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A Theory of Negotiations and Formation of Coalitions

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Abstract

This paper proposes a new solution concept to three-player coalitional bargaining problems where the underlying economic opportunities are described by a partition function. This classic bargaining problem is modeled as a dynamic non-cooperative game in which players make conditional or unconditional offers, and coalitions continue to negotiate as long as there are gains from trade. The theory yields a unique stationary perfect equilibrium outcome–the negotiation value–and provides a unified framework that selects an economically intuitive solution and endogenous coalition structure. For such games as pure bargaining games the negotiation value coincides with the Nash bargaining solution, and for such games as zero-sum and majority voting games the negotiation value coincides with the Shapley value. However, a novel situation arises where the outcome is determined by pairwise sequential bargaining sessions in which a pair of players forms a natural match. In addition, another novel situation exists where the outcome is determined by one pivotal player bargaining unconditionally with the other players, and only the pairwise coalitions between the pivotal player and the other players can form.

KEYWORDS: Coalitional bargaining, subgame perfect equilibrium, existence, uniqueness, partition functions, externalities, conditional and unconditional offers.

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1. Introduction

We study three-player coalitional bargaining games in which players can form pairwise or multilateral coalitions and write binding agreements dividing the worth of coalitions among their members. What coalitions will form? What will be the final outcome of negotiations?

This classical problem, a natural generalization of the two-player bargaining problem, was first formulated by von Neumann and Morgenstern (1944), the founders of game theory, and indeed was the central focus of their seminal work. Since the early days of game theory these questions have interested and challenged many researchers but still remain largely unanswered.[1] The existing solution concepts have at least one of three shortcomings: there are games of economic interest for which there exists no solution, or the solution set is too big (multiple solutions), or the solution is counter-intuitive and/or intractable. Building on the existing literature, this paper proposes a new theory that yields a solution that always exists, is unique and economically intuitive, and is expressed by a simple analytical formula.

Cooperative game theory has developed numerous solution concepts to the coalition bargaining problem. These include stable sets (von Neumann and Morgenstern (1944)), the core, the Shapley value (Shapley (1953)), the Nash bargaining solution (Nash (1950, 1953)), the bargaining set (Aumann and Maschler (1964)), the kernel (Davis and Maschler (1965)), and the nucleoulus (Schmeidler (1969)). In addition, Shenoy (1979), Aumann and Dreze (1974), Hart and Kurz (1983), and Aumann and Myerson (1988) have developed cooperative models of valuation and coalition formation.[2]

The focus of this paper, though, is on the non-cooperative foundations of negotiations. This other strand of the literature models the negotiation process using a dynamic non-cooperative game in extensive form, such as those of Rubinstein (1982), Gul (1989), Selten (1981), Chatterjee et al. (1993), Hart and Mas-Colell (1996), and Ray and Vohra (1999), among many others.[3]

The starting point of our analysis is an economic situation in which the underlying opportunities are captured in a *partition function*. Partition functions generalize the concept of *characteristic function*, which is the cornerstone of most cooperative and non-cooperative models of coalitional bargaining. While characteristic functions describe in a single number the worth of each coalition of players, with partition functions the worth of coalitions

[1] See, for example, Selten (1987, p. 42), Maschler (1992, p. 595), and Binmore, Osborne, and Rubisntein (1992, p.204).

[2] See Gomes (1999a) for a discussion of these cooperative game theory solutions, and also the cooperative foundations of the solution concept introduced in this paper.

[3] Several papers such as Bennett (1997), Moldovanu (1990), and Gomes, Hart, and Mas-Colell (1999) address the non-cooperative n-person coalitional bargaining problem using a more general non-transferable utility (NTU) framework.

depends not only on the coalition itself but also on the coalitions formed by non-members, i.e., the coalition structure. We believe that this generalization is important because it allows us to capture any positive or negative externalities that formation of coalitions creates on non-members (for example, see Ray and Vohra (1999) and Bloch (1996)).

In our model, similar to Gul (1989), each player owns an asset and can buy or sell assets in exchange for cash payments. The values of combinations of assets are specified by the partition function, and thus depend on the coalition structure. As in Rubinstein (1982), the negotiation game evolves with players making proposals to acquire assets–with a certain lapse of time between proposals–followed by players that have received offers making their response (whether or not to accept the offer). The equilibrium concept used is stationary perfect Nash equilibrium. Our analysis has two main points of departure from the existing literature in strategic bargaining. First, we allow for the possibility of coalitions, once formed, to remain negotiating with other players or other coalitions until all gains from trade have been exploited. Second, we allow for a richer set of offers that includes offers that are both *conditional* and *unconditional* on the acceptance decision of other players.

One very satisfactory feature of our model is that it provides a *unique* stationary perfect equilibrium outcome, which we name the *negotiation value*. We also prove that this solution is Pareto efficient and is continuous on the parameters of the game. A large part of this paper is dedicated to the proof of the uniqueness, drawing on concepts of convex geometry that are introduced in the paper. We show that the space of partition function games is divided into twenty-six distinct regions (polyhedral cones) and that in each region the negotiation value is given by a linear function of the parameters of the game. In the limit, as the interval between offers shrinks to zero, the different regions into which the space of partition function games is divided collapses into eight regions.[4] Another, important property of the model is that the negotiation value for games in each region have very intuitive economic properties. We now describe the particular features and characteristics of the solution for each of the eight regions.

First, the Nash bargaining solution coincides with the negotiation value for games in the region that contains pure bargaining games.[5] Interestingly, this extends the Nash bargaining solution to a broad class of multilateral bargaining problems, where the worth of any pairwise coalition is less than a third–but not necessarily equal to zero, as in pure bargaining games–of the grand coalition value. Naturally, for these games, we obtain that the equilibrium coalition structure is such that only the grand coalition can form, but not

[4]Similarly to the negotiation value, the nucleolus also have linearity regions (see Schmeidler (1969) and Brune (1983)).

[5]In a pure bargaining game the cooperation of all players is needed to achieve gains from trade; otherwise all players get their reservation value.

any pairwise coalition.

Second, the Shapley value coincides with the negotiation value for games in the region that contains zero-sum, simple majority voting, and one-seller two-buyer market games. More generally, the set of games for which the Shapley value is the solution is characterized by one simple linear inequality, and corresponds to situations where any coalition can form in equilibrium.

Moreover, there is a third and novel situation where the outcome of negotiations is determined by *pairwise sequential* bargaining sessions: a first pairwise bargaining session in which a pair of players forms a *natural coalition*, strengthening their bargaining position vis à vis the non-member, followed by a second pairwise bargaining session between the natural coalition and the non-member. We provide an example of an oligopolistic industry where there are gains from merging, in which negotiations develop in sequential bargaining sessions. Negotiations develop sequentially in three regions because there are three possible natural coalitions of players.

In addition, there is a fourth novel situation where the outcome of negotiations is determined by one *pivotal* player bargaining *unconditionally* with the other two players. Any pairwise coalition between the pivotal player and the other players can form, but not the pairwise coalition between the non-pivotal players. We provide an example of a labor market game with one firm–the pivotal player–and two workers, where the firm negotiates individually with each worker and the workers are better off not forming a union to collectively bargain for wages. The last three of the eight regions correspond to situations where each of the three players can be pivotal.

Situations where there are natural coalitions or pivotal players have an equilibrium allocation and coalition structure that are intrinsically different from the classical Nash bargaining solution and Shapley value. A strong feature of our theory is that it yields not only a unique solution concept, but also a *unified* framework that selects an intuitive equilibrium concept for all coalitional bargaining games in partition function.

Our solution concept is different from other coalitional bargaining models for essentially two reasons. First, we allow for coalitions that reach an agreement to be able to further negotiate with other players (see also Gul (1989) and Hart and Kurz (1983)). In contrast to the strategic models of Selten (1981), Chatterjee et al. (1993), Ray and Vohra (1999), and Hart and Mas-Colell (1996), once a coalition reaches an agreement, it cannot be further renegotiated and the coalition leaves the game. Second, proposals in our model can be both conditional or unconditional, while most papers on strategic bargaining only allow for conditional offers (an exception is Krishna and Serrano (1996)). With conditional offers a rejection by even only one of the players receiving the offer blocks other players that have accepted the offer from exiting the game with the amount offered to them. Expanding the strategy set adds a new degree of realism to strategic bargaining, and allows for a Pareto efficient equilibrium outcome to arise. We show that when unconditional offers are ruled out, inefficiencies due to delays in reaching agreements can arise, and such inefficiencies can be significant for strictly super-additive games with small discount factors.[6]

The remainder of the paper is organized as follows: Section 2 presents the negotiation model, Section 3 develops the game-theoretic analysis of negotiations, Section 4 establishes the uniqueness of the solution, Section 5 studies the economic properties of the solution, and the Appendix contains the proofs of the theorems.

2. The Negotiation Game

Let $N = \{1, 2, 3\}$ be the set of players, and define a *coalition structure* (c.s.) as a *partition* π of the set of players N.[7] Our analysis starts with a cooperative game in *partition function* (see Thrall and Lucas (1963)). This cooperative game captures the worth $v(C, \pi) \in R$ of a *coalition* C belonging to a coalition structure π.

Definition 1: A *partition function game* v assigns a value $v(C, \pi) \in R$ to all coalitions $C \in \pi$ belonging to a partition π of the set of players N. A partition function game is *weakly super-additive* if the grand coalition is (weakly) efficient,

$$v(N, \{N\}) \geq \sum_{C \in \pi} v(C, \pi), \text{ for all } \pi \text{ and } C \in \pi. \tag{1}$$

All partition function games considered in this paper are weakly super-additive. Note that this assumption is weaker than the super-additivity assumption used in most of the literature on cooperative and non-cooperative bargaining.

Throughout the paper, i, j, and k always refers to distinct elements in N, and, for convenience, we define the following variables associated with a game in partition function v: $v_i = v(i, \{\{1\}, \{2\}, \{3\}\})$, $V_i = v(i, \{\{i\}, \{j, k\}\})$, $V_{ij} = v(\{i, j\}, \{\{i, j\}, \{k\}\})$, and $V = v(N, \{N\})$. Also, $e_i \in R^3$ is defined as the unit row vector with ith coordinate equal to 1, and remaining values equal to zero.

We model negotiations as an infinite horizon non-cooperative game with complete information, utilizing the cooperative game in partition function as the basic underlying

[6] Chatterjee et al. (1993) focus on the study of inefficiencies of equilibrium allocations of coalitional games.

[7] For example, $\pi = \{\{1, 2\}, \{3\}\}$ is the partition where players 1 and 2 form a coalition and 3 is left alone, $\pi = \{\{1, 2, 3\}\}$ is the partition where all the players form the grand coalition N, and $\pi = \{\{1\}, \{2\}, \{3\}\}$ is the partition where no players form any coalitions.

structure.

The players' utility function over a stream of random cash flows $(x_t)_{t=0}^{\infty}$ is equal to $\sum_{t=0}^{\infty} \delta^t E(x_t)$. So players are risk-neutral and have a common per period discount factor equal to $\delta \in (0,1)$. Also, there exists a common medium of exchange, money, that players can use to transfer utility to other players.

In our negotiations, similar to Gul (1989), each player owns an asset that generates a constant stream of cash flow at each period. Players at every period of the game can buy or sell assets in exchange for cash payments. If player i buys j's asset, player j leaves the game with the offered amount, and player i is now the coalition $\{i,j\}$, representing the fact that he now owns the initial assets of both i and j.[8] The partition function v describes the cash flows generated by the assets for all their possible combinations. Specifically, the coalition C generates a total per-period cash flow equal to $(1-\delta)\, v(C,\pi)$, which is worth $v(C,\pi)$, when the coalition $C \in \pi$ belonging to the coalition structure π is formed. For example, i's asset generates a per-period cash flow equal to either $(1-\delta)\, v_i$ or $(1-\delta)\, V_i$ depending on whether the c.s. is $\{\{i\},\{j\},\{k\}\}$ or $\{\{i\},\{j,k\}\}$.

Note that games in partition function are more general than games in *characteristic function*, because they allow for the possibility of coalitions imposing externalities on other players. The characteristic function game satisfies $v(C) := v(C,\pi) = v(C,\pi')$ for all $C \in \pi \cap \pi'$, which implies that $v_i = V_i$ for all $i \in N$. However, the partition function game can capture any positive ($v_i \leq V_i$) or negative ($v_i \geq V_i$) externalities that the coalition $\{j,k\}$ creates for player i.

The negotiation game evolves with players making proposals to acquire the assets of other players followed by those players that have received offers accepting them or not, as in Rubinstein (1982). However, unlike most models on non-cooperative bargaining, we allow for a richer set of offers that includes both *conditional* and *unconditional* offers.

Specifically, the *strategy set* of i's offers, denoted by S_i, includes the following types of offers: (1) Offers to only one player, such as an offer to buy player j at a price p_j. Player j's asset is exchanged for the offered amount conditional only on j's acceptance. (2) Joint offers to both players, such as an offer to buy both j and k at a price p_j and p_k, respectively. The joint offer must also specify the order in which the players sequentially respond to the offer, and one of the four types of conditions: (i) conditional on both player j's and k's acceptance decisions, (ii) conditional only on j's acceptance decision and unconditional on k's acceptance decision, (iii) conditional only on k's acceptance decision and unconditional

[8] Alternatively, i can sell his assets to player j. Also, equivalently, we can interpret the transaction as players (firms) i and j agreeing to merge their assets into a new firm in exchange for a fraction of the ownership stake of the new firm.

on j's acceptance decision, or (iv) unconditional. In particular, if the offer to player j is unconditional on k's acceptance decision then if j accepts the offer j's asset is transferred to i and j leaves the game with the amount offered, regardless of the response of player k (see also Krishna and Serrano (1996)). If the offer to player j is conditional on k's acceptance decision, then j leaves the game with the amount offered in exchange for his asset if and only if both j and k accept the offer. Proposals can also be behavior strategies, $\Delta(S_i)$, which is a probability distribution over the set of offer strategies S_i.

The *coalition bargaining game* or *negotiation game* (v, δ) is the game with the following extensive form: At the beginning of each period one of the players belonging to the current coalition structure π is randomly chosen, with equal probability, to be the proposer.[9] If player $i \in \pi$ is the proposer he then chooses an offer from the strategy set S_i, and players receiving the offer respond in the order specified, either accepting or rejecting the offer. An exchange of ownership of assets and cash takes place according to the responses of the offer and the precise conditions attached to it (see previous paragraph). This defines the new current coalition structure π, and the game is repeated, after a lapse of one period of time, with a new proposer being randomly chosen as described.

Our notion of equilibrium is *stationary subgame perfect Nash equilibrium (SSPNE).* A strategy profile is SSPNE if it is a subgame perfect equilibrium and the strategies are such that the choice at each stage of the game depends only on the current coalition structure and the current proposer, but neither on the history of the game nor on calendar time.

3. Game-theoretic Analysis of Negotiations

The analysis of the three-player coalitional bargaining game is consistent with the Rubinstein (1982) and Ståhl (1972) two-player bargaining model. Suppose that players j and k form coalition $\{j, k\}$ and player i remains independent, so that the coalition structure (c.s.) is $\{\{i\}, \{j, k\}\}$. In the next period of the game coalition $\{j, k\}$ and player i proceed with a bilateral bargain as in Rubinstein (1982), in which both players i and $\{j, k\}$ can be chosen to propose offers with probability equal to $\frac{1}{2}$. It is a well-known result (e.g., see Osborne and Rubinstein (1990) and Sutton (1986)) that the bilateral bargaining game has a unique stationary perfect equilibrium (indeed, according to Rubinstein (1982) there is a unique subgame perfect equilibrium), in which the two players with reservation values $V_i = v(\{i\}, \{\{i\}, \{j, k\}\})$ and $V_{jk} = v(\{j, k\}, \{\{i\}, \{j, k\}\})$ split by half the surplus $V - V_i - V_{jk}$.

[9] The model could be naturally extended to the case where the probability of i being chosen the proposer is equal to $w(i, \pi) \geq 0$, such that $\sum_{i \in \pi} w(i, \pi) = 1$.

Theorem 0: *Let σ be an SSPNE of the negotiation game (v, δ). There is a unique SSPNE outcome of the subgame starting with a coalition structure $\{\{i\}, \{j, k\}\}$, where the equilibrium outcomes of player i and coalition $\{j, k\}$ are, respectively, equal to*

$$X_i = V_i + \frac{1}{2}(V - V_i - V_{jk}) \text{ and } X_{jk} = V_{jk} + \frac{1}{2}(V - V_i - V_{jk}). \tag{2}$$

Throughout the paper, we represent the equilibrium outcome associated with an SSPNE σ by the variables (ϕ_{ij}) and (ϕ_i) for all $i, j \in \{1, 2, 3\}$, where ϕ_i represents the *expected equilibrium outcome of player i unconditional on the choice of the proposer*, and ϕ_{ij} represents the *expected equilibrium outcome of player i conditional on player j being chosen to be the proposer*. Our main goal in the paper is to solve for the values of (ϕ_i) and (ϕ_{ij}).

Suppose that we are given an SSPNE with equilibrium outcome (ϕ_i). What are the expected equilibrium payoffs in the subgame starting in the node after the acceptance-decision stage with c.s. equal to $\{1, 2, 3\}$? Player i's utility in the current period is the flow $(1 - \delta) v_i$. His utility in the subgame starting next period with the proposal stage is ϕ_i (because this subgame is just like the original game and the equilibrium is stationary), which has a present value equal to $\delta\phi_i$. This implies that player i's expected utility is equal to

$$y_i = \delta\phi_i + (1 - \delta) v_i. \tag{3}$$

Similarly, what are the expected equilibrium payoffs in the subgame starting in the node after the acceptance-decision stage with c.s. equal to $\{\{i, j\}, \{k\}\}$? Player k and the coalition $\{i, j\}$ derive utility $(1 - \delta) V_{ij}$ and $(1 - \delta) V_k$ in the current period, and their utilities in the subgame starting next period with the proposal stage are equal to X_{ij} and X_k (according to Theorem 0), which have a present value equal to δX_{ij} and δX_k. Their expected utilities are then

$$Y_{ij} = \delta X_{ij} + (1 - \delta) V_{ij} \text{ and } Y_k = \delta X_k + (1 - \delta) V_k. \tag{4}$$

3.1. *Conditional and Unconditional Offers*

Our first result provides a mathematical characterization of the SSPNE outcome of the negotiation game. Theorem 1 allow us to map the game theory problem of finding subgame perfect equilibria into an equivalent and more tractable mathematical problem of solving a system of equations subject to inequality constraints.

Theorem 1: *If σ is an SSPNE of the negotiation game (v, δ) then the equilibrium outcome (ϕ_{ij}) and (ϕ_i) associated with σ satisfies conditions (1) to (5) below:*

(1) If $Y_j \geq y_j$ and $Y_k \geq y_k$ then $\phi_{ii} = V - y_j - y_k$, $\phi_{ji} = y_j$, and $\phi_{ki} = y_k$. It is a best response strategy for player i to offer y_j to j, and offer y_k to k conditional on their joint acceptance.

(2) If $Y_j \geq y_j$ and $Y_k \leq y_k$ then $\phi_{ii} = V - y_j - Y_k$, $\phi_{ji} = y_j$, and $\phi_{ki} = Y_k$. It is a best response strategy for player i to offer y_j to j unconditional on k's acceptance, and offer Y_k to k conditional or unconditional on j's acceptance.

(3) If $Y_j - y_j < Y_k - y_k \leq 0$ then $\phi_{ii} = V - Y_j - y_k$, $\phi_{ji} = Y_j$, and $\phi_{ki} = y_k$. It is a best response strategy for player i to offer Y_j to j conditional or unconditional on k's acceptance, and offer y_k to k unconditional on j's acceptance.

(4) If $Y_j - y_j = Y_k - y_k \leq 0$ then $\phi_{ii} = V - Y_j - y_k$, $\phi_{ji} = \mu_i Y_j + (1 - \mu_i) y_j$ and $\phi_{ki} = (1 - \mu_i) Y_k + \mu_i y_k$, where $\mu_i \in [0, 1]$. It is a best response strategy for player i to offer, with probability μ_i, Y_j to j conditional or unconditional on k's acceptance, and offer y_k to k unconditional on j's acceptance and to offer, with probability $1 - \mu_i$, y_j to j unconditional on k's acceptance, and offer Y_k to k conditional or unconditional on j's acceptance.

(5) The equilibrium outcome is related by the system of equations:

$$\phi_i = \frac{1}{3} (\phi_{1i} + \phi_{2i} + \phi_{3i}) \text{ for all } i \in \{1, 2, 3\}. \tag{5}$$

Conversely, if there exists a set of numbers (ϕ_{ij}) and (ϕ_i) for $i, j \in \{1, 2, 3\}$ satisfying conditions (1) to (5) above, then there exists an SSPNE σ of the negotiation game (v, δ) with an associated equilibrium outcome equal to (ϕ_{ij}) and (ϕ_i).

We provide an outline of the proof. We first look at the necessary part of the theorem followed by the converse (sufficient part).

The decision whether or not to accept an offer is dependent on the payoffs (y_i), which are equal to the expected equilibrium payoff of players conditional on the c.s. remaining equal to $\{1, 2, 3\}$ after the acceptance decision, and the payoffs Y_{ij} and Y_k, which are equal to the expected equilibrium payoffs of coalition $\{i, j\}$ and player k conditional on the c.s. remaining equal to $\{\{i, j\}, k\}$ after the acceptance decision. With this in mind we can show that the *best acceptance strategy* for players j and k for any given offer in the strategy set S_i is as follows:

(i) If i offers p_j to player j then j's best response is to accept if and only if $p_j \geq y_j$.

(ii) If i offers p_j to j and p_k to k conditional on their joint acceptance then j and k's best response is to accept if and only if $p_j \geq y_j$ and $p_k \geq y_k$, regardless of the order of response.

(iii) If i offers p_j to j unconditional on k's acceptance, and offer p_k to k conditional on j's acceptance then the best responses are as follows. Player j's best response, regardless of whether he is the first or last to respond, is to accept if and only if $p_j \geq y_j$. Player k's best response, also independent of the order of response, is to accept if and only if $p_k \geq Y_k$.

Note that the strategy above is a best response because if j accepts the offer and k rejects the offer then k's payoff is equal to Y_k, and if j rejects the offer then k's acceptance decision is irrelevant for his payoff.

(iv) If i offers p_j to j unconditional on k's acceptance and offers p_k to k unconditional on j's acceptance then the best responses are as follows, where the order of response is relevant for the strategies that each player chooses. Say that j is the first player to respond followed by k. Player k's best response is to accept if and only if $p_k \geq Y_k$ and player j has previously accepted the offer, or accept if $p_k \geq y_k$ and player j has previously rejected the offer. Player j's best response is to accept if and only if $p_j \geq Y_j$ and $p_k \geq y_k$, or accept if $p_j \geq y_j$ and $p_k < y_k$.

