CARESS Working Paper #97-15

"Cooperation and Computability in N-Player Games"

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Cooperation and Computability in N-Player Games *

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November 1997

ABSTRACT. A Common Interest game is a game in which there exists a unique vector of payoffs which strictly Pareto-dominates all other payoffs. We consider the undiscounted repeated game obtained by the infinite repetition of such an n-player Common Interest game. We restrict supergame strategies to be computable within Church's thesis, and we introduce computable trembles on these strategies. If the trembles have sufficiently large support, the only equilibrium vector of payoffs which survives is the Pareto-efficient one.

The result is driven by the ability of the players to use the early stages of the game to communicate their intention to play cooperatively in the future. The players 'take turns' to reveal their cooperative intentions, and the result is proved by 'backwards induction' on the set of players.

We also show that our equilibrium selection result fails when there are a countable infinity of players.

JEL CLASSIFICATION: C72, C73, C79.

KEYWORDS: Cooperation, Computability, Repeated Common Interest Games, Perturbations.

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* We are grateful to Nabil Al-Najjar and to seminar participants at the University of Pennsylvania and Lund University for helpful discussions and comments. Of course, any remaining errors are our own responsibility.

1. INTRODUCTION

1.1. *Motivation*

The 'Folk Theorem' of repeated games states that, if players are sufficiently patient, any vector of long-run payoffs which is individually rational can be sustained as an equilibrium.

This is true regardless of the structure of the stage game which is being repeated. In other words, any vector of pay-offs can be sustained in equilibrium regardless of how more or less appealing such outcome might be in terms of the stage game. In particular, the stage game might possess a 'cooperative equilibrium' which constitutes a natural focus for the players. Nevertheless, cooperation will only be one of the very many possible long-run equilibrium outcomes.

The Folk Theorem of repeated games is an extremely robust result. In general, it holds when we impose subgame-perfection, when incomplete information is allowed (Fudenberg and Maskin 1986), when we consider a finite but 'long' repeated game (Benoit and Krishna 1985), or when we put bounds on the memory of the players (Sabourian 1989).

The literature on repeated games is vast and we do not even attempt a survey here. We simply mention the surveys of Aumann (1981) and Sabourian (1989), and the pioneering contributions of Aumann and Shapley (1976) and Rubinstein (1979).

In this paper, we are concerned with undiscounted infinitely-repeated games obtained from the repetition of a stage game which belongs to a class of strategic-form n-player games known as Common Interest games. These are those games in which a unique vector of payoffs strictly Pareto-dominates all other feasible payoff vectors.

We show that, when the repeated game is 'perturbed' in an appropriate way and certain computability conditions hold, cooperation is the only possible equilibrium outcome of such a repeated Common Interest game. Our results, generalize the results in Anderlini and Sabourian (1995) (henceforth A-S) for 2-players repeated Common Interest games to n-player games when n is finite. We also show that our equilibrium selection result no longer holds when the number of players is countably infinite.

1.2. Intuition

Consider the following 3-player Common Interest game in matrix form in which player one chooses rows, player two chooses columns, and player three chooses matrices.

A_3	A_2	B_2
A_1	$3,3,3$	$-1,2,0$
B_1	$2,-1,0$	$0,0,0$

B_3	A_2	B_2
A_1	$1,1,1$	$-1,-1,-1$
B_1	$-1,-1,-1$	$0,0,0$

Figure 1

Consider next the game obtained by the infinite undiscounted repetition of this stage game. The Folk Theorem tells us that in this repeated game any vector of strictly positive payoffs in the convex hull of the feasible payoff space of the stage game can be supported by a subgame-perfect equilibrium.

However, intuitively inefficient outcomes come from a lack of coordination. Since the three players interact for a long time, it should be possible for each player to take actions early in the repeated game to signal and convince the others that he will 'play cooperatively'[1] in the future. Therefore, the efficient payoff vector $(3,3,3)$ should emerge if players do not discount the future.

The results of this paper formalize the intuition we have just given. More precisely, we show the following. Suppose that the pure strategies of the repeated game are perturbed[2] in a 'computable' way. Suppose also that players choose repeated game strategies which can be implemented by finite programs or computing devices (Turing

[1]Here, and throughout the paper, we will refer to a choice of action which (for some actions of the other players) leads to the efficient payoff vector as 'playing cooperatively'.

[2]As in many other settings, the perturbations can be interpreted in two different ways. They can represent actual 'mistakes' made by the players when they select their repeated game strategy. Or they can represent the fact that each player is uncertain about the 'identity' (the motivations of) the other players.

machines). Then the set of equilibrium payoffs shrinks to the efficient payoff vector as the perturbations vanish.

In any Common Interest game, if a player is 'sufficiently convinced' that all other players will play cooperatively, then it is in the player's interest to reply by playing cooperatively as well. Therefore in a 2-player Common Interest games the selection of the Pareto-efficient outcome becomes intuitively clear once it is shown that one player can convince the other of his cooperative intentions. In an n-player Common Interest game more is needed. For instance, the signalling mechanism we have described intuitively will select the efficient outcome if it is the case that any player i, by signalling his cooperative intentions, can induce (motivate) another player $i+1$ to reveal his cooperative intentions.

More formally, to select the efficient equilibria for the n-player case, we need to generalize the signalling mechanism, by 'backwards induction' on the set of all players as follows. Suppose that a history of play, say $h_{t_{n-1}}$, has taken place such that all players but one, say player n, are (almost) sure to play cooperatively after $h_{t_{n-1}}$. Then, since the game is one of Common Interest, it pays the n-th player to play cooperatively after $h_{t_{n-1}}$ and as a result all players can earn (approximately) the efficient payoff after this point. Now suppose that a history $h_{t_{n-2}}$ has taken place such that all players other than player $n-1$ and player n are (almost) sure to play cooperatively after $h_{t_{n-2}}$. Then player $n-1$, by signalling its cooperative intentions, can ensure that at some point in the future, a history like $h_{t_{n-1}}$ will have taken place. Since the players earn (approximately) the efficient payoff after $h_{t_{n-1}}$, it follows that after history $h_{t_{n-2}}$, player $n-1$, by signalling, can obtain (approximately) the efficient payoff and thus the equilibrium continuation payoff must also be (approximately) efficient for all players. This argument can be continued by 'backwards induction' to show that for all $i < n$ the continuation payoffs are (approximately) efficient after any history h_{t_i} such that all players $j = 1, .., i$ are certain to play cooperatively in the future. Thus the possibility of signalling ensures that the efficient equilibrium is selected in the overall repeated game.

As in A-S, the two key features of our analysis in this paper are the role of the perturbations of the repeated game, and the of restriction of the players to computable strategies. They enable us to show that a player can use the early stages of play to convince the others that he intends to play cooperatively in the long-run. We start with an intuitive explanation of the need to introduce perturbations of the repeated game.

Unless perturbations of the players supergame strategies are introduced, it is possible that each player will attach probability one to strategies which result in inefficient outcomes. If this is the case any strategy which attempts to signal an intention to play cooperatively will take the players 'off the equilibrium path' (the histories h_{t_i} defined above may be 'off the equilibrium path'). Once the history of play is off the equilibrium path, the players' beliefs can only be defined in an *ad hoc* way. Introducing perturbations of the repeated game as we do below guarantees that the the set of strategies which will eventually play the cooperative outcome has positive probability. This, in turn, makes the players' beliefs well defined according to Bayes' rule after any attempt to signal that a player intends to play cooperatively in the future.

The restriction to computable strategies plays a three-fold role in our analysis. First of all, it implies that each player's strategy space in the infinitely repeated game is a *countably infinite* set. The set of finite programs can be numbered (cf. Section 3 below), and from this fact it follows directly that there is a countable infinity of supergame strategies which can be played by any algorithm (finite program).

The second implication of computable strategies is a consequence of the fact that general programs can 'simulate' other programs. It follows that computable strategies can make their own action depend on the action which any other computable strategy would take in the same situation. A computable strategy which 'simulates' other strategies as part of its program is a well defined computable strategy.

The third consequence of our assumption of computable strategies is that we can invoke a particular 'pseudo-fixed point' result in the space of computable functions. We expand on this point below.

To illustrate the role of the computability restrictions within the signalling mechanism described above, consider a 3-player repeated Common Interest game (such as the one in Figure 1) and let (x_1^E, x_2^E, x_3^E) be an equilibrium strategy profile. Consider first for each player $i = 1, 2$ the set of computable non-cooperative supergame strategies $\overline{\mathcal{Q}}_i$: the set of strategies which fail to cooperate in the long-run. This is a countable set since the entire set of computable strategies is countable. For each i it is useful to visualize this set as being on the horizontal axis as in Figure 2 below, together with the probabilities (on the vertical axis) assigned to strategies in this set by the perturbation of the player's equilibrium strategy.

Now for each $i = 1, 2$ consider a strategy, say x_i^*, constructed in the following way. Firstly, x_i^* 'enumerates' sufficiently many non-cooperative strategies in $\overline{\mathcal{Q}}_i$, so that the 'tail' left after such enumeration has 'sufficiently small' probability compared with the probability which the perturbation assigns to x_i^* itself. Because the set of non-cooperative strategies is countable, this can be achieved enumerating *finitely many* non-cooperative strategies, say $\tilde{t}_i$ of them (in Figure 2 we have set $\tilde{t}_i = 9$).

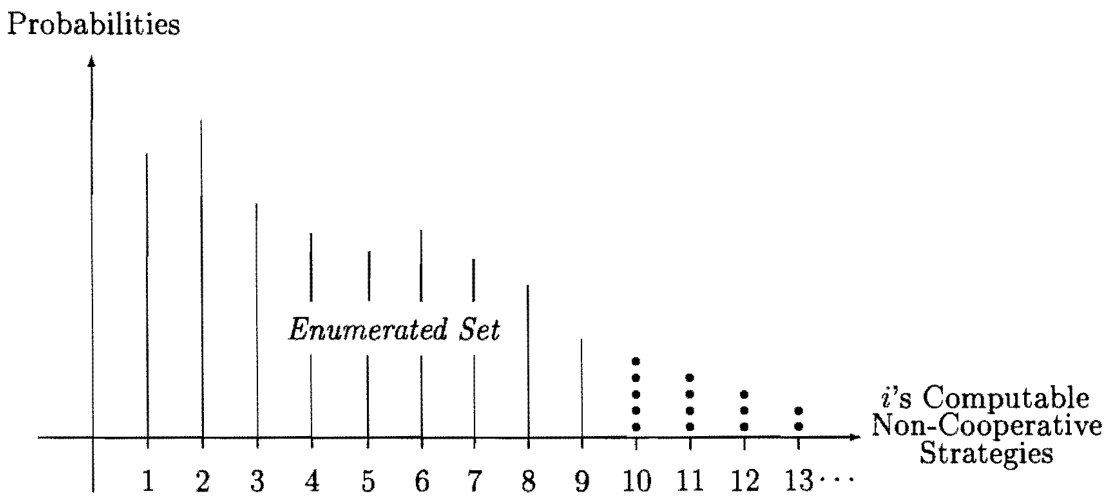

Figure 2

Strategy x_i^* can then use the first $\tilde{t}_i$ stages of play after some some fixed period v_i to signal its intention to play cooperatively in the long-run as follows. In the first period of the signalling phase, at $v_i + 1$, simulate what action the non-cooperative strategy '1' would take in period $v_i + 1$, and then make sure that the action taken by x_i^* is *different* from the action of non-cooperative strategy '1'. In the second period of the signalling phase, at $v_i + 2$, do the same operation with non-cooperative strategy '2', in the third period of the signalling phase with strategy '3', and so on until period $\tilde{t}_i + v_i$ included. After period $t_i = \tilde{t}_i + v_i$ simply take the cooperative action, regardless of the previous history of play.

