = 1, 2, \ldots, n$. Hence,

$$
\begin{aligned}
&Z_1(\boldsymbol{\varphi}) - Z(\boldsymbol{\varphi}) \\
=& \sum_{i=1}^{n} \int_{\underline{v}_i}^{\overline{v}_i} \left(v_i - \frac{k_i}{c_i}\right) [P_i(v_i) - P_i^*(v_i)] \, \mathrm{d}F_i(v_i) \\
=& \sum_{i=1}^{n} \int_{\underline{v}_i}^{d^l + \frac{k_i}{c_i}} \left(v_i - \frac{k_i}{c_i}\right) [P_i(v_i) - P_i^*(v_i)] \, \mathrm{d}F_i(v_i) + \sum_{i=1}^{n} \int_{d^l + \frac{k_i}{c_i}}^{\overline{v}_i} \left(v_i - \frac{k_i}{c_i}\right) [P_i(v_i) - P_i^*(v_i)] \, \mathrm{d}F_i(v_i) \\
\geq& \sum_{i=1}^{n} \int_{\underline{v}_i}^{d^l + \frac{k_i}{c_i}} d^l \, [P_i(v_i) - P_i^*(v_i)] \, \mathrm{d}F_i(v_i) + \sum_{i=1}^{n} \int_{d^l + \frac{k_i}{c_i}}^{\overline{v}_i} \left(v_i - \frac{k_i}{c_i}\right) [P_i(v_i) - P_i^*(v_i)] \, \mathrm{d}F_i(v_i) \\
=& \int_{\underline{v}_1}^{\overline{v}_i} \cdots \int_{\underline{v}_n}^{\overline{v}_i} \sum_{i=1}^{n} \left(\max\left\{v_i - \frac{k_i}{c_i}, d^l\right\}\right) [p_i(\boldsymbol{v}) - p_i^*(\boldsymbol{v})] \, \mathrm{d}F_1(v_1) \ldots \mathrm{d}F_n(v_n) \geq 0.
\end{aligned}
$$

This completes the proof. ■

C.4 Omitted proofs in Section 5.1

Proof of Theorem 5. Part 1 of Theorem 5 follows from Theorem 8 immediately. Assume for the rest of the proof that $n_H(1-c)F_H(d_H^* + \rho) + n_L(1-c)F_L(d_H^* + \rho) > F_H(d_H^* + \rho)^{n_H} F_L(d_H^* + \rho)^{n_L}$. Hence, in optimum, $\min\{\varphi_L, \varphi_H\} \leq 1 - c$.

Let

$$
v_H^\natural := \sup\left\{v \,\middle|\, \frac{F_H(v)^{n_H - 1}(1 - F_H(v))}{1 - c} - 1 + F_H(v)^{n_H} \leq 0\right\}.
$$

Given φ_H, let $v_H^l(\varphi_H)$ be such that $F\left(v_H^l(\varphi_H)\right)^{n_H - 1} = n\varphi_H$ and

$$
v_H^u(\varphi_H) := \inf\left\{v \,\middle|\, 1 - F_H(v)^{n_H} - \frac{n_H \varphi_H}{1 - c}[1 - F_H(v)] \geq 0\right\}.
$$

I prove part 2 of Theorem 5 by proving the following claims:

1. If $F_H(d_H^* + \rho)^{n_H - 1} \geq n_H(1 - c)$, the optimal $\boldsymbol{\varphi}^* = (F_H(d_H^* + \rho)^{n_H - 1}/n_H, 0)$, the optimal inspection rule satisfies $\boldsymbol{Q}^* = ((1 - \varphi_H^*/P_H^*)/c, 0)$ and the following allocation rule is optimal:

$$
P_H^*(v) = \begin{cases} F_H(v)^{n_H - 1} & \text{if } v - \rho \geq d_H^* \\ \varphi_H^* & \text{if } v - \rho < d_H^* \end{cases}
$$

and $P_L^*(v) = 0$.

2. If $F_H(d_H^* + \rho)^{n_H - 1} < n_H(1 - c)$ and $d_H^{**} + \rho \leq v_H^\natural$, the optimal $\boldsymbol{\varphi}^* = ((1 - c)/n_H(1 - cF_H(d_H^{**} + \rho)), 0)$, the optimal inspection rule satisfies $\boldsymbol{Q}^* =$

$((1-\varphi_H^*/P_H^*)/c, 0)$ and the following allocation rule is optimal:

$$P_H^*(v) := \begin{cases} \frac{\varphi_H^*}{1-c} & \text{if } v \geq d_H^{**} + \rho \\ \varphi_H^* & \text{if } v < d_H^{**} + \rho \end{cases}.$$

and $P_L^*(v) = 0$.

3. If $F_H(d_H^* + \rho)^{n_H - 1} < n_H(1-c)$ and $d_H^{**} + \rho > v_H^{\natural}$, the optimal φ_H^* is defined by

$$\int_{\underline{v}}^{\overline{v}} (v - \min\{v, v_H^u(\varphi_H^*)\})\, \mathrm{d}F_H(v) + (1-c)\left[\int_{\underline{v}}^{\overline{v}} \left(v - \max\left\{v, v_H^l(\varphi_H^*)\right\}\right) \mathrm{d}F_H(v) + \rho\right] = 0$$

and the optimal $\varphi_L^* = 0$, the optimal inspection rule satisfies $\boldsymbol{Q}^* = ((1-\varphi_H^*/P_H^*)/c, 0)$ and the following allocation rule is optimal:

$$P_H^*(v) := \begin{cases} \frac{\varphi_H^*}{1-c} & \text{if } v \geq v_H^u(\varphi_H^*) \\ F_H(v)^{n_H - 1} & \text{if } v_H^l(\varphi_H^*) < v < v_H^u(\varphi_H^*) \\ \varphi_H^* & \text{if } v \leq v_H^l(\varphi_H^*) \end{cases},$$

and $P_L^*(v) = 0$.

Here, I only provide a proof for the first claim. The proofs of the rest two claims are similar and neglected here.

I abuse notation a bit and let $n_g, g \in \{H, L\}$ denote the set of agents in group g as well. Since we focus on group symmetric mechanisms, d_L^u and d_H^u exist such that $d_i^u = d_g^u$ for all $i \in n_g$ and $g \in \{H, L\}$. Given d^l, d_L^u and d_H^u, let $\Phi(d^l, d_L^u, d_H^u) \subset \{\boldsymbol{\varphi} | n_L\varphi_L + n_H\varphi_H \leq 1\}$ denote the feasible set of $\boldsymbol{\varphi}$. I often abuse notation and use Φ to denote the feasible set when its meaning is clear. We consider two cases in turn: $\varphi_L \leq 1-c$ and $\varphi_H \leq 1-c$.

Case 1: Suppose that $\varphi_L \leq 1-c$. Suppose that $d_H^u = \overline{v} - \rho$ and $d_L^u = d^l$. By a similar argument to that of Lemma 20, $\boldsymbol{\varphi} \in \Phi$ if and only if

$$\frac{\varphi_H}{1-c} \geq 1, \tag{35}$$

$$n_L\varphi_L F_L(d^l + \rho) + n_H\varphi_H F_H(d^l + \rho) + \frac{n_L\varphi_L}{1-c}\left[1 - F_L(d^l + \rho)\right] = F_H(d^l + \rho)^{n_H}, \tag{36}$$

$$\frac{n_L\varphi_L}{1-c} \leq \frac{1 - F_L(d^l + \rho)^{n_L}}{1 - F_L(d^l + \rho)} F_H(d^l + \rho)^{n_H}. \tag{37}$$

Then Φ is non-empty if and only if

$$F_H(d^l + \rho)^{n_H - 1} \geq n_H(1-c). \tag{38}$$

Let $\Phi'(d^l, d^u_L, d^u_H)$ denote the set of $\boldsymbol{\varphi}$ that satisfies (35) and (36). Clearly, $\Phi \subset \Phi'$, and Φ' is non-empty if and only if (38) holds. It is not hard to see that Φ' is convex.

Given $\boldsymbol{\varphi}$, let $Z(\boldsymbol{\varphi})$ denote the principal's optimal payoff. Consider the principal's relaxed problem of $\max_{\boldsymbol{\varphi}\in\Phi'} Z(\boldsymbol{\varphi})$. Because the principal's objective function is linear in $\boldsymbol{\varphi}$ and Φ' is convex, there is an optimal $\boldsymbol{\varphi}$ which is an extreme point of Φ'. The set Φ' has two extreme points: $\boldsymbol{\varphi}^H := (\frac{F_H(d^l+\rho)^{n_H-1}}{n_H}, 0)$ and

$$\boldsymbol{\varphi}^L := \left(1-c, \frac{1-c}{n_L}\frac{F_H(d^l+\rho)^{n_H} - n_H(1-c)F_H(d^l+\rho)}{1-cF_L(d^l+\rho)}\right).$$

Next we calculate the principal's optimal payoffs at the two extreme points in turn.