This strategy is indeed a best response. If player j rejects he knows that player k's best response is to accept if $p_k \geq y_k$, in which case j gets a payoff of Y_j. Therefore, player j's best response is to accept any $p_j \geq Y_j$, whenever $p_k \geq y_k$. On the other hand, if $p_k < y_k$, player j knows that the best response of player k is to reject any offer $p_k < y_k$ if player j has previously rejected the offer, in which case j's payoff is equal to y_j. Therefore, player j's best response is to accept if and only if $p_j \geq y_j$, whenever $p_k < y_k$.

We prove in the Appendix that the maximum expected utility that player i can achieve choosing offers in the strategy set S_i is $\phi_{ii} = \max\{V - y_j - y_k, V - y_j - Y_k, V - Y_j - y_k\}$. Using both this result and the best acceptance strategies above we can easily see that conditions (1) to (4) of the theorem are true: simply observe that the best response strategy proposed in the statement of the theorem implements the maximum payoff for each of the cases.

Condition (5) of the theorem also must hold. The equilibrium payoff of player i satisfies $\phi_i = \frac{1}{3}\sum_{j=1}^{3} \phi_{ij}$, because each player $j \in \{1, 2, 3\}$ is chosen to propose with probability equal to $\frac{1}{3}$, and player i's payoff when j is chosen to propose is, by definition, equal to ϕ_{ij}.

In order to prove the converse we show that the stationary strategy profile constructed using conditions (1) to (5) satisfies the one-stage deviation principle and that the infinite-horizon negotiation game is continuous at infinity (see Fudenberg and Tirole (1991)).

We remark that the ability to use behavior strategies is important, and we will see later on that there may not exist pure strategy equilibrium (see Example 2). Observe that the best-response set of a player proposing offers may not be single-valued. However, any choice of strategy from the best-response set yields the same equilibrium outcome $\left(\phi_{ji}\right)$ in all but one important case. Whenever $Y_j - y_j = Y_k - y_k < 0$, the probability that player i chooses each of his best response strategies determines the payoff of players j and k conditional on the event that player i is the proposer: for example, $\phi_{ji} = \mu_i Y_j + (1 - \mu_i)\, y_j$ is decreasing in μ_i (because $Y_j < y_j$) and $\phi_{ki} = (1 - \mu_i)\, Y_k + \mu_i y_k$ is increasing in μ_i, where μ_i is as given in case 4 of the theorem.

3.2. *Discussion of Conditional and Unconditional Offers*

We believe that expanding the strategy set to allow for the possibility of both conditional or unconditional offers adds a new degree of realism to strategic bargaining that is worth exploring. Surprisingly, we will see in the following sections of the paper that even when the new choices are added to the players' strategy set, the mathematical problem of solving for the equilibrium outcome is still very tractable.

The ability to make both conditional and unconditional offers is an important point of departure between our theory and other non-cooperative bargaining models. One one hand, for example, in Selten (1981), Gul (1989), Chatterjee et al. (1993), Ray and Vohra (1996), and Hart and Mas-Colell (1996), offers are, implicitly, conditional on the acceptance of all players: if even one player receiving an offer rejects it, then the offer is canceled and a new player is chosen to become the new proposer. On the other hand, in Krishna and Serrano (1996) offers are, implicitly, unconditional on the acceptance decision of other players, and thus a player that accepts an offer can exit the game with the offered amount, regardless of whether or not the offer is accepted by the other players.

We now intuitively discuss the implications of allowing for conditional and unconditional offers. We first show that when unconditional offers are ruled out, inefficiencies in the negotiation outcome can arise that can be significant for strictly super-additive games with small discount factors. However, in negotiations with very patient players (large discount factors), there is no significant change in the equilibrium allocation when we restrict attention to only conditional offers, as most papers in the literature do. We then discuss the impact of ruling out conditional offers (conditional on the acceptance of other players) and show that our results would change in an essential way for several games of interest. We also provide a reinterpretation of Krishna and Serrano's (1996) results in the framework of conditional and unconditional offers.

First, consider how our results change if players are restricted to using only conditional offers. Say for example that the equilibrium outcome is either as in case 2, $Y_k \leq y_k$ and $Y_j \geq y_j$, or as in case 3, $Y_k - y_k < Y_j - y_j \leq 0$ (see Theorem 1). What is the best response strategy for player i if he is restricted to using conditional offers? Consider the strategy of making an offer conditional on acceptance by both players. It is easy to see that the maximum payoff that player i can achieve is $\phi_{ii} = V - y_j - y_k$ and the other players get $\phi_{ji} = y_j$ and $\phi_{ki} = y_k$. Alternatively, consider the strategy of proposing an offer y_j, exclusively to player j. Upon acceptance by player j, a coalition $\{i, j\}$ forms and continues negotiating in the next round with player k. The payoff to all players then is equal to $\phi_{ii} = Y_{ij} - y_j = V - y_j - Y_k - (1-\delta)(V - V_k - V_{ij})$, $\phi_{ji} = y_j$, and $\phi_{ki} = Y_k$. Therefore, whenever $y_k - Y_k \geq (1-\delta)(V - V_k - V_{ij})$ player i is better off with an offer exclusively to player j. Note, however, that there is a loss of efficiency equal to $(1-\delta)(V - V_k - V_{ij})$, because there is a delay of one period until all gains from trade are fully exploited. We have just intuitively seen that inefficiencies can arise in equilibrium when we rule out unconditional offers, and those inefficiencies can be significant for strictly super-additive games with small discount factors. The analysis also shows that the equilibrium allocation is approximately the same with or without the ability to make unconditional offers in negotiations with very patient players.

Second, consider now how our results change if, as Krishna and Serrano (1996), and unlike in most of the existing literature, players are restricted to using only unconditional offers. This restriction does not alter the results whenever the conditions of cases 2, 3, and 4 of Theorem 1 are satisfied. This is so because there is a best response strategy where offers are unconditional in such cases. For example, if $Y_k \leq y_k$ and $Y_j \geq y_j$, or if $Y_k - y_k < Y_j - y_j \leq 0$, then it is a best response for player i to propose a joint unconditional offer y_j and Y_k, where player j is chosen to be the first to respond to the offer, followed by k. However, the restriction of using only unconditional offers does significantly alter the results in case 1, where $Y_j \geq y_j$ and $Y_k \geq y_k$, and the payoff of player i with conditional offers is equal to $V - y_j - y_k$. The highest payoff that player i can achieve with unconditional offers is equal to $\max\{V - y_j - Y_k, V - Y_j - y_k\}$ (see proof of Theorem 1), which is strictly smaller than the payoff he can achieve with conditional offers. Also, offers to only one player will only result in a worse outcome for player i : suppose that $Y_j - y_j \geq Y_k - y_k$ (which implies that $\max\{V - y_j - Y_k, V - Y_j - y_k\} = V - y_j - Y_k$), and assume that i makes an offer y_j to player j, forming the coalition $\{i, j\}$, which then negotiates as a group with player k. The payoff to player i is equal to $\phi_{ii} = V - y_j - Y_k - (1-\delta)(V - V_k - V_{ij}) \leq V - y_j - Y_k$, and the payoff to player k is Y_k. Note also that in the case where all offers are unconditional, in contrast with the case where all offers are conditional, the equilibrium allocation is Pareto efficient because a player would rather make a joint offer that is accepted by all players than make an offer to form a coalition with one player and then negotiate with the remaining player as a group.

Krishna and Serrano (1996) analyze a multilateral bargaining problem of dividing a dollar among n players, and show that there is a unique subgame perfect equilibrium if players are restricted to using only unconditional offers. This result seems surprising in light of our discussion in the previous paragraph, and we provide a reconciliatory explanation. First, there is an added technological feature in Krishna and Serrano's model: not only can players accepting an offer unconditionally leave the game with the proposed offer, *but also* the total amount over which the other players remain bargaining is reduced by the offered amount. In a sense, players in this set-up are given a degree of power–giving away a share of the dollar without unanimous agreement–that seems too strong for many economic problems of interest. However, Krishna and Serrano also naturally extend their results to situations where the unanimous agreement of all players is needed. This is accomplished by interpreting proposals as offers to purchase the right to represent the remaining players in future negotiations, in exchange for an immediate cash payment. In order for the model to work, they also need to assume that the offeror is able to borrow the offered amount from a third party at no cost (interest) and with repayment due only when the offeror is able to reach an agreement with other players in the negotiation.[10]

Interestingly, this arrangement in Krishna and Serrano (1996) can be reinterpreted in a third way, using the framework of this paper. This is accomplished considering proposals as offers to purchase the right to represent another player for a price to be paid conditional on the offeror reaching an agreement with the other remaining players in the negotiation. This reformulation also underscores that the offers in Krishna and Serrano (1996), although they might seem at first to be unconditional offers, are in reality conditional offers.

3.3. *Strategic Equivalence of Games*

Von Neumann and Morgenstern (1944) introduced the concept of strategic equivalence for characteristic function games. This concept can be straightforwardly extended to partition function games.

Definition 2: A partition function game v with n players is *strategically equivalent* to the

[10] Krishna and Serrano (1996) do not explicitly describe the repayment terms of the borrowed funds; this interpretation is inferred from the analysis that follows.

partition function game v' if and only if there exists $n+1$ constants $(c, a_1, \cdots, a_n)$ with $c > 0$ such that $v'(C, \pi) = cv(C, \pi) + \sum_{i \in C} a_i$ for all $C \in \pi$ and all c.s. π.

The idea behind the concept of strategic equivalence is that the strategic motivations of the players in both games are exactly the same, and also the equilibrium outcomes for both games are directly related. An immediate corollary of Theorem 1 establishes the relationship of SSPNE solutions of strategic equivalent games.

Corollary 1: *Let (ϕ_{ij}) and (ϕ_i) be an SSPNE outcome of the negotiation game (v, δ) and let v' be a game that is strategically equivalent to v: $v'(C, \pi) = cv(C, \pi) + \sum_{i \in C} a_i$ for all $C \in \pi$ and all c.s. π and $c > 0$. Then (ϕ'_{ij}) and (ϕ'_i), where $\phi'_i = a_i + c\phi_i$ and $\phi'_{ij} = a_i + c\phi_{ij}$, is an SSPNE outcome of the negotiation game (v', δ).*

When considering games in characteristic function only *essential* games are of interest. A game is said to be essential if $v(N) > \sum_{i \in N} v(i)$. It can be easily shown that *inessential* characteristic function games satisfy $v(C) = \sum_{i \in C} v(i)$ for all coalitions C, and thus players have no interesting strategic motivations (i.e., the equilibrium is $\phi_i = v(i)$; see Von Neumann and Morgenstern (1944)). Essential games have a convenient (0,1)-*normalization* associated with the game v,

$$\frac{v(C) - \sum_{i \in C} v(i)}{v(N) - \sum_{i \in N} v(i)}.$$

However, partition function games allow for a richer set of strategic considerations. For example, it can well be the case that $v(N, \{N\}) = \sum_{i \in N} v_i$, but due to externality effects $v(C, \pi) \neq \sum_{i \in C} v_i$ for some $C \in \pi$. For example, a coalition between players $\{j, k\}$ might be able to impose negative externalities on player i, $V_i < v_i$, and for this reason be able to strategically demand some compensation from player i, even though $v(N, \{N\}) = \sum_{i \in N} v_i$.

In this paper we assume only that $v(N, \{N\}) \geq \sum_{i \in N} v_i$, including the possibility of strict inequality. Therefore, in general, the game does not have a strategic equivalent (0,1)-normalization. Nevertheless, the 0-*normalization* associated with the game v, $v(C, \pi) - \sum_{i \in C} v_i$, is useful because it is relatively easier to compute the equilibrium of the 0-normalized game than the equilibrium of the original game.

3.4. *Existence of Equilibrium*

Our next result establishes the existence and efficiency of SSPNE outcomes.

Theorem 2: *There exists an SSPNE for all games (v, δ). Furthermore, all SSPNE are Pareto efficient.*

The proof of the theorem is an application of the Kakutani fixed point theorem. First, note that the Pareto efficiency of all equilibrium allocations is an immediate consequence of Theorem 1. In equilibrium, proposals by player i are such that $\sum_{j=1}^{3} \phi_{ji} = V$, and such proposals are accepted with probability one. This implies that the sum of expected payoffs of players equals $\sum_{i=1}^{3} \left(\frac{1}{3} \sum_{j=1}^{3} \phi_{ij} \right) = \frac{1}{3} \sum_{j=1}^{3} \sum_{i=1}^{3} \phi_{ij} = V$ and thus is on the Pareto frontier.

We first define the subset X,

$$X = \left\{ x \in R^3 \text{ such that } \sum_{i=1}^{3} x_i = V \text{ and } x_i \geq \underline{x}_i = \min \{v_i, V_i\} \right\}, \tag{6}$$

of agreements that are in the Pareto frontier and satisfy the individual rationality constraint for all players. Also, consider the correspondence $\Phi : X \to R^3 \times R^3$, where $\Phi_{ji}(x)$ represents the expected payoff of player j when i is the proposer, which is defined using the conditions (1) to (4) of Theorem 1, and consider the function $F : R^3 \times R^3 \to R^3$ where $F(x) = (F_1(x), F_2(x), F_3(x))$ and $F_i(x) = \frac{1}{3} \left(\sum_{j=1}^{3} x_{ij} \right)$, which corresponds to condition (5) of Theorem 1. We show that the correspondence $F \circ \Phi : X \to R^3$ satisfies all the conditions of the Kakutani fixed point theorem ($F \circ \Phi$ is u.h.c., X is compact, convex, and a non-empty subset of the Euclidean space R^3, and $F \circ \Phi(x)$ is non-empty and convex for all x) and thus there is a fixed point $x \in X$ such that $x \in F \circ \Phi(x)$. Using the converse of Theorem 1 the set of payoffs $\Phi_{ij}(x)$, with x fixed point of $F \circ \Phi$, satisfies all the conditions (1)-(5) of Theorem 1 and thus there exists an SSPNE σ with expected equilibrium payoff equal to x.

4. The Negotiation Polyhedra

The results of the previous section encode the game theory and strategic elements of the coalitional bargaining game into equations and inequalities. In this section we develop the mathematics of the coalitional bargaining problem, and prove the main theorems of the paper. Using concepts and theorems from convex geometry, introduced throughout the section, we show that there is a *unique* SSPNE outcome for all negotiation games. We show that the space of partition function games is divided into twenty-six distinct polyhedral cones, and that in each region the SSPNE outcome is given by a distinct linear function of the parameters of the game.

4.1. *Stationary Equilibrium and Polyhedral Cones*

What are the SSPNE outcomes of the negotiation game? Suppose that $x \in R^3$ is an SSPNE outcome, and let $y \in R^3$ be given by $y_i = \delta x_i + (1 - \delta) v_i$ (see expression (3)). It must then be true that $Y-y$ belongs to at least one of the following cases, where the triple $(i,j,k)\in\Pi$ belongs to the set of permutations Π of the N players (the motivation for the choice of labels for the cases will be clear later on):

$$\begin{aligned}
&I: Y_1-y_1\geq 0,\ Y_2-y_2\geq 0, \text{ and } Y_3-y_3\geq 0,\\
&II\,(i): Y_i-y_i\leq 0,\ Y_j-y_j\geq 0, \text{ and } Y_k-y_k\geq 0,\\
&III_1\,(i,j,k): Y_i-y_i<Y_j-y_j\leq 0, \text{ and } Y_k-y_k\geq 0,\\
&III_2\,(k): Y_i-y_i=Y_j-y_j<0, \text{ and } Y_k-y_k\geq 0,\\
&IV_1\,(i,j,k): Y_i-y_i<Y_j-y_j<Y_k-y_k\leq 0,\\
&IV_2\,(i): Y_i-y_i<Y_j-y_j=Y_k-y_k<0,\\
&IV_3: Y_i-y_i=Y_j-y_j=Y_k-y_k<0,\\
&IV_4\,(k): Y_i-y_i=Y_j-y_j<Y_k-y_k\leq 0.
\end{aligned}$$

Define for each triple (i,j,k) the set of eight cases above as

$$\mathbb{Q}(i,j,k)=\{I,II(i),III_1(i,j,k),III_2(k),IV_1(i,j,k),IV_2(i),IV_3,IV_4(k)\}\,.$$

The set of all cases is given by $\mathbb{Q}=\underset{(i,j,k)\in\Pi}{\cup}\mathbb{Q}(i,j,k)$, when we consider all permutations of N. Note that there are a total of twenty-six cases: one case I and IV_3; three cases $II\,(i)$, $III_2\,(k)$, $IV_2\,(i)$, and $IV_4\,(k)$; and six cases $III_1\,(i,j,k)$ and $IV_1\,(i,j,k)$. Because of the symmetry of the problem we can concentrate on the analysis of the eight cases in $\mathbb{Q}(i,j,k)$.

The following theorem is the result of the separate analysis of each case.

THEOREM 3: *Let $r\in R^3$ and $\omega\in R^3$ be defined by*

$$\begin{aligned}
(r\,(v))_i &= v_i+\frac{1}{3}\,(V-v_1-v_2-v_3)\,,\\
(\omega\,(v,\delta))_i &= \delta\,(X_i-r_i)+(1-\delta)\,(V_i-v_i)\,.
\end{aligned}\tag{7}$$

A payoff $\phi=(\phi_i)$ is an SSPNE equilibrium outcome if and only if there exists $Q\in\mathbb{Q}(i,j,k)$, such that $\phi=r\,(v)+\Phi_Q\cdot\omega\,(v,\delta)$ and $\Omega_Q\cdot\omega\,(v,\delta)\leq 0$, where (i,j,k) is any permutation of N, P is the 3x3 matrix $P=\begin{bmatrix} e_i\\ e_j\\ e_k\end{bmatrix}$, and Φ_Q and Ω_Q are given by the following matrices:

I: Ω_I and Φ_I are equal to

$$\begin{bmatrix} -1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix} P \text{ and } \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} P.$$

II(i): $\Omega_{II(i)}$ and $\Phi_{II(i)}$ are equal to

$$\begin{bmatrix} 1 & 0 & 0 \\ -\delta & -(3-\delta) & 0 \\ -\delta & 0 & -(3-\delta) \end{bmatrix} P \text{ and } \frac{1}{(3-\delta)}\begin{bmatrix} 2 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} P.$$

$III_1(i,j,k)$: $\Omega_{III_1(i,j,k)}$ and $\Phi_{III_1(i,j,k)}$ are equal to

$$\begin{bmatrix} -\left(3\delta-2\delta^2\right) & -\delta^2 & -\left(9+\delta^2-9\delta\right) \\ \delta & (3-\delta) & 0 \\ 3(1-\delta) & -(3-2\delta) & 0 \end{bmatrix} P \text{ and } \frac{1}{\left(9+\delta^2-9\delta\right)}\begin{bmatrix} (6-5\delta) & -3(1-\delta) & 0 \\ -3(1-\delta) & (3-2\delta) & 0 \\ -(3-2\delta) & -\delta & 0 \end{bmatrix} P.$$

$III_2(k)$: $\Omega_{III_2(k)}$ and $\Phi_{III_2(k)}$ are equal to

$$\begin{bmatrix} -\delta & -\delta & -(6-5\delta) \\ (3-2\delta) & -3(1-\delta) & 0 \\ -3(1-\delta) & (3-2\delta) & 0 \end{bmatrix} P \text{ and } \frac{1}{\delta(6-5\delta)}\begin{bmatrix} (3-2\delta) & -3(1-\delta) & 0 \\ -3(1-\delta) & (3-2\delta) & 0 \\ -\delta & -\delta & 0 \end{bmatrix} P.$$

$IV_1(i,j,k)$: $\Omega_{IV_1(i,j,k)}$ and $\Phi_{IV_1(i,j,k)}$ are equal to

$$\begin{bmatrix} 3(1-\delta) & -(3-2\delta) & 0 \\ -\delta^2 & -\left(12\delta-2\delta^2-9\right) & -\left(9+\delta^2-9\delta\right) \\ \delta(3-2\delta) & \delta^2 & \left(9-9\delta+\delta^2\right) \end{bmatrix} P \text{ and } \frac{1}{\left(9+\delta^2-9\delta\right)}\begin{bmatrix} (6-5\delta) & -3(1-\delta) & 0 \\ -3(1-\delta) & (3-2\delta) & 0 \\ -(3-2\delta) & -\delta & 0 \end{bmatrix} P.$$

$IV_2(i)$: $\Omega_{IV_2(i)}$ and $\Phi_{IV_2(i)}$ are equal to

$$\begin{bmatrix} -(7\delta-6) & -(3-2\delta) & -(3-2\delta) \\ \delta^2 & \left(12\delta-2\delta^2-9\right) & \left(9+\delta^2-9\delta\right) \\ \delta^2 & \left(9+\delta^2-9\delta\right) & \left(12\delta-2\delta^2-9\right) \end{bmatrix} P \text{ and } \frac{1}{\delta(6-5\delta)}\begin{bmatrix} 4\delta & -\delta & -\delta \\ -2\delta & 3 & -(3-\delta) \\ -2\delta & -(3-\delta) & 3 \end{bmatrix} P.$$

IV_3: Ω_{IV_3} and Φ_{IV_3} are equal to

$$\begin{bmatrix} (7\delta-6) & (3-2\delta) & (3-2\delta) \\ (3-2\delta) & (7\delta-6) & (3-2\delta) \\ (3-2\delta) & (3-2\delta) & (7\delta-6) \\ (6-5\delta) & (4\delta-3) & (4\delta-3) \\ (4\delta-3) & (6-5\delta) & (4\delta-3) \\ (4\delta-3) & (4\delta-3) & (6-5\delta) \end{bmatrix} P \text{ and } \frac{1}{3\delta}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & 0 & 2 \end{bmatrix} P.$$

$IV_4(k)$: $\Omega_{IV_4(k)}$ and $\Phi_{IV_4(k)}$ are equal to

$$\begin{bmatrix} -(4\delta-3) & -(4\delta-3) & -(6-5\delta) \\ \delta & \delta & (6-5\delta) \\ -3(1-\delta) & (3-2\delta) & 0 \\ (3-2\delta) & -3(1-\delta) & 0 \end{bmatrix} P \text{ and } \frac{1}{\delta(6-5\delta)} \begin{bmatrix} (3-2\delta) & -3(1-\delta) & 0 \\ -3(1-\delta) & (3-2\delta) & 0 \\ -\delta & -\delta & 0 \end{bmatrix} P.$$

We remark that all systems of linear inequalities $\Omega_Q \cdot \omega \leq 0$ are feasible for all $Q \in \mathbb{Q}$ and $\delta \in (0,1)$. This implies that we can find a negotiation game (v, δ) that has an SSPNE outcome satisfying any of the twenty-six cases above (see Examples in Section 5 for equilibria in each of the regions).[11] Note that $P\omega = [\omega_i, \omega_j, \omega_k]^T$ and thus the effect of multiplying the matrices above by P is just to order the vector ω according to (i, j, k). Also, the inequalities have been conveniently expressed in such a way that none of the inequalities in $\Omega_Q \cdot \omega \leq 0$ are redundant, and thus the expressions cannot be further simplified. Remarkably, the result above holds for all values of the discount rate $\delta \in (0,1)$.