Playing strategy x_i^* will clearly ensure that by period $t_i = \tilde{t}_i + v_i$ the opposing players will know that i is *not* using any of the non-cooperative strategies $1, 2, \cdots, \tilde{t}_i$ in the 'enumerated set' of Figure 2. Therefore, they must conclude that if he is playing a non-cooperative strategy, he must be using one of the strategies in the 'tail' of Figure 2. Since this 'tail' has 'sufficiently small' probability relative to the cooperative strategy x_i^*, using Bayes' rule the opposing players must know, by time $t_i = \tilde{t}_i + v_i$, that they are facing a cooperative strategy with 'sufficiently high' probability. Using strategy x_i^* reveals, by time $t_i = \tilde{t}_i + v_i$, the player's intention to cooperate in the long-run, up to a 'sufficiently high' degree of precision. Throughout the rest of the paper, we will refer to a strategy like x_i^* as a 'revealing' or a 'signaling' strategy for i.

Now construct x_1^* and x_2^* so that x_1^* signals in the first $\tilde{t}_1$ periods and x_2^*'s signalling phase starts after x_1^* has finished its signalling. Moreover, ensure that x_2^* behaves like the equilibrium machine x_2^E in the first $\tilde{t}_1$ periods (during the signalling phase of x_1^*). In other words suppose that $v_1 = 0$, $t_1 = \tilde{t}_1$, $v_2 = \tilde{t}_1$ and the output of x_2^* is the same as the output of x_2^E at any stage before and including $v_2 = \tilde{t}_1$. (Notice that we are setting $t_2 = v_2 + \tilde{t}_1 = \tilde{t}_1 + \tilde{t}_2$.)

Next, assume that x_i^* is in the support of the perturbation for $i = 1, 2$. Denote the histories of play after t_1 and t_2 periods, if players 1 and 2 choose x_1^* and x_2^*, and player 3 chooses the equilibrium strategy x_3^E, by $h_{t_1}^*$ and $h_{t_2}^*$ respectively.[3] Then if $h_{t_2}^*$ occurs (the signalling phase of x_2^* ends), player 3 is almost sure that players 1 and 2 are cooperative and thus x_3^E will play cooperatively after this history.[4] Now

[3]Notice that all histories of length t_1 generated by x_2^* and x_2^E are identical because x_2^*'s behaviour is assumed to be identical to that of x_2^E for the first t_1 periods of the game.

[4]This follows from the fact that equilibrium strategies must be optimal in expected terms.

if $h^*_{t_1}$ is observed (the signalling phase of x^*_1 has ended) then player 2 by choosing x^*_2 can induce a history $h^*_{t_2}$ and thus earn the efficient payoff after period t_2. Since the equilibrium strategy x^E_2 for player 2 is optimal on the 'equilibrium path' and x^E_2 and x^*_2 behave identically up to and including period t_1, it follows that after player 1's signalling phase (after history $h^*_{t_1}$), the payoff to x^E_2 (and thus the payoffs to all players) must be close to the efficient one(s). Finally, since player 1, by choosing x^*_1, can induce $h^*_{t_1}$ and thus the efficient payoff when other players follow their equilibrium strategies[5] x^E_2 and x^E_3, it follows from optimality of x^E_1 that the equilibrium payoff of 1 (and thus those of all players) in the entire game must be close to the efficient one.

It is at this point that the pseudo-fixed point theorem we mentioned above comes into play. The construction yielding our revealing strategies clearly is open to a potential circularity. As we construct each x^*_i, we take as *given* the probability which x^*_i has according to the perturbation. But since we are constructing x^*_i, its 'number' and therefore its probability may vary. We avoid this potential circularity using the pseudo-fixed point which we mentioned above. (This is a Corollary of the Recursion Theorem, which for completeness is stated as Theorem A.6). This makes our signalling strategies x^*_i well defined.

Our equilibrium selection result revolves on a backwards induction argument on the set of players. The logic of this argument breaks down in the case of a countable infinity of players. In Section 9 we show that our equilibrium selection result no longer holds in this case.

1.3. Related Literature

There is a sizeable literature on repeated games played by computing machines. We do not attempt a survey here, but simply refer to the early contributions of Abreu and Rubinstein (1988), Rubinstein (1986), Neyman (1985) and Aumann (1981). The class of computing devices most often considered in this literature is the set of finite automata (Moore machines). The focus of most of the finite automata literature

[5]This follows from the fact that the history of play generated for the first t_1 periods by the strategy triple (x^*_1, x^E_2, x^E_3) is identical to the one generated by (x^*_1, x^*_2, x^E_3) — see footnote 3.

are the Nash equilibria of the machine game — a profile of machines (one for each player) such that no player can improve his payoff by unilaterally deviating to a different machine.

We depart from the automata literature in several ways. First of all the class of computing devices which we consider (Turing machines) is in some sense 'wider' than the class of finite automata. Binmore (1987), in a pioneering paper, adopts the stance that Turing machines are the 'correct' class of computing devices to consider since they embody what is widely accepted in mathematics as the appropriate notion of effective computability in the widest possible sense. They represent a 'most powerful' class of computing devices. For reasons of space, we refer the reader to A-S for further references and discussion of this point.

As we mentioned above, in A-S we consider 2-player repeated Common Interest games in which strategies are restricted to be computable. The game is perturbed in a way similar to that used below, and the efficient payoff pair is selected as the unique surviving equilibrium outcome. Much of the intuition described in Section 1.2 is true for the results of A-S as well as for the results of this paper. This is not surprising since both papers can be viewed as proposing a *technique* which appears to be quite powerful in modelling signalling/communication among players in strategic situations in which there is a 'common interest' among the players.[6],[7]

There is a large recent literature on equilibrium selection in evolutionary/learning games,[8] in games played by machines or with restricted strategies,[9] and in games

[6] Anderlini (1990) applies a similar technique to select the efficient equilibrium in 2-player one-shot Common Interest games with pre-play communication.

[7] The proofs of the results which we present in this paper have the same structure as in A-S. However, they are substantially more complex since in this paper we need to ensure the possibility of $n-1$ players signalling (sequentially) on the equilibrium path, whereas in A-S the possibility of one player signalling is sufficient to select the efficient equilibrium. In particular, in this paper, the Communication Lemma in section 6.1 and the optimality proof in section 6.2 below are a great deal more intricate than their counterparts in A-S.

[8] See for instance Fudenberg and Maskin (1990), Binmore and Samuelson (1992), Kandori, Mailath, and Rob (1993), Young (1993), and the recent surveys by Kandori (1997), and Marimon (1997).

[9] See Cho (1994), Piccione and Rubinstein (1993), Aumann and Sorin (1989), Abreu and Rubinstein (1988) and Rubinstein (1986), to name a few.

with pre-play communication.[10,11] Almost all the results available in the literature are valid only for 2-player games (and sometimes only for 2×2 games) and they do not extend to n-player games[12]. By contrast, this paper demonstrates that the *techniques* used in A-S do generalize to n-player games by applying a 'backwards induction' type argument on the set of players.

1.4. Overview

In the next section we set up the standard notation for an infinitely repeated n-player Common Interest game with no discounting. In Section 3 we briefly introduce the notion of computability and the associated notation. In Section 4 we describe the model in detail and introduce our equilibrium concept. Section 5 contains the main result of the paper. In Sections 6 and 7 we present the proof of the main result and discuss one generalization of it. In Section 8 we discuss what happens to our main result when correlation across player types is allowed. Section 9 generalizes our previous model to the case of a Common Interest game with a countable infinity of players. In this case the equilibrium selection result we have described above fails. Section 10 briefly concludes the paper. All proofs are in the Appendix except for the Communication Lemma, a result which is central to the analysis of the entire paper. The Appendix also contains some additional material which we have removed from the main body of the paper for ease of exposition. In the numbering of equations, Theorems etc., a prefix 'A' indicates that the relevant item is in the Appendix.

2. UN-DISCOUNTED INFINITELY REPEATED COMMON INTEREST GAMES

The stage game of the repeated game we consider will be denoted by $G = \{\mathcal{A}_i, \pi_i\}_{i=1}^n$. We take G to be a finite-action, n-player, strategic-form game. A generic player will be denoted by $i = 1, \ldots, n$. Player i's finite action set is denoted by $\mathcal{A}_i$, and

[10]See Farrell (1988), Farrell (1993), Kim and Sobel (1995), among others.

[11]See A-S for further references on equilibrium selection models and on the differences the analysis carried out here and the existing literature.

[12]There are some exceptions. For example, Chatterjee and Sabourian (1997) analyze the equilibrium set in n-person bargaining games played by finite automata.

$\mathcal{A} \equiv \mathcal{A}_1 \times, \ldots, \times \mathcal{A}_n$ is the players' joint action set. Typical elements of $\mathcal{A}_i$ and $\mathcal{A}$ are denoted by a_i and a respectively. Following standard notation, $\pi_i : \mathcal{A} \to \mathbb{R}$ denotes player i's payoff function, while $\pi : \mathcal{A} \to \mathbb{R}^n$ yields a payoff vector given an action profile $a \in \mathcal{A}$. Let V, with typical element $\pi = (\pi_1, \ldots, \pi_n)$, be the payoff space of G. In other words, $V \equiv \pi(\mathcal{A})$.

We can now define the class of Common Interest games.

DEFINITION 1: *A strategic-form game G is said to be of Common Interest if and only if it has a unique vector of feasible payoffs (which may be associated with more than one action profile) which strictly Pareto-dominates all other payoff vectors. G is assumed to be a Common Interest game and π^e is its unique Pareto-efficient payoff vector. The action profile $a^e \in \mathcal{A}$ is one (arbitrarily fixed) action profile which yields such a payoff vector to the players.*

For the sake of simplicity only we will focus attention on Common Interest games in which each player has at least three pure strategies available. In Section 7 we indicate why this property is not needed for our results.

ASSUMPTION 1: *For all $i = 1, \ldots, n$, the cardinality of $\mathcal{A}_i$ is at least three.*

Next, we define the infinitely repeated game, G^∞, obtained from G. Let a_{it} be player i's action at time $t = 0, \cdots, \infty$, and a_t the players' joint action at t. Let $\mathcal{H}_t$ be the set of all possible *finite* histories of play of length t, with typical element $h_t = (a_0, \cdots, a_{t-1})$ (define h_0 to be the empty set, denoted by $\emptyset$). The set of all possible finite histories of play, regardless of length, is denoted by $\mathcal{H} \equiv \bigcup_{t=0}^{\infty} \mathcal{H}_t$. A strategy for player i in G^∞ is a map $\sigma_i : \mathcal{H} \to \mathcal{A}_i$. The action profile which players take at time t along the outcome path induced by σ will be indicated by $\mathrm{a}_t(\sigma) = (\mathrm{a}_{1t}(\sigma), \ldots, \mathrm{a}_{nt}(\sigma))$. The history of length t generated by a vector of supergame strategies σ is denoted by $\mathrm{h}_t(\sigma) \equiv (\mathrm{a}_0(\sigma), \cdots, \mathrm{a}_{t-1}(\sigma))$. The long-run undiscounted payoff to player i is $\liminf_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \pi_i[\mathrm{a}_t(\sigma)]$.

3. TURING MACHINES

A Turing machine is an abstract computing device. Each machine is identified by its 'program' which consists of a finite set of symbols obeying some syntactical rules. For reasons of space, we do not specify these rules in detail here.[13] Using a standard technique known as Gödel numbering, the natural numbers can be put in a one-to-one (computable) correspondence with Turing machines. Gödel numbering can also be applied to code and decode the machines' inputs and outputs. This is because these are also assumed to be finite strings drawn from a fixed alphabet. $\mathbb{N}$ will denote the set of natural numbers throughout the paper. Using notation which is standard in the computability literature we will denote by $\varphi_x(y)$ the result of the computation of the Turing machine with Gödel number $x \in \mathbb{N}$ on the input string coded by the Gödel number $y \in \mathbb{N}$. By $\varphi_x(y) \uparrow$ and $\varphi_x(y) \downarrow$ we will respectively indicate that the computation $\varphi_x(y)$ does not halt (it 'loops'), and that it does halt.