Case 1.1: $\boldsymbol{\varphi} = \boldsymbol{\varphi}^H$. In this case, the principal's optimal payoff is given by $Z(\boldsymbol{\varphi}^H) = Z_{2,H}(d^l(\boldsymbol{\varphi}^H))$, where $d^l(\boldsymbol{\varphi}^H)$ is given by (36) and

$$\begin{aligned} Z_{1,H}(d^l) =& F_H(d^l+\rho)^{n_H}\left[\int_{\underline{v}}^{\max\{\rho, d^l+\rho\}}(v-\rho)\mathrm{d}F_H(v)+\rho\right] \\ &+ n_H\int_{\max\{\rho,d^l+\rho\}}^{\overline{v}}(v-\rho)F_H(v)^{n_H-1}\mathrm{d}F_H(v). \end{aligned}$$

If $d^l < 0$, then it is not hard to see that $Z_{1,H}$ is strictly increasing in d^l. If $d^l \geq 0$, then, after some algebra, we have

$$Z'_{1,H}(d^l) = (n_H-1)F_H(d^l+\rho)^{n_H-2}f_H(d^l+\rho)\left[\int_{\underline{v}}^{\overline{v}}\big(v-\max\{v, d^l+\rho\}\big)\,\mathrm{d}F_H(v)+\rho\right].$$

Note that the term in front of the brackets is strictly positive, and the term inside the bracket is strictly decreasing in d^l and is equal to zero if and only if $d^l = d^*_H$, where d^*_H is given by (19). Hence, $Z_{1,H}$ achieves its maximum at d^*_H.

Case 1.2: $\boldsymbol{\varphi} = \boldsymbol{\varphi}^L$. If (38) holds with equality, then the two extreme points coincide and $Z(\boldsymbol{\varphi}^L) = Z(\boldsymbol{\varphi}^H)$. Suppose that (38) holds with strictly inequality so that the two extreme points do not coincide, then the principal's optimal payoff is given by $Z(\boldsymbol{\varphi}^L) = Z_{2,L}(d^l(\boldsymbol{\varphi}^L))$, where $d^l(\boldsymbol{\varphi}^L)$ is given by (36), and, for $g \in \{H, L\}$ and $g \neq g'$,

$$
\begin{aligned}
Z_{2,g}(d^l) =& n_{g'}(1-c)\left[\int_{\underline{v}}^{\max\{\rho, d^l+\rho\}} (v-\rho)\mathrm{d}F_{g'}(v) + \rho\right] \\
&+ n_{g'}\int_{\max\{\rho, d^l+\rho\}}^{\overline{v}} (v-\rho)F_{g'}(v)^{n_{g'}-1}\mathrm{d}F_{g'}(v) \\
&+ (1-c)\frac{F_{g'}(d^l+\rho)^{n_{g'}} - n_{g'}(1-c)F_{g'}(d^l+\rho)}{1-cF_g(d^l+\rho)} \\
&\quad \cdot \left[\int_{\underline{v}}^{\max\{\rho, d^l+\rho\}} (v-\rho)\mathrm{d}F_g(v) + \rho + \frac{1}{1-c}\int_{\max\{\rho, d^l+\rho\}}^{\overline{v}} (v-\rho)\mathrm{d}F_g(v)\right].
\end{aligned}
$$

Clearly, if $d^l < 0$, then $Z_{2,L}$ is strictly increasing in d^l. If $d^l \geq 0$, then, after some algebra, we have

$$
\begin{aligned}
Z'_{2,L}(d^l) =& \frac{F_H(v+d^l)^{n_H-1} - n_H(1-c)}{[1-cF_L(d^l+\rho)]^2} \\
&\cdot \left\{cF_H(d^l+\rho)f_L(d^L+\rho) + [1-cF_L(d^l+\rho)]f_H(d^L+\rho)\right\} \\
&\cdot \left[\begin{array}{l} (1-c)\int_{\underline{v}}^{\overline{v}} \left(v - \max\{v, d^l+\rho\}\right)\mathrm{d}F_L(v) + (1-c)\rho \\ + \int_{\underline{v}}^{\overline{v}} \left(v - \min\{v + d^l + \rho\}\right)\mathrm{d}F_L(v) \end{array}\right].
\end{aligned}
$$

Cleary, the second line in the above equation is also strictly positive. If (38) holds with strictly inequality, then the first line in the above equation is strictly positive. Furthermore, the third line is strictly decreasing in d^l and is equal to zero if and only if $d^l = d_L^{**}$, where d_L^{**} is given by (20). Hence, $Z_{2,L}$ achieves its maximum at d_L^{**}.

Because the left-hand side of (38) is strictly increasing in d^l, $\Phi'(d^l, d^l, \overline{v}-\rho) \neq \emptyset$ implies that $\Phi'(d^{l'}, d^{l'}, \overline{v}-\rho) \neq \emptyset$ for all $d^{l'} \geq d^l$. Fix d^l such that $\Phi'(d^l, d^l, \overline{v}-\rho) \neq \emptyset$, then

$$
\begin{aligned}
& Z_{2,H}(d^l) - Z_{2,L}(d^l) \\
=& \left[F_H(d^l+\rho)^{n_H-1} - n_H(1-c)\right]\left[\int_{\underline{v}}^{\overline{v}} \left(v - \max\{v, d^l+\rho\}\right)\mathrm{d}F_H(v) + \rho\right] \\
&- \frac{\left[F_H(d^l+\rho)^{n_H-1} - n_H(1-c)\right]F_H(d^l+\rho)}{1-cF_L(d^l+\rho)} \\
&\cdot \left[\begin{array}{l} (1-c)\int_{\underline{v}}^{\overline{v}} \left(v - \max\{v, d^l+\rho\}\right)\mathrm{d}F_L(v) + (1-c)\rho \\ + \int_{\underline{v}}^{\overline{v}} \left(v - \min\{v, d^l+\rho\}\right)\mathrm{d}F_L(v) \end{array}\right].
\end{aligned}
$$

Suppose that $\Phi'(\max\{d_L^{**}, d_H^*\}, \max\{d_L^{**}, d_H^*\}, \overline{v}-\rho) \neq \emptyset$. Since $d_H^* \geq d_L^{**}$, then by construction $Z_{2,H}(d_L^{**}) - Z_{2,L}(d_L^{**}) \geq 0$. Hence,

$$Z_{2,H}(d_H^{*}) - Z_{2,L}(d_L^{**}) = Z_{2,H}(d_H^{*}) - Z_{2,H}(d_L^{**}) + Z_{2,H}(d_L^{**}) - Z_{2,L}(d_L^{**}) \geq 0.$$

By Lemma 19, if $\varphi_L \leq 1-c$, then $Z(\boldsymbol{\varphi}) \leq Z_2(\boldsymbol{\varphi})$, where Z_2 is defined by

$$\begin{aligned}
Z_2(\boldsymbol{\varphi}) =& n_H\varphi_H\left[\int_{\underline{v}}^{\max\{\rho,d^l+\rho\}}(v-\rho)\mathrm{d}F_H(v)+\rho\right] \\
&+ n_H\int_{\max\{\rho,d^l+\rho\}}^{\overline{v}}(v-\rho)F_H(v)^{n_H-1}\mathrm{d}F_H(v) \\
&+ n_L\varphi_L\left[\int_{\underline{v}}^{\max\{\rho,d^l+\rho\}}(v-\rho)\mathrm{d}F_L(v)+\rho+\frac{1}{1-c}\int_{\max\{\rho,d^l+\rho\}}^{\overline{v}}(v-\rho)\mathrm{d}F_L(v)\right],
\end{aligned}$$

and $d^l = d^l(\boldsymbol{\varphi})$ is defined by (36). Hence,

$$Z(\boldsymbol{\varphi}) \leq Z_2(\boldsymbol{\varphi}) \leq \max\left\{Z_{1,H}(d^l(\boldsymbol{\varphi})), Z_{2,L}(d^l(\boldsymbol{\varphi}))\right\} \leq \max\left\{Z_{1,H}(d_H^{*}), Z_{2,L}(d_L^{**})\right\} \leq Z_{1,H}(d_H^{*}).$$

Finally, note that $\boldsymbol{\varphi}^H \in \Phi'$ implies that $\boldsymbol{\varphi}^H \in \Phi$. Hence, if $\boldsymbol{\varphi}^H \in \Phi'$ or $F_H(d_H^{*}+\rho)^{n_H-1} \geq n_H(1-c)$, it is optimal to set $\boldsymbol{\varphi}^{1*} = (F_H(d_H^{*}+\rho)^{n_H-1}/n_H, 0)$.