We provide a brief outline of the steps involved in the proofs. The point of most interest in the proof is the application of the Fourier-Motzkin elimination method (Dantzig (1963) and Ziegler (1994)) to obtain the system of linear inequalities $\Omega_Q \cdot \omega \leq 0$. First, assume that the statement of case Q is true. Conditions (1) to (4) of Theorem 1 define explicit expressions for the equilibrium payoff of player j when i is the proposer, ϕ_{ji}, as a function of the equilibrium payoff x. Substituting these expressions into condition (5) of Theorem 1 gives us a system of linear equations, which has a unique solution equal to $x = r + \Phi_Q \cdot \omega$.

In the next step of the proof, we substitute the expression for the equilibrium outcome into the conditions imposed by case Q, resulting in a system of linear inequalities. This substitution yields, directly, that the system of linear inequalities is equal to $\Omega_Q \cdot \omega \leq 0$, for all cases Q, except for the cases $IV_2(i)$, IV_3, and $IV_4(k)$. These three cases are more involved because the equilibrium strategies use behavior strategies, and the system of linear inequalities depends on the probability distribution parameters. However, using the Fourier-Motzkin elimination method we are able to eliminate the probability distribution parameters from the system of linear inequalities, and obtain an equivalent system of inequalities equal to $\Omega_Q \cdot \omega \leq 0$ (see, for example, case IV_3 in the proof).

The system of inequalities $\Omega_Q \cdot \omega \leq 0$ define a polyhedral cone $H(Q)$, where $H(Q) = \left\{\omega \in R^3 : \Omega_Q \cdot \omega \leq 0\right\}$. A *polyhedral cone* is a cone[12] determined by the intersection of a finite number of half-spaces containing the origin 0. The representation of the polyhedral

[11] Note that the equilibrium outcome is the same in the regions $III_2(k)$ and $IV_4(k)$, and in the regions $III_1(i,j,k)$ and $IV_1(i,j,k)$. The equilibrium strategies, though, are different and for this reason we describe these regions separately.

[12] A cone is set that contains all non-negative finite linear combinations of points belonging to it.

cones $H(Q)$ using the matrices Ω_Q, or intersection of half-spaces, is known as the H-representation of the polyhedral cones.

4.2. *Uniqueness of Equilibrium*

The goal of this section is to prove that all negotiation games have a unique SSPNE outcome. By Theorem 3 uniqueness is an immediate consequence of the following claim: for all Q and Q' in $\mathbb{Q}$ if $\omega(v,\delta) \in H(Q) \cap H(Q')$ then $\Phi_Q \cdot \omega(v,\delta) = \Phi_{Q'} \cdot \omega(v,\delta)$.

The proof of the claim, though, is not straightforward. We proceed in steps and use some key results from convex geometry (see Ziegler (1994)) that we now introduce.

We start by recalling some basic definitions. Given any finite set of points $V \subset R^3$, we denote its conical hull by $cone(V) = \{\sum_{i=1}^{n} \lambda_i v_i : \ \lambda_i \geq 0 \text{ and } v_i \in V\}$. It is straightforward that $cone(V)$ is a convex cone. An extremal ray of cone $H \subset R^3$ is any point $\omega \in H$, $\omega \neq 0$, such that there exists a vector $p \in R^3$ where p is a supporting hyperplane to the cone H, and $H \cap \left\{x \in R^3 : p \cdot x = 0\right\} = \{\lambda\omega : \lambda \geq 0\}$. A vector p defines a supporting hyperplane if for all $x \in H$ then $p \cdot x \leq 0$. We denote the set of all extremal rays of a cone H as $ext(H)$. Also, the lineality space of a cone $H = \left\{x \in R^3 : \Omega \cdot x = 0\right\}$ is equal to the linear space $lineal(H) = \left\{x \in R^3 : \Omega \cdot x = 0\right\}$, and the lineality of a cone is the dimension of the lineality space.

With these basic definitions in place we formally state the key representation result from the theory of polytopes: a polyhedral cone H with lineality zero can be represented as $H = cone(ext(H))$, the convex hull of its extremal rays. The representation of the cone as the convex hull of its extremal rays is also known as the V-representation of the cone.

We now define the set of points that are the candidates to be the extremal rays of the polyhedral cones $H(Q)$.

Definition 3: Define the set of points $\mathbb{V} = \bigcup_{(i,j,k)\in\Pi} \{a_i, b_i, c_{ijk}, d_{ijk}\}$ and the set of values $(\Phi(\nu))_{\nu\in\mathbb{V}}$ where,

$$\begin{aligned} a_i &= e_i, & \Phi(a_i) &= 0, \\ b_i &= -(3-\delta)e_i + \delta e_j + \delta e_k, & \Phi(b_i) &= -2e_i + e_j + e_k, \\ c_{ijk} &= -(3-2\delta)e_i - 3(1-\delta)e_j + \delta e_k, & \Phi(c_{ijk}) &= -e_i + e_k, \\ d_{ijk} &= -(3-2\delta)e_i - 3(1-\delta)e_j + (4\delta-3)e_k, & \Phi(d_{ijk}) &= -e_i + e_k, \end{aligned}$$

and associate (one-to-one correspondence) each element $Q \in \mathbb{Q}$ to a subset of $\mathbb{V}$ as follows:

$$\begin{array}{ll}
I = \{a_1, a_2, a_3\}, & II(i) = \{b_i, a_j, a_k\}, \\
III_1(i,j,k) = \{b_i, c_{ijk}, a_k\}, & III_2(k) = \{c_{ijk}, c_{jik}, a_k\}, \\
IV_1(i,j,k) = \{d_{ijk}, c_{ijk}, b_i\}, & IV_2(i) = \{d_{ijk}, d_{ikj}, b_i\}, \\
IV_3 = \{d_{ijk}, d_{ikj}, d_{jik}, d_{jki}, d_{kij}, d_{kji}\}, & IV_4(k) = \{c_{ijk}, c_{jik}, d_{ijk}, d_{jik}\}.
\end{array}$$

Note that the set $\mathbb{V}$ has a total of eighteen points, and we remind that $e_i \in R^3$ is the ith unit vector. Also, from now on we refer to $Q \in \mathbb{Q}$, interchangeably, as a subset of $\mathbb{V}$ using the one-to-one correspondence above. The next lemma shows that the set of points in Q are the extremal rays of the cone $H(Q)$. The proof of the lemma is based on the direct application of the results of convex geometry just introduced.

Lemma 1: *For all $Q \in \mathbb{Q}$ then $H(Q) = cone(Q)$ and Q is the set of extremal rays of the cone $H(Q)$. Furthermore, for all $\omega \in H(Q)$ there is a unique representation of $\omega = \sum_{\nu \in Q} \alpha_\nu \nu$ as a non-negative linear combination of points in Q, and for all $\omega \in H(Q)$, $\Phi_Q \cdot \omega = \sum_{\nu \in Q} \alpha_\nu \Phi(\nu)$.*

This result allows us to represent the polyhedral cones $H(Q)$ simply as a non-negative linear combination of the points in Q. Also, note that we can compute the SSPNE outcomes $\Phi_Q \cdot \omega = \sum_{\nu \in Q} \alpha_\nu \Phi(\nu)$, given any $\omega \in H(Q)$, simply by computing the linear combination of the values $\Phi(\nu)$, where the parameters α_ν are given uniquely by $\omega = \sum_{\nu \in Q} \alpha_\nu \nu$. This allows us to summarize all the information contained in Theorem 3 in the set of extremal points $\mathbb{V}$, the set of values $(\Phi(\nu))_{\nu \in \mathbb{V}}$, and the set of subsets $\mathbb{Q} \subset 2^{\mathbb{V}}$.

The other key step in the proof of the claim is to establish the relationship among the polyhedral cones $H(Q)$. Lemma 2 shows that the family $H(Q)$, $Q \in \mathbb{Q}$, is a *polyhedral complex*: for all $Q \in \mathbb{Q}$, $H(Q)$ is a polyhedron; and for all $Q, Q' \in \mathbb{Q}$, $H(Q) \cap H(Q')$ is a face of both $H(Q)$ and $H(Q')$.[13] This result is important because it shows that any given $\omega(v, \delta)$ belongs either to a unique polyhedral cone $H(Q)$ or to a common face of two polyhedral cones $H(Q)$ and $H(Q')$.

Lemma 2: *The family $H(Q)$, $Q \in \mathbb{Q}$, is a polyhedral complex. For any Q and Q' in $\mathbb{Q}$ with $Q' \neq Q$ then $H(Q) \cap H(Q') = cone(Q \cap Q')$, which is a common face of $H(Q)$ and $H(Q')$. Furthermore, any $\omega \in R^3$ is represented uniquely as $\omega = \sum_{\nu \in Q} \alpha_\nu \nu$, for some $Q \in \mathbb{Q}$.*

With these results in place we are ready to prove the main result of the paper.

[13] The *faces* of a polyhedral cone are all the intersections of the polyhedron with hyperplanes for which the polyhedron is entirely contained in one of the two half-spaces determined by the hyperplanes. The faces of $H(Q)$ are given by *cone* (Q') where $Q' \subseteq Q$.

THEOREM 4: *There exists a unique SSPNE outcome for any negotiation game* (v, δ). *This unique equilibrium outcome is defined as the negotiation value, and is a continuous function of the parameters of the game. The negotiation value is equal to*

$$\Phi(v, \delta) = r(v) + \sum_{\nu \in Q} \alpha_\nu \Phi(\nu),$$

where $Q \in \mathbb{Q}$ *and the coefficients* $(\alpha_\nu)_{\nu \in Q}$ *are the unique non-negative numbers such that* $\omega(v, \delta) = \sum_{\nu \in Q} \alpha_\nu \nu$, *where* $\omega(v, \delta)$ *and* $r(v)$ *are given by expression (7).*

PROOF: From Theorem 3 we have that a payoff ϕ is an SSPNE outcome if and only if there exists $Q \in \mathbb{Q}$ such that $\phi = r(v) + \Phi_Q \cdot \omega(v, \delta)$ and $\omega(v, \delta) \in H(Q)$. Lemma 1 implies that $\Phi_Q \cdot \omega(v, \delta) = \sum_{\nu \in Q} \alpha_\nu \Phi(\nu)$ where $\omega(v, \delta) = \sum_{\nu \in Q} \alpha_\nu \nu$. Suppose now that there exists another $Q' \in \mathbb{Q}$ with $Q \neq Q'$ such that $\omega(v, \delta) \in H(Q) \cap H(Q')$. Lemma 1 also implies that $\omega(v, \delta) = \sum_{\nu \in Q'} \alpha'_\nu \nu$, and $\Phi_{Q'} \cdot \omega(v, \delta) = \sum_{\nu \in Q'} \alpha'_\nu \Phi(\nu)$. However, Lemma 2 implies that there is a unique representation of $\omega(v, \delta)$ as a non-negative combination of ν with all $\nu \in Q$ and $Q \in \mathbb{Q}$. Therefore, $\alpha_\nu = \alpha'_\nu$ for all $\nu \in Q \cap Q'$ and $\alpha_\nu = \alpha'_\nu = 0$ for all $\nu \in \mathbb{V} \backslash (Q \cap Q')$. This implies that $\Phi_Q \cdot \omega(v, \delta) = \Phi_{Q'} \cdot \omega(v, \delta) = \sum_{\nu \in Q} \alpha_\nu \Phi(\nu) = \sum_{\nu \in Q'} \alpha'_\nu \Phi(\nu)$, and thus there is a unique SSPNE outcome. Q.E.D.

We refer from now on to the unique SSPNE outcome as the *negotiation value*. Existence and uniqueness results, such as the one in this paper, are uncommon in most coalitional bargaining models in the literature.[14]

4.3. *The Limit Negotiation Value*

Of particular interest is the limit negotiation value as the time interval between proposals becomes arbitrarily small, and thus the discount rate δ converges to one. The limit negotiation value has a particular simple and intuitive mathematical characterization that we present in this section. Our next theorem shows that the twenty-six distinct cases (polyhedral cones) collapse into only eight distinct cases (also polyhedral cones) in the limit when $\delta \to 1$.

THEOREM 5: *Let* X *and* r *be given by expressions (2) and (7). The limit negotiation value of* (v, δ) *as* δ *converges to* 1 *exists and is uniquely given by:*

[14] For example, in the non-cooperative bargaining models of Selten (1981), and the related models of Bennett (1997) and Moldovanu (1992), there can be multiple stationary solutions. Also, Ray and Vohra (1999) show that there is always a stationary equilibrium, but there is no general uniqueness result. Cooperative models of coalitional bargaining also commonly do not have both existence and uniqueness results.

Case I. *The negotiation value is* $\phi_i = r_i$ *if* $X_i \geq r_i$ *for all* $i = 1, 2, 3$.

Cases $II\,(i)$. *The negotiation value is* $\phi_i = X_i$, $\phi_j = r_j - \frac{(X_i - r_i)}{2}$, *and* $\phi_k = r_k - \frac{(X_i - r_i)}{2}$, *if* $X_i \leq r_i$, $(X_i - r_i) + 2\,(X_j - r_j) \geq 0$, *and* $(X_i - r_i) + 2\,(X_k - r_k) \geq 0$.

Cases $III\,(k)$. *The negotiation value is* $\phi_i = X_i$, $\phi_j = X_j$, *and* $\phi_k = V - X_i - X_j$, *if* $X_1 + X_2 + X_3 \geq V$, $(X_i - r_i) + 2\,(X_j - r_j) \leq 0$, *and* $2\,(X_i - r_i) + (X_j - r_j) \leq 0$.

Case IV. *The negotiation value is* $\phi_i = X_i + \frac{1}{3}\left(V - \sum_{j=1}^{3} X_j\right)$ *if* $X_1 + X_2 + X_3 \leq V$.

This result can be easily proved using Theorems 3 and 4 by taking the limits of Ω_Q, $H\,(Q)$, and Φ_Q when $\delta \to 1$. First, note that since $\omega_i \to X_i - r_i$ the results of cases I and $II(i)$ immediately follows. Also, note that all cases $III_1(i,j,k)$, $III_1(j,i,k)$, and $III_2(k)$ have the same limit negotiation value and that case $III\,(k)$ above is equivalent to $H\,(III_1(i,j,k)) \cup H\,(III_1(j,i,k)) \cup H\,(III_2(k))$ (see proof). Finally, all the different polyhedral cones of type IV collapse into the polyhedral cone $H\,(IV_3)$: $H\,(IV_1(i,j,k)) \cup H\,(IV_2(i)) \cup H\,(IV_4(k)) \subset H\,(IV_3)$. The unique limit case IV is simply determined by one linear inequality $\omega_1 + \omega_2 + \omega_3 \leq 0$, which is equivalent to $X_1 + X_2 + X_3 \leq V$.

Therefore, the twenty-six distinct cases (polyhedral cones) collapse in the limit into only eight distinct cases. The next section develops the economic content of Theorems 4 and 5.

5. The Economics of Negotiations and Formation of Coalitions

In this section we explore the economic properties of the negotiation value. We analyze several problems of economic interest and conclude that the negotiation value is intuitive and captures well the essence of negotiations.

5.1. *The Nash Bargaining Solution and the Negotiation Value*

The classical solution concept for n-person pure bargaining situations is the Nash bargaining solution. In such games the cooperation of all players is needed to achieve gains from trade; otherwise all players get their reservation value. The Nash bargaining solution has been obtained using both axiomatic cooperative game theory concepts (Nash (1950)) and non-cooperative game theory concepts (see Rubinstein (1982) and Osborne and Rubinstein (1990)).

We will see not only that the negotiation value proposed in this paper coincides with the Nash bargaining solution for three-person pure bargaining games, but also that it generalizes the Nash bargaining solution to a class of multilateral bargaining problems, in which the cooperation of any pair of players creates only a fraction of the grand coalition gains. The following proposition, though an immediate application of Theorem 4, formalizes this idea.

Proposition 6: *The negotiation value of the game* (v, δ) *is the Nash bargaining solution,* $r(v)$, *whenever* $(\omega(v, \delta))_i = \delta(X_i - r_i) + (1-\delta)(V_i - v_i) \geq 0$. *Furthermore, in equilibrium, only the grand coalition, but not any pairwise coalition, can form. In the limit when* $\delta \to 1$, *there is no advantage from being the proposer.*

The interest of this proposition is perhaps best illustrated with an example.

Example 1: *Multilateral bargaining games*: Consider the negotiation game (v, δ) where $\delta \in (0,1)$, $v_i = V_i = 0$, $V_{ij} \leq \frac{1}{3}$ for all pairs $\{i, j\}$, and $V = 1$.

Note that this multilateral bargaining game is more general than a pure bargaining game, in which $v_i = V_i = 0$, $V_{ij} = 0$, and $V = 1$. The negotiation value of the multilateral bargaining game can be obtained by computing $r_i(v) = \frac{1}{3}$, and $\omega_i(v, \delta) = \frac{\delta}{6}(1 - 3V_{jk})$. Because $\omega(v, \delta) \geq 0$, Proposition 6 implies that the negotiation value is $\phi_i(v, \delta) = \frac{1}{3}$, which is the Nash bargaining solution of the pure bargaining game. This generalizes the Nash bargaining solution as the appropriate solution concept for games where all coalitions $\{i, j\}$ can achieve only a value V_{ij} less than or equal to $\frac{1}{3}$ of the grand coalition gains.

It is worth explaining the economic intuition of this result. The threat of any pair of players i and j to form a coalition $\{i, j\}$ is not credible because the most the coalition $\{i, j\}$ can get, alienating player k, is $Y_{ij} = \frac{\delta}{2}(1 + V_{ij}) \leq \frac{2\delta}{3} \leq \frac{2}{3}$, which is less than $\frac{2}{3}$, the amount they can get by agreeing to split the dollar equally. In other words, the ability of coalitions to demand more than an equal split of the dollar is an outside option that is not credible (see Sutton (1986) and Osborne and Rubinstein (1990)).

We believe that the negotiation value is a reasonable prediction of how players play a multilateral bargaining model. Note that this prediction is in contrast to the prediction of both Hart and Mas-Colell (1996) and Gul (1989) that the Shapley value will be the outcome of the multilateral bargaining game. Their models have different comparative statics implications than ours: in their model, a change in the values V_{ij} changes the outcome of the negotiation, while in our model the outcome remains unchanged, as long as $V_{ij} \leq \frac{1}{3}$. Which solution concept is more appropriate for the multilateral bargaining game above is an interesting issue that experimental economic analysis can settle.

5.2. *The Shapley Value and the Negotiation Value*

Another well-established solution concept is the Shapley value. This is a central solution concept that has also been derived using both axiomatic cooperative game theory (Shapley (1953)) and non-cooperative game theory (Gul (1989), Hart and Moore (1990), and Hart and Mas-Colell (1996)). We now show that the negotiation value is also related to the Shapley value.

The Shapley value has been generally studied in the context of characteristic function games. We first provide a more general definition of the Shapley value that extends the concept to games in partition function.

Definition 4: Let v be a game in partition function. Define the marginal contribution of player i to the coalition S, $i \notin S \subset N$, as $m(S,i) = v(S \cup i, \{S \cup i, N \backslash (S \cup i)\}) - v(S, \{S, N \backslash S\})$. Also, let $\pi \in \Pi$ denote any permutation of the three players, and let $S(\pi, i)$ denote the coalition of players that come before player i. The *Shapley value* of player i is defined as the average marginal contribution of player i to his predecessors:

$$Sh_i(v) = \frac{1}{3!} \sum_{\pi \in \Pi} m(S(\pi, i), i).$$

First note that for games in characteristic function, $v(S \cup i, \{S \cup i, N \backslash (S \cup i)\}) = v(S \cup i)$ and $v(S, \{S, N \backslash S\}) = v(S)$, and thus the definition above coincides with the standard definition of the Shapley value for games in characteristic function. More generally, for a three-player game in partition function we have that

$$Sh_i(v) = \frac{1}{6}(2(V - V_{jk}) + 2V_i + (V_{ij} - V_j) + (V_{ik} - V_k)). \tag{8}$$

In two of the six permutations player i is the last, and his marginal contribution is thus $V - V_{jk}$; in two of the permutations he is the first player, and his marginal contribution is thus V_i; in one permutation he comes second, just after player j, and his marginal contribution is thus $V_{ij} - V_j$; and in one permutation he comes second, just after player k, and his marginal contribution is thus $V_{ik} - V_k$.

An important result of this paper establishes the relationship between the negotiation value and the Shapley value.

Proposition 7: *The negotiation value of game (v, δ) is*

$$\phi_i = Sh_i(v) + \frac{(1-\delta)}{3\delta}(2(V_i - v_i) - (V_j - v_j) - (V_k - v_k)),$$

if $\Omega_{IV_3} \cdot \omega(v, \delta) \leq 0$. In the limit when $\delta \to 1$, if

$$X_1 + X_2 + X_3 \leq V, \tag{9}$$

the negotiation value is equal to the Shapley value

$$\phi_i = X_i + \frac{1}{3}\left(V - \sum_{j=1}^{3} X_j\right) = Sh_i(v),$$

and there is an advantage from being the proposer equal to

$$\phi_{ii} = Sh_i(v) + \frac{1}{3}\left(V - \sum_{j=1}^{3} X_j\right).$$

Furthermore, any pairwise coalition can form in equilibrium.