DEFINITION 2: *A partial function f from $\mathbb{N}^m$ to $\mathbb{N}$ is called computable if and only if*

$$\exists x \in \mathbb{N} \quad \textit{such that} \quad f(y_1, \cdots, y_m) \simeq \varphi_x(y_1, \cdots, y_m) \;\; \forall \, (y_1, \cdots, y_m) \; \in \mathbb{N}^m$$

Since the output of a Turing machine need not be defined for all possible inputs, special care must be taken in asserting 'equalities'. The symbol '$\simeq$' used between two Turing machines, two computable functions or any combination of these (as in Definition 2) means 'defined on the same set of inputs and equal whenever defined'.

We conclude this section with an observation. The computability framework which we have just described imposes 'weak' restrictions in the following sense. The notion of Turing-computability is widely agreed to embody the widest possible intuitive notion of *effective computability*. Intuitively, a function is effectively computable if and only

[13]There are a large number of texts on computability. These range from textbooks such as Cutland (1980) and Hopcroft and Ullman (1979) to classic references such as Davis (1958) and Rogers (1967). Anderlini (1989) contains a brief exposition of the basic details.

if its values can be computed in a finite 'number of steps' using some conceivable 'finite device'. In the mathematical literature this claim is known as *Church's Thesis.*[14]

4. THE MODEL

4.1. Computable Strategies

Recall that G is a *finite-action* game and that we do not consider mixed strategies [15] within the stage game. Therefore, we can use the numbering technique mentioned in Section 3 above to assign (in a computable way) a code in $\mathbb{N}$ to any element of $\mathcal{H}$. In a completely analogous way, the elements of $\mathcal{A}_i$ can also be coded in $\mathbb{N}$. From these two coding operations, it is immediate that a strategy in G^∞ can be thought of as a function from $\mathbb{N}$ to $\mathbb{N}$. Since this does not cause any ambiguity, now and throughout the rest of the paper, we use the same symbol for $h_t \in \mathcal{H}$ and a_i, and for their 'codes' in $\mathbb{N}$.

DEFINITION 3: *A strategy σ_i for player i in G^∞ is called a computable strategy if and only if*

$$\exists\, x_i \in \mathbb{N} \quad \textit{such that} \quad \sigma_i(h_t) = \varphi_{x_i}(h_t) \;\; \forall\, h_t \in \mathcal{H}$$

The action profile at time t corresponding to a given vector of computable strategies $x = (x_1, \ldots, x_n)$, is denoted by $\mathrm{a}_t(x)$. The history of length t generated by x is denoted by $\mathrm{h}_t(x) \equiv \{\mathrm{a}_0(x), \ldots, \mathrm{a}_{t-1}(x)\}$. As we mentioned above, the computation of a Turing Machine on a given input may or may not be defined (the computation may or may not *halt*). For a variety of technical reasons, all somehow related to the so-called

[14]Notice further that the restriction to countable domain and range of computable functions is in some sense 'without loss of generality'. Very loosely speaking, this is because any 'language of first order logic' with a countable set of 'sentences' must have a 'countable model'. See, for instance, Bell and Machover (1977), Ch. 5 on Model Theory, particularly the Löwenheim-Skolem Theorem.

[15]Because of our computability restrictions, it is hard to consider explicitly a ***continuum*** of randomizations within the stage game G. Notice however that a ***finite*** set of mixed strategies can easily be considered within the stage game itself by adding appropriate entries to its normal form. Adding mixed strategies in this way to a Common Interest game always yields another Common Interest game.

'halting problem', we are not able to exclude from our analysis all Turing machines which do not halt on all possible histories of the repeated game. To keep matters as simple as possible in this respect, we will only consider either Turing machines which halt on all possible histories and yield an output in $\mathcal{A}_i$, or Turing machines which *never* halt on any possible history of the repeated game. This motivates the next Definition.

DEFINITION 4: *The set of allowable Turing machines for player i in G^∞ is the set of all Turing machines which either halt for all $h_t \in \mathcal{H}$ and yield an output in the action set $\mathcal{A}_i$, or which do not halt for any possible history of the repeated game. Formally, define*

$$\mathcal{S}_i^H \equiv \{x_i \in \mathbb{N} \mid \varphi_{x_i}(h_t) \downarrow \in \mathcal{A}_i \; \forall\, h_t \in \mathcal{H}\}$$

and

$$\mathcal{S}_i^{\overline{H}} \equiv \{x_i \in \mathbb{N} \mid \varphi_{x_i}(h_t) \uparrow \; \forall\, h_t \in \mathcal{H}\}$$

We call $\mathcal{S}_i^H$ and $\mathcal{S}_i^{\overline{H}}$ the set of halting and non-halting strategies for player i respectively. We also define $\mathcal{S}^H \equiv \mathcal{S}_1^H \times, \ldots, \times \mathcal{S}_n^H$, $\mathcal{S}^{\overline{H}} \equiv \mathcal{S}_1^{\overline{H}} \times, \ldots, \times \mathcal{S}_n^{\overline{H}}$, $\mathcal{S}_i \equiv \mathcal{S}_i^H \cup \mathcal{S}_i^{\overline{H}}$ and $\mathcal{S} \equiv \mathcal{S}_1 \times, \ldots, \times \mathcal{S}_n$.

We extend the players' payoff functions in the repeated game so that they are defined for any vector $x \in \mathcal{S}$. To 'neutralize' the role of non-halting strategies we will assume that they are dominated and that any halting strategy is a best response to any strategy profile which contains one or more non-halting strategies.

We can now define the long-run undiscounted payoffs yielded by any vector of Turing machines in $\mathcal{S}$.

ASSUMPTION 2: *The long-run payoffs in the undiscounted infinitely repeated game are defined for any vector of Turing machines in $\mathcal{S}$. Moreover, all non-halting strategies are dominated by some halting strategy, and any halting strategy is a best re-*

sponse to any strategy profile which contains one or more non-halting strategies.[16] *Formally, let $\Pi_i : \mathcal{S} \to \mathbb{R}$ be the long-run payoff to player i. Then*

$$\Pi_i(x) \equiv \liminf_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \pi_i[\mathrm{a}_t(x))] \quad \forall\, x \in \mathcal{S}^H$$

and Π_i satisfies the following two conditions[17]

$$\forall\, x_i \in \mathcal{S}_i^{\overline{H}},\ \exists\, x_i' \in \mathcal{S}_i^H \text{ s.t. } \Pi_i(x_i', x_{-i}) > \Pi_i(x_i, x_{-i})\ \forall\, x_{-i} \in \mathcal{S}_{-i} \qquad \text{(dominance)}$$

$$\Pi_i(x_i, x_{-i}) \geq \Pi_i(x_i', x_{-i})\ \forall\, x_i' \in \mathcal{S}_i,\ x_i \in \mathcal{S}_i,\ x_{-i} \notin \mathcal{S}_{-i}^H \qquad \text{(best response)}$$

Notice that, of course, we could have stated the best response part of Assumption 2 in an equivalent way assuming that all halting strategies for a given player yield the *same* payoff against any strategy profile which contains one or more non-halting strategies.

Assumption 2 stipulates that 'not playing' G^∞ is dominated by playing it, whatever the outcome. Moreover, playing a halting strategy is a best response whenever one or more of the other players refuse to play the game. To use the standard example of Chess, not playing the game is like overturning the board instead of making a legal move. As a result the player loses the game. Any strategy consisting entirely of legal moves is a best response to a player who overturns the chess-board.

[16]There are two issues about the 'best response' part of Assumption 2 which are worth emphasizing at this point. The first is that the best response property of halting machines which we are assuming is only used in our proof of existence of perturbed equilibria, and not in the proof of our equlibrium selection result. The second point to notice is that our best response assumption, as stated, is stronger than needed for our existence result. More precisely, the existence part of Theorem 1 below also holds if we assume only that any 'cooperative strategy' (see Definition 5 below) — rather than any halting strategy, cooperative or not — is a best response to any strategy profile which contains one or more non-halting strategies. In A-S only this weaker version of the best response property is used. In this paper we choose to use this stronger version of Assumption 2 because it saves a considerable amount of extra notation and space in the anaylsis of our model with a countable infinity of players which we present in Section 9 below.

[17]Throughout the paper a subscript of $-i$ attached to any symbol, say z, indicates the array $z_{-i} = (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n)$.

Our main result is an equilibrium selection result which singles out the cooperative equilibria of the repeated game. We conclude this section by defining what is meant by a cooperative strategy in the repeated game.

DEFINITION 5: *A strategy x_i for player i is said to be 't-cooperative' if and only if it is guaranteed to play cooperatively at t and in all subsequent periods, regardless of the history of play. Formally, let $\mathcal{C}_i^t$ denote the set of cooperative strategies at t for player i, then*

$$\mathcal{C}_i^t \equiv \{x_i \in \mathcal{S}_i \quad | \quad t' \geq t \Rightarrow \varphi_{x_i}(h_{t'}) = a_i^e \quad \forall\, h_{t'} \in \mathcal{H}_{t'}\}$$

The complement of $\mathcal{C}_i^t$ in $\mathcal{S}_i$ is denoted by $\overline{\mathcal{C}}_i^t$. The strategies for player i in the set $\mathcal{C}_i \equiv \cup_{t=0}^{\infty}\mathcal{C}_i^t$ are called simply cooperative strategies. The complement of $\mathcal{C}_i$ in $\mathcal{S}_i$ is denoted by $\overline{\mathcal{C}}_i$.

4.2. Admissible Trembles

In Section 1.2 we described informally how a strategy which is capable of revealing a player's intention to cooperate in the long run may be constructed by 'simulating' some non-cooperative strategies, 'keeping track' of their probability and of the probability of the 'tail' of non-cooperative strategies which have not yet been enumerated and simulated. Implicitly we were assuming that the proposed algorithm has access to the probabilities of individual strategies and of the 'tail'. To make the construction rigorous, we need some assumptions (as in A-S) on the perturbations of G^∞ which guarantee the feasibility of these operations.

The first assumption we need is that the probabilities which the perturbations assign to strategies must be computable. It is possible to state this assumption in a variety of different ways. We choose the formulation below mainly for the sake of simplicity. [18] It is convenient to state Definitions 6 and 8 for an abstract probability distribution over $\mathbb{N}$, before using them to define the actual perturbations of G^∞.

[18] As in Anderlini and Sabourian (1990), we could state Definition 6 below more weakly in terms of *approximate* computability and derive the same results as in this paper.

anism described above, consider a 3-player repeated Common Interest game (such as the one in Figure 1) and let (x_1^E, x_2^E, x_3^E) be an equilibrium strategy profile. Consider first for each player $i = 1, 2$ the set of computable non-cooperative supergame strategies $\overline{Q}_i$: the set of strategies which fail to cooperate in the long-run. This is a countable set since the entire set of computable strategies is countable. For each i it is useful to visualize this set as being on the horizontal axis as in Figure 2 below, together with the probabilities (on the vertical axis) assigned to strategies in this set by the perturbation of the player's equilibrium strategy.

Now for each $i = 1, 2$ consider a strategy, say x_i^*, constructed in the following way. Firstly, x_i^* 'enumerates' sufficiently many non-cooperative strategies in $\overline{Q}_i$, so that the 'tail' left after such enumeration has 'sufficiently small' probability compared with the probability which the perturbation assigns to x_i^* itself. Because the set of non-cooperative strategies is countable, this can be achieved enumerating *finitely many* non-cooperative strategies, say $\tilde{t}_i$ of them (in Figure 2 we have set $\tilde{t}_i = 9$).