Case 2: Suppose that $\varphi_H \leq 1-c$. Since $d_H^{**} \geq d_H^{*} \geq d_L^{*}$, by a similar argument to that in Case 1, we can show that if $\varphi_H \leq 1-c$, then $Z(\boldsymbol{\varphi}) \leq Z_{2,H}(d_H^{**})$, where the maximum is achieved if

$$\boldsymbol{\varphi}^{2*} = \left(\frac{1-c}{n_H}\frac{F_L(d_H^{**}+\rho)^{n_L} - n_L(1-c)F_L(d_H^{**}+\rho)}{1-cF_H(d_H^{**}+\rho)}, 1-c\right).$$

Note that in this case $\varphi_L^{2*} \leq 1-c$. Hence, by Lemma 19, $Z(\varphi) \leq Z(\boldsymbol{\varphi}^{2*}) \leq Z(\boldsymbol{\varphi}^{1*})$.

Hence, it is optimal to set $\boldsymbol{\varphi}^{*} = \boldsymbol{\varphi}^{1*} = (F_H(d_H^{*}+\rho)^{n_H-1}/n_H, 0)$. This completes the proof of the first claim. ■

Lemma 19 *Let Z and Z_2 be defined as in the proof of Theorem 5. If $\varphi_L \leq 1-c$, then $Z(\boldsymbol{\varphi}) \leq Z_2(\boldsymbol{\varphi})$.*

Proof. Fix $\boldsymbol{\varphi}$ and the corresponding d^l. Assume, without loss of generality, that $d^l \geq 0$. Note that $Z_2(d^l)$ is attained by the following interim allocation rule $\boldsymbol{P}$:

$$P_H(v) = \begin{cases} F_H(v-\rho)^{n_H-1} & \text{if } v-\rho > d^l \\ \varphi_H & \text{if } v-\rho \leq d^l \end{cases},$$

and

$$P_L(v) = \begin{cases} \frac{\varphi_L}{1-c} & \text{if } v - \rho > d^l \\ \varphi_L & \text{if } v - \rho \leq d^l \end{cases}.$$

$\boldsymbol{P}$ can be implemented by the following ex post allocation rule $\boldsymbol{p}$: For $i \in n_H$,

$$p_i(\boldsymbol{v}) = \begin{cases} \frac{1}{|\{j \in n_H | v_j = v_i\}|} & \text{if } v_i - \rho \geq d^l \text{ and } v_i = \max_{j \in n_H} \{v_j\} \\ \frac{\varphi_H}{\prod_{j \neq i} F_j\left(d^l + \rho_j\right)} & \text{if } \max_j \{v_j - \rho_j\} < d^l \\ 0 & \text{otherwise} \end{cases},$$

and, for $i \in n_L$,

$$p_i(\boldsymbol{v}) = \begin{cases} \frac{1}{\left|\left\{j \in n_L \middle| v_j - \rho \geq d^l\right\}\right|} & \text{if } v_i - \rho \geq d^l \text{ and } \max_{j \in n_H} \{v_j - \rho\} < d^l \\ \frac{\varphi_L}{\prod_{j \neq i} F_j\left(d^l + \rho_j\right)} & \text{if } \max_j \{v_j - \rho_j\} \leq d^l \\ 0 & \text{otherwise} \end{cases}.$$

Let $\boldsymbol{P}^*$ denote an optimal interim allocation rule given $\boldsymbol{\varphi}$. By Theorem 4, there exists d^{l*} and d_i^{u*} for $i = 1, \ldots, n$ such that $\boldsymbol{P}^*$ is given by (17). Let $\boldsymbol{p}^*$ denote the corresponding ex post allocation rule that implements $\boldsymbol{P}^*$.

By construction, $P_g(v) = \varphi_g \leq P_g^*(v)$ for all $v \leq d^l + \rho$ and $g \in \{H, L\}$, and $P_L(v) = \frac{\varphi_L}{1-c} \geq P_L^*(v)$ for all $v \geq d^l + \rho$. Hence,

$$\begin{aligned}
& Z_2\left(d^l(\boldsymbol{\varphi})\right) - Z(\boldsymbol{\varphi}) \\
=& n_H \int_{\underline{v}_H}^{\overline{v}_H} (v - \rho)\left[P_H(v) - P_H^*(v)\right] \mathrm{d}F_H(v) + n_L \int_{\underline{v}_L}^{\overline{v}_L} (v - \rho)\left[P_L(v) - P_L^*(v)\right] \mathrm{d}F_L(v) \\
\geq& n_H \int_{\underline{v}_H}^{\overline{v}_H} \left(\max\{d^l, v - \rho\}\right)\left[P_H(v) - P_H^*(v)\right] \mathrm{d}F_H(v) + n_L \int_{\underline{v}_L}^{\overline{v}_L} d^l \left[P_L(v) - P_L^*(v)\right] \mathrm{d}F_L(v) \\
=& \int_{\underline{v}_1}^{\overline{v}_1} \cdots \int_{\underline{v}_n}^{\overline{v}_n} \left\{\sum_{i \in n_H} \left(\max\left\{v_i - \rho, d^l\right\}\right)\left[p_i(\boldsymbol{v}) - p_i^*(\boldsymbol{v})\right] + \sum_{i \in n_L} d^l \left[p_i(\boldsymbol{v}) - p_i^*(\boldsymbol{v})\right]\right\} \mathrm{d}F_1(v_1) \ldots \mathrm{d}F_n(v_n) \\
\geq& 0.
\end{aligned}$$

This completes the proof. ■

Proof of Theorem 6. Part 1 of Theorem 6 follows from Theorem 8 immediately. It remains to prove part 2 of Theorem 6. Assume for the rest of the proof that $n_H(1-c)F_H(d_H^* + \rho)F_L(\overline{v}_H)^{n_L} + n_L(1-c)F_L(d_H^* + \rho) > F_H(d_H^* + \rho)^{n_H} F_L(d_H^* + \rho)^{n_L}$, $n_{g'}(1-c)F_{g'}(d_g^{**} + \rho) + n_g\left[1 - cF_g(d_g^{**} + \rho)\right] \geq F_{g'}(d_g^{**} + \rho)^{n_{g'}}$ for $g, g' \in \{H, L\}$ and $g \neq g'$, and $n_L \geq n_L^{**}(\rho, c)$. By a similar argument to that of Corollary 2, $n_L \geq n_L^{**}(\rho, c)$ if and only if

$$\frac{1-F_L\left(d_L^{**}+\rho\right)}{1-cF_L\left(d_L^{**}+\rho\right)} \leq 1-F_L\left(d_L^{**}+\rho\right)^{n_L}. \tag{39}$$

Hence, in optimum $\min\{\varphi_L, \varphi_H\} \leq c$. I abuse notation a bit and let $n_g, g \in \{H, L\}$ denote the set of agents in group g as well. Since we focus on group symmetric mechanisms, d_L^u and d_H^u exist such that $d_i^u = d_g^u$ for all $i \in n_g$ and $g \in \{H, L\}$. Let $\Phi(d^l, d_L^u, d_H^u) \subset \{\boldsymbol{\varphi} | n_L\varphi_L + n_H\varphi_H \leq 1\}$ denote the feasible set of $\boldsymbol{\varphi}$ given d^l, d_L^u and d_H^u. I often abuse notation and use Φ to denote the feasible set when its meaning is clear. We consider three cases in turn: $\max\{\varphi_L, \varphi_H\} \leq 1-c$, $\varphi_L \leq 1-c$, $\varphi_H \leq 1-c$.