Note that the condition in which the negotiation value is equal to the Shapley value, $X_1 + X_2 + X_3 \leq V$, is equivalent to

$$(V_{12} - V_3) + (V_{13} - V_2) + (V_{23} - V_1) \geq V, \tag{10}$$

which is a generalization of the value-additivity condition (VA) of Gul (1989) to partition function games.[15]

The analysis of the equilibrium strategies is revealing. In equilibrium, a proposer i randomly chooses a player, say j, and offers him his Shapley value, $Sh_j(v)$, while player k is left with only X_k. Note that player i's payoff, ϕ_{ii}, is greater than his Shapley value by $\frac{1}{3}\left(V - \sum_{j=1}^{3} X_j\right)$, and player k's payoff is smaller than his Shapley value by the same amount. Therefore, there is an advantage from being the proposer and a disadvantage from being the last player to form a coalition. Thus the Shapley value arises as the equilibrium in situations where players want to rush to form any pairwise coalition as soon as possible. This property helps explain why both Gul (1989) and Hart and Mas-Colell (1996) get the Shapley value as the equilibrium of their non-cooperative models; in both models the framework is such that any two pairs of players are equally likely to meet and form a coalition. Interestingly, note that this is the opposite of what happens in the case where the Nash bargaining is the solution, because in this case no pairwise coalition can arise in equilibrium.

We finalize this section with the analysis of several games of economic interest.

Example 2: *One-seller and two-buyer market game*: Consider the negotiation game (v, δ) where $\delta \in (0,1)$, $v_i = V_i = 0$, $V_{12} = v_H = 1$, $V_{13} = v_L$, $V_{23} = 0$, and $V = v_H = 1$, with

[15] Note also that for characteristic function games the negotiation value is exactly equal to the Shapley value for any $\delta \in (0,1)$ such that $\Omega_{IV_3} \cdot \omega(v, \delta) \leq 0$.

$v_L < v_H = 1$. In this game player 1 is the seller, player 2 is the high valuation buyer, and player 3 is the low valuation buyer.

The negotiation value can be easily obtained by first computing the fundamental parameters of the game according to our theory: $r_i = \frac{v_H}{3}$, $\omega_1 = \frac{\delta}{6}(1 - 3V_{23}) = \frac{\delta}{6}$, $\omega_2 = \frac{\delta}{6}(1 - 3V_{13}) = \frac{\delta}{6}(1 - 3v_L)$, and $\omega_3 = \frac{\delta}{6}(1 - 3V_{12}) = -\frac{\delta}{3}$. Application of Proposition 7 shows that the conditions of case IV_3 are satisfied,

$$\begin{aligned}
(7\delta - 6)\,\omega_1 + (3 - 2\delta)\,(\omega_2 + \omega_3) &= -\frac{1}{2}\delta\,(3 - 3\delta + (3 - 2\delta)\,v_L) \leq 0, \\
(3 - 2\delta)\,(\omega_1 + \omega_3) + (7\delta - 6)\,\omega_2 &= -\frac{1}{2}\delta\,(3 - 3\delta + (7\delta - 6)\,v_L) \leq 0, \\
(3 - 2\delta)\,(\omega_1 + \omega_2) + (7\delta - 6)\,\omega_3 &= -\frac{1}{2}\delta\,(-6\,(1 - \delta) + (3 - 2\delta)\,v_L) \leq 0, \\
(6 - 5\delta)\,\omega_1 + (4\delta - 3)\,(\omega_2 + \omega_3) &= -\frac{1}{2}\delta\,(-3\,(1 - \delta) + (4\delta - 3)\,v_L) \leq 0, \\
(4\delta - 3)\,(\omega_1 + \omega_3) + (6 - 5\delta)\,\omega_2 &= -\frac{1}{2}\delta\,(-3\,(1 - \delta) + (6 - 5\delta)\,v_L) \leq 0, \\
(4\delta - 3)\,(\omega_1 + \omega_2) + (6 - 5\delta)\,\omega_3 &= -\frac{1}{2}\delta\,(6\,(1 - \delta) + (4\delta - 3)\,v_L) \leq 0,
\end{aligned}$$

if $\delta \geq \frac{6}{7}$ and $v_H > v_L \geq \max\left\{\frac{3(1-\delta)}{6-5\delta}, \frac{6(1-\delta)}{3-2\delta}, \frac{3(1-\delta)}{4\delta-3}\right\}$.[16] Note that this condition is equivalent to $v_H > v_L \geq 0$, when δ approaches one, which is also equivalent to condition (10). By Proposition 7 we have that the negotiation value is the Shapley value

$$\phi_1 = \frac{v_H}{2} + \frac{v_L}{6},\ \phi_2 = \frac{v_H - v_L}{2} + \frac{v_L}{6} = \frac{1}{2}v_H - \frac{1}{3}v_L,\ \text{and } \phi_3 = \frac{v_L}{6},$$

if the conditions above are satisfied. This solution generalizes the solution of the one-seller two-buyer market game in Osborne and Rubinstein (1990) when players are allowed to use contracts and resell the asset.

It is instructive to interpret the equilibrium strategies of the game. The one-seller two-buyer game is an example of a game that has no equilibrium in pure strategy. The behavior strategies are as follows, where $\Delta = \frac{1}{6}v_L = \frac{1}{3}\left(V - \sum_{j=1}^{3} X_j\right)$, and the probability distribution parameters are p_1, p_2, and p_3 given by (see proof of Theorem 3) $p_1 = p_3 + 0.25$, $p_2 = 0.25 - p_3$, and $p_3 \in [0, 0.25]$:

Buyer 2 offers to buy the seller's asset for $\phi_1 - \Delta$ and pays buyer 3 $\phi_3 = \Delta$ to leave the market (with probability p_2), and offers to buy the seller's asset for ϕ_1 (with probability $1 - p_2$); buyer 2's payoff is $\phi_2 + \Delta$. Buyer 3 offers to buy the seller's asset for $\phi_1 - \Delta$ and resell it to buyer 2 for $\phi_1 + \Delta$ (with probability p_3), and offers to buy the seller's asset for

[16] Note that the value of the right-hand-side function of δ is contained in the interval $(0, 1)$ for $\delta \in (6/7, 1)$, is decreasing in δ, and converges to zero when δ approaches one.

ϕ_1 and resell it to buyer 2 for $\phi_1 + 2\Delta$ (with probability $1 - p_3$); buyer 3's payoff is 2Δ. The seller offers to sell his asset to buyer 3 for $\phi_1 + \Delta$, who then resell the asset to buyer 2 for ϕ_1 (with probability p_1), and offers to sell his asset to buyer 2 for $\phi_1 + \Delta$ (with probability $1 - p_1$); the seller's payoff is $\phi_1 + \Delta$. Note that in all cases the player with the initiative benefits by Δ.

In general we can readily tell when the negotiation value converges to the Shapley value by inspecting whether condition (10) is satisfied or not. The next examples illustrate the applications to some other games.

EXAMPLE 3: *Three-person majority voting game*: In this game $v(S) = 1$ if $\#S \geq 2$, and $v(S) = 0$ otherwise.

Note that this game satisfies condition (10) because $(V_{12} - V_3)+(V_{13} - V_2)+(V_{23} - V_1) = 3 > V = 1$ and thus the negotiation value coincides with the Shapley value, which is just an equal split of the dollar.

EXAMPLE 4: *Three-person zero-sum game*: In this game $v_i = 0$, $V_{jk} = -V_i = c_i > 0$, and $V = 0$.

This game also satisfies condition (10) because $(V_{12} - V_3) + (V_{13} - V_2) + (V_{23} - V_1) = 2(c_1 + c_2 + c_3) > 0$, and thus the negotiation value also coincides with the Shapley value which is

$$\phi_1 = -c_1 + \frac{(c_1 + c_2 + c_3)}{3}, \; \phi_2 = -c_3 + \frac{(c_1 + c_2 + c_3)}{3}, \text{ and } \phi_3 = -c_3 + \frac{(c_1 + c_2 + c_3)}{3}.$$

Interestingly, this solution coincides with the solution proposed by von Neumann and Morgenstern (1944) for three-person zero-sum games.

5.3. *Natural Coalitions and Pivotal Players*

We have seen that the Nash bargaining is the equilibrium outcome when no pairwise coalition can form, and that the Shapley value is the equilibrium outcome when all pairwise coalitions can form. A novel element of our theory is that neither the Nash bargaining solution nor the Shapley value seems to be the right solution concept for a broad class of games.

First, there are situations where the outcome of a negotiation is better determined by sequential pairwise bargaining sessions: a first pairwise bargaining session in which a specific pairwise coalition is formed–natural coalition–followed by a second pairwise bargaining session between the natural coalition and the player that was left alone. We provide an example of an oligopolistic industry where there are gains from merging, in which the consolidation occur in two stages, with the first stage being a combination of two natural merger partners.

Second, there are other types of situations where the outcome of a negotiation is better determined by one pivotal player bargaining unconditionally with the other players, where any pairwise coalition between the pivotal player and the other players can form, but not the pairwise coalition between the non-pivotal players. We provide an example of a labor market game with one firm (the pivotal player) and two workers, where the workers are better off bargaining individually with the firm rather than forming a union to collectively bargain for wages.

These two new situations have an equilibrium allocation that is intrinsically different from the well-known Nash bargaining solution and the Shapley value. As we will see, the situation where $\{j,k\}$ is a natural coalition corresponds to case $II(i)$ and the situation where player k is pivotal corresponds to case $III(k)$. The next proposition provide an intuitive interpretation of each situation using outside options.

PROPOSITION 8: *The equilibrium coalition structures and outcome of the negotiation game* (v,δ), *for* δ *close to one, are as follows:*

(1) Natural coalition $\{j,k\}$*: If* $X_i \leq r_i$, $X_j \geq v_j + \frac{(X_{jk}-v_j-v_k)}{2}$, *and* $X_k \geq v_k + \frac{(X_{jk}-v_j-v_k)}{2}$ *then the equilibrium is the result of sequential bargaining between players* j *and* k *to form coalition* $\{j,k\}$, *followed by bargaining between coalition* $\{j,k\}$ *and player* i *to form the grand coalition. The outcome is* $\phi_i = X_i$, $\phi_j = v_j + \frac{(X_{jk}-v_j-v_k)}{2}$, *and* $\phi_k = v_k + \frac{(X_{jk}-v_j-v_k)}{2}$, *and there is no advantage from being the proposer.*

(2) Pivotal player k*: If* $X_i + X_j \geq X_{ij}$, $X_j \leq v_j + \frac{(X_{jk}-v_j-v_k)}{2}$, *and* $X_i \leq v_i + \frac{(X_{ik}-v_i-v_k)}{2}$ *then equilibrium is the result of player* k *bargaining unconditionally with players* i *and* j, *and the only pairwise coalitions that can form are either* $\{k,i\}$ *or* $\{k,j\}$. *The outcome is* $\phi_i = X_i$, $\phi_j = X_j$, *and* $\phi_k = V - X_i - X_j$, *and there is no advantage from being the proposer.*

First, consider the case where $\{j,k\}$ is a natural coalition. The outcome of negotiations can be obtained proceeding backwards in the sequential bargaining sessions. In the last bargaining session, the values of player i and the coalition $\{j,k\}$ are, respectively, X_i and X_{jk}, as we know from Theorem 0. In the first bargaining session, players j and k bargain over X_{jk} using as disagreement points v_j and v_k, which are their status quo values if they do not reach an agreement, and the coalition $\{j,k\}$ does not form. Note that indeed only the pairwise coalition $\{j,k\}$ will form because neither j nor k want to form coalitions $\{j,i\}$ and

$\{k, i\}$: the payoffs of the players left out when coalitions $\{j, i\}$ and $\{k, i\}$ forms are equal to, respectively, X_k and X_j, and we have that $X_k \geq v_k + \frac{(X_{jk} - v_j - v_k)}{2}$ and $X_j \geq v_j + \frac{(X_{jk} - v_j - v_k)}{2}$.

Now, consider the situation where player k is pivotal. The outcome of negotiations can be obtained observing that players i and j cannot demand a higher payoff than X_i and X_j from player k by threatening to form the coalition $\{i, j\}$ because they would be worse off pursuing this strategy as $X_{ij} \leq X_i + X_j$. Also, note that players i and j are not willing to accept any offer lower than X_i and X_j because they can guarantee this amount by credibly holding out. This is so because if i holds out then k would successfully bargain with j to form a coalition; j's gain are $v_j + \frac{(X_{jk} - v_j - v_k)}{2} \geq X_j$, and thus j does not want to hold out when i holds out.

Our next result compares the predictions of the negotiation value, the Nash bargaining solution, and the Shapley value in situations where there are natural coalitions and pivotal players.

Corollary 2: *The Nash bargaining and the Shapley value bias the limit negotiation value in the following systematic ways:*
(1) Whenever $\{j, k\}$ forms a natural coalition then $r_i \geq \phi_i \geq Sh_i(v)$, $\phi_j \geq r_j$, and $\phi_k \geq r_k$.
(2) Whenever player k is a pivotal player then $r_i \geq \phi_i \geq Sh_i(v)$, $r_j \geq \phi_j \geq Sh_j(v)$, and $r_k \leq \phi_k \leq Sh_k(v)$.

As we expected the players j and k when forming a natural coalition are able to strengthen their bargaining position and get more than their Nash bargaining value (the opposite happening with the player left out). Interestingly, the Shapley value underestimates the equilibrium outcome of player i, because we have that $V \leq \sum_{j=1}^{3} X_j$ and $Sh_i(v) = X_i + \frac{1}{3}\left(V - \sum_{j=1}^{3} X_j\right) \leq X_i = \phi_i$.

In the situation where k is a pivotal player then player k gets more than the Nash bargaining solution, but less then the Shapley value, and the opposite happens with players j and k. This is so because both players i and j get more than the Shapley value, $Sh_i(v) = X_i + \frac{1}{3}\left(V - \sum_{j=1}^{3} X_j\right) \leq X_i = \phi_i$ and, similarly, $\phi_j \geq Sh_j(v)$, which implies that $\phi_k \leq Sh_k(v)$, because $\sum Sh_i(v) = \sum \phi_i = V$.

A better understanding of negotiations can be grasped by analyzing more closely two examples, each one illustrating one of the two new situations.

Example 5: *Labor market game*: In this game $\delta \in (0, 1)$, $v_i = V_i = 0$, $V_{12} = v$, $V_{13} = v$, $V_{23} \leq 1 - 2v$ where $v \in \left[\frac{1}{3}, 1\right]$, and $V = 1$. The firm is player 1, and players 2 and 3 are the two workers.

How much are the firm's profits and the employee wages? Are the workers better off forming a union to collectively bargain for wages?

As we have done so far, we can solve the game by looking directly at the parameters

$$\omega_1 = \frac{\delta}{6}\left(1 - 3V_{23}\right) \geq \frac{\delta}{3}\left(3v - 1\right), \; \omega_2 = \frac{\delta}{6}\left(1 - 3v\right), \text{ and } \omega_3 = \frac{\delta}{6}\left(1 - 3v\right).$$

Since the conditions of case $III_2(1)$ hold,

$$\begin{aligned} -\delta\left(\omega_2 + \omega_3\right) - \left(6 - 5\delta\right)\omega_1 &\leq -2\delta\left(1 - \delta\right)\left(3v - 1\right) \leq 0 \\ \left(3 - 2\delta\right)\omega_2 - 3\left(1 - \delta\right)\omega_3 &= -\frac{1}{6}\delta^2\left(3v - 1\right) \leq 0 \\ -3\left(1 - \delta\right)\omega_2 + \left(3 - 2\delta\right)\omega_3 &= -\frac{1}{6}\delta^2\left(3v - 1\right) \leq 0 \end{aligned}$$

the negotiation value is equal to

$$\phi_1 = \frac{2\left(1 - \delta\right) + \delta v}{6 - 5\delta}, \; \phi_2 = \frac{1}{2}\left(\frac{4 - \left(3 + v\right)\delta}{6 - 5\delta}\right), \text{ and } \phi_3 = \frac{1}{2}\left(\frac{4 - \left(3 + v\right)\delta}{6 - 5\delta}\right),$$

and the limit negotiation value ($\delta \to 1$) is equal to

$$\phi_1 = v, \; \phi_2 = \frac{1 - v}{2}, \text{ and } \phi_3 = \frac{1 - v}{2},$$

where $\phi_1 \geq \phi_2 = \phi_3$. Therefore, in equilibrium, the firm hires both employees for a wage of $\frac{1-v}{2}$, and the firm's profit is v.

Note that the firm is not able to hire any employee at a wage lower than $\frac{1-v}{2}$. Otherwise, the employee could just wait until the firm signs a contract with the other employee and bargain with the firm for a wage equal to half of the extra profits that he could bring to the firm, which results in a wage equal to $\frac{1}{2}\left(1 - v\right)$. Also, the firm is not willing to sign a wage contract above $\frac{1}{2}\left(1 - v\right)$ with any employee. The threat of forming a union to bargain for higher wages is not credible. The union can bargain for a total wage package equal to half of the surplus that the union brings to the firm, which is equal to $\frac{1}{2}\left(1 + V_{23}\right) \leq 1 - v$.[17] Therefore, collective bargaining results in a wage per worker lower than the amount the firm is willing to offer in the first place.

Note that, contrary to what we assumed so far, if $V_{23} \geq 1 - 2v$ then a different equilibrium would arise. In this new situation condition (10) holds, and thus the Shapley value is the limit equilibrium outcome. Interestingly, in this case the workers would collectively benefit from forming a union. However, the union is not stable, because the firm would be tempted

[17] Note that we assume $V_{23} \leq 1 - 2v$, and thus it can be either positive, if $v \leq \frac{1}{2}$ (e.g., workers derive utility from participating in a union), or negative (e.g., there are costs associated with operating a union).

to make a higher individual offer to only one member of the union. This stylized example illustrates that the theory of negotiations in this paper can bring new insights to collective bargaining and unionization models.

EXAMPLE 6: *Oligopolistic industries and mergers and acquisitions*: Consider the game $v_i = V_i = 0$, $V = 1$, $V_{12} = v_H$, $V_{13} = v_{L_1}$, $V_{23} = v_{L_2}$ where $v_H \in \left[\frac{1}{3}, 1\right]$ and $v_{L_1} \leq v_{L_2} \leq \frac{1-\delta v_H}{3-\delta} \leq v_H$.

In this example there are three firms competing in an industry where there are gains from consolidation. What are the prices at which firms merge? Are there any natural merger partners in this industry?

The negotiation value and strategies provide a direct answer to the questions above. The parameters of the mergers and acquisitions game are

$$\omega_1 = \frac{\delta}{6}\left(1 - 3v_L\right), \ \omega_2 = \frac{\delta}{6}\left(1 - 3v_L\right), \text{ and } \omega_3 = \frac{\delta}{6}\left(1 - 3v_H\right).$$

The conditions of case $II(3)$ hold,

$$\begin{aligned} \omega_3 &= \frac{\delta}{6}\left(1 - 3v_H\right) \leq 0, \\ -\delta\omega_3 - (3-\delta)\,\omega_1 &= -\frac{1}{2}\delta\left(1 - \delta v_H + (3-\delta)\,v_L\right) \leq 0, \\ -\delta\omega_3 - (3-\delta)\,\omega_2 &= -\frac{1}{2}\delta\left(1 - \delta v_H + (3-\delta)\,v_L\right) \leq 0, \end{aligned}$$

and thus the negotiation value is

$$\phi_1 = \frac{1}{2}\left(\frac{2-\delta+\delta v_H}{3-\delta}\right), \ \phi_2 = \frac{1}{2}\left(\frac{2-\delta+\delta v_H}{3-\delta}\right), \text{ and } \phi_3 = \frac{1-\delta v_H}{3-\delta},$$

and the limit negotiation value is equal to

$$\phi_1 = \frac{1+v_H}{4}, \ \phi_2 = \frac{1+v_H}{4}, \text{ and } \phi_3 = \frac{1-v_H}{2},$$

where $\phi_1 = \phi_2 \geq \phi_3$, whenever $v_L \leq \frac{1-v_H}{2}$ and $v_H \in \left[\frac{1}{3}, 1\right]$.

It is worth exploring several issues that are behind this solution. Note that the industry does not consolidate in a random fashion. If firms 1 and 3 merge their profitability increases by v_L. However, there are still gains from further consolidation with firm 2. What are the gains for each merging firm? Assume that the initial merger between firms 1 and 3 is irreversible or divesting is too costly to be considered a viable option.[18] Firm 2 and

[18] We do not explore further the role of this assumption in this paper, although we believe the results are

conglomerate $\{1,3\}$ will then split the merger gains in a Nash bargaining way, each getting, respectively, $\frac{1}{2}(1-v_L)$ and $\frac{1}{2}(1+v_L)$. Note that the value of the conglomerate $\{1,3\}$ is $\frac{1}{2}(1+v_L) \leq \frac{1}{4}(3-v_H) = \phi_1 + \phi_3$. Therefore, one can predict that firms 1 and 3 are not going to merge and, by the same reasoning, one can also rule out a merger between firms 2 and 3.

Consider now a merger between firms 1 and 2. The value of the conglomerate $\{1,2\}$ is equal to $\frac{1}{2}(1+v_H)$ and the value of firm 3 is $\frac{1}{2}(1-v_H)$ (see previous paragraph). How should the value of the conglomerate $\{1,2\}$ be split among firms 1 and 2? Firm 2 has an apparent stronger bargaining position than firm 1 because $v_{L_1} \leq v_{L_2}$ and thus it seems reasonable that firm 2 should receive a higher share of the value than firm 1. This intuitive idea is wrong, however. Firm 2 does not have any credible outside options other than to merge with firm 1, and thus the Nash bargaining solution is an equal split of the value of the conglomerate $\{1,2\}$.

5.4. *Externalities in Negotiations*

We discuss in this section the role of externalities in negotiations. The formation of coalitions may impose externalities on the non-members: in a partition function game coalition $\{i,j\}$ creates an externality worth $V_k - v_k$ for player k. How important are these externalities for the outcome of negotiations?

PROPOSITION 9: *The negotiation game (v,δ), where v is a game in partition function, and the negotiation game $(\overline{v},\delta)$, where $\overline{v}$ is the characteristic function game*

$$\overline{v}_i = v_i,\ \overline{v}_{ij} = \delta^{-1}\left((2-\delta)\,v_k - (2-\delta)\,V_k + \delta V_{ij}\right),\ \overline{V} = V,$$

have the same SSPNE equilibrium outcome. In particular, in the limit as $\delta \to 1$, both negotiation games v and $\overline{v}$, where $\overline{v}$ is the characteristic function game

$$\overline{v}_i = v_i,\ \overline{v}_{ij} = V_{ij} - (V_k - v_k),\ \overline{V} = V,$$

have the same SSPNE equilibrium outcome.

This proposition illustrates that the worth of a coalition is equally dependent on how much value it creates, V_{ij}, and how much negative externality it imposes on the non-members of the coalition, $-(V_k - v_k)$. It is an interesting issue for experimental economics to assess

not dependent on it.

whether the presence of externalities influences the equilibrium allocations as predicted by the negotiation value.