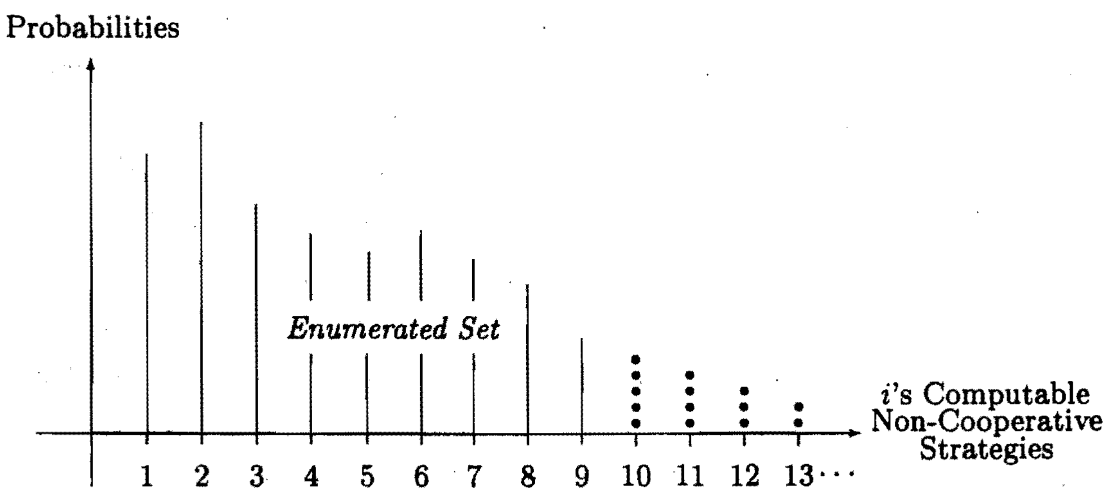

Figure 2

Strategy x_i^* can then use the first $\tilde{t}_i$ stages of play after some some fixed period v_i to signal its intention to play cooperatively in the long-run as follows. In the first period of the signalling phase, at $v_i + 1$, simulate what action the non-cooperative

DEFINITION 6: *For each player i, a probability distribution $\{P_i(1), P_i(2), \ldots, P_i(x_i), \ldots\}$ over $\mathbb{N}$ (denoted by P_i) is said to be 'computable' if and only if there exists a Turing machine which computes (at least) all non-zero values of P_i as a function of x_i. Formally, let Δ^∞ represent the unit simplex in $\mathbb{R}^\infty$ and* $\mathrm{supp}(P_i) \equiv \{x_i \in \mathbb{N} \mid P_i(x_i) > 0\}$*, then $P_i \in \Delta^\infty$ is said to be computable if and only if $\exists\, p_i \in \mathbb{N}$ such that* $x_i \in \mathrm{supp}(P_i)$ *implies*

$$\varphi_{p_i}(x_i) = P_i(x_i)$$

and $\varphi_{p_i}(x_i) \downarrow \Rightarrow \varphi_{p_i}(x_i) = P_i(x_i)$.

The second computability property which we require our perturbations to satisfy concerns the possibility of computing the probability of the 'tail' of the probability distribution to which we referred intuitively in Section 1.2. We must be careful as to precisely what set of strategies we put on the horizontal axis of Figure 2 since being able to 'enumerate' and compute the probability of the tail of a set as we described in Section 1.2 is equivalent to some 'regularity' properties for the set itself which we will discuss shortly. There are a variety of ways to proceed. Again, we choose what seems intuitively the simplest formulation, even though it is by no means the most general.[19].

Consider the set of computable supergame strategies for player i which have the property that *if* action a_i^e is played at any stage, then action a_i^e is played forever at all later stages. A convenient set to put on the horizontal axis of Figure 2 is the complement of this set. Signalling that a player's strategy is *not* in this latter set and then playing cooperatively is clearly a good way to signal the player's intention to cooperate in the long-run. Our next step is to define formally this set of computable supergame strategies and its complement. As in A-S, for want of a better term, we call it the set of 'quasi-cooperative' strategies.

[19]In Section 7 we describe one alternative way to formulate Definition 7 below.

DEFINITION 7: *A supergame strategy x_i for player i is said to be 'quasi-cooperative' if and only if it has the property that after cooperating once it will cooperate forever, regardless of the opponents' play. Formally, let $\mathcal{Q}_i$ denote the set of quasi-cooperative strategies for player i, then*[20]

$$\mathcal{Q}_i \equiv \{x_i \in \mathcal{S}_i \quad | \quad \varphi_{x_i}(h_t) \downarrow = a_i^e \text{ and } t' > t \;\; \Rightarrow \;\; \varphi_{x_i}(h_{t'}) \downarrow = a_i^e \;\; \forall \, h_{t'} \in \mathcal{H}_{t'}\} \quad (1)$$

The complement of $\mathcal{Q}_i$ in $\mathcal{S}_i$ is denoted by $\overline{\mathcal{Q}}_i$.[21]

We are interested in trembles which guarantee that the probability of $\overline{\mathcal{Q}}_i$ is computable. Since both $\mathcal{Q}_i$ and $\overline{\mathcal{Q}}_i$ are infinite sets, this is not a property which follows automatically from computability of the probability distribution in the sense of Definition 6.

DEFINITION 8: *A Probability distribution P_i over the natural numbers is said to be $\overline{\mathcal{Q}}_i$-computable if and only if the probability which P_i assigns to $\overline{\mathcal{Q}}_i$ is a 'computable real number' in the sense that it can be approximated by a Turing machine up to any arbitrarily given degree of precision. Formally, let $P_i(\overline{\mathcal{Q}}_i) \equiv \sum_{x \in \overline{\mathcal{Q}}_i} P_i(x)$ then $P_i \in \Delta^\infty$ is said to be $\overline{\mathcal{Q}}_i$-computable if and only if $\exists q_i \in \mathbb{N}$ such that*

$$| \;\; \varphi_{q_i}(c) - P_i(\overline{\mathcal{Q}}_i) \; | < \; \frac{1}{c} \quad \forall \, c \in \mathbb{N}$$

The equilibrium notion which we define in the next Section will involve perturbations which are both computable and $\overline{\mathcal{Q}}_i$-computable for all players. We call these distributions 'admissible' probability distributions.

[20]Notice that the set $\mathcal{C}_i$ of cooperative strategies neither contains nor is it contained in the set $\mathcal{Q}_i$ of quasi-cooperative strategies. For instance, a strategy which outputs the cooperative action a_i^e on history h_0, then outputs some non-cooperative action for all possible h_1 and again outputs the cooperative action on all histories of length two or more, is cooperative but not quasi-cooperative. A strategy which *never* outputs the cooperative action a_i^e (including any strategy in $\mathcal{S}_i^{\overline{H}}$) is quasi-cooperative but clearly does not belong to the set of cooperative strategies $\mathcal{C}_i$.

[21]Notice that (1) implies that all strategies in $\overline{\mathcal{Q}}_i$ are halting strategies.

DEFINITION 9: *A Probability distribution P_i over $\mathbb{N}$ is said to be admissible for player i if and only if a) it gives positive probability only to machines in $\mathcal{S}_i$ (in other words* $\text{supp}(P_i) \subseteq \mathcal{S}_i$*), b) it is computable according to Definition 6, and c) it is $\overline{\mathcal{Q}}_i$-computable according to Definition 8. Throughout the rest of the paper we will denote by $\mathcal{P}_i$ the set of probability distributions which are admissible for player i.*

Lemma A.2 shows that the set $\overline{\mathcal{Q}}_i^{P_i} \equiv \text{supp}(P_i) \cap \overline{\mathcal{Q}}_i$ is *recursively enumerable* in the sense of Definition A.2 whenever P_i is admissible in the sense of Definition 9. Intuitively, a subset of $\mathbb{N}$ is recursively enumerable if and only if its elements can be exhaustively enumerated by a Turing machine. Therefore, there are three Turing machines associated with each $P_i \in \mathcal{P}_i$. One which computes the probabilities of individual machines, one which computes the probability of $\overline{\mathcal{Q}}_i$, and a third one which 'enumerates' the elements of $\overline{\mathcal{Q}}_i^{P_i}$. We will refer to such a triple of Turing machines as a 'basis' for P.

DEFINITION 10: *A triple $(p_i, q_i, m_i) \in \mathbb{N}^3$ is said to be a 'basis' for an admissible $P_i \in \mathcal{P}_i$ if and only if $\varphi_{p_i}(\cdot)$ computes the values of P_i as in Definition 6, $\varphi_{q_i}(\cdot)$ computes (approximately) the value of $P_i(\overline{\mathcal{Q}}_i)$ as in Definition 8, and $\varphi_{m_i}(\cdot)$ 'enumerates' $\overline{\mathcal{Q}}_i^P$ 'without repetitions' as in Theorems A.4 and A.5.* [22]

4.3. Equilibrium

The equilibrium concept we use is that of Trembling Hand Perfect (Selten 1975, Myerson 1978) with the restriction that supergame strategies must be computable and perturbations must be admissible.

From a formal point of view Definitions 11 and 12 below of Trembling Hand Perfect equilibrium are standard except for the treatment of the support of the trembles. We find it convenient to 'parameterize' classes of possible perturbations by a 'lower bound' on their support. Some notation is necessary. Throughout the rest of the paper we

[22] If $\overline{\mathcal{Q}}_i^{P_i}$ is empty, we require m_i to compute the function 'nowhere defined'.

denote the set of admissible probability distributions for player i which have support *at least* as large as a given set $\mathcal{R}_i$, by $\mathcal{P}_i(\mathcal{R}_i)$. Formally $\forall \mathcal{R}_i \subseteq \mathbb{N}$ we let

$$\mathcal{P}_i(\mathcal{R}_i) \equiv \{ P_i \in \Delta^\infty \quad | \quad \mathcal{R}_i \subseteq \mathrm{supp}(P_i) \text{ and } P_i \text{ is admissible for } i \}$$

As is standard, we require equilibrium strategies to be optimal against the opposing players' equilibrium strategies played with large probability, and the perturbation played with arbitrarily small probability. It is useful to establish some notation for the set of computable strategies which are best responses to a given profile of strategies and perturbations. Formally we let $\forall\, x^*_{-i} \in \mathcal{S}_{-i}$, $\forall\, P_{-i}$ and $\forall\, \epsilon > 0$

$$\begin{gathered}
\mathcal{B}_i(x^*_{-i}, P_{-i}, \epsilon) \equiv \arg\max_{x_i \in \mathcal{S}_i} \Bigg\{ (1-\epsilon)^{n-1}\Pi_i(x_i, x^*_{-i}) + \dots \\
(1-\epsilon)^{n-1-k}\epsilon^k \Bigg[\sum_{(x_1,..,x_k)\in\mathcal{S}_1\times\cdot\cdot\times\mathcal{S}_k} P_1(x_1)\cdot\cdot P_k(x_k)\Pi_i(x_i, x_1,.., x_k, x^*_{k+1},.., x^*_n) + \dots \\
\sum_{(x_{n-k-1},..,x_n)\in\mathcal{S}_{n-k-1}\times\cdot\cdot\times\mathcal{S}_n} P_{n-k-1}(x_{n-k-1})\cdot\cdot P_n(x_n)\Pi_i(x_i, x^*_1,.., x^*_{n-k-2}, x_{n-k-1},.., x_n) \Bigg] + \dots \\
\epsilon^{n-1}\sum_{x_{-i}\in\mathcal{S}_{-i}} P_1(x_1)\cdots P_{i-1}(x_{i-1})P_{i+1}(x_{i+1})\cdots P_n(x_n)\Pi_i(x_i, x_{-i}) \Bigg\}
\end{gathered} \tag{2}$$

where each of the terms within square brackets is the addition of the $(n-1)!/[(n-k-1)!k!]$ terms which represent all possible combinations of $n-k-1$ strategies x^*_i, and k strategies x_i in the perturbations. Moreover, (2) contains $n-2$ such square-bracketed terms, obtained as k varies from 1 to $n-2$.