Case 1: Suppose that $\max\{\varphi_L, \varphi_H\} \leq 1-c$**.** Suppose that $d^l = d_H^u = d_L^u = \hat{d}$. Fix $\hat{d}$. Then $\boldsymbol{\varphi}$ is feasible if and only if it satisfies:

$$n_H\varphi_H F_H\left(\hat{d}+\rho\right) + n_L\varphi_L F_L\left(\hat{d}+\rho\right) + \frac{n_H\varphi_H}{1-c}\left[1-F_H\left(\hat{d}+\rho\right)\right] + \frac{n_L\varphi_L}{1-c}\left[1-F_L\left(\hat{d}+\rho\right)\right] = 1. \tag{40}$$

$$\frac{n_g\varphi_g}{1-c}\left[1-F_g\left(\hat{d}+\rho\right)\right] \leq 1-F_g\left(\hat{d}+\rho\right)^{n_g}, \ \forall g \in \{H, L\}, \tag{41}$$

$$\frac{n_H\varphi_H}{1-c}\left[1-F_H\left(\hat{d}+\rho\right)\right] + \frac{n_L\varphi_L}{1-c}\left[1-F_L\left(\hat{d}+\rho\right)\right] \leq 1-F_H\left(\hat{d}+\rho\right)^{n_H} F_L\left(\hat{d}+\rho\right)^{n_L}. \tag{42}$$

Let $\Phi'(d^l, d_H^u, d_L^u)$ denote the set of $\boldsymbol{\phi}$ that satisfies (40). Clearly, $\Phi \subset \Phi'$. It is not hard to see that Φ' is non-empty and convex. Given $\boldsymbol{\varphi}$, let $Z(\boldsymbol{\varphi})$ denote the principal's optimal payoff. Consider the principal's relaxed problem of $\max_{\boldsymbol{\varphi}\in\Phi'} Z(\boldsymbol{\varphi})$. Because the principal's objective function is linear in $\boldsymbol{\varphi}$ and Φ' is convex, there is an optimal $\boldsymbol{\varphi}$ which is an extreme point of Φ'. Φ' contains two extreme points: for $g \in \{H, L\}$, $\boldsymbol{\varphi}^g$ is given by

$$\frac{n_g\varphi_g}{1-c} = \frac{1}{1-cF_g\left(\hat{d}+\rho\right)}, \ \varphi_{g'} = 0, \ g \neq g'.$$

If $\boldsymbol{\varphi} = \boldsymbol{\varphi}^g$, then $Z(\boldsymbol{\varphi}^g) = Z_{3,g}(\hat{d}(\boldsymbol{\varphi}^g))$, where $\hat{d}(\boldsymbol{\varphi}^g)$ is given by (40) and $Z_{3,g}$ is defined by

$$Z_{3,g}(\hat{d}) := \frac{1}{1-cF_g\left(\hat{d}+\rho\right)}\left\{(1-c)\left[\int_{\underline{v}_g}^{\max\{\rho,\hat{d}+\rho\}}(v-\rho)\mathrm{d}F_g(v)+\rho\right] + \int_{\max\{\rho,\hat{d}+\rho\}}^{\overline{v}_g}(v-\rho)\mathrm{d}F_g(v)\right\}.$$

By the same argument as that in Theorem 3, $Z_{3,g}$ is maximized at $d_g^{**} \geq 0$.

For $\hat{d} \geq 0$, we have

$$
\begin{aligned}
&Z_{3,H}(\hat{d}) - Z_{3,L}(\hat{d}) \\
=&\frac{1}{1 - cF_H\left(\hat{d} + \rho\right)} \left\{ \begin{array}{l} (1-c)\left[\int_{\underline{v}_H}^{\overline{v}_H} \left(v - \max\{v, \hat{d}+\rho\}\right) \mathrm{d}F_H(v) + \rho\right] \\ + \int_{\underline{v}_H}^{\overline{v}_H} \left(v - \min\{v, \hat{d}+\rho\}\right) \mathrm{d}F_H(v) \end{array} \right\} \\
&- \frac{1}{1 - cF_L\left(\hat{d} + \rho\right)} \left\{ \begin{array}{l} (1-c)\left[\int_{\underline{v}_L}^{\overline{v}_L} \left(v - \max\{v, \hat{d}+\rho\}\right) \mathrm{d}F_L(v) + \rho\right] \\ + \int_{\underline{v}_L}^{\overline{v}_L} \left(v - \min\{v, \hat{d}+\rho\}\right) \mathrm{d}F_L(v) \end{array} \right\}.
\end{aligned}
$$

By construction, if $d_H^{**} \leq d_L^{**}$, then $Z_{3,H}(d_H^{**}) - Z_{3,L}(d_H^{**}) \leq 0$. Hence, $Z_{3,H}(\hat{d}_H^{**}) \leq Z_{3,L}(\hat{d}_H^{**}) \leq Z_{3,L}(\hat{d}_L^{**})$.

It is easy to show that if $\varphi_g \leq 1-c$ for all $g \in \{H, L\}$, then $Z(\boldsymbol{\varphi}) \leq Z_3(\boldsymbol{\varphi})$, where

$$
Z_3(\boldsymbol{\varphi}) := \sum_{g \in \{H,L\}} \frac{1}{1 - cF_g\left(\hat{d}(\boldsymbol{\varphi}) + \rho\right)} \left\{ \begin{array}{l} (1-c)\left[\int_{\underline{v}_g}^{\max\{\rho, \hat{d}(\boldsymbol{\varphi})+\rho\}} (v-\rho)\mathrm{d}F_g(v) + \rho\right] \\ + \int_{\max\{\rho, \hat{d}(\boldsymbol{\varphi})+\rho\}}^{\overline{v}_g} (v-\rho)\mathrm{d}F_g(v) \end{array} \right\},
$$

and $\hat{d}(\boldsymbol{\varphi})$ is defined by (40). Hence, if $\varphi_g \leq 1-c$ for all $g \in \{H, L\}$, then

$$
Z(\boldsymbol{\varphi}) \leq Z_3(\boldsymbol{\varphi}) \leq \max\{Z_{3,H}(\hat{d}(\boldsymbol{\varphi})), Z_{3,L}(\hat{d}(\boldsymbol{\varphi}))\} \leq \max\{Z_{3,H}(d_H^{**}), Z_{3,L}(d_L^{**})\} \leq Z_{3,L}(d_L^{**}).
$$

Finally, note that if $\varphi_H = 0$, then (41) implies that (42). Hence, $\boldsymbol{\varphi}^L \in \Phi\left(d_L^{**}, d_L^{**}, d_L^{**}\right)$ if and only if φ_L^L satisfies inequality (41). Furthermore, φ_L^L satisfies inequality (41) if and only if d_L^{**} satisfies (39). Hence, if d_L^{**} satisfies (39), it is optimal to set $\boldsymbol{\varphi}^{1*} = (0, (1-c)/\{n_L[1 - cF_L(d_L^{**} + \rho)]\})$.

Case 2: $\varphi_L \leq 1-c$**.** Since $d_L^{**} \geq d_H^{**} \geq d_H^{*} \geq d_L^{*}$, by a similar argument to that of Case 1 in the proof of Theorem 5, $Z(\varphi) \leq Z_{2,L}(d_L^{**})$, where the maximum is achieved at

$$
\boldsymbol{\varphi}^{2*} = \left(1-c, \frac{1-c}{n_L} \frac{F_H(d_L^{**}+\rho)^{n_H} - n_H(1-c)F_H(d_L^{**}+\rho)}{1 - cF_L(d_L^{**}+\rho)}\right).
$$

Case 3: $\varphi_H \leq 1-c$**.** Since $d_L^{**} \geq d_H^{**} \geq d_H^{*} \geq d_L^{*}$, by a similar argument to that of Case 1 in the proof of Theorem 5, $Z(\varphi) \leq Z_{2,H}(d_H^{**})$, where the maximum is achieved at

$$
\boldsymbol{\varphi}^{3*} = \left(\frac{1-c}{n_H} \frac{F_L(d_H^{**}+\rho)^{n_L} - n_L(1-c)F_L(d_H^{**}+\rho)}{1 - cF_H(d_H^{**}+\rho)}, 1-c\right).
$$

By assumption, $\varphi_g^{2*} \leq 1-c$ and $\varphi_g^{3*} \leq 1-c$ for $g \in \{H, L\}$. Hence, $Z(\boldsymbol{\varphi}^{2*}) \leq Z(\boldsymbol{\varphi}^{1*})$ and $Z(\boldsymbol{\varphi}^{3*}) \leq Z(\boldsymbol{\varphi}^{1*})$. Hence, it is optimal to set $\boldsymbol{\varphi}^{*} = \boldsymbol{\varphi}^{1*}$. This completes the proof. ■

C.5 Omitted proofs in Section 5.2

Fix $\boldsymbol{\varphi}$ and let $\{d_i^u\}_i$ and d^l be the associated optimal thresholds. Assume, without loss of generality, that $d_1^u \geq \cdots \geq d_n^u \geq d^l$. Let $1 \leq \xi_1 < \cdots < \xi_L \leq n$ be such that $d_1^u = \cdots = d_{\xi_1}^u$, $d_{\xi_\iota}^u > d_{\xi_\iota+1}^u = \cdots = d_{\xi_{\iota+1}}^u$ for $\iota = 1, \ldots, L-1$ and $d_{\xi_L}^u > d_{\xi_L+1}^u = \cdots = d_n^u = d^l$. Note that in the symmetric environment $d_i^u \geq d_j^u$ only if $\varphi_i \geq \varphi_j$. The proof of Theorem 7 uses the following properties of d^l and d_i^u:

Lemma 20 *If $\frac{\varphi_i}{1-c} \geq 1$ for all $i \leq \xi_1$, then $d_{\xi_1}^u = \overline{v} - \frac{k}{c}$; otherwise $\frac{\varphi_i}{1-c} < 1$ for all $i \leq \xi_1$ and $d_{\xi_1}^u$ satisfies*

$$1 - F\left(d_{\xi_1}^u + \frac{k}{c}\right)^{\xi_1} = \sum_{i=1}^{\xi_1} \frac{\varphi_i}{1-c_i}\left[1 - F\left(d_{\xi_1}^u + \frac{k}{c}\right)\right],$$

$$1 - F(v)^{\xi_1} \lesseqgtr \sum_{i=1}^{\xi_1} \frac{\varphi_i}{1-c}\left[1 - F(v)\right] \text{ if } v \lesseqgtr d_{\xi_1}^u + \frac{k}{c},$$

$$\frac{\varphi_i}{1-c} \geq F\left(d_{\xi_1}^u + \frac{k}{c}\right)^{\xi_1 - 1}, \forall i = 1, \ldots, \xi_1.$$

For $\iota = 1, \ldots, L-1$, $\frac{\varphi_i}{1-c} < 1$ for $\xi_\iota + 1 \leq i \leq \xi_{\iota+1}$ and $d_{\xi_{\iota+1}}^u$ satisfies

$$F\left(d_{\xi_{\iota+1}}^u + \frac{k}{c}\right)^{\xi_\iota} - F\left(d_{\xi_{\iota+1}}^u + \frac{k}{c}\right)^{\xi_{\iota+1}} = \sum_{i=\xi_\iota+1}^{\xi_{\iota+1}} \frac{\varphi_i}{1-c_i}\left[1 - F\left(d_{\xi_{\iota+1}}^u + \frac{k}{c}\right)\right],$$

$$F(v)^{\xi_\iota} - F(v)^{\xi_{\iota+1}} \lesseqgtr \sum_{i=\xi_\iota+1}^{\xi_{\iota+1}} \frac{\varphi_i}{1-c}\left[1 - F(v)\right] \text{ if } v \lesseqgtr d_{\xi_{\iota+1}}^u + \frac{k}{c},$$

$$\frac{\varphi_i}{1-c} \geq F\left(d_{\xi_{\iota+1}}^u + \frac{k}{c}\right)^{\xi_{\iota+1} - 1}, \forall i = \xi_\iota + 1, \ldots, \xi_{\iota+1}.$$

Finally, d^l satisfies

$$F\left(d^l + \frac{k}{c}\right)^{\xi_L} = \sum_{i=1}^{n} \varphi_i F\left(d^l + \frac{k}{c}\right) + \sum_{i=\xi_L+1}^{n} \frac{\varphi_i}{1-c}\left[1 - F\left(d^l + \frac{k}{c}\right)\right],$$

$$F(v)^{\xi_L} \lesseqgtr \sum_{i=1}^{n} \varphi_i F(v) + \sum_{i=\xi_L+1}^{n} \frac{\varphi_i}{1-c}\left[1 - F(v)\right] \text{ if } v \lesseqgtr d^l + \frac{k}{c}.$$

The arguments used to prove Lemma 20 are similar to that used to show that $\overline{P}_i^m$ converges to $\overline{P}_i$ if $d^l < v_i - \frac{k_i}{c_i} < d_i^u$, and are neglected here.

Proof of Theorem 7. The first part of the theorem directly follows from Theorem 8.

Assume, for the rest of the proof, that $F(v^*)^{n-1} < n(1-c)$. Consider an optimal $\boldsymbol{\varphi}$. Let $\{d_i^u\}_i$ and d^l be the associated optimal thresholds. Assume, without loss of generality, that $d_1^u \geq \cdots \geq d_n^u \geq d^l$. Let ξ_ι ($\iota = 1, \ldots, L$) be defined as in the beginning of this subsection.

First, I show that $L = 1$. Suppose, to the contrary, that $L \geq 2$. Suppose that $d_{\xi_2}^u < 0$, then the principal's objective function is strictly increasing in φ_i for $i > \xi_2$. Hence, in optimum, it must be that $d_{\xi_2}^u \geq 0$. Construct a new $\boldsymbol{\varphi}^*$ as follows: Let

$$\varphi_i^* = \frac{1}{\xi_2} \sum_{j=1}^{\xi_2} \varphi_j, \text{ for all } i = 1, \ldots, \xi_2,$$

and $\varphi_i^* = \varphi_i$ for all $i > \xi_2$. Let d_i^{u*} and d^{l*} be the optimal thresholds associated with $\boldsymbol{\varphi}^*$. Then $d_1^{u*} = \ldots d_{\xi_2}^{u*}$ and $d_i^{u*} = d_i^u$ for all $i > \xi_2$. There are two cases: (1) $\varphi_i < 1-c$ for all $i \leq \xi_1$ and (2) $\varphi_i \geq 1-c$ for all $i \leq \xi_1$.

Case 1: $\varphi_i < 1-c$ for all $i \leq \xi_1$. In this case, $\varphi_1^* < 1-c$. Then $d_{\xi_2}^{u*}$ is defined by

$$\left[1 - F\left(d_{\xi_2}^{u*} + \frac{k}{c}\right)\right] \sum_{i=1}^{\xi_2} \frac{\varphi_i^*}{1-c} = 1 - F\left(d_{\xi_2}^{u^*} + \frac{k}{c}\right)^{\xi_2}. \tag{43}$$

Hence, $d^u_{\xi_2} < d^{u^*}_{\xi_2} < d^u_{\xi_1}$. Let $Z(\boldsymbol{\varphi})$ denote the principal's payoff given $\boldsymbol{\varphi}$. Then

$$
\begin{aligned}
& Z(\boldsymbol{\varphi}^*) - Z(\boldsymbol{\varphi}) \\
=& \sum_{i=1}^{\xi_2}\left[\int_{d^{u*}_{\xi_2}+\frac{k}{c}}^{\overline{v}} \left(v_i - \frac{k}{c}\right)\frac{\varphi^*_i}{1-c}\mathrm{d}F(v_i) + \int_{d^u_{\xi_2}+\frac{k}{c}}^{d^{u^*}_{\xi_2}+\frac{k}{c}}\left(v_i - \frac{k}{c}\right)F(v_i)^{\xi_2-1}\mathrm{d}F(v_i)\right] \\
& - \sum_{i=\xi_1+1}^{\xi_2}\int_{d^u_{\xi_2}+\frac{k}{c}}^{\overline{v}}\left(v_i - \frac{k}{c}\right)\frac{\varphi_i}{1-c}\mathrm{d}F(v_i) \\
& - \sum_{i=1}^{\xi_1}\left[\int_{d^u_{\xi_2}+\frac{k}{c}}^{d^u_{\xi_1}+\frac{k}{c}}\left(v_i - \frac{k}{c}\right)F(v_i)^{\xi_1-1}\mathrm{d}F(v_i) + \int_{d^u_{\xi_1}+\frac{k}{c}}^{\overline{v}}\left(v_i - \frac{k}{c}\right)\frac{\varphi_i}{1-c}\mathrm{d}F(v_i)\right] \\
=& \int_{d^u_{\xi_2}+\frac{k}{c}}^{d^{u*}_{\xi_2}+\frac{k}{c}}\left(v-\frac{k}{c}\right)\xi_2F(v)^{\xi_2-1}\mathrm{d}F(v) + \int_{d^{u^*}_{\xi_2}+\frac{k}{c}}^{\overline{v}}\left(v-\frac{k}{c}\right)\sum_{i=1}^{\xi_2}\frac{\varphi^*_i}{1-c}\mathrm{d}F(v) \\
& - \int_{d^u_{\xi_2}+\frac{k}{c}}^{d^u_{\xi_1}+\frac{k}{c}}\left(v-\frac{k}{c}\right)\left(\sum_{i=\xi_1+1}^{\xi_2}\frac{\varphi_i}{1-c}+\xi_1F(v)^{\xi_1-1}\right)\mathrm{d}F(v) - \int_{d^u_{\xi_1}+\frac{k}{c}}^{\overline{u}}\left(v-\frac{k}{c}\right)\sum_{i=1}^{\xi_2}\frac{\varphi_i}{1-c}\mathrm{d}F(v) \\
=& \int_{d^u_{\xi_2}+\frac{k}{c}}^{d^{u*}_{\xi_2}+\frac{k}{c}}\left(v-\frac{k}{c}\right)\xi_2F(v)^{\xi_2-1}\mathrm{d}F(v) + \int_{d^{u^*}_{\xi_2}+\frac{k}{c}}^{d^u_{\xi_1}+\frac{k}{c}}\left(v-\frac{k}{c}\right)\sum_{i=1}^{\xi_2}\frac{\varphi_i}{1-c}\mathrm{d}F(v) \\
& - \int_{d^u_{\xi_2}+\frac{k}{c}}^{d^u_{\xi_1}+\frac{k}{c}}\left(v-\frac{k}{c}\right)\left(\sum_{i=\xi_1+1}^{\xi_2}\frac{\varphi_i}{1-c}+\xi_1F(v)^{\xi_1-1}\right)\mathrm{d}F(v) \\
=& d^u_{\xi_1}\left[F\left(d^u_{\xi_1}+\frac{k}{c}\right)\sum_{i=1}^{\xi_1}\frac{\varphi_i}{1-c} - F\left(d^u_{\xi_1}+\frac{k}{c}\right)^{\xi_1}\right] + d^{u*}_{\xi_2}\left[F\left(d^{u*}_{\xi_2}+\frac{k}{c}\right)^{\xi_2} - F\left(d^{u*}_{\xi_2}+\frac{k}{c}\right)\sum_{i=\xi_1+1}^{\xi_2}\frac{\varphi_i}{1-c}\right] \\
& - d^u_{\xi_2}\left[F\left(d^u_{\xi_2}+\frac{k}{c}\right)^{\xi_2} - F\left(d^u_{\xi_2}+\frac{k}{c}\right)\sum_{i=\xi_1+1}^{\xi_2}\frac{\varphi_i}{1-c} - F\left(d^u_{\xi_2}+\frac{k}{c}\right)^{\xi_1}\right] \\
& - \int_{d^u_{\xi_2}+\frac{k}{c}}^{d^{u*}_{\xi_2}+\frac{k}{c}}\left[F(v)^{\xi_2} - F(v)\sum_{i=\xi_1}^{\xi_2}\frac{\varphi_i}{1-c} - F(v)^{\xi_1}\right]\mathrm{d}v - \int_{d^{u^*}_{\xi_2}+\frac{k}{c}}^{d^u_{\xi_1}+\frac{k}{c}}\left[F(v)\sum_{i=1}^{\xi_1+1}\frac{\varphi_i}{1-c} - F(v)^{\xi_1}\right]\mathrm{d}v,
\end{aligned}
$$