Another interesting point illustrated by the proposition is that all equilibrium outcomes spanned by partition function games can also be replicated with characteristic function games. Therefore, the characteristic functions introduced by von Neumann and Morgenstern (1944) are sufficient to capture most of the interesting strategic elements of negotiations.

6. Conclusions

This paper introduces a new concept of value for three-player coalition bargaining games– the negotiation value. The negotiation value is Pareto efficient and is the unique subgame perfect Nash equilibrium of a dynamic non-cooperative game in which players make conditional or unconditional offers, and coalitions remain negotiating as long as there are gains from trade. The underlying economic opportunities in the model are described by a partition function game, which allows us to capture the role of externalities in negotiations: the value of a coalition depends not only on its own worth but also on how much negative externality it imposes on its non-members.

The theory developed in the paper provides a unified framework that selects an economically intuitive solution for all partition function games. This solution can either be the Nash bargaining solution (case in which no pairwise coalition forms), the outcome of sequential bargaining sessions (case in which only the pairwise coalition between two natural partners forms), the outcome of a pivotal player bargaining with the other players (case in which only the two pairwise coalitions with the pivotal player forms), or the Shapley value (case in which all pairwise coalitions forms).

We conclude with a discussion of several natural extensions of this paper. First, although we restricted our analysis to three-player coalitional bargaining games, the framework of this paper is suitable to generalizations, and the concept of negotiation value can be extended to n-player games (see Gomes (1999b)). Second, even though we favor the non-cooperative approach to bargaining, analyzing the cooperative game theory foundations of the negotiation value can enhance our understanding of this solution concept. Furthermore, comparison of the cooperative formulation of this new solution with well-established solutions from cooperative game theory such as the nucleolus, the core, and the Shapley value allows for a deeper understanding of each solution concept (see Gomes (1999a)).

Moreover, we note that most of the literature in coalitional bargaining (including this paper) follows the von Neumann and Morgenstern (1944) approach of describing the coalitional game using characteristic or partition function forms. However, given a game in strategic form in which players have a given strategy set and are allowed to write binding agreements, how is the characteristic or partition function generated? One major gap in the literature in coalitional bargaining is the lack of a theory for the determination of the partition function, starting from a game in strategic form (see Shubik (1983) and Ray and Vohra (1997, 1999)). Gomes (1999c) shows that the solution of the strategic form game and the solution of the coalitional bargaining game in partition function are naturally linked, and provides a consistent way to obtain the partition function form given a strategic form game.

Appendix

Proof of Theorem 1: Suppose that σ is an SSPNE of the game (v, δ). We have already seen in the discussion following the theorem what the best acceptance strategy is. Given that we already know how players respond to offers, let us determine the best proposal strategy. Obviously, if i proposes an offer that is unacceptable to all other players (or if he chooses not to propose), then his payoff is equal to y_i.

(i) If i proposes an offer only to player j, then the highest payoff i can get is $Y_{ij} - y_j$. This is so because in the continuation game the equilibrium payoff of the coalition $\{i, j\}$ is Y_{ij} and the minimum that player j accepts is $p_j = y_j$.

(ii) If i proposes an offer to both players j and k conditional on their joint acceptance, then the highest payoff i can get is $V - y_j - y_k$. This is immediately true because the minimum that players j and k accept is $p_j = y_j$ and $p_k = y_k$, respectively.

(iii) If i proposes an offer both to players j and k conditional on j accepting but unconditional on k accepting, then the highest payoff i can get is $V - y_j - Y_k$. This is an immediate implication of the best response strategies of j and k, because players j and k accept a minimum offer of $p_j = y_j$ and $p_k = Y_k$, respectively.

(iv) If i proposes an offer to both players j and k unconditional on the acceptance of both, then the highest payoff i can achieve is

$$\begin{cases} \max\{V - y_j - Y_k, V - Y_j - y_k, Y_{ij} - y_j, Y_{ik} - y_k, y_i\}, & \text{if } Y_k \leq y_k \text{ or } Y_j \leq y_j \\ \max\{V - Y_j - Y_k, Y_{ij} - y_j, Y_{ik} - y_k, y_i\}, & \text{if } Y_k \geq y_k \text{ and } Y_j \geq y_j \end{cases}. \tag{11}$$

Let us prove this claim. Obviously, player i can achieve $Y_{ij} - y_j$, $Y_{ik} - y_k$ or y_i with an offer that only one or none of the players accepts. Consider then the best offer that i can make that is acceptable by both j and k. Assume that i chooses player j to be the first player to respond to the offer. We have seen before that j accepts the offer if and only if $p_j \geq Y_j$ and $p_k \geq y_k$, or $p_j \geq y_j$ and $p_k < y_k$, and that player k accepts the offer if and only if $p_k \geq Y_k$, if player j has previously accepted the offer, or if player j has previously rejected the offer, if $p_k \geq y_k$. Therefore, player i can buy players' j and k assets at a minimum cost equal to $\min\{Y_j + y_k, y_j + Y_k\}$ if $Y_k \leq y_k$, and equal to $Y_j + Y_k$ if $Y_k \geq y_k$, when player j is the first player to respond. But since player i can choose either j or k to be the first player to respond then he can buy both players' assets at a minimum cost equal to $\min\{Y_j + y_k, y_j + Y_k\}$ if $Y_k \leq y_k$ or $Y_j \leq y_j$, and equal to $Y_j + Y_k$ if $Y_k \geq y_k$ and $Y_j \geq y_j$, which proves the claim.

We have shown, so far, that player i's expected utility when chosen to propose, ϕ_{ii}, is equal to

$$\phi_{ii} = \max\{y_i, Y_{ij} - y_j, Y_{ik} - y_k, V - y_j - y_k, V - y_j - Y_k, V - Y_j - y_k\}.$$

We now show that $Y_{ij} - y_j \leq V - y_j - Y_k$ (and also $Y_{ik} - y_k \leq V - y_k - Y_j$) and that $y_i \leq V - y_j - y_k$. Alternatively, it is always a best response strategy for player i to make an offer that is acceptable by all players.

First, let us prove that $Y_{ij} - y_j \leq V - y_j - Y_k$, and that the inequality holds strictly if $V_k + V_{ij} < V$ holds strictly (similarly we also have that $Y_{ik} - y_k \leq V - y_k - Y_j$). Note first that $Y_{ij} - y_j \leq V - y_j - Y_k$ is equivalent to $Y_{ij} + Y_k \leq V$, and

$$\begin{aligned} Y_{ij} + Y_k &= \delta X_{ij} + (1-\delta) V_{ij} + \delta X_k + (1-\delta) V_k \\ &= \delta (X_k + X_{ij}) + (1-\delta)(V_k + V_{ij}) \\ &= \delta V + (1-\delta)(V_k + V_{ij}), \end{aligned}$$

However, since $V_k + V_{ij} \leq V$ then $Y_{ij} + Y_k \leq V$, and if $V_k + V_{ij} < V$ then $Y_{ij} - y_j < V - y_j - Y_k$.

Now, let us show that $y_i \leq V - y_j - y_k$, and that the inequality holds strictly if $v_1 + v_2 + v_3 < V$ holds strictly. This holds because $v_i + v_j + v_k \leq V$ and $\phi_i + \phi_j + \phi_k \leq V$, which then implies that $y_i + y_j + y_k = \delta\left(\phi_i + \phi_j + \phi_k\right) + (1-\delta)\left(\ v_i +\ v_j +\ v_k\right) \leq V$.

We thus have that the highest expected utility that player i can achieve, conditional on being chosen to be the proposer, is equal to

$$\phi_{ii} = \max\left\{V - y_j - y_k, V - y_j - Y_k, V - Y_j - y_k\right\}. \tag{12}$$

Using this result and the best acceptance strategies we can easily prove that conditions (1) to (4) of the theorem are true: just observe that the best response strategy proposed in the statement of the theorem implements the maximum payoff for each of the cases. Condition (5) of the theorem also holds (see discussion following the theorem), which completes the necessary part of the theorem.

We now prove the converse of the theorem. Suppose that we are given payoffs $\left(\phi_{ji}\right)$ and $\left(\phi_i\right)$ for $i, j \in \{1, 2, 3\}$ satisfying all the conditions above. We claim that the stationary strategy profile σ of proposals and responses considered above is a stationary subgame perfect equilibrium. We use the one-stage deviation principle for infinite-horizon games to prove the claim. This proposition states that in any infinite-horizon game with observed actions that is continuous at infinity, a strategy profile σ is subgame perfect if and only if there is no player i and strategy σ_i' that agrees with σ_i except at a single stage t of the game and history h^t, such that σ_i' is a better response to σ_{-i} than σ_i conditional on history h^t being reached (see Fudenberg and Tirole (1991)).

Note first that the game is continuous at infinity: for each player i his utility function is such that, for any two histories h and h' such that the restrictions of the histories to the first t periods coincides, then the payoff of player i, $\left|u_i(h) - u_i(h')\right|$, converges to zero as t converge to infinity. It is immediately clear that the negotiation game is continuous at infinity because $\left|u_i(h) - u_i(h')\right| \leq M\left(\delta^{t+1} + \delta^{t+2} + \cdots\right) = \frac{M}{1-\delta}\delta^{t+1}$, where $M \geq V - \min\{v_j, V_j\} - \min\{v_k, V_k\}$.

The strategy profile σ_i is such that, by construction, no single deviation σ_i' at both the proposal and response stage can lead to a better response than σ_i. Therefore, by the one-stage deviation principle, the stationary strategy profile σ is a subgame perfect Nash equilibrium. Q.E.D.

Proof of Theorem 2: For the proof of Pareto efficiency of the SSPNE, see the discussion following the statement of Theorem 2.

Consider the subset X, defined by expression (6), of all agreements that are in the Pareto frontier and satisfy the individual rationality constraint for all players. Any equilibrium payoff belongs to the set X because the minimum that a player i can get is equal to $\min\{v_i, V_i\}$ where v_i is his payoff if no coalition is formed, and V_i is his minimum payoff if the coalition $\{j, k\}$ is formed. Note that the set X is non-empty because $v_1 + v_2 + v_3 \leq V$ and $\underline{x}_i = \min\{v_i, V_i\} \leq v_i$. Obviously the set X is also compact and convex.

We now define a correspondence $\Phi : X \to R^3 \times R^3$, where $\Phi_{ji}(x)$ represents the expected payoff of player j when i is the proposer. Given any payoff $x = (x_1, x_2, x_3)$, let $\Phi(x) \subseteq R^3 \times R^3$ be a system of payoffs that satisfy conditions (1) to (4) of Theorem 1, defined as follows:

1) If $Y_j \geq y_j$ and $Y_k \geq y_k$ then $\Phi_{ii}(x) = V - y_j - y_k$, $\Phi_{ji}(x) = y_j$, and $\Phi_{ki}(x) = y_k$.
2) If $Y_j \geq y_j$ and $Y_k \leq y_k$ then $\Phi_{ii}(x) = V - y_j - Y_k$, $\Phi_{ji}(x) = y_j$, and $\Phi_{ki}(x) = Y_k$.
3) If $Y_j - y_j < Y_k - y_k \leq 0$ then $\Phi_{ii}(x) = V - Y_j - y_k$, $\Phi_{ji}(x) = Y_j$, and $\Phi_{ki}(x) = y_k$.
4) If $Y_j - y_j = Y_k - y_k \leq 0$ then $\Phi_{ii}(x) = V - Y_j - y_k = V - y_j - Y_k$, and $\Phi_{ji}(x) = [Y_j, y_j]$, and $\Phi_{ki}(x) = [Y_k, y_k]$.

Note that the correspondence is single-valued for all elements $x \in X$, except at the points x where $Y_j - y_j = Y_k - y_k < 0$, where $\Phi_{ji}(x) = [Y_j, y_j]$ and $\Phi_{ki}(x) = [Y_k, y_k]$, two convex sets (closed intervals). We thus have that for $x \in X$, $\Phi(x)$ is a convex (and non-empty) set. Also, the correspondence Φ is upper hemi-continuous (u.h.c.). The correspondence Φ is obviously continuous at any point $x \in X - D$ where

$$D = \{x \in X \text{ such that } Y_i - y_i = Y_j - y_j < 0 \text{ for some pair } (i,j) \text{ with } i \neq j\},$$

and is u.h.c. at any $x \in D$. Note that Φ is not lower hemi-continuous (and therefore discontinuous) at a point $x \in D$.

Define now a function $F : R^3 \times R^3 \to R^3$ as $F(x) = (F_1(x), F_2(x), F_3(x))$ with $F_i(x) = \frac{1}{3}\left(\sum_{j=1}^{3} x_{ij}\right)$ for any $x = (x_{ij})_{i,j \in \{1,2,3\}}$. This function is obviously continuous. Consider now the composition $F \circ \Phi : X \to R^3$. The composition is u.h.c. because Φ is u.h.c. and F is continuous. Also note that $F \circ \Phi(x)$ is convex for all x.

We prove that the image of $F \circ \Phi(X) \subseteq X$. This is so because, for any $x \in X$, $\sum_{j=1}^{3} \Phi_{ji}(x) = V$, and thus $\sum_{i=1}^{3} (F \circ \Phi)_i(x) = \sum_{i=1}^{3} \frac{1}{3} \sum_{j=1}^{3} \Phi_{ij}(x) = \frac{1}{3} \sum_{j=1}^{3} \sum_{i=1}^{3} \Phi_{ij}(x) = V$. Also, $\Phi_{ij}(x) \geq \min\{y_i, Y_i\} \geq \underline{x}_i$ for any $i \neq j$ because $x_i \geq \underline{x}_i$ and $y_i = \delta x_i + (1-\delta) v_i \geq \underline{x}_i$. Finally, note that $\Phi_{ii}(x) \geq V - y_j - y_k \geq \underline{x}_i$ because $x_i + x_j + x_k = V$, and $v_i + v_j + v_k \leq V$; thus $y_i + y_j + y_k = \delta x_i + (1-\delta) v_i + \delta x_j + (1-\delta) v_j + \delta x_k + (1-\delta) v_k \leq V$. Then $\underline{x}_i \leq y_i \leq V - y_j - y_k \leq \Phi_{ii}(x)$, implying that $\Phi_{ii}(x) \geq \underline{x}_i$. This then implies that $(F \circ \Phi)_i(x) = \frac{1}{3} \sum_{j=1}^{3} \Phi_{ij}(x) \geq \underline{x}_i$, which shows that $F \circ \Phi(x) \in X$.

All the conditions of the Kakutani fixed point theorem hold for the correspondence $F \circ \Phi : X \to X$ and thus there is a fixed point $x \in X$ such that $x \in F \circ \Phi(x)$: X is compact, convex, and a non-empty subset of the Euclidean space R^3; $F \circ \Phi(x)$ is non-empty and convex for all x; and $F \circ \Phi$ is u.h.c.

From Theorem 1 the set of payoffs $\Phi_{ij}(x)$ for x fixed point of $F \circ \Phi$ satisfies conditions (1) to (5) of Theorem 1. The theorem then implies that there exists an SSPNE σ with expected equilibrium payoff equal to x. Q.E.D.

Proof of Theorem 3: Consider the 0-normalized game u: $u_i = 0$, $U_i = V_i - v_i$, $U_{jk} = V_{jk} - v_j - v_k$, and $U = V - v_1 - v_2 - v_3$. By Corollary 1 the equilibrium outcome (ϕ_i) of game (v, δ) and the equilibrium outcome (x_i) of game (u, δ) are related by $x_i = \phi_i - v_i$, and thus the set of equilibrium outcomes satisfies $\Phi(u, \delta) = \Phi(v, \delta) - (v_1, v_2, v_3)$.

Moreover, the definitions of Ω_Q and Φ_Q are independent of the game, $r(u) = r(v) - v_i$, and $\omega(u, \delta) = \omega(v, \delta)$, because

$$(\omega(u,\delta))_i = U_i + \frac{\delta}{2}(U - U_i - U_{jk}) - \frac{\delta}{3}U = V_i + \frac{\delta}{2}(V - V_i - V_{jk}) - \left(v_i + \frac{\delta}{3}\left(V - \sum_{j=1}^{3} v_j\right)\right) = (\omega(v,\delta))_i.$$

Therefore, proving the theorem for the 0-normalized game (u, δ) is equivalent to proving it for the game (v, δ).

In the remainder of this proof we consider only the 0-normalized game (u, δ). The variables (y_i), (Y_i), and (Y_{ij}) for the normalized game (u, δ) are $y_i := \delta x_i$, $Y_i := U_i + \frac{\delta}{2}(U - U_i - U_{jk})$, and $Y_{ij} := U_{jk} + \frac{\delta}{2}(U - U_i - U_{jk})$, and we also consider the transformation of variables

$$r = \frac{U}{3} \text{ and } \omega = \omega(u, \delta) = Y - \frac{\delta}{3}U. \tag{13}$$

We now analyze separately each of the following cases $Q \in \mathbb{Q}$.

I. $Y_1 - \delta x_1 \geq 0$, $Y_2 - \delta x_2 \geq 0$, and $Y_3 - \delta x_3 \geq 0$.

Conditions (1)-(4) of Theorem 1 give us the following system of equilibrium payoffs:

$$\begin{array}{ccc} \phi_{11}=U-\delta x_2-\delta x_3 & \phi_{21}=\delta x_2 & \phi_{31}=\delta x_3 \\ \phi_{12}=\delta x_1 & \phi_{22}=U-\delta x_1-\delta x_3 & \phi_{32}=\delta x_3 \\ \phi_{13}=\delta x_1 & \phi_{23}=\delta x_2 & \phi_{33}=U-\delta x_1-\delta x_2 \end{array}$$

and by condition (5) these equilibrium payoffs satisfy the following system of equations:

$$\begin{aligned} x_1 &= \frac{1}{3}\left(U-\delta x_2-\delta x_3+2\delta x_1\right), \\ x_2 &= \frac{1}{3}\left(U-\delta x_1-\delta x_3+2\delta x_2\right), \\ x_3 &= \frac{1}{3}\left(U-\delta x_1-\delta x_2+2\delta x_3\right). \end{aligned}$$

The unique solution of the system of linear equations above is $x_i=\frac{U}{3}=r_i$ for all i. The conditions that must be satisfied by the solution $x_i=\frac{U}{3}$ are $Y_i-\delta x_i\geq 0$ for all i, which are equivalent to,

$$Y_i \geq \frac{\delta U}{3} \text{ for all } i. \tag{14}$$

Using the transformation of variables in (13), we obtain the expressions for Φ_I and Ω_I. By the converse of Theorem 1 we have that if the game satisfies the system of inequalities (14) then there is an equilibrium with outcome given by Φ_I. This final step of the proof is similar for each of the next following cases, and for this reason will be omitted in the remaining cases.

$II(i)$. $Y_i-\delta x_i\leq 0$, $Y_j-\delta x_j\geq 0$, and $Y_k-\delta x_k\geq 0$.

Conditions (1)-(4) of Theorem 1 give us the following system of equilibrium payoffs:

$$\begin{array}{ccc} \phi_{ii}=U-\delta x_k-\delta x_j & \phi_{ji}=\delta x_j & \phi_{ki}=\delta x_k \\ \phi_{ij}=Y_i & \phi_{jj}=U-\delta x_k-Y_i & \phi_{kj}=\delta x_k \\ \phi_{ik}=Y_i & \phi_{jk}=\delta x_j & \phi_{kk}=U-\delta x_j-Y_i \end{array}$$

Substituting these values in the system of linear equations (5) we have

$$\begin{aligned} x_i &= \frac{1}{3}\left(U-\delta x_k-\delta x_j+2Y_i\right), \\ x_j &= \frac{1}{3}\left(U-\delta x_k-Y_i+2\delta x_j\right), \\ x_k &= \frac{1}{3}\left(U-\delta x_j-Y_i+2\delta x_k\right). \end{aligned}$$

The unique solution of the system is

$$\begin{aligned} x_i &= \frac{2Y_i+U\left(1-\delta\right)}{3-\delta}, \\ x_j &= \frac{U-Y_i}{3-\delta}, \\ x_k &= \frac{U-Y_i}{3-\delta}. \end{aligned}$$

This solution is the equilibrium of the normalized game if the system of inequalities, $Y_i - \delta x_i \leq 0$, $Y_j - \delta x_j \geq 0$, and $Y_k - \delta x_k \geq 0$, holds. This system of inequalities corresponds to

$$\begin{aligned}
Y_i - \delta\left(\frac{2Y_i + U(1-\delta)}{3-\delta}\right) &\leq 0, \\
Y_j - \delta\left(\frac{U - Y_i}{3-\delta}\right) &\geq 0, \\
Y_k - \delta\left(\frac{U - Y_i}{3-\delta}\right) &\geq 0,
\end{aligned}$$

which can be simplified to

$$\begin{aligned}
3Y_i &\leq U\delta, \\
(3-\delta)Y_j + \delta Y_i &\geq U\delta, \\
(3-\delta)Y_k + \delta Y_i &\geq U\delta.
\end{aligned}$$

Note that after the transformation of variables in (13), we obtain the expressions for $\Phi_{II(i)}$ and $\Omega_{II(i)}$. Similarly, we also can establish the converse result.

$III_1(i,j,k)$. $Y_i - \delta x_i < Y_j - \delta x_j \leq 0$, and $Y_k - \delta x_k \geq 0$.

Conditions (1)-(4) are:

$$\begin{array}{ccc}
\phi_{ii} = U - \delta x_k - Y_j & \phi_{ji} = Y_j & \phi_{ki} = \delta x_k \\
\phi_{ij} = Y_i & \phi_{jj} = U - \delta x_k - Y_i & \phi_{kj} = \delta x_k \\
\phi_{ik} = Y_i & \phi_{jk} = \delta x_j & \phi_{kk} = U - \delta x_j - Y_i
\end{array}$$

and the system of equation (5) is

$$\begin{aligned}
x_i &= \frac{1}{3}\left(2Y_i + U - \delta x_k - Y_j\right), \\
x_j &= \frac{1}{3}\left(\delta x_j + Y_j + U - \delta x_k - Y_i\right), \\
x_k &= \frac{1}{3}\left(U - Y_i - \delta x_j + 2\delta x_k\right).
\end{aligned}$$

Solving the system of equations yields:

$$\begin{aligned}
x_i &= \frac{(1-\delta)\left((3-\delta)U - 3Y_j\right) + (6-5\delta)Y_i}{9+\delta^2-9\delta}, \\
x_j &= \frac{3(1-\delta)(U-Y_i) + (3-2\delta)Y_j}{9+\delta^2-9\delta}, \\
x_k &= \frac{(3-2\delta)U - (3-2\delta)Y_i - Y_j\delta}{9+\delta^2-9\delta}.
\end{aligned}$$

This solution is the equilibrium of the normalized game if the system of inequalities, $Y_i - \delta x_i \leq Y_j - \delta x_j \leq 0$, and $Y_k - \delta x_k \geq 0$, holds. This system of inequalities corresponds, after some simplifications, to:

$$
\begin{aligned}
\left(9+\delta^2-9\delta\right)Y_k+\left(3\delta-2\delta^2\right)Y_i+Y_j\delta^2 &\geq \left(3\delta-2\delta^2\right)U,\\
\left(3-\delta\right)Y_j+Y_i\delta &\leq \delta U,\\
\left(9-6\delta\right)Y_j-9\left(1-\delta\right)Y_i &> \delta^2 U.
\end{aligned}
$$

The transformation of variables in (13) gives us the expressions for $\Phi_{III_1(i,j,k)}$ and $\Omega_{III_1(i,j,k)}$. Note that due to the upper hemi-continuity of the Nash equilibrium correspondence (see Fudenberg and Tirole (1991)), we claim that if all the inequalities $\Omega_{III_1(i,j,k)} \cdot \omega \leq 0$ hold, and the last inequality is strict, then $r(u) + \Phi_{III_1(i,j,k)} \cdot \omega$ is an SSPNE outcome of the normalized game. Therefore, we can replace the last inequality, $\left(9-6\delta\right)Y_j-9\left(1-\delta\right)Y_i > \delta^2 U$, by $\left(9-6\delta\right)Y_j-9\left(1-\delta\right)Y_i \geq \delta^2 U$.