Intuitively, an $(\epsilon, \mathcal{R})$ Computable Trembling Hand Equilibrium (with $\mathcal{R} = (\mathcal{R}_1,\dots, \mathcal{R}_n)$) is a vector of computable strategies and an array of perturbations, each with support at least $\mathcal{R}_i$, such that the given strategies are a best response to each other given the perturbations. Formally, we state

DEFINITION 11: *An $(\epsilon, \mathcal{R})$ Computable Trembling Hand Equilibrium (abbreviated $(\epsilon, \mathcal{R})$-CTHE) is a 2n-tuple $\{x^\epsilon_i, , P^\epsilon_i\}^n_{i=1}$ with $x^\epsilon_i \in \mathcal{S}_i$ and $P^\epsilon_i \in \mathcal{P}_i(\mathcal{R}_i)$ such that for* *all* $i = 1, \ldots, n$

$$x_i^\epsilon \in \mathcal{B}_i(x_{-i}^\epsilon, P_{-i}^\epsilon, \epsilon)$$

The set of equilibrium $2n$*-tuples for a given pair* $(\epsilon, \mathcal{R})$ *will be denoted by* $E(\epsilon, \mathcal{R})$*, and the set of corresponding equilibrium long-run payoff vectors will be denoted by* $\Pi^E(\epsilon, \mathcal{R})$.

The next Definition is simply the limit of the perturbed equilibrium of Definition 11 as ϵ vanishes.

DEFINITION 12: *A* $\mathcal{R}$*-CTHE is the limit of any sequence of* $\Pi^E(\epsilon, \mathcal{R})$ *as* ϵ *vanishes. The set of* $\mathcal{R}$*-CTHE is denoted by* $\Pi^E(\mathcal{R})$.

Our main result below selects the cooperative payoff vector as the only possible equilibrium outcome, provided that the perturbations have sufficiently large support. As in A-S, there are three features of our model that are crucial for our results: computable supergame strategies, computable perturbations, and sufficiently large support. In our view there are two possible interpretations of our equilibrium concept: meta-players choosing machines to implement their strategies (Rubinstein 1986, Abreu and Rubinstein 1988, among others) and a process of learning/evolution on populations of machines (Binmore and Samuelson 1992, Anderlini and Sabourian 1997). In A-S, we discuss in detail the two different interpretations of our equilibrium concept and how each of the three features of our model should be understood within the two different interpretations of our equilibrium concept.

A serious drawback of the meta-players interpretation of our results is that it may be impossible for a computing device of the class considered to choose an optimal machine. In other words, the task of choosing an optimal Turing machine may not be computable. For this reason and because of the lack of realism of the meta-players interpretation, we prefer to interpret our equilibria as steady states of an learning/evolutionary process on populations of machines[23]. In particular, we simply assume that Trembling Hand Perfect equilibria are the limit points of some interesting evolutionary dynamics.[24]

5. OPTIMALITY AND EXISTENCE OF EQUILIBRIA

We are now ready to present the main formal result of the paper. It states that in the limit as the noise vanishes all Computable Trembling Hand Equilibria of an infinitely repeated n-player Common Interest game with no discounting are cooperative, provided that the perturbations have 'sufficiently large' support.

So far we have not imposed any requirement that the support of our perturbations be large, or even non-degenerate. From the intuition given in Section 1.2 it is clear, however, that some large support assumption is necessary to produce cooperation in the long-run. The 'lower bound' on the support of the trembles in Theorem 1 below has a simple intuitive interpretation. As a function of the 'parameters' of the model (the exact 'shape' and the 'intensity' of the perturbations), the 'revealing' strategies informally described in Section 1.2 may change. We then need to ensure that the relevant revealing strategies are always given positive probability by the perturbations. We achieve this by stipulating that the support of the perturbations should contain all possible (for all 'parameter configurations' that is) revealing strategies in the first place.

The formal statement of the Theorem includes both existence and optimality of equilibria.

THEOREM 1: *There exists a 'sufficiently large' set of n-tuples of machines $\mathcal{R}$ such that the set of $\mathcal{R}$-Computable Trembling Hand Equilibria of an infinitely Repeated*

[23]In Anderlini and Sabourian (1997), we study explicitly a learning/evolutionary system, with an algorithmic initial distribution and algorithmic dynamics. The dynamics shape the distribution of algorithmic learning rules which play an action in a one-shot normal form game as a function of (some statistic of) the past history of play. Under some conditions we find that the system converges *globally* to a Nash equilibrium of the underlying game.

[24]See the two recent surveys by Kandori (1997) and Marimon (1997) on evolution and learning.

Common Interest game with no discounting is not empty and all equilibria are cooperative. Formally, $\exists\, \mathcal{R} = (\mathcal{R}_1, .., \mathcal{R}_n)$ *such that*

$$E(\epsilon, \mathcal{R}) \neq \emptyset \quad \forall\, \epsilon \in [0, 1]$$

and

$$\Pi^E(\mathcal{R}) = \pi^e$$

We conclude this Section with an observation.

REMARK 1: *Recall that, in our definition of equilibrium, the sets* $\mathcal{R}_i$ *are 'lower bounds' on the support of the perturbations. It follows that whenever* $\mathcal{R}_i \subseteq \mathcal{R}'_i$ *for all* $i = 1, \ldots, n$, *we must have* $\Pi^E(\mathcal{R}') \subseteq \Pi^E(\mathcal{R})$. *Therefore Theorem 1 implies that all Computable Trembling Hand Equilibria with perturbations having supports larger than the sets* $\mathcal{R}_i$ *of Theorem 1 are cooperative.*

6. PROOF OF THEOREM 1

The proof of Theorem 1 can be divided into three separate arguments. The first is the formalization of the intuitive argument presented in Section 1.2; we call this the Communication Lemma (Lemma 3 below). The second part of the argument shows that, assuming the equilibrium set is not empty, since the stage game is a Common Interest game, the Communication Lemma implies that all equilibrium payoffs are in fact cooperative. We present this argument in Section 6.2 below. The last part of the argument shows that the equilibrium set is not empty. This is presented as Lemma 8 in Section 6.3 below. Theorem 1 is then an immediate consequence of Lemma 7 and lemma 8 below.

In the arguments which follow and in the Appendix, we make use of a technique accepted as standard in this area of mathematics known as *proof by Church's thesis*: it is assumed that whenever a 'clear procedure' exists for computing a function then it follows that such function is computable by a Turing machine.[25]

6.1. A Lemma on Communication

We start by showing that for each player $i = 1, \ldots, n$ it is possible to construct a computable function which will take as inputs two arbitrary Turing machines x_i and y_i, the 'parameters' (the 'basis' as in Definition 10) of a profile of admissible distributions $P = (P_1, .., P_n)$, a level of precision parameter $k \in \mathbb{N}$, and a history $h_t \in \mathbb{N}$, with the following properties.

First, for each i, compute a number $\tilde{t}_i$ so that the probability of the 'tail' of $\overline{\mathcal{Q}}_i^P$ (recall that $\overline{\mathcal{Q}}_i^{P_i} \equiv \overline{\mathcal{Q}}_i \cap \text{supp}(P_i)$) after the first $\tilde{t}_i$ elements have been taken out is small relative to the probability of x_i. Secondly, if h_t has $t \geq \tilde{t}_i + v_i + 1$, where $v_1 = 0$ and $v_i = i - 1 + \sum_{j<i} \tilde{t}_j$ for all $i = 2, \ldots, n$, the output of the computation is the cooperative action a_i^e. Thirdly, if h_t has $v_i + 1 \leq t < \tilde{t}_i + v_i + 1$, the output of the computation is an action $a_i \in \mathcal{A}_i$ which is *different* from the action taken by the $(t - v_i - 1)$-th strategy in the 'enumeration' of $\overline{\mathcal{Q}}_i^{P_i}$ given h_t. Fourthly, if h_t has $t = v_i$ then the output of the computation is an action in $\mathcal{A}_i$ which is different from the action taken by machine y_i given h_t. Fifthly, if h_t has $t < v_i$, the output of the computation is the same as the output of y_i given the input h_t. For the sake of clarity, we present this part of the argument as two separate Lemmas.

LEMMA 1: *There exists a computable function d_i from $\mathbb{N}^5$ to $\mathbb{N}$ such that for all $(x_i, p_i, q_i, m_i, k) \in \mathbb{N}^5$, whenever (p_i, q_i, m_i) is a basis (as in Definition 10) for an admissible probability distribution $P_i \in \mathcal{P}_i$, and $P_i(x_i) > 0$ we have $d_i(x_i, p_i, q_i, m_i, k) = \tilde{t}_i$, where $\tilde{t}_i$ satisfies*

$$\frac{1}{k}\varphi_{p_i}(x_i) > P_i(\overline{\mathcal{Q}}_i) - \sum_{\tau=0}^{\tilde{t}_i - 1} \varphi_{p_i}(\varphi_{m_i}(\tau)) \tag{3}$$

Lemma 1 is identical to Lemma 1 in A-S. The reader should refer to A-S for the proof. Here we only notice that the left-hand side of the above inequality is the

[25]Thorough discussions of this way of proceeding are in, for instance, Cutland (1980) or Rogers (1967).

probability of x_i according to $P_i \in \mathcal{P}_i$, multiplied by a 'small number' $1/k$. Therefore k is our 'degree of precision' parameter which quantifies how small the probability of the tail must be relative to the probability of x_i. The first term on the right-hand side of the inequality is the probability assigned by $P_i \in \mathcal{P}_i$ to the set $\overline{\mathcal{Q}}_i^{P_i}$. The second term on the right-hand side of the inequality is the sum of the probabilities of the first $\tilde{t}_i$ terms of the set $\overline{\mathcal{Q}}_i^{P_i}$ as enumerated by m_i. Therefore the right-hand side of the inequality is the probability of the tail of $\overline{\mathcal{Q}}_i^{P_i}$ after the first $\tilde{t}_i$ elements have been taken out.

LEMMA 2: *For each player i, there exists a computable function g_i from $\mathbb{N}^{4n+3}$ to $\mathbb{N}$ such that $\forall\,(x,p,q,m) = \{x_i,p_i,q_i,m_i\}_{i=1}^n \in \mathbb{N}^{4n}$ and $\forall\,(y_i,k,h_t) \in \mathbb{N}^3$, whenever it is the case that for all i (p_i,q_i,m_i) forms a basis for an admissible probability distribution $P_i \in \mathcal{P}_i$, $P_i(x_i) > 0$, and $y_i \in \mathcal{S}_i^H$, we have*

$$g_i(x,y_i,p,q,m,k,h_t) = g_i(x_i,x_{-i},y_i,p,q,m,k,h_t) = \begin{cases} a_i^e \in \mathcal{A}_i & \text{if } h_t \text{ has } t \geq \tilde{t}_i + v_i + 1 \\ a_i \in \mathcal{A}_i \text{ s.t. } a_i \neq a_i^e, a_i \neq \varphi_{\varphi_{m_i}(t-v_i-1)}(h_t) & \text{if } h_t \text{ has } v_i < t \leq \tilde{t}_i + v_i \\ a_i \in \mathcal{A}_i \text{ s.t. } a_i \neq a_i^e, a_i \neq \varphi_{y_i}(h_t) & \text{if } h_t \text{ has } t = v_i \\ \varphi_{y_i}(h_t) & \text{if } h_t \text{ has } t < v_i \end{cases} \tag{4}$$

where $\tilde{t}_i$ is as in Lemma 1, $v_1 = 0$ and $v_i = i - 1 + \sum_{j<i}\tilde{t}_i$ for $i = 2,\ldots,n$. Moreover, for any $(x,y_i,p,q,m,k) \in \mathbb{N}^{4n+2}$, either $g_i(x,y_i,p,q,m,k,h_t)\uparrow$ for all h_t or $g_i(x,y_i,p,q,m,k,h_t)\downarrow$ for all h_t.

PROOF: See Appendix.

Before proceeding further note that, since m_i enumerates the set $\overline{\mathcal{Q}}_i^{P_i}$, the term $\varphi_{\varphi_{m_i}(t-v_i-1)}(h_t)$ in the right-hand side of (4) is precisely the output of the $(t-v_i)$-th strategy in the enumeration of $\overline{\mathcal{Q}}_i^{P_i}$ on input h_t. The computation performed by g_i formalizes the simulation step intuitively described in Section 1.2.