where the third equality holds because $\sum_{i=1}^{\xi_2}\varphi_i = \sum_{i=1}^{\xi_2}\varphi^*_i$, and the last equality holds by integration by parts. Because $d^{u*}_{\xi_2}$ satisfies (43), $d^u_{\xi_1}$ satisfies that

$$
1 - F\left(d^u_{\xi_1}+\frac{k}{c}\right)^{\xi_1} = \left[1 - F\left(d^u_{\xi_1}+\frac{k}{c}\right)\right]\sum_{i=1}^{\xi_1}\frac{\varphi_i}{1-c}
$$

$$
1 - F(v)^{\xi_1} < [1 - F(v)]\sum_{i=1}^{\xi_1}\frac{\varphi_i}{1-c}, \forall v < d^u_{\xi_1}+\frac{k}{c},
$$

and $d^u_{\xi_2}$ satisfies that

$$1-F\left(d^u_{\xi_2}+\frac{k}{c}\right)^{\xi_2}=\left[1-F\left(d^u_{\xi_2}+\frac{k}{c}\right)\right]\sum_{i=\xi_1+1}^{\xi_2}\frac{\varphi_i}{1-c}+1-F\left(d^u_{\xi_2}+\frac{k}{c}\right)^{\xi_1}$$

$$1-F\left(v\right)^{\xi_2}>\left[1-F\left(v\right)\right]\sum_{i=\xi_1+1}^{\xi_2}\frac{\varphi_i}{1-c}+1-F\left(v\right)^{\xi_1},\forall v>d^u_{\xi_2}+\frac{k}{c},$$

we have

$$\begin{aligned}
&Z(\boldsymbol{\varphi}^*)-Z(\boldsymbol{\varphi})\\
>&\left(d_{\xi_1}-d^{u^*}_{\xi_2}\right)\left(\sum_{i=1}^{\xi_1}\frac{\varphi_i}{1-c}-1\right)-\left(d^{u*}_{\xi_2}-d_{\xi_2}\right)\sum_{\xi_1+1}^{\xi_2}\frac{\varphi_i}{1-c}\\
&+\int_{d^u_{\xi_2}+\frac{k}{c}}^{d^{u*}_{\xi_2}+\frac{k}{c}}\sum_{i=\xi_1+1}^{\xi_2}\frac{\varphi_i}{1-c}\mathrm{d}v-\int_{d^{u*}_{\xi_2}+\frac{k}{c}}^{d^u_{\xi_1}+\frac{k}{c}}\left(\sum_{i=1}^{\xi_1}\frac{\varphi_i}{1-c}-1\right)\mathrm{d}v=0,
\end{aligned}$$

which is a contradiction to the optimality of $\boldsymbol{\varphi}$.

Case 2: $\varphi_i\geq 1-c$ for all $i\leq\xi_1$. If $\varphi^*_1\geq 1-c$, then $d^{u*}_{\xi_2}=\overline{d}$. In this case, we have

$$\begin{aligned}
&Z(\boldsymbol{\varphi}^*)-Z(\boldsymbol{\varphi})\\
=&\sum_{i=1}^{\xi_2}\int_{d^u_{\xi_2}+\frac{k}{c}}^{\overline{v}}\left(v_i-\frac{k}{c}\right)F(v_i)^{\xi_2-1}\mathrm{d}F(v_i)-\sum_{i=\xi_1+1}^{\xi_2}\int_{d^u_{\xi_2}+\frac{k}{c}}^{\overline{v}}\left(v_i-\frac{k}{c}\right)\frac{\varphi_i}{1-c}\mathrm{d}F(v_i)\\
&-\sum_{i=1}^{\xi_1}\int_{d^u_{\xi_2}+\frac{k}{c}}^{\overline{v}}\left(v_i-\frac{k}{c}\right)F(v_i)^{\xi_1-1}\mathrm{d}F(v_i)\\
=&\int_{d^u_{\xi_2}+\frac{k}{c}}^{\overline{v}}\left(v-\frac{k}{c}\right)\xi_2F(v)^{\xi_2-1}\mathrm{d}F(v)-\int_{d^u_{\xi_2}+\frac{k}{c}}^{\overline{v}}\left(v-\frac{k}{c}\right)\left(\sum_{i=\xi_1+1}^{\xi_2}\frac{\varphi_i}{1-c}+\xi_1F(v)^{\xi_1-1}\right)\mathrm{d}F(v)\\
=&\left(v-\frac{k}{c}\right)\left[F(v)^{\xi_2}-F(v)\sum_{i=\xi_1+1}^{\xi_2}\frac{\varphi_i}{1-c}-F(v)^{\xi_1}\right]\Bigg|_{d^u_{\xi_2}+\frac{k}{c}}^{\overline{v}}\\
&-\int_{d^u_{\xi_2}+\frac{k}{c}}^{\overline{v}}\left[F(v)^{\xi_2}-F(v)\sum_{i=\xi_1+1}^{\xi_2}\frac{\varphi_i}{1-c}-F(v)^{\xi_1}\right]\mathrm{d}v,
\end{aligned}$$

where the last equality holds by integration by parts. Because $d^u_{\xi_2}$ satisfies that

$$\left[1-F\left(d^u_{\xi_2}+\frac{k}{c}\right)\right]\sum_{i=\xi_1+1}^{\xi_2}\frac{\varphi_i}{1-c}+1-F\left(d^u_{\xi_2}+\frac{k}{c}\right)^{\xi_1}=1-F\left(d^u_{\xi_2}+\frac{k}{c}\right)^{\xi_2},$$

$$[1-F(v)]\sum_{i=\xi_1+1}^{\xi_2}\frac{\varphi_i}{1-c}+1-F(v)^{\xi_1}<1-F(v)^{\xi_2},\forall v>d^u_{\xi_2}+\frac{k}{c},$$

we have

$$Z(\boldsymbol{\varphi}^*)-Z(\boldsymbol{\varphi})>-\left(\bar{v}-\frac{k}{c}-d^u_{\xi_2}\right)\sum_{i=1}^{\xi_2}\frac{\varphi_i}{1-c}+\int_{d^u_{\xi_2}+\frac{k}{c}}^{\bar{v}}\sum_{i=\xi_1+1}^{\xi_2}\frac{\varphi_i}{1-c}\mathrm{d}v=0,$$

which is a contradiction to the optimality of $\boldsymbol{\varphi}$. If $\varphi_1^*<1-c$, then let $d_1^{u*}=\cdots=d_{\xi_2}^{u*}$ be defined by (43). Note that if $\xi_1=1$ and $\varphi_1/(1-c)=1$, then the new mechanism using $\boldsymbol{\varphi}^*$ coincides with the old mechanism using $\boldsymbol{\varphi}$. In this case, we can redefine $d_1^u:=d^u_{\xi_2}$ without changing the mechanism. Except for this case, we can show, by a similar argument to that in Case 1, that $Z(\boldsymbol{\varphi}^*)-Z(\boldsymbol{\varphi})>0$, which is a contradiction to the optimality of $\boldsymbol{\varphi}$.