$III_2\left(k\right).$ $Y_i-\delta x_i = Y_j-\delta x_j < 0$, and $Y_k-\delta x_k \geq 0$.

Conditions (1)-(4) are:

$$
\begin{array}{ccc}
\phi_{ii} = U - Y_j - \delta x_k & \phi_{ji} = Y_j & \phi_{ki} = \delta x_k \\
\phi_{ij} = Y_i & \phi_{jj} = U - \delta x_k - Y_i & \phi_{kj} = \delta x_k \\
\phi_{ik} = \left(1-p\right)Y_i + p\delta x_i & \phi_{jk} = pY_j + \left(1-p\right)\delta x_j & \phi_{kk} = U - \delta x_i - Y_j
\end{array}
$$

where $p \in [0,1]$ is the probability that player k chooses an offer unconditional on player j accepting and conditional on player i accepting, and $(1-p)$ is the probability that he chooses an offer unconditional on player i accepting and conditional on player j accepting.

The system of equations (5) is

$$
\begin{aligned}
x_k &= \frac{1}{3}\left(U-\delta x_i - Y_j + 2\delta x_k\right),\\
x_j &= \frac{1}{3}\left(pY_j + \left(1-p\right)\delta x_j + U - \delta x_k - Y_i + Y_j\right),\\
x_i &= \frac{1}{3}\left(\left(1-p\right)Y_i + p\delta x_i + Y_i + U - \delta x_k - Y_j\right),\\
Y_i - \delta x_i &= Y_j - \delta x_j,
\end{aligned}
$$

where the last equation, $Y_i-\delta x_i = Y_j-\delta x_j$, corresponds to one of the conditions of case $III_2\left(k\right)$. The unique solution of the system of equations is

$$
\begin{aligned}
x_i &= \frac{\left(3-2\delta\right)Y_i + \left(1-\delta\right)\left(2\delta U - 3Y_j\right)}{\delta\left(6-5\delta\right)},\\
x_j &= \frac{\left(3-2\delta\right)Y_j + \left(1-\delta\right)\left(2\delta U - 3Y_i\right)}{\delta\left(6-5\delta\right)},\\
x_k &= \frac{\left(2-\delta\right)U - Y_i - Y_j}{6-5\delta},\\
p &= \frac{\left(9\delta-9\right)Y_i + \left(9-6\delta\right)Y_j - \delta^2 U}{\left(3Y_i+3Y_j-2\delta U\right)\delta}.
\end{aligned}
$$

This solution is the equilibrium of the normalized game if the system of inequalities, $Y_i-\delta x_i = Y_j-\delta x_j < 0$, and $Y_k-\delta x_k \geq 0$, holds (note that the condition $Y_j-\delta x_j = Y_i-\delta x_i$ is already satisfied), and in addition $p \in [0,1]$. This is equivalent to

$$\begin{aligned}
\delta\left(\frac{(2-\delta)\,U - Y_i - Y_j}{6-5\delta}\right) &\leq Y_k \\
\delta\left(\frac{(3-2\delta)\,Y_j + (1-\delta)\,(2\delta U - 3Y_i)}{\delta\,(6-5\delta)}\right) &> Y_j
\end{aligned}$$

and the inequalities arising from the restriction $p \in [0,1]$ are

$$\begin{aligned}
\frac{(9\delta-9)\,Y_i + (9-6\delta)\,Y_j - \delta^2 U}{(3Y_j + 3Y_i - 2\delta U)\,\delta} &\leq 1, \\
\frac{(9\delta-9)\,Y_i + (9-6\delta)\,Y_j - \delta^2 U}{(3Y_j + 3Y_i - 2\delta U)\,\delta} &\geq 0.
\end{aligned}$$

Note that the first two inequalities can be restated as

$$\begin{aligned}
\delta\,((2-\delta)\,U - Y_i - Y_j) &\leq (6-5\delta)\,Y_i, \\
3Y_j + 3Y_i &< 2\delta U.
\end{aligned}$$

But since $3Y_j + 3Y_i - 2\delta U < 0$ then the denominator of the conditions imposed on p is negative and the inequalities are equivalent to

$$\begin{aligned}
(9-6\delta)\,Y_i - 9\,(1-\delta)\,Y_j &\leq \delta^2 U, \\
(9-6\delta)\,Y_j - 9\,(1-\delta)\,Y_i &\leq \delta^2 U.
\end{aligned}$$

Therefore, the system of inequalities is equivalent to

$$\begin{aligned}
(6-5\delta)\,Y_k + \delta\,(Y_i + Y_j) &\geq \delta\,(2-\delta)\,U, \\
3Y_j + 3Y_i &< 2\delta U, \\
(9-6\delta)\,Y_i - 9\,(1-\delta)\,Y_j &\leq \delta^2 U, \\
(9-6\delta)\,Y_j - 9\,(1-\delta)\,Y_i &\leq \delta^2 U.
\end{aligned}$$

The upper hemi-continuity of the Nash equilibrium correspondence implies that the same equilibrium outcome given by the formula above also holds when the second inequality holds strictly, $3Y_j + 3Y_i \leq 2\delta U$. But note that if we add up both of the two last inequalities we obtain $\delta\,(3Y_i + 3Y_j) \leq 2\delta^2 U$, and therefore the second inequality can be dropped from the system of inequalities. Finally, the transformation of variables in (13) gives the desired expressions.

$IV_1(i,j,k)$. $Y_i - \delta x_i < Y_j - \delta x_j < Y_k - \delta x_k \leq 0$

Conditions (1)-(4) are:

$$\begin{array}{ccc}
\phi_{ii} = U - Y_j - \delta x_k & \phi_{ji} = Y_j & \phi_{ki} = \delta x_k \\
\phi_{ij} = Y_i & \phi_{jj} = U - \delta x_k - Y_i & \phi_{kj} = \delta x_k \\
\phi_{ik} = Y_i & \phi_{jk} = \delta x_j & \phi_{kk} = U - Y_i - \delta x_j
\end{array}$$

The system of equation (5) is

$$\begin{aligned}
x_i &= \frac{1}{3}\left(U - Y_j - \delta x_k + 2Y_i\right), \\
x_j &= \frac{1}{3}\left(U - \delta x_k - Y_i + Y_j + \delta x_j\right), \\
x_k &= \frac{1}{3}\left(U - Y_i - \delta x_j + 2\delta x_k\right).
\end{aligned}$$

The solution of the system is,

$$\begin{aligned}
x_i &= \frac{\left(3+\delta^2-4\delta\right)U - 3\left(1-\delta\right)Y_j + \left(6-5\delta\right)Y_i}{9+\delta^2-9\delta}, \\
x_j &= \frac{3\left(1-\delta\right)\left(U-Y_i\right) + \left(3-2\delta\right)Y_j}{9+\delta^2-9\delta}, \\
x_k &= \frac{\left(3-2\delta\right)U - \left(3-2\delta\right)Y_i - Y_j\delta}{9+\delta^2-9\delta},
\end{aligned}$$

and the conditions are $Y_i - \delta x_i < Y_j - \delta x_j < Y_k - \delta x_k \leq 0$. The conditions can be expressed, after some algebra, as:

$$\begin{aligned}
\left(9\delta-9\right)Y_i + \left(9-6\delta\right)Y_j &> \delta^2 U, \\
\left(2\delta^2-12\delta+9\right)Y_j - Y_i\delta^2 + \left(-9-\delta^2+9\delta\right)Y_k &< -\delta^2 U, \\
\left(9-9\delta+\delta^2\right)Y_k + \left(3\delta-2\delta^2\right)Y_i + Y_j\delta^2 &\leq \left(3\delta-2\delta^2\right)U.
\end{aligned}$$

The upper hemi-continuity of the Nash equilibrium correspondence implies that the equilibrium outcome also holds when the first and second inequality hold strictly. Finally, the transformation of variables in (13) gives the desired expressions.

$IV_2(i)$. $Y_i - \delta x_i \leq Y_j - \delta x_j = Y_k - \delta x_k < 0$.

Conditions (1)-(4) are:

$$\begin{array}{ccc}
\phi_{ii} = U - Y_j - \delta x_k & \phi_{ji} = pY_j + \left(1-p\right)\delta x_j & \phi_{ki} = \left(1-p\right)Y_k + p\delta x_k \\
\phi_{ij} = Y_i & \phi_{jj} = U - Y_i - \delta x_k & \phi_{kj} = \delta x_k \\
\phi_{ik} = Y_i & \phi_{jk} = \delta x_j & \phi_{kk} = U - Y_i - \delta x_j
\end{array}$$

where $p \in [0,1]$ is the probability that player i chooses an offer unconditional on player j accepting and conditional on player k accepting, and $(1-p)$ is the probability that he chooses an offer unconditional on player k accepting and conditional on player j accepting.

The system of equation (5) is

$$\begin{aligned}
x_i &= \frac{1}{3}\left(U - Y_j - \delta x_k + 2Y_i\right), \\
x_j &= \frac{1}{3}\left(pY_j + \left(1-p\right)\delta x_j + U - Y_i - \delta x_k + \delta x_j\right), \\
x_k &= \frac{1}{3}\left(\left(1-p\right)Y_k + p\delta x_k + \delta x_k + U - Y_i - \delta x_j\right), \\
\delta x_j - Y_j &= \delta x_k - Y_k.
\end{aligned}$$

Note that the first and last equation determines the value of x_i, and $x_j = \delta^{-1}\left(\delta x_k + Y_j - Y_k\right)$ as a function of x_k. It remains then only to solve for the values of x_k and p using the second and third equations:

$$\begin{aligned}
\delta^{-1}\left(\delta x_k + Y_j - Y_k\right) &= \frac{1}{3}\left(pY_j + (2-p)\left(\delta x_k + Y_j - Y_k\right) + U - Y_i - \delta x_k\right), \\
x_k &= \frac{1}{3}\left((1-p)\,Y_k + (1+p)\,\delta x_k + U - Y_i - \left(\delta x_k + Y_j - Y_k\right)\right),
\end{aligned}$$

collecting all terms in x_k and p yields:

$$\begin{aligned}
(3-\delta)\,x_k - Y_k p + p\delta x_k + \frac{3}{\delta}\left(Y_j - Y_k\right) - U - 2Y_j + 2Y_k + Y_i &= 0, \\
-3x_k + \delta p x_k + (1-p)\,Y_k - Y_j + U - Y_i + Y_k &= 0.
\end{aligned}$$

Subtracting the first and second equations cancels all terms in p and gives us the solution for x_k :

$$x_k = \frac{2\delta U - 2Y_i\delta - (3-\delta)\,Y_j + 3Y_k}{(6-\delta)\,\delta}.$$

The last equation can be rewritten as $\left(\delta x_k - Y_k\right)p - 3x_k + U + 2Y_k - Y_j - Y_i = 0$. Since $\delta x_k - Y_k > 0$ then there is a unique solution for p. We have then found a unique solution:

$$\begin{aligned}
x_i &= \frac{(2-\delta)\,U - Y_j - Y_k + 4Y_i}{6-\delta}, \\
x_j &= \frac{2\delta U - 2Y_i\delta + 3Y_j - (3-\delta)\,Y_k}{(6-\delta)\,\delta}, \\
x_k &= \frac{2\delta U - 2Y_i\delta - (3-\delta)\,Y_j + 3Y_k}{(6-\delta)\,\delta},
\end{aligned}$$

with

$$p = \frac{\left(-9 - 2\delta^2 + 12\delta\right)Y_k + \left(9 + \delta^2 - 9\delta\right)Y_j + Y_i\delta^2 - \delta^2 U}{\delta\left((3-\delta)\,Y_k + (3-\delta)\,Y_j - 2\delta U + 2Y_i\delta\right)}.$$

This is the equilibrium of the game if the system of inequalities $Y_i - \delta x_i \leq Y_j - \delta x_j = Y_k - \delta x_k < 0$ holds (note that the condition $Y_j - \delta x_j = Y_k - \delta x_k$ is already satisfied) and $p \in [0,1]$, which is equivalent to:

$$\begin{aligned}
(3-2\delta)\,Y_j + (7\delta - 6)\,Y_i + (3-2\delta)\,Y_k &\geq \delta^2 U, \\
(3-\delta)\,Y_k + (3-\delta)\,Y_j + 2Y_i\delta &< 2\delta U, \\
\frac{\left(-9 - 2\delta^2 + 12\delta\right)Y_k + \left(9 + \delta^2 - 9\delta\right)Y_j + Y_i\delta^2 - \delta^2 U}{\delta\left((3-\delta)\,Y_k + (3-\delta)\,Y_j - 2\delta U + 2Y_i\delta\right)} &\leq 1, \\
\frac{\left(-9 - 2\delta^2 + 12\delta\right)Y_k + \left(9 + \delta^2 - 9\delta\right)Y_j + Y_i\delta^2 - \delta^2 U}{\delta\left((3-\delta)\,Y_k + (3-\delta)\,Y_j - 2\delta U + 2Y_i\delta\right)} &\geq 0,
\end{aligned}$$

But note that the second inequality implies that the denominators in the third and fourth inequalities are negative. Therefore the system of inequalities is equivalent to:

$$\begin{aligned}
(3-2\delta)\,Y_j + (7\delta-6)\,Y_i + (3-2\delta)\,Y_k &\geq \delta^2 U, \\
(3-\delta)\,Y_k + (3-\delta)\,Y_j + 2\delta Y_i &< 2\delta U, \\
\left(12\delta - 2\delta^2 - 9\right) Y_j + \left(9 + \delta^2 - 9\delta\right) Y_k + \delta^2 Y_i &\leq \delta^2 U, \\
\left(12\delta - 2\delta^2 - 9\right) Y_k + \left(9 + \delta^2 - 9\delta\right) Y_j + \delta^2 Y_i &\leq \delta^2 U.
\end{aligned}$$

The upper hemi-continuity of the Nash equilibrium correspondence implies that the equilibrium outcome also holds when the second inequality holds strictly, $(3-\delta)\,Y_k + (3-\delta)\,Y_j + 2\delta Y_i \leq 2\delta U$. But note that adding up the last two inequalities we get $\delta\left((3-\delta)\,Y_k + (3-\delta)\,Y_j + 2\delta Y_i\right) \leq 2\delta^2 U$ and thus the second inequality becomes redundant. Finally, the transformation of variables (13) gives the desired expressions.

IV_3. $Y_i - \delta x_i = Y_j - \delta x_j = Y_k - \delta x_k < 0$

Conditions (1)-(4) are:

$$\begin{array}{ccc}
\phi_{ii} = U - Y_j - \delta x_k & \phi_{ji} = p_i Y_j + (1-p_i)\,\delta x_j & \phi_{ki} = (1-p_i)\,Y_k + p_i \delta x_k \\
\phi_{ij} = p_j Y_i + (1-p_j)\,\delta x_i & \phi_{jj} = U - Y_i - \delta x_k & \phi_{kj} = (1-p_j)\,Y_k + p_j \delta x_k \\
\phi_{ik} = p_k Y_i + (1-p_k)\,\delta x_i & \phi_{jk} = (1-p_k)\,Y_j + p_k \delta x_j & \phi_{kk} = U - Y_i - \delta x_j
\end{array}$$

where p_i, p_j and p_j all belong to the interval $[0,1]$. The system of equation (5),

$$\begin{aligned}
x_i &= \frac{1}{3}\left(U - Y_j - \delta x_k + p_j Y_i + (1-p_j)\,\delta x_i + p_k Y_i + (1-p_k)\,\delta x_i\right), \\
x_j &= \frac{1}{3}\left(U - Y_i - \delta x_k + p_i Y_j + (1-p_i)\,\delta x_j + (1-p_k)\,Y_j + p_k \delta x_j\right), \\
x_k &= \frac{1}{3}\left(U - Y_i - \delta x_j + (1-p_i)\,Y_k + p_i \delta x_k + (1-p_j)\,Y_k + p_j \delta x_k\right),
\end{aligned}$$

implies that $x_i + x_j + x_k = U$. Imposing the condition $\delta x_i - Y_i = \delta x_j - Y_j = \delta x_k - Y_k$ we thus get the system of equations

$$\begin{aligned}
\delta x_i - Y_i &= \delta x_j - Y_j = \delta x_k - Y_k, \\
x_i + x_j + x_k &= U,
\end{aligned}$$

which has the unique solution:

$$\begin{aligned}
x_i &= \frac{Y_i}{\delta} + \frac{1}{3}\frac{\delta U - Y_i - Y_j - Y_k}{\delta}, \\
x_j &= \frac{Y_j}{\delta} + \frac{1}{3}\frac{\delta U - Y_i - Y_j - Y_k}{\delta}, \\
x_k &= \frac{Y_k}{\delta} + \frac{1}{3}\frac{\delta U - Y_i - Y_j - Y_k}{\delta}.
\end{aligned}$$

The restrictions $Y_i - \delta x_i = Y_j - \delta x_j = Y_k - \delta x_k < 0$ imply that

$$\delta\left(x_i + x_j + x_k\right) = \delta U > Y_i + Y_j + Y_k,$$

which is equivalent to

$$Y_i + Y_j + Y_k < \delta U.$$

We are now interested in solving the system of equations for p_i, p_j, and p_k:

$$\begin{aligned}
&U - Y_j - \delta x_k + p_j Y_i + (1 - p_j)\,\delta x_i + p_k Y_i + (1 - p_k)\,\delta x_i - 3x_i = 0, \\
&p_i Y_j + (1 - p_i)\,\delta x_j + U - \delta x_i - Y_k + (1 - p_k)\,Y_j + p_k \delta x_j - 3x_j = 0, \\
&(1 - p_i)\,Y_k + p_i \delta x_k + (1 - p_j)\,Y_k + p_j \delta x_k + U - Y_i - \delta x_j - 3x_k = 0.
\end{aligned}$$

After rearranging terms, the system is equivalent to:

$$\begin{aligned}
&(\delta x_i - Y_i)\,(p_j + p_k) = U - Y_j - \delta x_k + 2\delta x_i - 3x_i, \\
&(\delta x_j - Y_j)\,(p_i - p_k) = -\delta x_i + \delta x_j + Y_j + U - 3x_j - Y_k, \\
&(\delta x_k - Y_k)\,(p_i + p_j) = -2Y_k - U + Y_i + \delta x_j + 3x_k,
\end{aligned}$$

and substituting the expressions for x_i, x_j, x_k results in:

$$\begin{aligned}
-\delta\,(\delta U - Y_i - Y_j - Y_k)\,(p_j + p_k) &= 4\delta Y_k - 5Y_i\delta - 3Y_j - 3Y_k + 6Y_i + 4Y_j\delta - \delta^2 U, \\
\delta\,(\delta U - Y_i - Y_j - Y_k)\,(p_i - p_k) &= 3\,(Y_k + 2Y_j\delta - Y_i\delta - \delta Y_k + Y_i - 2Y_j)\,, \\
\delta\,(\delta U - Y_i - Y_j - Y_k)\,(p_i + p_j) &= -7\delta Y_k + 2Y_i\delta + 2Y_j\delta + \delta^2 U + 6Y_k - 3Y_i - 3Y_j.
\end{aligned}$$

We can immediately verify that any vector p_i, p_j and p_k is the unique solution of the system of linear equations:

$$\begin{aligned}
p_i &= p_k + \frac{(1-\delta)\,(3Y_k - 6Y_j + 3Y_i)}{\delta\,(\delta U - Y_i - Y_j - Y_k)}, \\
p_j &= \frac{\delta^2 U + (4Y_k - 5Y_i + 4Y_j)\,(1-\delta) - (Y_i + Y_j + Y_k)}{\delta\,(\delta U - Y_i - Y_j - Y_k)} - p_k, \\
p_k &= p_k.
\end{aligned}$$

Note that p_i, p_j and p_j all belong to the interval $[0, 1]$ (we must also have $Y_i + Y_j + Y_k < \delta U$). This imposes the following six additional inequalities that must hold:

$$\begin{aligned}
p_k + \frac{(1-\delta)\,(3Y_k - 6Y_j + 3Y_i)}{\delta\,(\delta U - Y_i - Y_j - Y_k)} &\geq 0, \\
p_k + \frac{(1-\delta)\,(3Y_k - 6Y_j + 3Y_i)}{\delta\,(\delta U - Y_i - Y_j - Y_k)} &\leq 1, \\
\frac{\delta^2 U + (4Y_k - 5Y_i + 4Y_j)\,(1-\delta) - (Y_i + Y_j + Y_k)}{\delta\,(\delta U - Y_i - Y_j - Y_k)} - p_k &\geq 0, \\
\frac{\delta^2 U + (4Y_k - 5Y_i + 4Y_j)\,(1-\delta) - (Y_i + Y_j + Y_k)}{\delta\,(\delta U - Y_i - Y_j - Y_k)} - p_k &\leq 1, \\
p_k &\geq 0, \\
p_k &\leq 1.
\end{aligned}$$