Consider a Turing machine which computes the function g_i of Lemma 2. To clarify our next step, suppose that player i were allowed to use a Turing machine computing g_i as his 'strategy' in the repeated game. Then, if for every i the inputs (p_i, q_i, m_i) happened to be the basis of the actual perturbation P_i of G^∞, and the input x_i happened to be exactly the Gödel number of such strategy, as k becomes large, the hypothetical strategy g_i mimics the behaviour of y_i up to (and including) period v_i. In period $v_i + 1$ it then reveals itself to be different from strategy[26] y_i, and then proceeds to reveal itself (given P_i) as belonging to $\mathcal{Q}_i$ with a higher and higher degree of precision by time $\tilde{t}_i + v_i + 1$. Since g_i by construction implies that the action a_i^e will be played after $\tilde{t}_i + v_i + 1$, the definition of quasi-cooperative strategies $\mathcal{Q}_i$ then implies that player i would have revealed his cooperative intentions to an arbitrarily high degree of precision by period $\bar{t}_i = \tilde{t}_i + v_i + 2$.

However, to go from g_i to an actual strategy in $\mathcal{S}_i$ (the set of allowable Turing machines for i in the repeated game) which reveals its cooperative intentions to an arbitrary degree of precision for a given $P \in \mathcal{P}$, we face the following two difficulties (these difficulties appear also in A-S).

First of all, the function g_i of Lemma 2 takes as input not only a history of play h_t, but also a (4n+2)-tuple (x, y_i, p, q, m, k). Computable strategies in $\mathcal{S}_i$, only take h_t as an input. We solve this problem by invoking a 'parameterization' result known in the computability literature as the *s-m-n* Theorem (Theorem A.1). In essence, the *s-m-n* Theorem guarantees that, for each set of fixed values of the inputs of g_i other than h_t, it is possible to find (computably), a computable function which takes only h_t as input, and which gives the same output as g_i.

The second problem, which we anticipated in Section 1.2, is that the play yielded by g_i in Lemma 2 will manage to signal effectively a player's cooperative intentions only if the value of x_i happens to be precisely the Gödel number of the strategy defined by g_i itself. This potential circularity is avoided appealing to a pseudo-fixed point result (Theorem A.6), which is a Corollary of the Recursion Theorem.[27]

[26]Later on, in the proof of the optimality result, the arbitrary strategy y_i will be set equal to the *equilibrium* strategy for i. This ensures that our signalling machines reveal themselves to be different from the equilibrium strategies as well as revealing their cooperative intentions within the perturbation.

[27]See, for instance, Cutland (1980) Theorem 11.1.1.

The Communication Lemma 3 below states that, provided that for all i, $y_i \in \mathcal{S}_i^H$ and $P_i \in \Delta^\infty$ is admissible in the sense of Definition 9 and that its support is 'sufficiently large', then for each i there exist a machine $x_i^* \in \text{supp}(P_i)$ which mimics the behaviour of y_i up to (and including) period $v_i - 1$, at period v_i it reveals itself to be different from y_i, for the next $\tilde{t}_i$ periods it takes actions to reveal its cooperative intentions up to any arbitrary degree of precision ($\tilde{t}_i$ depends on the precision level) and thereafter it plays the cooperative action . The proof of Lemma 3 involves taking the function g_i of Lemma 2 and from it obtaining a revealing strategy, resolving the two difficulties above in the way we have outlined. This yields a revealing strategy, for each possible profile of admissible distributions P, for each arbitrary machine y_i and for each possible degree of precision k. The argument is then concluded by setting the 'minimum support' equal to the set of all possible (for all parameter configurations) revealing strategies.

Given that the essence of Lemma 3 below is that players can signal through the early stages of play their intention to play cooperatively in the long-run, it is useful to establish some notation on probability distributions over computable strategies updated on the basis of a given history of play.

DEFINITION 13: *Given a probability distribution $P_i \in \Delta^\infty$, the symbol $P_i|h_t \in \Delta^\infty$ stands for the distribution P_i updated on the basis of history h_t using Bayes' rule. The elements of $P_i|h_t$ are denoted by $P_i(x_i|h_t)$. The probability which $P_i|h_t$ assigns to a subset, say $\mathcal{W}$, of $\mathbb{N}$ is denoted by $P_i(\mathcal{W}|h_t)$.*

We are now ready to state formally the main Lemma on which the proof of Theorem 1 revolves.

LEMMA 3 [Communication Lemma]: *There exists an array of sets $\mathcal{R} = (\mathcal{R}_1, \ldots, \mathcal{R}_n)$ satisfying $\mathcal{R}_i \subset \mathcal{S}_i$ such that, for any $P = (P_1, \ldots, P_n)$ satisfying $P_i \in \mathcal{P}_i(\mathcal{R}_i)$ for all i, for all $y = (y_1, \ldots, y_n) \in \mathbb{N}^n$, and for all $c \in \mathbb{N}$, there exists a corresponding array of machines $(x_1^*, \ldots, x_{n-1}^*)$ and an array of time periods $(t_1, \ldots, t_{n-1})$ such that,*

$\forall\, i, j \leq n-1$ (by convention set $t_0 = -1$ so that $t_{i-1} + 1 = 0$ when $i = 1$)

$$\text{(i)}\quad \varphi_{x_i^*}(h_t) = \varphi_{y_i}(h_t) \qquad \forall t \leq t_{i-1}$$

$$\text{(ii)}\quad \varphi_{x_i^*}(h_{t_{i-1}+1}) \neq \varphi_{y_i}(h_{t_{i-1}+1}) \qquad \forall\, h_{t_{i-1}+1} \in \mathcal{H}_{t_{i-1}+1}$$

$$\text{(iii)}\quad P_i(C_i^t | \mathrm{h}_t(x_i^*, x_{-i})) > \frac{c}{c+1} \qquad \forall\, x_{-i} \in \mathcal{S}_{-i} \quad \forall\, t \geq t_i$$

$$\text{(iv)}\quad \varphi_{x_i^*}(h^t) = a_i^e \qquad \forall t \geq t_i$$

$$\text{(v)}\quad t_i < t_j \quad \Leftrightarrow \quad i < j$$

In other words, x_i^* simulates y_i up to and including period t_{i-1}, at period $t_{i-1}+1$ it reveals itself to be different from machine y_i, it then reveals its cooperative intentions with a degree of precision $c/(c+1)$ by time t_i, and finally at period t_i and thereafter it plays cooperatively.[28]

PROOF: See Appendix.

We conclude this Section with the observation that the sets $\mathcal{R}_i$ of the statement of the Communication Lemma have a clear intuitive interpretation.[29] Each $\mathcal{R}_i$ is the set of all possible signalling strategies for player i, for all possible arrays of admissible probability distributions, for any machine y_i and for all possible values of the degree of precision parameter k. The set $\mathcal{R}_i$ also contains some non-halting Turing machines. Intuitively, these correspond to the configurations of parameters (p, q, m) which do not form the basis of any array of admissible probability distributions or to a machine y_i which does not always halt. Finally, notice that since all halting Turing machines in $\mathcal{R}_i$ eventually cooperate forever we have that $\mathcal{R}_i \cap \mathcal{S}_i^H \subseteq \mathcal{C}_i$.

[28] Note that x_i^* and t_i depend on $(y_1, \ldots, y_n)$ and c (in fact, in the argument which follows t_i depends only on c). We suppress this from the notation whenever there is no risk of ambiguity.

[29] See also (A.15) in the proof of the Communication Lemma in which the sets $\mathcal{R}_i$ are defined.

6.2. *Optimality*

In this Section we show that, provided that the equilibrium set $\Pi^E(\mathcal{R})$ is not empty, the Communication Lemma (Lemma 3) is enough to ensure that the equilibrium set consists only of the efficient payoff vector π^e. The intuition behind the proof is the same as the backwards induction argument applied to the set of all players described informally in section 1.2.

Some extra notation is needed. Given any $2n$-tuple $\{x_i^\epsilon, P_i^\epsilon\}_{i=1}^n$, constituting an $(\epsilon, \mathcal{R}) - CTHE$, for each i let $\overline{P}_i^\epsilon$ be the 'overall' probability distribution obtained from the combination (weight $1-\epsilon$) of the degenerate distribution placing probability one on x_i^ϵ and the actual perturbation (weight ϵ). Thus

$$\overline{P}_i^\epsilon(x) = \begin{cases} (1-\epsilon) + \epsilon P_i^\epsilon(x_i) & \text{if } x_i = x_i^\epsilon \\ \epsilon P_i^\epsilon(x_i) & \text{if } x_i \neq x_i^\epsilon \end{cases} \tag{5}$$

Let any n-tuple of computable (halting) strategies $x = (x_1, \ldots, x_n) \in \mathcal{S}^H$ and any history of length t, $h_t \in \mathcal{H}_t$, be given. We can then define recursively the outcome path generated by x, given h_t, as follows. Let $\mathrm{a}_t(x|h_t) = (\varphi_{x_1}(h_t), \ldots, \varphi_{x_n}(h_t))$, and $\mathrm{a}_{t+1}(x|h_t) = (\varphi_{x_1}(h_t, \mathrm{a}_t(x|h_t)), \ldots, \varphi_{x_n}(h_t, \mathrm{a}_t(x|h_t)))$. Continuing by forward recursion in this way, we can clearly define the continuation of h_t generated by x at any $t' > t$. We denote this by $\mathrm{a}_{t'}(x|h_t)$.

The history of length $t' > t$ generated by $x \in \mathcal{S}^H$, given h_t, can be defined as $\mathrm{h}_{t'}(x|h_t) = (h_t, \mathrm{a}_t(x|h^t), \ldots, \mathrm{a}_{t'}(x|h_t))$. The *infinite* history generated by $x \in \mathcal{S}^H$, given h_t can also be defined in the obvious way and will be denoted by $\mathrm{h}_\infty(x|h_t)$.

Given any infinite outcome path h_∞, let $\mathrm{a}_t(h_\infty)$ denote the n-tuple of actions which the players take at t in h_∞. The long-run pay-off to player i along h_∞ is denoted by $\overline{\Pi}_i(h_\infty)$ and is given by $\liminf_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \pi_i(\mathrm{a}_t(h_\infty))$.

The *expected* long-run pay-off to player i at period t, given *overall* probabilities[30]

[30] See (5) above.

over machines $\overline{P}_i$ and $\overline{P}_{-i}$ and given a history h_t is denoted by[31] $\mathcal{E}_i(\overline{P}_i, \overline{P}_{-i}, h_t)$. Thus

$$\mathcal{E}_i(\overline{P}_i, \overline{P}_{-i}, h_t) = \begin{cases} \sum_{x\in\mathcal{S}^H} \overline{P}(x)\,\overline{\Pi}_i(\mathrm{h}_\infty(x|h_t)) & \text{if } h_t \neq \phi \\ \sum_{x\in\mathcal{S}} \overline{P}(x)\,\Pi_i(x) & \text{if } h_t = \phi \end{cases} \tag{6}$$

REMARK 2: *Notice that in (6), if h_t is not empty the expectation is taken over $\mathcal{S}^H$. This is because if the perturbations are admissible, the posterior probabilities attach positive weights only to machines that always halt and therefore we can ignore the payoff associated with machine profiles $x \notin \mathcal{S}^H$.*

REMARK 3: *Note that with some abuse of notation in what follows we may write one or more machines x_j as arguments of $\mathcal{E}_i$. In this case it is understood that we mean the 'degenerate' distribution assigning probability one to such machine(s).*

Before continuing any further, we now need to define formally two properties of histories and associated profiles of machines.