Hence, by induction, we have $L=1$. For ease of notation, let $j:=\xi_1$. Next, we show that $j=0$ or n. Suppose, to the contrary, that $0<j<n$. Suppose that $d^l<0$, then the principal's objective function is strictly increasing in φ_i for $i>j$. Hence, in optimum, it must be that $d^l\geq 0$. Construct a new $\boldsymbol{\varphi}^*$ as follows: Let

$$\varphi_i^*=\frac{1}{n}\sum_{j=1}^{n}\varphi_j,\ \text{for all } i=1,\ldots,n.$$

Case 1: $\varphi_i<1-c$ for all $i\leq j$. In this case, $\varphi_1^*<1-c$. Let $d_1^{u*}=\cdots=d_n^{u*}$ be such that

$$1-F\left(d_j^{u*}+\frac{k}{c}\right)^n=\sum_{i=1}^{n}\frac{\varphi_i^*}{1-c}\left[1-F\left(d_j^{u*}+\frac{k}{c}\right)\right]. \tag{44}$$

Then $d_j^{u*}<d_j^u$. Let d^{l*} be such that

$$F\left(d^{l*}+\frac{k}{c}\right)^n=\sum_{i=1}^{n}\varphi_i^*F\left(d^{l*}+\frac{k}{c}\right). \tag{45}$$

Then $d^l<d^{l*}$. There are two subcases to consider: (i) $d^{l*}\leq d_j^{u*}$ and (ii) $d^{l*}>d_j^{u*}$.

(i) Suppose that $d^{l*} \leq d_j^{u*}$. Then

$$
\begin{aligned}
& Z(\boldsymbol{\varphi}^*) - Z(\boldsymbol{\varphi}) \\
=& \sum_{i=1}^{n} \left[\int_{\underline{v}}^{d^{l^*}+\frac{k}{c}} \left(v_i - \frac{k}{c}\right) \varphi_i^* \mathrm{d}F(v_i) + \int_{d^{l*}+\frac{k}{c}}^{d_j^{u*}+\frac{k}{c}} \left(v_i - \frac{k}{c}\right) F(v_i)^{n-1} \mathrm{d}F(v_i) + \int_{d_j^{u*}+\frac{k}{c}}^{\overline{v}} \left(v_i - \frac{k}{c}\right) \frac{\varphi_i^*}{1-c} \mathrm{d}F(v_i) \right] \\
& - \sum_{i=1}^{n} \int_{\underline{v}}^{d^{l}+\frac{k}{c}} \left(v_i - \frac{k}{c}\right) \varphi_i \mathrm{d}F(v_i) - \sum_{i=j+1}^{n} \int_{d^{l}+\frac{k}{c}}^{\overline{v}} \left(v_i - \frac{k}{c}\right) \frac{\varphi_i}{1-c} \mathrm{d}F(v_i) \\
& - \sum_{i=1}^{j} \left[\int_{d^{l}+\frac{k}{c}}^{d_j^{u}+\frac{k}{c}} \left(v_i - \frac{k}{c}\right) F(v_i)^{j-1} \mathrm{d}F(v_i) + \int_{d_j^{u}+\frac{k}{c}}^{\overline{v}} \left(v_i - \frac{k}{c}\right) \frac{\varphi_i}{1-c} \mathrm{d}F(v_i) \right] \\
=& \int_{d^{l}+\frac{k}{c}}^{d^{l^*}+\frac{k}{c}} \left(v - \frac{k}{c}\right) \sum_{i=1}^{n} \varphi_i \mathrm{d}F(v) + \int_{d^{l*}+\frac{k}{c}}^{d_j^{u*}+\frac{k}{c}} \left(v - \frac{k}{c}\right) nF(v)^{n-1} \mathrm{d}F(v) + \int_{d_j^{u*}+\frac{k}{c}}^{d_j^{u}+\frac{k}{c}} \left(v - \frac{k}{c}\right) \sum_{i=1}^{n} \frac{\varphi_i}{1-c} \mathrm{d}F(v) \\
& - \int_{d^{l}+\frac{k}{c}}^{d_j^{u}+\frac{k}{c}} \left(v - \frac{k}{c}\right) \sum_{i=j+1}^{n} \frac{\varphi_i}{1-c} \mathrm{d}F(v) - \int_{d^{l}+\frac{k}{c}}^{d_j^{u}+\frac{k}{c}} \left(v - \frac{k}{c}\right) jF(v)^{j-1} \mathrm{d}F(v) \\
=& d_j^u \left[F\left(d_j^u + \frac{k}{c}\right) \sum_{i=1}^{j} \frac{\varphi_i}{1-c} - F\left(d_j^u + \frac{k}{c}\right)^j \right] + d_j^{u*} \left[F\left(d_j^{u*} + \frac{k}{c}\right)^n - F\left(d_j^{u*} + \frac{k}{c}\right) \sum_{i=1}^{n} \frac{\varphi_i}{1-c} \right] \\
& + d^{l*} \left[F\left(d^{l*} + \frac{k}{c}\right) \sum_{i=1}^{n} \varphi_i - F\left(d^{l*} + \frac{k}{c}\right)^n \right] \\
& + d^{l} \left[-F\left(d^{l} + \frac{k}{c}\right) \sum_{i=1}^{n} \varphi_i + F\left(d^{l} + \frac{k}{c}\right) \sum_{i=j+1}^{n} \frac{\varphi_i}{1-c} + F\left(d^{l} + \frac{k}{c}\right)^j \right] \\
& - \int_{d^{l}+\frac{k}{c}}^{d^{l*}+\frac{k}{c}} F(v) \sum_{i=1}^{n} \varphi_i \mathrm{d}v - \int_{d^{l*}+\frac{k}{c}}^{d_j^{u*}+\frac{k}{c}} F(v)^n \mathrm{d}v - \int_{d_j^{u}+\frac{k}{c}}^{d_j^{u*}+\frac{k}{c}} F(v) \sum_{i=1}^{n} \frac{\varphi_i}{1-c} \mathrm{d}v \\
& + \int_{d^{l}+\frac{k}{c}}^{d_j^{u}+\frac{k}{c}} \left[F(v) \sum_{i=j+1}^{n} \frac{\varphi_i}{1-c} + F(v)^j \right] \mathrm{d}v,
\end{aligned}
$$

where the second equality holds because $\sum_{i=1}^n \varphi_i^* = \sum_{i=1}^n \varphi_i$ and the last equality holds by integration by parts. Because d_j^{u*} satisfies (44), d^{l*} satisfies (45), d_j^u satisfies that

$$
1 - F\left(d_j^u + \frac{k}{c}\right)^j = \sum_{i=j+1}^{n} \frac{\varphi_i}{1-c} \left[1 - F\left(d_j^u + \frac{k}{c}\right)\right],
$$

$$
1 - F\left(v\right)^j < \sum_{i=j+1}^{n} \frac{\varphi_i}{1-c} \left[1 - F\left(v\right)\right], \forall v < d_j^u,
$$

and d^l satisfies that

$$1-F\left(d^l+\frac{k}{c}\right)^j+\sum_{i=j+1}^{n}\frac{\varphi_i}{1-c}\left[1-F\left(d^l+\frac{k}{c}\right)\right]+\sum_{i=1}^{n}\varphi_i F\left(d^l+\frac{k}{c}\right)=1,$$

$$1-F(v)^j+\sum_{i=j+1}^{n}\frac{\varphi_i}{1-c}\left[1-F(v)\right]+\sum_{i=1}^{n}\varphi_i F(v)<1,\forall v>d^l$$

$$F(v)^j-F(v)^n>\left[1-F(v)\right]\sum_{i=j+1}^{n}\frac{\varphi_i}{1-c},\forall v>d^l=d_{j+1}^u,$$

we have

$$\begin{aligned}&Z(\boldsymbol{\varphi}^*)-Z(\boldsymbol{\varphi})\\&>d_j^u\left(\sum_{i=1}^{j}\frac{\varphi_i}{1-c}-1\right)+d_j^{u*}\left(1-\sum_{i=1}^{n}\frac{\varphi_i}{1-c}\right)+d^l\sum_{i=j+1}^{n}\frac{\varphi_i}{1-c}\\&-\int_{d^l+\frac{k}{c}}^{d^{l*}+\frac{k}{c}}\sum_{i=j+1}^{n}\frac{\varphi_i}{1-c}\mathrm{d}v-\int_{d^{l*}+\frac{k}{c}}^{d_j^{u*}+\frac{k}{c}}\sum_{i=j+1}^{n}\frac{\varphi_i}{1-c}\mathrm{d}v-\int_{d_j^u+\frac{k}{c}}^{d_j^{u*}+\frac{k}{c}}\left(\sum_{i=1}^{j}\frac{\varphi_i}{1-c}-1\right)\mathrm{d}v=0,\end{aligned}$$

which is a contradiction to the optimality of $\boldsymbol{\varphi}$.