We use the Fourier-Motzkin elimination method (see Dantzig (1963) and Ziegler (1994)) to eliminate the parameter p_k from the above system of inequalities. We first rewrite the system of inequalities as follows:

$$\begin{aligned}
p_k + \frac{(1-\delta)\left(3Y_k - 6Y_j + 3Y_i\right)}{\delta\left(\delta U - Y_i - Y_j - Y_k\right)} &\geq 0, \\
p_k - \frac{\delta^2 U + \left(4Y_k - 5Y_i + 4Y_j\right)(1-\delta) - \left(Y_i + Y_j + Y_k\right)}{\delta\left(\delta U - Y_i - Y_j - Y_k\right)} &\geq -1, \\
p_k &\geq 0, \\
-p_k - \frac{(1-\delta)\left(3Y_k - 6Y_j + 3Y_i\right)}{\delta\left(\delta - Y_i - Y_j - Y_k\right)} &\geq -1, \\
-p_k + \frac{\delta^2 U + \left(4Y_k - 5Y_i + 4Y_j\right)(1-\delta) - \left(Y_i + Y_j + Y_k\right)}{\delta\left(\delta U - Y_i - Y_j - Y_k\right)} &\geq 0, \\
-p_k &\geq -1,
\end{aligned}$$

where in the first three inequalities the coefficient of p_k is $+1$ and in the last three inequalities the coefficient of p_k is -1. By the Fourier-Motzkin elimination method we can eliminate the variable p_k by adding each of the first three inequalities to each of the last three inequalities. Then the system is equivalent to

$$\begin{gathered}
0 \geq -1, \\
\frac{(1-\delta)\left(3Y_k - 6Y_j + 3Y_i\right)}{\delta\left(\delta U - Y_i - Y_j - Y_k\right)} + \frac{\delta^2 U + \left(4Y_k - 5Y_i + 4Y_j\right)(1-\delta) - \left(Y_i + Y_j + Y_k\right)}{\delta\left(\delta U - Y_i - Y_j - Y_k\right)} \geq 0, \\
\frac{(1-\delta)\left(3Y_k - 6Y_j + 3Y_i\right)}{\delta\left(\delta U - Y_i - Y_j - Y_k\right)} \geq -1, \\
-\frac{\delta^2 U + \left(4Y_k - 5Y_i + 4Y_j\right)(1-\delta) - \left(Y_i + Y_j + Y_k\right)}{\delta\left(\delta U - Y_i - Y_j - Y_k\right)} - \frac{(1-\delta)\left(3Y_k - 6Y_j + 3Y_i\right)}{\delta\left(\delta U - Y_i - Y_j - Y_k\right)} \geq -2, \\
0 \geq -1, \\
-\frac{\delta^2 U + \left(4Y_k - 5Y_i + 4Y_j\right)(1-\delta) - \left(Y_i + Y_j + Y_k\right)}{\delta\left(\delta U - Y_i - Y_j - Y_k\right)} \geq -2, \\
-\frac{(1-\delta)\left(3Y_k - 6Y_j + 3Y_i\right)}{\delta\left(\delta U - Y_i - Y_j - Y_k\right)} \geq -1, \\
\frac{\delta^2 U + \left(4Y_k - 5Y_i + 4Y_j\right)(1-\delta) - \left(Y_i + Y_j + Y_k\right)}{\delta\left(\delta U - Y_i - Y_j - Y_k\right)} \geq 0, \\
0 \geq -1.
\end{gathered}$$

Note that the first, fifth, and last inequality are always satisfied, and the remaining six inequalities can be simplified to:

$$\begin{aligned}
(3-2\delta)\left(Y_i + Y_j + Y_k\right) - 9(1-\delta) Y_i &\leq \delta^2 U, \\
(3-2\delta)\left(Y_i + Y_j + Y_k\right) - 9(1-\delta) Y_j &\leq \delta^2 U, \\
(3-2\delta)\left(Y_i + Y_j + Y_k\right) - 9(1-\delta) Y_k &\leq \delta^2 U, \\
(4\delta-3)\left(Y_i + Y_j + Y_k\right) + 9(1-\delta) Y_i &\leq \delta^2 U, \\
(4\delta-3)\left(Y_i + Y_j + Y_k\right) + 9(1-\delta) Y_j &\leq \delta^2 U, \\
(4\delta-3)\left(Y_i + Y_j + Y_k\right) + 9(1-\delta) Y_k &\leq \delta^2 U.
\end{aligned}$$

The upper hemi-continuity of the Nash equilibrium correspondence implies that the equilibrium outcome also holds when the inequality $Y_i + Y_j + Y_k \leq \delta U$ holds strictly. Also, adding up the first three equations (or the last three) yields $Y_i + Y_j + Y_k \leq \delta U$. Finally, the transformation of variables (13) gives the desired expressions.

$IV_4(k)$. $Y_i - \delta x_i = Y_j - \delta x_j < Y_k - \delta x_k \leq 0$.

Conditions (1)-(4) are:

$$\begin{array}{ccc} \phi_{ii} = U - Y_j - \delta x_k & \phi_{ji} = Y_j & \phi_{ki} = \delta x_k \\ \phi_{ij} = Y_i & \phi_{jj} = U - Y_i - \delta x_k & \phi_{kj} = \delta x_k \\ \phi_{ik} = pY_i + (1-p)\,\delta x_i & \phi_{jk} = (1-p)\,Y_j + p\delta x_j & \phi_{kk} = U - Y_i - \delta x_j \end{array}$$

where $p \in [0,1]$ is the probability that player k chooses an offer unconditional on player j accepting and conditional on player i accepting, and $(1-p)$ is the probability that he chooses an offer unconditional on player i accepting and conditional on player j accepting. The system of equation (5) is

$$\begin{aligned} x_i &= \frac{1}{3}\left(U - \delta x_k - Y_j + Y_i + pY_i + (1-p)\,\delta x_i\right), \\ x_j &= \frac{1}{3}\left(U - \delta x_k - Y_i + Y_j + (1-p)\,Y_j + p\delta x_j\right), \\ x_k &= \frac{1}{3}\left(U - Y_i - \delta x_j + 2\delta x_k\right), \\ \delta x_i - Y_i &= \delta x_j - Y_j. \end{aligned}$$

As before, we obtain the unique solution

$$\begin{aligned} x_i &= \frac{(3-2\delta)\,Y_i - 3\,(1-\delta)\,Y_j + 2\delta\,(1-\delta)\,U}{\delta\,(6-5\delta)}, \\ x_j &= \frac{(3-2\delta)\,Y_j - 3\,(1-\delta)\,Y_i + 2\delta\,(1-\delta)\,U}{\delta\,(6-5\delta)}, \\ x_k &= \frac{(2-\delta)\,U - Y_i - Y_j}{6-5\delta}, \\ p &= \frac{9\,(\delta-1)\,Y_j + (9-6\delta)\,Y_i - \delta^2 U}{(3Y_i + 3Y_j - 2\delta U)\,\delta}. \end{aligned}$$

This is the equilibrium of the game if the system of inequalities $Y_i - \delta x_i = Y_j - \delta x_j < Y_k - \delta x_k \leq 0$ holds (note that the condition $Y_i - \delta x_i = Y_j - \delta x_j$ is already satisfied) and $p \in [0,1]$:

$$\begin{aligned} Y_j - \frac{(3-2\delta)\,Y_j - 3\,(1-\delta)\,Y_i + 2\delta\,(1-\delta)\,U}{6-5\delta} &< Y_k - \delta\left(\frac{(2-\delta)\,U - Y_i - Y_j}{6-5\delta}\right), \\ \delta\left(\frac{(2-\delta)\,U - Y_i - Y_j}{6-5\delta}\right) &\geq Y_k, \\ \frac{9\,(\delta-1)\,Y_j + (9-6\delta)\,Y_i - \delta^2 U}{(3Y_i + 3Y_j - 2\delta U)\,\delta} &\leq 1, \\ \frac{9\,(\delta-1)\,Y_j + (9-6\delta)\,Y_i - \delta^2 U}{(3Y_i + 3Y_j - 2\delta U)\,\delta} &\geq 0. \end{aligned}$$

The first two inequalities simplify to

$$\begin{aligned} (4\delta - 3)(Y_i + Y_j) + (6 - 5\delta) Y_k &> \delta^2 U, \\ (6 - 5\delta) Y_k + \delta (Y_i + Y_j) &\leq \delta (2 - \delta) U. \end{aligned}$$

But adding up $(4\delta - 3)(Y_i + Y_j) + (6 - 5\delta) Y_k - \delta^2 U > 0$ and $-(6 - 5\delta) Y_k - \delta (Y_i + Y_j) + \delta (2 - \delta) U \geq 0$ implies that $3 (Y_i + Y_j) - 2\delta U < 0$. We can then simplify the third and fourth inequalities to:

$$\begin{aligned} -\delta^2 U + 9Y_j\delta - 6Y_i\delta - 9Y_j + 9Y_i &\geq (-2\delta + 3Y_i + 3Y_j)\delta, \\ -\delta^2 U + 9Y_j\delta - 6Y_i\delta - 9Y_j + 9Y_i &\leq 0, \end{aligned}$$

which after further simplifications yields:

$$\begin{aligned} (4\delta - 3)(Y_i + Y_j) + (6 - 5\delta) Y_k &> \delta^2 U, \\ (6 - 5\delta) Y_k + \delta (Y_i + Y_j) &\leq \delta (2 - \delta) U, \\ \delta^2 U + (6\delta - 9) Y_j + (9 - 9\delta) Y_i &\geq 0, \\ \delta^2 U + (6\delta - 9) Y_i + (9 - 9\delta) Y_j &\geq 0. \end{aligned}$$

The upper hemi-continuity of the Nash equilibrium correspondence implies that the equilibrium outcome also holds when the first inequality holds strictly, $(4\delta - 3)(Y_i + Y_j) + (6 - 5\delta) Y_k \geq \delta^2 U$. Finally, the transformation of variables (13) gives the desired expressions. Q.E.D.

Proof of Lemma 1: We will use the following result in order to obtain the set of extremal rays of the cone $H = \{\omega : \Omega \cdot \omega \leq 0\}$. A vector $x \in H$ is an extremal ray of the cone H if and only if $x \in H$ and $\Omega_i x = 0$ and $\Omega_j x = 0$, for Ω_i and Ω_j two linearly independent row vectors of the matrix Ω.

First note that any two ν and ν' in $\mathbb{V}$ are linearly independent. This is true for all $\delta \in (0,1)$ because

$$-(3 - \delta) < -(3 - 2\delta) < -3(1 - \delta) < (4\delta - 3) < \delta < 1$$

for $\delta \in (0,1)$ and the definitions of $\nu \in \mathbb{V}$.

Also, note that once we establish that $Q = ext(H(Q))$, then from the theory of cones for any $\omega \in H(Q) = cone(Q)$ there is a unique representation as a non-negative combination of the extremal rays, $\omega = \sum_{\nu \in Q} \alpha_\nu \nu$ with $\alpha_\nu \geq 0$.

Also, note that it is immediately clear that for all $Q \in \mathbb{Q}$ the matrix Ω_Q of the H-representation of the cones $H(Q)$ have rank equal to 3 (full rank). Therefore, $lineal(H(Q)) = \{x \in R^3 : \Omega_Q \cdot x = 0\} = \{0\}$ and thus all cones $H(Q)$ have lineality zero.

Note that if $\Phi_Q \cdot \nu = \Phi(\nu)$ for all $\nu \in Q$ and all $Q \in \mathbb{Q}$, then if $\omega = \sum_{\nu \in Q} \alpha_\nu \nu$ we have that $\Phi_Q \cdot \omega = \sum_{\nu \in Q} \alpha_\nu \Phi(\nu)$ because Φ_Q is a linear transformation. Therefore, it is enough to prove that $\Phi_Q \cdot \nu = \Phi(\nu)$ for all $\nu \in Q$ and all $Q \in \mathbb{Q}$.

With these results in place we now analyze each of the cones $H(Q)$ for all $Q \in \mathbb{Q}$.

For $Q = I$ it is straightforward that $H(Q) = \{\omega \in R^3_+\} = cone(a_1, a_2, a_3)$. Note also that $\Phi_Q \cdot a_i = 0 = \Phi \cdot a_i$, for all $i \in \{1, 2, 3\}$.

For $Q = II(i)$ the extremal rays of the cone $H(Q)$ are b_i, a_j, and a_k because $\Omega_Q \cdot b_i$, $\Omega_Q \cdot a_j$, and $\Omega_Q \cdot a_k$ are, respectively, equal to

$$\begin{bmatrix} -(3-\delta) \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -(3-\delta) \\ 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 \\ 0 \\ -(3-\delta) \end{bmatrix}.$$

Furthermore, we have that $\Phi_Q \cdot b_i$, $\Phi_Q \cdot a_j$, and $\Phi_Q \cdot a_k$ are, respectively, equal to

$$\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

For $Q = III_1(i, j, k)$ we have that the extremal rays of $H(Q)$ are b_i, c_{ijk}, and a_k because $\Omega_Q \cdot b_i$, $\Omega_Q \cdot c_{ijk}$, and $\Omega_Q \cdot a_k$ are, respectively, equal to

$$\begin{bmatrix} 0 \\ 0 \\ -\left(9 + \delta^2 - 9\delta\right) \end{bmatrix}, \quad \begin{bmatrix} 0 \\ -\left(9 + \delta^2 - 9\delta\right) \\ 0 \end{bmatrix}, \text{ and } \begin{bmatrix} -\left(9 + \delta^2 - 9\delta\right) \\ 0 \\ 0 \end{bmatrix}.$$

Furthermore, $\Phi_Q \cdot b_i$, $\Phi_Q \cdot c_{ijk}$,and $\Phi_Q \cdot a_k$ are, respectively, equal to

$$\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \text{ and } \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}.$$

For $Q = III_2(k)$ the extremal rays of $H(Q)$ are c_{ijk}, c_{jik}, and a_k because $\Omega_Q \cdot c_{ijk}$, $\Omega_Q \cdot c_{jik}$, and $\Omega_Q \cdot a_k$ are, respectively, equal to

$$\begin{bmatrix} 0 \\ (-6 + 5\delta)\,\delta \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ (-6 + 5\delta)\,\delta \end{bmatrix}, \text{ and } \begin{bmatrix} -6 + 5\delta \\ 0 \\ 0 \end{bmatrix}.$$

Furthermore, $\Phi_Q \cdot c_{ijk}$, $\Phi_Q \cdot c_{jik}$,and $\Phi_Q \cdot a_k$ are equal to

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

For $Q = IV_1(i, j, k)$ the extremal rays of $H(Q)$ are d_{ijk}, c_{jik}, and b_i because $\Omega_Q \cdot d_{ijk}$, $\Omega_Q \cdot c_{jik}$, and $\Omega_Q \cdot b_i$ are, respectively, equal to

$$\begin{bmatrix} 0 \\ 0 \\ -3\,(1-\delta)\left(9 + \delta^2 - 9\delta\right) \end{bmatrix}, \begin{bmatrix} 0 \\ -3\,(1-\delta)\left(9 + \delta^2 - 9\delta\right) \\ 0 \end{bmatrix}, \text{ and } \begin{bmatrix} -9 - \delta^2 + 9\delta \\ 0 \\ 0 \end{bmatrix}.$$

Furthermore, $\Phi_Q \cdot d_{ijk}$, $\Phi_Q \cdot c_{jik}$, and $\Phi_Q \cdot b_i$ are equal to

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \text{ and } \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}.$$

For $Q = IV_2(i)$ the extremal rays of $H(Q)$ are d_{ijk}, d_{jik}, and b_i because $\Omega_Q \cdot d_{ijk}$, $\Omega_Q \cdot d_{ikj}$, and $\Omega_Q \cdot b_i$, are, respectively, equal to

$$\begin{bmatrix} 0 \\ 0 \\ -3\delta\,(1-\delta)\,(6-\delta) \end{bmatrix}, \quad \begin{bmatrix} 0 \\ -3\delta\,(1-\delta)\,(6-\delta) \\ 0 \end{bmatrix}, \text{ and } \begin{bmatrix} -3\,(1-\delta)\,(6-\delta) \\ 0 \\ 0 \end{bmatrix}.$$

Furthermore, $\Phi_Q \cdot d_{ijk}$, $\Phi_Q \cdot d_{ikj}$, $\Phi_Q \cdot b_i$, are equal to

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}.$$

For $Q = IV_3$ the extremal rays of $H(Q)$ are $d_{ijk}, d_{ikj}, d_{jik}, d_{kij}, d_{jki}, d_{kji}$ because $\Omega_Q \cdot d_{ijk}$, $\Omega_Q \cdot d_{ikj}$, $\Omega_Q \cdot d_{jik}$, $\Omega_Q \cdot d_{kij}$, $\Omega_Q \cdot d_{jki}$, $\Omega_Q \cdot d_{kji}$, are equal to, respectively, $9\delta(1-\delta)$ multiplied by

$$\begin{bmatrix} 0 \\ -1 \\ -2 \\ -2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ -1 \\ -2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ -1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \\ 0 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \\ 0 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 0 \\ -1 \\ -2 \end{bmatrix}.$$

Furthermore, $\Phi_Q \cdot d_{ijk}$, $\Phi_Q \cdot d_{ikj}$, $\Phi_Q \cdot d_{jik}$, $\Phi_Q \cdot d_{kij}$, $\Phi_Q \cdot d_{jki}$, $\Phi_Q \cdot d_{kji}$, are equal to

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Finally, for $Q = IV_4(k)$ the extremal rays of $H(Q)$ are $c_{ijk}, c_{jik}, d_{ijk}$, and d_{jik} because $\Omega_Q \cdot c_{ijk}$, $\Omega_Q \cdot c_{jik}, \Omega_Q \cdot d_{ijk}$, $\Omega_Q \cdot d_{jik}$, are equal to, respectively, $(6-5\delta)$ multiplied by

$$\begin{bmatrix} -3(1-\delta) \\ 0 \\ 0 \\ -\delta \end{bmatrix}, \begin{bmatrix} -3(1-\delta) \\ 0 \\ -\delta \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3(1-\delta) \\ 0 \\ -\delta \end{bmatrix}, \begin{bmatrix} 0 \\ -3(1-\delta) \\ -\delta \\ 0 \end{bmatrix}.$$

Furthermore, $\Phi_Q \cdot c_{ijk}$, $\Phi_Q \cdot c_{jik}$, $\Phi_Q \cdot d_{ijk}$, $\Phi_Q \cdot d_{jik}$ are equal to

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

Q.E.D.

PROOF OF LEMMA 2:

We first show that the first part of the lemma (for any Q and Q' in $\mathbb{Q}$ with $Q' \neq Q$ then $H(Q) \cap H(Q') = cone(Q \cap Q')$) implies that any $\omega \in R^3$ can be represented uniquely as $\omega = \sum_{\nu \in Q} \alpha_\nu \nu$, for some $Q \in \mathbb{Q}$.

By Theorems 2 and 3 we have that $\underset{Q\in\mathbb{Q}}{\cup} H(Q) = R^3$ and therefore there exists a $Q \in \mathbb{Q}$ such that $\omega \in H(Q)$. But since $H(Q) = cone(Q)$ then $\omega \in R^3$ can be represented as $\omega = \sum_{\nu \in Q} \alpha_\nu \nu$ with $\alpha_\nu \geq 0$. Conversely, this representation is unique. Suppose that $\omega = \sum_{\nu \in Q} \alpha_\nu \nu = \sum_{\nu \in Q'} \alpha'_\nu \nu$ where Q,$Q' \in \mathbb{Q}$ and $Q' \neq Q$, with $\alpha_\nu \geq 0$ for all $\nu \in Q$ and $\alpha'_\nu \geq 0$ for all $\nu \in Q'$. Extend, for convenience, the definitions of α_ν and α'_ν to $\nu \in \mathbb{V}$, setting $\alpha_\nu = 0$ for all $\nu \in \mathbb{V} \backslash Q$ and $\alpha'_\nu = 0$ for all $\nu \in \mathbb{V} \backslash Q'$. The first part of Lemma 2 implies that $\omega = \sum_{\nu \in Q \cap Q'} \alpha''_\nu \nu$ (extend the definitions of α''_ν to $\nu \in \mathbb{V}$ setting $\alpha''_\nu = 0$ for all $\nu \in \mathbb{V} \backslash (Q \cap Q')$). But then, since the representation of $\omega \in H(Q)$ and $\omega \in H(Q')$ is unique, it must be the case that $\alpha''_\nu = \alpha'_\nu = \alpha_\nu$ for all $\nu \in \mathbb{V}$. This proves the uniqueness part of the lemma.

We proceed to prove the first part of the lemma. It is obvious that $cone\,(Q \cap Q') \subset H\,(Q) \cap H\,(Q')$. The difficult part is the converse: $cone\,(Q \cap Q') \supset H\,(Q) \cap H\,(Q')$.

We use the following claim to prove the converse. Recall that any vector $p \in R^3$ can be associated with the hyperplane H, where $H = \{\omega : p.\omega = 0\}$. A hyperplane H is separating if and only if for all $\omega \in H\,(Q)$ and $\omega' \in H\,(Q')$ then $p.\omega \leq 0$ and $p.\omega' \geq 0$.

Claim 1: *Suppose that for any two cones $H\,(Q) = cone\,(Q)$ and $H\,(Q') = cone\,(Q')$, there exists a separating hyperplane H such that if $\nu \in (Q \cup Q') \setminus (Q \cap Q')$ then $\nu \notin H$. Then $H\,(Q) \cap H\,(Q') = cone\,(Q \cap Q')$.*

Proof of Claim 1: We need to prove that $cone\,(Q \cap Q') \supset H\,(Q) \cap H\,(Q')$. It is obvious that $H\,(Q) \cap H\,(Q')$ is a cone and that $H\,(Q) \cap H\,(Q') \subset cone\,(Q \cup Q')$. But the separating hyperplane H is such that $H\,(Q) \cap H\,(Q') \subset H$ and all $\nu \in (Q \cup Q') \setminus (Q \cap Q')$ are such that $\nu \notin H$, and thus $\nu \notin H\,(Q) \cap H\,(Q')$. Therefore, $H\,(Q) \cap H\,(Q') \subset cone\,(Q \cap Q')$. Q.E.D.

We now proceed showing that for each pair Q and Q' in $\mathbb{Q}$ with $Q' \neq Q$ there exists a separating hyperplane H, associated with a vector p, such that $\nu \notin Q \cap Q'$ implies that $\nu \notin H$. Note that throughout the remaining of the proof we rely heavily on the variables in Definition 3.

1. First, consider the case with $Q = I$.

Consider the separating hyperplane associated with $p = e_i$. By Claim 1 it is straightfoward that $H\,(I) \cap H\,(II(i)) = cone\,(a_j, a_k)$, $H\,(I) \cap H\,(III_1(i,j,k)) = H\,(I) \cap H\,(III_2(k)) = cone\,(a_k)$, $H\,(I) \cap H\,(IV_1\,(i,j,k)) = H\,(I) \cap H\,(IV_2\,(i)) = H\,(I) \cap H\,(IV_4\,(k)) = \{0\}$. Also, $H\,(I) \cap H\,(IV_3) = \{0\}$, because $p = e_i + e_j + e_k$ defines a separating hyperplane: $p \cdot \nu > 0$ for $\nu \in I$ and $p \cdot \nu < 0$ for all $\nu \in IV$.