DEFINITION 14: *A machine x_i for player i is said to be* consistent *with a given finite history $h_t = (a_0, \ldots, a_{t-1})$ if and only if $\varphi_{x_i}(\emptyset) = a_{i0}$, and for all $\tau = 0, 1, \ldots, t-2$ we have that $\varphi_{x_i}(a_0, \ldots, a_\tau) = a_{i\tau+1}$. In other words, x_i is consistent with h_t if player i's component of history h_t could possibly have been generated by machine x_i.*

Given a profile of machines $x = (x_1, \ldots, x_n)$, two finite histories h_t and $h_{t'}$ with $t' > t$ are said to be consecutive *histories given x if and only if the following two conditions hold. (a) For every $i = 1, \ldots, n$, $h_{t'}$ is consistent with x_i as defined above, and (b) for some finite history $h_{t'-t}$ of length $t' - t$ we have that $h_{t'} = (h_t, h_{t'-t})$. In other words, two histories are consecutive given x if and only if they are both consistent with x and they are a continuation of each other.*

Given a profile of machines x and k distinct finite histories, these histories are said to be consecutive given x if and only they can be ordered into $k-1$ pairs of histories which are consecutive given x according to the definition above.

[31] Notice that the distinction between 'continuation' pay-offs and pay-offs conditional on a particular finite history of play is immaterial since we assume that players do not discount the future, and long-run pay-offs are ranked using the 'limit of the mean' criterion.

To ease the exposition, we have divided the rest of the argument into four separate Lemmas. The first Lemma tells us that the signaling strategies of the Communication Lemma yield a set of consecutive histories along which the signaling strategies 'take turns' to reveal their cooperative intentions, up to any arbitrary degree of precision. Moreover, before a particular player starts his signaling phase, the signaling strategy is consistent with the history generated by the equilibrium machines.

LEMMA 4: *Consider any $(\epsilon, \mathcal{R})$-CTHE with $\mathcal{R} = (\mathcal{R}_1, \ldots, \mathcal{R}_n)$ where, for each i, $\mathcal{R}_i$ is the set defined by (A.15) in the Proof of the Communication Lemma. Let the $2n$-tuple constituting*[32] *such equilibrium be $\{x_i^\epsilon, P_i^\epsilon\}_{i=1}^n$. For each i, let also $\overline{P}_i^\epsilon$ be the 'overall' probability distribution as defined in (5).*

Then, for all $c \in \mathbb{N}$ there exist a set of signalling machines $x_1^{c,\epsilon}, \ldots, x_{n-1}^{c,\epsilon}$ with $x_i^{c,\epsilon} \neq x_i^\epsilon$ for all $i = 1, \ldots n-1$, and a set of histories $h_{t_1^{c,\epsilon}}^{c,\epsilon}, \ldots, h_{t_{n-1}^{c,\epsilon}}^{c,\epsilon}$ with $t_1^{c,\epsilon} < \ldots < t_{n-1}^{c,\epsilon}$ which are consecutive given $(x_1^c, \ldots, x_{n-1}^c, x_n^\epsilon)$, and which occur with positive probability, such that the following conditions hold.

For all $i = 1, \ldots, n-1$, and for all $j \geq i$

$$h_{t_i^{c,\epsilon}}^{c,\epsilon} = \mathrm{h}_{t_i^{c,\epsilon}}(x_1^{c,\epsilon}, \ldots, x_{n-1}^{c,\epsilon}, x_n^c) = \mathrm{h}_{t_i^{c,\epsilon}}(x_1^{c,\epsilon}, \ldots, x_j^{c,\epsilon}, x_{j+1}^\epsilon, \ldots, x_n^\epsilon) \tag{7}$$

For all $i = 1, \ldots, n-1$ and for any h_t with $t \geq t_i^{c,\epsilon}$ which occurs with positive probability,

$$\overline{P}_i^\epsilon(C_i^t | h_t) > \frac{c}{c+1} \tag{8}$$

And finally, for all $i = 1, \ldots, n-1$ and for any h_t with $t \geq t_i^{c,\epsilon}$,

$$\varphi_{x_i^{c,\epsilon}}(h_t) = a_i^e \tag{9}$$

[32] Recall that according to Assumption 2 (dominance), for each i, all machines in $\mathcal{S}_i^{\overline{H}}$ are *dominated* by some machine in $\mathcal{S}_i^H$. It follows that it must be that all equilibrium machines are in fact halting machines so that $x_i^\epsilon \in \mathcal{S}_i^H$ for all $i = 1, \ldots, n$.

PROOF: See Appendix.

Our next Lemma asserts that, once all players up to and including $n-1$ have revealed their cooperative intentions, the expected payoff obtained by the equilibrium strategy for player n must be (approximately) the cooperative one.

LEMMA 5: *Let any $(\epsilon, \mathcal{R})$-CTHE be given as in Lemma 4. Let $t_{n-1}^{c,\epsilon}$ and $h_{t_{n-1}^{c,\epsilon}}^{c,\epsilon}$ also be as in Lemma 4. Then*

$$\lim_{c\to\infty} \mathcal{E}_n(x_n^\epsilon, \overline{P}_{-n}^\epsilon | h_{t_{n-1}^{c,\epsilon}}^{c,\epsilon}, h_{t_{n-1}^{c,\epsilon}}^{c,\epsilon}) = \pi_n^e$$

PROOF: See Appendix.

Our next Lemma formalizes the fact that we can carry out our revelation argument 'by induction on the set of players'. More precisely, Lemma 6 below asserts that, along the consecutive histories generated by the signaling machine yielded by the Communication Lemma, *if* the equilibrium strategy for player $i+1$ achieves (approximately) the cooperative payoff (conditional on $h_{t_i^{c,\epsilon}}^{c,\epsilon}$), *then* the payoff achieved by the equilibrium strategy for player i (conditional on $h_{t_{i-1}^{c,\epsilon}}^{c,\epsilon}$) must also be (approximately) the cooperative one.

LEMMA 6: *Consider any sequence of $(\epsilon, \mathcal{R})$-CTHE with $\epsilon \to 0$ such that for each i, $\mathcal{R}_i$ is the set yielded by the Communication Lemma. Let the $2n$-tuple constituting such equilibria for each given ϵ be given by $\{x_i^\epsilon, P_i^\epsilon\}_{i=1}^n$. For each given ϵ and each given c let $\overline{P}_i^\epsilon$, $h_{t_i^{c,\epsilon}}^{c,\epsilon}$, $t_i^{c,\epsilon}$ and $x_i^{c,\epsilon}$ be the overall probabilities over machines, consecutive histories, dates and signalling machines yielded by Lemma 4. Then, for all $i < n$*

$$\lim_{c\to\infty,\, \epsilon\to 0} \mathcal{E}_{i+1}(x_{i+1}^\epsilon, \overline{P}_{-(i+1)}^\epsilon | h_{t_i^{c,\epsilon}}^{c,\epsilon}, h_{t_i^{c,\epsilon}}^{c,\epsilon}) = \pi_{i+1}^e \tag{10}$$

implies that

$$\lim_{c\to\infty,\, \epsilon\to 0} \mathcal{E}_i(x_i^\epsilon, \overline{P}_{-i}^\epsilon | h_{t_{i-1}^{c,\epsilon}}^{c,\epsilon}, h_{t_{i-1}^{c,\epsilon}}^{c,\epsilon}) = \pi_i^e \tag{11}$$

PROOF: See Appendix

We can now use Lemmas 5 and 6, by backwards induction on the set of players. Our next Lemma is a direct result of this operation, and it finally closes the proof of the optimality result.

LEMMA 7: *Let $\mathcal{R} = (\mathcal{R}_1, \ldots, \mathcal{R}_n)$ be as in (A.15) of the Communication Lemma. Then $\Pi^E(\mathcal{R}) = \pi^e$.*

PROOF: See Appendix.

6.3. *Existence*

The argument which shows that $\Pi^E(\mathcal{R})$ is not empty is constructive. Recall that the strategies in $\mathcal{R}_i$ of the Communication Lemma are all either cooperative or non-halting. Recall also that by Assumption 2 any halting strategy is a best response to any strategy profile which contains one or more non-halting strategies. Therefore, since the underlying game is a Common Interest game, all halting strategies in $\mathcal{R}_i$ are a best response to all strategies in $\mathcal{R}_{-i}$. It follows that we can construct an equilibrium in which only strategies in $(\mathcal{R}_1, .., \mathcal{R}_n)$ are given positive probability.

LEMMA 8: *Let $\mathcal{R} = (\mathcal{R}_1, .., \mathcal{R}_n)$ be as in (A.15) of the proof of the Communication Lemma. Then $\Pi^E(\epsilon, \mathcal{R}) \neq \emptyset$ for any ϵ with $0 \leq \epsilon \leq 1$.*

The proof of Lemma 8 is identical to the existence result (Lemma 7) in A-S. The reader should refer to A-S for the proof.

7. TWO-ACTION GAMES

In Assumption 1 we stipulated that G should have at least *three* pure strategies for each player. In the arguments we have used to prove Theorem 1, we appealed to this property only in the proof of Lemma 2, which in turn we used to prove the Communication Lemma 3. We now explain why Theorem 1, as in A-S does not depend on the stage game G having this property.

Let us go back to the intuition for the Communication Lemma provided in Section 1.2. Consider again Definition 7 of a quasi-cooperative strategy. In our arguments so far, we have put on the horizontal axis of Figure 2 the set $\overline{\mathcal{Q}}_i$ — the set of strategies which *do not* have the property that if they play a_i^e at any one time they are guaranteed to play a_i^e forever after. It follows that our revealing strategies x_i^* of Lemma 3 must have the property, besides being capable of 'revelation' of course, that if they ever play a_i^e then they *will* be guaranteed to play a_i^e forever after that date.

Suppose now that G only allows player i exactly *two* distinct pure strategies — a_i^e and a_i'. Then it is possible that in order to reveal player i's cooperative intentions in the long-run, strategy x_i^* has to distinguish itself from a strategy which plays a_i' well before the end of the signalling phase at time $\tilde{t}_i + v_i + 1$. Strategy x_i^* can only distinguish itself from such strategy by actually playing the cooperative action a_i^e, since the cardinality of $\mathcal{A}_i$ is precisely two. The problem with two actions is now obvious since strategy x_i^* is supposed to play a_i^e only at and after $\tilde{t}_i + v_i + 1$, but signalling may require it to play a_i^e earlier, *during* the signalling phase.

The problem we have outlined can be resolved by changing what is put on the horizontal axis of Figure 2 — by changing Definition 7 of quasi-cooperative and non quasi-cooperative computable strategies in $\mathcal{Q}_i$ and $\overline{\mathcal{Q}}_i$. For example, we could, as in A-S, construct our signalling strategies x_i^* so that they signal only 'every other period'; say only when t is an odd number. In all even numbered periods we ensure that an action $a_i' \neq a_i^e$ is played. The end of the signalling phase can now easily be 'marked' by the fact that x_i^* will play a_i^e twice in a row, at $\tilde{t}_i+v_i+1$ and $\tilde{t}_i+v_i+2$. This by construction cannot happen during the signalling phase, and therefore is enough to signal to the opposing player that cooperative behaviour from the part of x_i^* has begun.

To conclude, we need to specify exactly what set should be put on the horizontal axis of Figure 2. This is implicit in the construction for x_i^* we have just outlined. We need to place on the horizontal axis of Figure 2 the set of machines which *do not* have the property that if they play a_i^e *twice* in a row then they are guaranteed to play the cooperative action a_i^e in all subsequent periods. Formally, we need to give a new definition to the set $\mathcal{Q}_i$ as follows.

$$\mathcal{Q}_i \equiv \{x_i \in \mathcal{S}_i | \varphi_{x_i}(h_t) \downarrow = a_i^e,\ \varphi_{x_i}(h_{t+1}) \downarrow = a_i^e \text{ and } t' > t \Rightarrow \varphi_{x_i}(h_{t'}) \downarrow = a_i^e\ \forall h_{t'} \in H_{t'}\}$$

As in A-S, the results of this paper this paper still hold for the the two action case if we adopt this new definition of the set $\mathcal{Q}_i$ and if the players signal 'every other period' as described above (and more fully in A-S). The formal statement of the result and the proofs are omitted for the sake of brevity.[33]

8. CORRELATED TYPES

Throughout our analysis so far we have assumed that the players' computable supergame strategies are perturbed in a way which is *independent* across players. The probability of a given n-tuple of Turing machines x, given the profile of perturbations P is simply given by $P_1(x_1) \cdots P_n(x_n)$.