(ii) Suppose that $d^{l*}>d_j^{u*}$. In this case, redefine $d^{l*}=d_j^{u*}$ such that

$$\sum_{i=1}^{n}\left[\varphi_i^*F\left(d^{l*}+\frac{k}{c}\right)+\frac{\varphi_i^*}{1-c}\left(1-F\left(d^{l*}+\frac{k}{c}\right)\right)\right]=1.$$

Then $d^l<d^{l*}=d_j^{u*}<d_j^u$. By a similar argument to that in case (ii), we can show that $Z(\boldsymbol{\varphi}^*)-Z(\boldsymbol{\varphi})>0$, which is a contradiction to the optimality of $\boldsymbol{\varphi}$.

Case 2: $\varphi_i\geq 1-c$ **for all** $i\leq j$**.** If $\varphi_1^*\geq 1-c$, then let $d_1^{u*}=\dots d_n^{u*}=\overline{d}$ and d^{l*} be defined by (45). By a similar argument to that in Case 1, we can show that $Z(\boldsymbol{\varphi}^*)-Z(\boldsymbol{\varphi})>0$, which is a contradiction to the optimality of $\boldsymbol{\varphi}$.

If $\varphi_1^*<1-c$, then let $d_1^{u*}=\dots d_n^{u*}=\overline{d}$ be defined by (44) and d^{l*} be defined by (45). Note that if $j=1$ and $\varphi_1/(1-c)=1$, then the new mechanism using $\boldsymbol{\varphi}^*$ coincides with the old mechanism using $\boldsymbol{\varphi}$. In this case, we can redefine $d_1^u:=d^l$ without changing the mechanism. Except for this case, we can show, by a similar argument to that in Case 1, that $Z(\boldsymbol{\varphi}^*)-Z(\boldsymbol{\varphi})>0$, which is a contradiction to the optimality of $\boldsymbol{\varphi}$.

Hence, $j=0$ or n.

Case 1: $j = 0$**.** In this case, for all i,

$$P_i^*(v_i) = \begin{cases} \frac{\varphi_i}{1-c} & \text{if } v_i \geq d^l + \frac{k}{c} \\ \varphi_i & \text{if } v_i < d^l + \frac{k}{c} \end{cases} .$$

is an optimal mechanism given $\boldsymbol{\varphi}$. Furthermore, $\boldsymbol{\varphi}$ and d^l must satisfy

$$\left[1 - F\left(d^l + \frac{k}{c}\right)\right] \sum_{j=1}^{i} \frac{\varphi_j}{1-c} \leq 1 - F\left(d^l + \frac{k}{c}\right)^i, \forall i \leq n, \tag{46}$$

$$F\left(d^l + \frac{k}{c}\right) \sum_{i=1}^{n} \varphi_i + \left[1 - F\left(d^l + \frac{k}{c}\right)\right] \sum_{i=1}^{n} \frac{\varphi_i}{1-c} = 1. \tag{47}$$

In particular, (46) holds for $i = n$, which implies

$$\sum_{i=1}^{n} \frac{\varphi_i}{1-c} \leq \frac{1 - F\left(d^l + \frac{k}{c}\right)^n}{1 - F\left(d^l + \frac{k}{c}\right)}.$$

Substituting this into (47) yields

$$F\left(d^l + \frac{k}{c}\right)^{n-1} \leq \sum_{i=1}^{n} \varphi_i \leq \frac{(1-c)\left[1 - F\left(d^l + \frac{k}{c}\right)^n\right]}{1 - F\left(d^l + \frac{k}{c}\right)}.$$

By the proof of the second part in Theorem 3, $j = 0$ is optimal if $v^{**} \leq v^{\natural}$, in which case the optimal $d^l = d_1^u = \cdots = d_n^u = v^{**} - \frac{k}{c}$. The set of optimal $\boldsymbol{\varphi}$ is given by $\Phi(d^l, d_1^u, \ldots, d_n^u)$. Clearly, $\boldsymbol{\varphi} \in \Phi$ if and only if $\boldsymbol{\varphi}$ satisfies conditions (46) and (47). Because $v^{**} \leq v^{\natural}$ implies that

$$1 \leq \frac{1}{1 - cF(v^{**})} \leq \frac{1 - F(v^{**})^n}{1 - F(v^{**})},$$

there exists $1 \leq h \leq n$ such that

$$\frac{1 - F(v^{**})^{h-1}}{1 - F(v^{**})} \leq \frac{1}{1 - cF(v^{**})} < \frac{1 - F(v^{**})^h}{1 - F(v^{**})}.$$

Hence, for all $i > h$, (46) holds if (47) holds. Given this, it is easy to see that the set of optimal $\boldsymbol{\varphi}$ is the convex hull of

$$\left\{ \boldsymbol{\varphi} \,\middle|\, \begin{array}{l} \varphi_{i_j} = (1-c)F(v^{**})^{j-1} \text{ if } j \leq h-1, \varphi_{i_h} = \frac{1-c}{1-cF(v^{**})} - \sum_{j=1}^{h-1}(1-c)F(v^{**})^{j-1}, \\ \varphi_{i_j} = 0 \text{ if } j \geq h+1 \text{ and } (i_1, \ldots, i_n) \text{ is a permutation of } (1, \ldots, n) \end{array} \right\}.$$

Case 2: $j = n$**.** In this case, let $d^u := d_1^u = \cdots = d_n^u$, and

$$P_i^*(v_i) = \begin{cases} \frac{\varphi_i}{1-c} & \text{if } v_i \geq d^u + \frac{k}{c} \\ F(v)^{n-1} & \text{if } d^l + \frac{k}{c} < v_i < d^u + \frac{k}{c} \\ \varphi_i & \text{if } v_i \leq d^l + \frac{k}{c} \end{cases} .$$

Furthermore, $\boldsymbol{\varphi}$, d^l and d^u must satisfy that

$$\left[1 - F\left(d^u + \frac{k}{c}\right)\right] \sum_{j=1}^{i} \frac{\varphi_j}{1-c} \leq 1 - F\left(d^u + \frac{k}{c}\right)^i, \forall i \leq n-1, \tag{48}$$

$$\left[1 - F\left(d^u + \frac{k}{c}\right)\right] \sum_{i=1}^{n} \frac{\varphi_i}{1-c} = 1 - F\left(d^u + \frac{k}{c}\right)^n, \tag{49}$$

$$F\left(d^l + \frac{k}{c}\right) \sum_{i=1}^{n} \varphi_i = F\left(d^l + \frac{k}{c}\right)^n. \tag{50}$$

(49) and (50) imply that d^l and d^u satisfy that

$$\frac{1 - F\left(d^u + \frac{k}{c}\right)^n}{1 - F\left(d^u + \frac{k}{c}\right)} = \frac{F\left(d^l + \frac{k}{c}\right)^{n-1}}{1-c}.$$

By the proof of the third part in Theorem 3, $j = n$ is optimal if $v^{**} > v^{\natural}$, in which case the optimal $d^l = v^l(\varphi^*) - \frac{k}{c}$ and the optimal $d_1^u = \cdots = d_n^u = v^u(\varphi^*) - \frac{k}{c}$. The set of optimal $\boldsymbol{\varphi}$ is given by $\Phi(d^l, d_1^u, \ldots, d_n^u)$. Clearly, $\boldsymbol{\varphi} \in \Phi$ if and only if $\boldsymbol{\varphi}$ satisfies conditions (48)-(50). It is easy to see that Φ is the convex hull of

$$\left\{\boldsymbol{\varphi} \,\middle|\, \varphi_{i_j} = (1-c)F(v^u(\varphi^*))^{j-1}\ \forall j \text{ and } (i_1, \ldots, i_n) \text{ is a permutation of } (1, \ldots, n)\right\}.$$

■

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