2. Now consider the case with $Q = II(i)$.

The intersection $H\,(II(i)) \cap H\,(III_1(i,j,k)) = cone\,(b_i, a_k)$, because $p = \delta e_i + (3-\delta)\,e_j$ defines a separating hyperplane: for cone $H\,(II(i))$, $p \cdot b_i = 0$, $p \cdot a_k = 0$, $p \cdot a_j = 3 - \delta > 0$, and for cone $H\,(III_1(i,j,k))$, $p \cdot c_{ijk} = 9\delta - \delta^2 - 9 < 0$. Similarly, we have that $H\,(II(j)) \cap H\,(III_1(i,j,k)) = cone\,(a_k)$, and $H\,(II(k)) \cap H\,(III_1(i,j,k)) = \{0\}$ (separating hyperplane is $p = e_k$).

The intersection $H\,(II(i)) \cap H\,(III_2\,(k)) = cone\,(a_k)$ and $H\,(II(i)) \cap H\,(III_2\,(i)) = \{0\}$ because $p = e_i$ defines a separating hyperplane.

The intersection $H\,(II(i)) \cap H\,(IV_1\,(i,j,k)) = cone\,(b_i)$, because $p = \delta e_i + (3-\delta)\,e_j$ defines a separating hyperplane: for the cone $H\,(II(i))$ see the previous paragraph, and for the cone $H\,(IV_1\,(i,j,k))$ we have that $p \cdot b_i = 0$, $p \cdot d_{ijk} = 9\delta - \delta^2 - 9 < 0$, $p \cdot c_{jik} = 9\delta - \delta^2 - 9 < 0$. Similarly, we also have that $H\,(II(i)) \cap H\,(IV_1\,(j,i,k)) = H\,(II(i)) \cap H\,(IV_1\,(j,k,i)) = \{0\}$.

The intersection, $H\,(II(i)) \cap H\,(IV_2\,(i)) = cone\,(b_i)$, because $p = \left(3\delta - 2\delta^2\right) e_i + \delta^2 e_j + \left(9 + \delta^2 - 9\delta\right) e_k$ defines a separating hyperplane: for cone $H\,(II(i))$ we have $p \cdot b_i = 0$, $p \cdot a_j = \delta^2 > 0$, and $p \cdot a_k = 9 + \delta^2 - 9\delta > 0$, and for cone $H\,(IV_2\,(i))$ we have $p \cdot d_{ijk} = -3\,(1-\delta)\left(9 + \delta^2 - 9\delta\right) < 0$, and $p \cdot d_{ikj} = -3\,(1-\delta)\,(\delta - 3)^2 < 0$.

The intersection $H\,(II(i)) \cap H\,(IV_3) = \{0\}$ because $p = \left(3\delta - 2\delta^2\right) e_i + \delta^2 e_j + \left(9 + \delta^2 - 9\delta\right) e_k$ defines a separating hyperplane: for cone $H\,(II(i))$ see the previous paragraph, and for cone $H\,(IV_3)$, $p \cdot d_{ijk} = -3\,(1-\delta)\left(9 + \delta^2 - 9\delta\right) < 0$, $p \cdot d_{ikj} = -3\,(1-\delta)\,(3-\delta)^2 < 0$, $p \cdot d_{jki} = -3\,(1-\delta)\left(9 - \delta^2 - 3\delta\right) < 0$, $p \cdot d_{kji} = -3\,(1-\delta)\left(9 - \delta^2 - 6\delta\right) < 0$, $p \cdot d_{jik} = -27\,(1-\delta)^2 < 0$, $p \cdot d_{kij} = -9\,(1-\delta)\,(3-\delta) < 0$.

Finally, the intersection $H\,(II(i)) \cap H\,(IV_4\,(k)) = \{0\}$ because $p = \delta e_i + (3-\delta)\,e_j$ defines a separating hyperplane: for cone $H\,(II(i))$ see the first paragraph of item 2, and for cone $H\,(IV_4\,(k))$, $p \cdot c_{ijk} = 9\delta - \delta^2 - 9 < 0$, $p \cdot c_{jik} = 6\delta + \delta^2 - 9 < 0$, $p \cdot d_{ijk} = 9\delta - \delta^2 - 9 < 0$, and $p \cdot d_{jik} = 6\delta + \delta^2 - 9 < 0$.

3. Now consider the case with $Q = III_1(i,j,k)$.

The intersection $H(III_1(i,j,k)) \cap H(III_2(k)) = \mathit{cone}(c_{ijk}, a_k)$ because $p = -9(1-\delta)e_i + (9-6\delta)e_j$ defines a separating hyperplane: for cone $H(III_1(i,j,k))$, $p \cdot c_{ijk} = 0$, $p \cdot a_k = 0$, and $p \cdot b_i = -27\delta + 27 + 3\delta^2 > 0$, and for cone $H(III_2(k))$, $p \cdot c_{jik} = 3\delta(-6+5\delta) < 0$. Similarly, we also have that $H(III_1(i,j,k)) \cap H(III_2(i)) = H(III_1(i,j,k)) \cap H(III_2(j)) = \{0\}$.

The intersection $H(III_1(i,j,k)) \cap H(IV_1(i,j,k)) = \mathit{cone}(c_{ijk}, b_i)$ because $p = \left(3\delta - 2\delta^2\right)e_i + \delta^2 e_j + \left(9 + \delta^2 - 9\delta\right)e_k$ defines a separating hyperplane: for cone $H(III_1(i,j,k))$, $p \cdot b_i = 0$, $p \cdot c_{ijk} = 0$, $p \cdot a_k = 9 + \delta^2 - 9\delta > 0$ and for cone $H(IV_1(i,j,k))$, $p \cdot d_{ijk} = -3(1-\delta)\left(9+\delta^2-9\delta\right) < 0$. Similarly, we also have that $H(III_1(i,j,k)) \cap H(IV_1(i,k,j)) = \mathit{cone}(b_i)$ and $H(III_1(i,j,k)) \cap H(IV_1(j,i,k)) = H(III_1(i,j,k)) \cap H(IV_1(j,k,i)) = \{0\}$.

The intersection $H(III_1(i,j,k)) \cap H(IV_2(i)) = \mathit{cone}(b_i)$ because $p = \left(3\delta - 2\delta^2\right)e_i + \delta^2 e_j + \left(9 + \delta^2 - 9\delta\right)e_k$ defines a separating hyperplane: see items 2 and 3 above. Similarly, $H(III_1(i,j,k)) \cap H(IV_2(j)) = H(III_1(i,j,k)) \cap H(IV_2(k)) = \{0\}$. We also have that $H(III_1(i,j,k)) \cap H(IV_3) = \{0\}$, because $p = \left(3\delta - 2\delta^2\right)e_i + \delta^2 e_j + \left(9 + \delta^2 - 9\delta\right)e_k$ also defines a separating hyperplane: see items 2 and 3 above.

The intersection $H(III_1(i,j,k)) \cap H(IV_4(k)) = \mathit{cone}(c_{ijk})$ because $p = 3(1-\delta)e_i + (2\delta - 3)e_j$ defines a separating hyperplane: for cone $H(III_1(i,j,k))$, $p \cdot c_{ijk} = 0$, $p \cdot a_k = 0$, and $p \cdot b_i = 9\delta - \delta^2 - 9 < 0$, and for cone $H(IV_4(k))$, $p \cdot c_{jik} = \delta(6-5\delta) > 0$, $p \cdot d_{ijk} = 0$, and $p \cdot d_{jik} = \delta(6-5\delta) > 0$. Similarly, we have that $H(III_1(i,j,k)) \cap IV_4(j) = H(III_1(i,j,k)) \cap H(IV_4(k)) = \{0\}$.

4. Now consider the case with $Q = III_2(k)$.

The intersection $H(III_2(k)) \cap H(IV_1(i,j,k)) = \mathit{cone}(c_{ijk})$, because $p = \left(3\delta - 2\delta^2\right)e_i + \delta^2 e_j + \left(9 + \delta^2 - 9\delta\right)e_k$ defines a separating hyperplane: for cone $H(III_2(k))$, $p \cdot c_{ijk} = 0$, $p \cdot c_{jik} = 3\delta^2(1-\delta) > 0$, $p \cdot b_k = 9 + \delta^2 - 9\delta > 0$, and for cone $H(IV_1(i,j,k))$, see item 3 above. Similarly, $H(III_2(j)) \cap H(IV_1(i,j,k)) = H(III_2(i)) \cap H(IV_1(i,j,k)) = \{0\}$.

The intersection $H(III_2(k)) \cap H(IV_2(i)) = \{0\}$ and $H(III_2(k)) \cap H(IV_3) = \{0\}$ because $p = \left(3\delta - 2\delta^2\right)e_i + \delta^2 e_j + \left(9 + \delta^2 - 9\delta\right)e_k$ defines a separating hyperplane: see previous the paragraph for cone $H(III_2(k))$ and item 3 for cones $H(IV_2(i))$ and $H(IV_3)$.

The intersection $H(III_2(k)) \cap H(IV_4(k)) = \mathit{cone}(c_{ijk}, c_{jik})$ because $p = \delta e_i + \delta e_j + (6 - 5\delta)e_k$ defines a separating hyperplane: for cone $H(III_2(k))$, $p \cdot c_{ijk} = p \cdot c_{jik} = 0$, $p \cdot b_k = 6 - 5\delta > 0$, and for cone $H(IV_4(k))$, $p \cdot d_{ijk} = p \cdot d_{jik} = -3(1-\delta)(6-5\delta) < 0$.

5. Now consider the case with $Q = IV_1(i,j,k)$.

The intersection $H(IV_1(i,j,k)) \cap H(IV_2(i)) = \mathit{cone}(b_i, d_{ijk})$ because $p = \delta^2 e_i + \left(12\delta - 2\delta^2 - 9\right)e_j + \left(9 + \delta^2 - 9\delta\right)e_k$ defines a separating hyperplane: for cone $H(IV_1(i,j,k))$, $p \cdot d_{ijk} = 0$, $p \cdot c_{ijk} = 3(1-\delta)\left(9+\delta^2-9\delta\right) > 0$, $p \cdot b_i = 0$, and for cone $H(IV_2(i))$, $p \cdot d_{ikj} = -3\delta(1-\delta)(6-\delta) < 0$.

The intersection $H(IV_1(i,j,k)) \cap H(IV_3) = \mathit{cone}(d_{ijk})$ because $p = (4\delta - 3)e_i + (4\delta - 3)e_j + (6 - 5\delta)e_k$ defines a separating hyperplane: for cone $H(IV_1(i,j,k))$, $p \cdot d_{ijk} = 0$, $p \cdot c_{ijk} = 3(\delta - 1)(5\delta - 6) > 0$, $p \cdot b_i = 3(\delta-1)(\delta-3) > 0$, and for cone $H(IV_3)$, $p \cdot d_{ikj} = 9\delta(\delta - 1) < 0$, $p \cdot d_{kji} = 18\delta(\delta-1) < 0$, $p \cdot d_{jki} = 9\delta(\delta-1) < 0$, $p \cdot d_{jik} = 0$, $p \cdot d_{kij} = 18\delta(\delta-1) < 0$.

The intersection $H(IV_1(i,j,k)) \cap H(IV_4(k)) = \mathit{cone}(c_{ijk}, d_{ijk})$ because $p = 3(1-\delta)e_i + (2\delta - 3)e_j$ defines a separating hyperplane: for cone $H(IV_1(i,j,k))$, $p \cdot d_{ijk} = 0$, $p \cdot c_{ijk} = 0$, $p \cdot b_i = 9\delta - 9 - \delta^2 < 0$, and for cone $H(IV_4(k))$, $p \cdot c_{jik} = p \cdot d_{jik} = \delta(6-5\delta) > 0$. Similarly, $H(IV_1(i,j,k)) \cap H(IV_4(j)) = H(IV_1(i,j,k)) \cap H(IV_4(i)) = \{0\}$.

6. Now consider the case with $Q = IV_2(i)$.

The intersection $H(IV_2(i)) \cap H(IV_3) = cone(d_{ijk}, d_{ikj})$, because $p = (7\delta - 6)e_i + (3 - 2\delta)e_j + (3 - 2\delta)e_k$ defines a separating hyperplane: for cone $H(IV_2(i))$, $p \cdot d_{ijk} = 0$, $p \cdot d_{ikj} = 0$, and $p \cdot b_i = 3(\delta - 1)(\delta - 6) > 0$, and for cone $H(IV_3)$, $p \cdot d_{kji} = 18\delta(\delta - 1) < 0$, $p \cdot \omega_{IV(j,ki)} = 18\delta(\delta - 1) < 0$, $p \cdot d_{jik} = 9\delta(\delta - 1) < 0$, $p \cdot d_{kij} = 9\delta(\delta - 1) < 0$.

The intersection $H(IV_2(i)) \cap H(IV_4(k)) = cone(d_{ijk})$, because $p = \delta^2 e_i + \left(12\delta - 2\delta^2 - 9\right)e_j + \left(9 + \delta^2 - 9\delta\right)e_k$ defines a separating hyperplane: for cone $H(IV_2(i))$ see item 5 above, and for cone $H(IV_4(k))$, $p \cdot c_{ijk} = -3(\delta - 1)\left(9 + \delta^2 - 9\delta\right) > 0$, $p \cdot c_{jik} = 9(\delta - 1)(-3 + 2\delta) > 0$, $p \cdot d_{ijk} = 0$, $p \cdot d_{jik} = 3\delta(\delta - 1)(\delta - 3) > 0$. Similarly, $H(IV_2(i)) \cap H(IV_4(i)) = \{0\}$.

7. Finally, consider the case with $Q = IV_3$.

The intersection $H(IV_3) \cap H(IV_4(k)) = cone(d_{ijk}, d_{jik})$, because $p = (4\delta - 3)e_i + (4\delta - 3)e_j + (6 - 5\delta)e_k$ defines a separating hyperplane: for cone $H(IV_3)$ see item 5 above, and for cone $H(IV_4(k))$, $p \cdot c_{ijk} = 3(\delta - 1)(5\delta - 6) > 0$, $p \cdot c_{jik} = 3(\delta - 1)(5\delta - 6) > 0$, $p \cdot d_{ijk} = 0$, $p \cdot d_{jik} = 0$.

We have then proved for all possible pairs Q and Q' in $\mathbb{Q}$ with $Q' \neq Q$ that $H(Q) \cap H(Q') = cone(Q \cap Q')$. Q.E.D.

Proof of Theorem 5: From the discussion following the theorem the only non-obvious point remaining to be proved is that case $III(k)$, which is associated with the polyhedral cone

$$H(III(k)) = \left\{\omega \in R^3 : \sum \omega \geq 0,\ \omega_i + 2\omega_j \leq 0,\ 2\omega_i + \omega_j \leq 0\right\},$$

satisfies $H(III(k)) = H(III_1(i,j,k)) \cup H(III_1(j,i,k)) \cup H(III_2(k))$, where

$$\begin{aligned}
H(III_1(i,j,k)) &= \left\{\omega \in R^3 : \sum \omega \geq 0,\ \omega_i + 2\omega_j \leq 0,\ \omega_j \geq 0\right\}, \\
H(III_1(j,i,k)) &= \left\{\omega \in R^3 : \sum \omega \geq 0,\ 2\omega_i + \omega_j \leq 0,\ \omega_i \geq 0\right\}, \\
H(III_2(k)) &= \left\{\omega \in R^3 : \sum \omega \geq 0,\ \omega_i \leq 0,\ \omega_j \leq 0\right\}.
\end{aligned}$$

We first show that $H(III_1(i,j,k)) \cup H(III_1(j,i,k)) \cup H(III_2(k)) \subset H(III(k))$. Suppose that $\omega \in H(III_1(i,j,k)) \cup H(III_1(j,i,k)) \cup H(III_2(k))$. If $\omega \in H(III_1(i,j,k))$ then $2\omega_i + 4\omega_j \leq 0$ and $-3\omega_j \leq 0$, which imply $2\omega_i + \omega_j \leq 0$ and thus $\omega \in H(III_2(k))$ (a similar argument holds for $III_1(j,i,k)$). Obviously, if $\omega \in H(III_2(k))$ then $\omega \in H(III(k))$. Now we show that $H(III(k)) \subset H(III_1(i,j,k)) \cup H(III_1(j,i,k)) \cup H(III_2(k))$. Suppose that $\omega \in H(III(k))$. Then we have either $\omega_j \leq 0$ or $\omega_j \geq 0$, and either $\omega_k \leq 0$ or $\omega_k \geq 0$. If either $\omega_j \geq 0$ or $\omega_k \geq 0$ holds then either ω belongs either to case $III_1(i,j,k)$ or to case $III_1(j,i,k)$. Otherwise, we must have both $\omega_j \leq 0$ and $\omega_k \leq 0$, which then imply that ω belongs to case $III_2(k)$. Q.E.D.

Proof of Proposition 7: Consider the normalized game (u,δ). We have that

$$\omega_i(u,\delta) = U_i + \frac{\delta}{2}(U - U_i - U_{jk}) - \frac{\delta}{3}U \text{ and } r_i(u,\delta) = \frac{1}{3}U,$$

and thus $\omega_i + \omega_j + \omega_k = U_i + \frac{\delta}{2}(U - U_i - U_{jk}) - \frac{\delta}{3}U + U_j + \frac{\delta}{2}(U - U_j - U_{ik}) - \frac{\delta}{3}U + U_k + \frac{\delta}{2}(U - U_k - U_{ij}) - \frac{\delta}{3}U$.

We then have that $\phi_i(u,\delta) = r_i + \frac{\omega_i}{\delta} - \frac{1}{3}\frac{\omega_i + \omega_j + \omega_k}{\delta}$ is equal to

$$\phi_i(u,\delta) = \frac{1}{6}\left((2U + U_{ik} - 2U_{jk} + U_{ij}) + \frac{1}{\delta}((4 - 2\delta)U_i - (2 - \delta)U_j - (2 - \delta)U_k)\right).$$

But the Shapley value of i is $Sh_i(u) = \frac{1}{6}(2(U - U_{jk}) + 2U_i + (U_{ij} - U_j) + (U_{ik} - U_k))$, and we have that $\phi_i(u,\delta) - Sh_i(u) = \frac{(1-\delta)}{3\delta}(2U_i - U_j - U_k)$.

Now for a general game (not normalized) we have that $\phi_i(v,\delta) = \overline{\phi}_i(u,\delta) + v_i$ and $Sh_i(v) = v_i + Sh_i(u)$ because

$$Sh_i(v) = \frac{1}{6}\begin{pmatrix} 2(V - v_i - v_j - v_k - (V_{jk} - v_j - v_k)) \\ +2(V_i - v_i) + ((V_{ij} - v_i - v_j) - (V_j - v_j)) \\ +((V_{ik} - v_i - v_k) - (V_k - v_k)) \end{pmatrix},$$

and then we have that

$$\phi_i(v,\delta) - Sh_i(v) = \phi_i(u,\delta) - Sh_i(u) = \frac{(1-\delta)}{3\delta}(2(V_i - v_i) - (V_j - v_j) - (V_k - v_k)).$$

Note that in the limit as $\delta \to 1$ we have that $\phi_i(v,\delta) \to Sh_i(v)$. Note also that if there are no externalities, $V_i = v_i$ for all $i \in \{1,2,3\}$, and then we have that $\phi_i(v,\delta) = Sh_i(v)$ as the formula above shows. Q.E.D.

Proof of Proposition 8: First note that $r_i = v_i + \frac{1}{3}(V - v_1 - v_2 - v_3)$. We then have that $r_i = V - r_j - r_k$ for all i. Also note that $X_i = V - X_{jk}$. This implies $r_i - X_i = V - r_j - r_k - X_i = X_{jk} - r_j - r_k$.

From Theorem 5 we have that $II(i)$ is equivalent to $X_i - r_i \leq 0$, $(X_i - r_i) + 2(X_j - r_j) \geq 0 \Leftrightarrow (X_j - r_j) \leq \frac{1}{2}(X_{jk} - r_j - r_k)$, $(X_i - r_i) + 2(X_k - r_k) \geq 0 \Leftrightarrow (X_k - r_k) \leq \frac{1}{2}(X_{jk} - r_j - r_k)$.

Also, we have that $III(k)$ is equivalent to $X_1 + X_2 + X_3 \geq V \Leftrightarrow X_i + X_j \geq X_{ij}$, $(X_i - r_i) + 2(X_j - r_j) \leq 0 \Leftrightarrow (X_j - r_j) \leq \frac{1}{2}(X_{jk} - r_j - r_k)$, and $2(X_i - r_i) + (X_j - r_j) \leq 0 \Leftrightarrow (X_i - r_i) \leq \frac{1}{2}(X_{ik} - r_i - r_k)$. Q.E.D.

Proof of Proposition 9: Consider the negotiation game $(\overline{v},\delta)$. The game $\overline{v}$ is in characteristic function because $\overline{V}_i = \overline{v}_i$. By Theorem 4 if we show that $r(\overline{v},\delta) = r(v,\delta)$ and $\omega(v,\delta) = \omega(\overline{v},\delta)$ then the equilibrium outcome of both games (v,δ) and $(\overline{v},\delta)$ must be the same.

By definition we have that

$$\overline{r}_i = \overline{v}_i + \frac{1}{3}(V - \overline{v}_1 - \overline{v}_2 - \overline{v}_3) \text{ and } \overline{\omega}_i = \overline{v}_i + \frac{\delta}{2}(V - \overline{v}_i - \overline{v}_{jk}) - \left(\overline{v}_i + \frac{\delta}{3}(V - \overline{v}_1 - \overline{v}_2 - \overline{v}_3)\right),$$

It is thus obvious that $r(\overline{v},\delta) = r(v,\delta)$. Substituting the expression for the game $\overline{v}$ we have that

$$\overline{\omega}_i = v_i + \frac{\delta}{2}\left(V - v_i - \frac{(2-\delta)v_k - (2-\delta)V_k + \delta V_{ij}}{\delta}\right) - \left(v_i + \frac{\delta}{3}(V - v_1 - v_2 - v_3)\right),$$

and after simplifications we have that

$$\overline{\omega}_i = V_i + \frac{\delta}{2}(V - V_i - V_{jk}) - \left(v_i + \frac{\delta}{3}(V - v_1 - v_2 - v_3)\right) = \omega_i.$$

Taking the limit as $\delta \to 1$ of $\overline{v}$ proves the second part of the proposition. Q.E.D.

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