It is clearly possible to imagine that the perturbations of players's strategies be correlated across players. In this case the probability of a given n-tuple of machines x, would have to be written as $P(x)$ where P is not an array of probability distributions but an actual probability distribution over $\mathcal{S}$.

In this Section we argue that correlation across players' types as described above will, in general, destroy the equilibrium selection result (Theorem 1) we have proved above. For reasons of space, we argue this point informally rather than setting up all the extra notation and other preliminaries required.

It is enough to consider a 3 player Common Interest game to bring out the point clearly. Imagine now a perturbation of players' strategies $P(\cdot,\cdot,\cdot)$ as follows. Besides obeying the appropriate analogue of our assumption of admissibility above (cf. Definition 9), P has the property that (almost) all its probability mass is concentrated

[33] In Anderlini and Sabourian (1990) the assumption that G should have three actions for both players is never made. The version of the Communication Lemma, which is proved for the 2-player case in that paper uses precisely the signalling 'every other period' construction which we have just outlined. Therefore, the arguments in that paper can be adapted to prove our results for the n-player case.

on triples of machines which contain *one* cooperative machine (for either player 1 or 2 or 3), and *two* non-cooperative machines (for the other players) — machines which play some strategy which does not lead to the efficient payoff regardless of the history of play.

Let us now follow our signalling construction described informally in Section 1.2 above in this new case. Imagine that we try to select the efficient outcome by having players 1 and 2 signalling their cooperative intention in this sequence, and finally player 3 responding to their signals.

In the early stages of play, player 1 is taking a sequence of actions designed to signal that he (player 1) is playing a cooperative strategy with high probability. However, due to the correlation across players' types described above, this also signals that player 2 is in fact playing a non-cooperative strategy (since with high probability *at most one player* is cooperative).

When it comes to player 2 to start to signal his cooperative intentions, due to the signalling by player 1 before him, the posterior probability that player 2 is playing a non-cooperative strategy will be very high. More crucially though, as player 2 signals his cooperative intentions, he will *undo* the effect of player 1's signalling. In particular, due to correlation, as the posterior probability of player 2 playing cooperatively grows, the posterior probability that player 1 will play cooperatively declines. Clearly, we may never reach the point at which player 3 is sufficiently convinced that *both* players 1 and 2 will play cooperatively in the future. This (for some Common Interest games) is essential for player 3 to find it profitable to switch to playing cooperatively.

Thus, the signalling argument we developed for the un-correlated case does not work any longer in the case of correlated types. Observe, however, that this will be the case for some types of correlations across players types but not for others. For instance if the perturbation $P(\cdot,\cdot,\cdot)$ places (almost) all the probability mass on triples of machines which contain 2 cooperative strategies, correlation will 'help' to select cooperation as the unique equilibrium outcome in the model we have developed above.

9. A Countable Infinity of Players

9.1. Backwards Induction?

In this Section we show that the equilibrium selection result we have proved above does not extend to the case of a countable infinity of players. Thus, since Theorem 1 holds for any finite number of players n, our results below can be viewed as identifying a *discontinuity at infinity.* The 'backwards induction' argument described intuitively in Section 1.2 no longer applies when there are a countable infinity of players.

One way to think intuitively about the results of this Section is the following. The signalling possibility on which the proof of Theorem 1 rests involves $n - 1$ players signalling *sequentially* their intention to play cooperatively in the future. Each player takes at least one period to signal his cooperative intentions. It is then clear that if there are a countable infinity of players, there is 'not enough time' to complete the signalling phase which sustains the equilibrium selection result in the case of any finite number n of players. The logic of backwards induction breaks down in this case.

The intuition we have just given turns out to be correct. The rest of this Section is devoted to showing this rigorously.

9.2. Preliminaries

The logic which drives the 'counter-example' to Theorem 1 in the case of a countable infinity of players which we present below is very general as will become clear shortly. However, to fix ideas it is useful to focus on one particular stage game. Consider the following 'unanimity' Common Interest game with a countable infinity of players. Each player can either cooperate (choose C) or defect (choose D). The game is symmetric and the payoffs to each player are represented in Figure 3 below. So, for instance, if player i plays C and *all* other players also play C, then i gets a payoff of α. If, on the other hand, i plays C and one or more other players play D, then i gets a payoff of γ, and so on.

	All Others Play C	Some Others Play D
C	α	γ
D	δ	β

Figure 3

We assume that $\alpha > \delta$, $\beta > \gamma$ and $\alpha > \beta$, so that the game has only two (strict) pure strategy Nash equilibria, which are Pareto-ranked. These are 'all play C' yielding a payoff of α to all players, and 'all play D' yielding a payoff of β to all players. For the rest of this Section our interest is focussed on the infinite undiscounted repetition of the stage game in Figure 3.

The next preliminary issue of which we have to take care is the following. In the model which we developed in the previous sections, when there are finitely many players, it is quite obvious how a computable supergame strategy should be defined (cf. Definition 3). This is because any history of play of *finite length* is a *finite object* in a well defined sense. Therefore it can be coded in an appropriate way and then given as input to the Turing machine which embodies the given computable strategy. In the case of infinitely many players, matters are not so simple anymore. A history of play of *finite length* is an *infinite object* in a well defined sense. In the case of a countable infinity of players, histories of play of finite length *cannot* be coded into the natural numbers.

There are many ways to resolve this issue. For instance, we could postulate that each player is given as input some 'statistic' of the past history of play which can be coded into $\mathbb{N}$. Alternatively, we could assume that the players can use a particular specification of Turing machines known as 'two-tape' Turing machines.[34] In this case the entire history of finite length could be placed on the 'read-only' tape of the machine, thus giving it potential access to the entire history. Each machine would in

[34]Hopcroft and Ullman (1979) contains a full discussion of the operations of Turing machines with multiple tapes. For reasons of space we do not go into any further details here.

fact 'scan' only a finite portion of such history in any case.

Several other possibilities exist. However, they each involve a 'choice of model' which we want to avoid. Therefore, we present or results of this Section in a general framework which is capable of containing as a special case all possible modelling choices which can be made at this stage of which we are aware. The results of this Section are extremely robust from this point of view.

For each player $i = 1, 2, \ldots$ let $\tilde{\mathcal{S}}_i$ be the set of i's supergame strategies[35] in the infinite undiscounted repetition of the stage game described in Figure 3. Let $\hat{\mathcal{S}}_i^H$ be the 'augmented supergame strategy set' for i, derived from $\tilde{\mathcal{S}}_i$ as follows. For each element of s_i of $\tilde{\mathcal{S}}_i$, let $\hat{\mathcal{S}}_i^H$ contain a countable infinity of elements $s_{i1}, s_{i2}, \ldots$ which have distinct 'names' but which represent the same supergame strategy.[36] Thus s_{ij} and s_{im} represent the same supergame strategy for all i, j and m. Next, let $\hat{\mathcal{S}}_i^{\overline{H}}$ be a set containing a countable infinity of 'non-halting' supergame strategies,[37] and let $\hat{\mathcal{S}}_i$ be the union of $\hat{\mathcal{S}}_i^H$ and $\hat{\mathcal{S}}_i^{\overline{H}}$.

Notice that each set $\hat{\mathcal{S}}_i$ contains both computable and non-computable strategies. Our results of this Section hold both for the case in which players are not restricted at all in their choice of supergame strategy in $\hat{\mathcal{S}}$, and when they are restricted to an arbitrary sub-set of $\hat{\mathcal{S}}_i$, provided that this set contains at least some non-cooperative strategies. This motivates our next assumption.[38]

[35]For reasons of space, we do not re-define here all the standard pieces of notation concerning the repeated game at hand. They are the same as in Section 2, except for the notation used for repeated game strategies.

[36]We introduce these 'copies' of each strategy purely to ensure the formal consistency of the general model we develop here with the Turing computability framework we have used so far. Recall that for each computable function there is a countably infinity of Turing machines which compute the given function. Thus in each $\mathcal{S}_i$ described in Definition 4 there is a countable infinity of Turing machines computing the same computable supergame strategy.

[37]The reason we introduce non-halting supergame strategies is, again, purely to ensure the formal consistency of the general model we develop here with the Turing computability framework we have used so far. The non-halting supergame strategies introduced here should be thought of in exactly the same way as the non-halting supergame strategies which we introduced in Definition 4 above for the model with a finite number of players. These are supergame strategies computed by a Turing machine which does not halt on any input.

[38]Notice that our notation for strategy sets in this section is the same as the one we used in the model that we developed and analyzed in the previous sections. We do this to help the exposition,

ASSUMPTION 3: *Each player $i = 1, 2, \ldots$ is restricted to choose a supergame strategy in a set $\mathcal{S}_i \subseteq \hat{\mathcal{S}}_i$. Each $\mathcal{S}_i$ is assumed to be a (weak) subset of $\hat{\mathcal{S}}_i$ such that*

$$\exists\, s_i \in \mathcal{S}_i \quad \textit{such that} \quad s_i(h_t) = D \quad \forall\, h_t \in \mathcal{H}_t \tag{12}$$

We denote by $\mathcal{S}_i^H = \mathcal{S}_i \cap \hat{\mathcal{S}}^H$ and $\mathcal{S}^{\overline{H}} = \mathcal{S}_i \cap \hat{\mathcal{S}}^{\overline{H}}$ the sets of halting and non-halting strategies in $\mathcal{S}_i$ respectively. Finally, we let $\mathcal{S}^H \equiv \mathop{\times}\limits_{i=1}^{\infty} \mathcal{S}_i^H$, $\mathcal{S}^{\overline{H}} \equiv \mathop{\times}\limits_{i=1}^{\infty} \mathcal{S}_i^{\overline{H}}$, and $\mathcal{S} \equiv \mathop{\times}\limits_{i=1}^{\infty} \mathcal{S}_i$.

In what follows we assume that the supergame payoffs to each player are defined for any profile of strategies in $\mathcal{S}$. We assume that both the dominance and the best response parts of Assumption 2 hold for our model with a countable infinity of players. We do not repeat the formal details here. Simply recall that we are assuming that all non-halting strategies are dominated by halting strategies, and that any halting strategy is a best response to any strategy profile which contains one or more non-halting strategies.

In our analysis of the previous sections we have restricted attention to perturbations which are admissible according to Definition 9. Our results below hold regardless of any restriction imposed on the set of perturbations which are allowed. This motivates the next assumption we make.

ASSUMPTION 4: *For each $i = 1, 2, \ldots$, let $\hat{\Delta}_i$ be the set of all possible probability measures on $\mathcal{S}_i$.*[39] *For every i, let Δ_i be the set of probability distributions wich are allowed for player i. We call this the set of 'admissible' probability distributions for player i.*[40] *for player i. The set Δ_i is only assumed to be (weakly) contained in*

as the strategy sets (and their subsets) which we use in this section are the analogues of the ones in our previous model. Of course, formally we are actually defining new strategy sets for the players.

[39]We are implicitly assuming that $\hat{\Delta}_i$ is not empty for every i, and that all measures in $\hat{\Delta}_i$ are such that each player's expected payoffs are well defined. The details are irrelevant to our results below.

[40]Notice that, once again, we are using the same terminology as for our model of the previous sections. This is simply to ease the exposition since the set of admissible distributions defined here is the analogue of the set of admissible distribution in our previous model. Of course, formally the two definitions are distinct.

$\hat{\Delta}_i$. *The perturbation of each player's supergame strategy, from now on is assumed to be an element* P_i *of* Δ_i. *Given a set* $\mathcal{R}_i \subseteq \mathcal{S}_i$, *the set of admissible probability distributions for player* i *which satisfy* $\mathcal{R}_i \subseteq \text{supp}(P_i)$ *is denoted by* $\mathcal{P}_i(\mathcal{R}_i)$.[41]

Our last preliminary task is to modify the concepts of $(\epsilon, \mathcal{R})$-CTHE and of $\mathcal{R}$-CTHE to suit the model we have developed in this Section. The modified strategy sets and sets of admissible perturbations do not present a problem. On the other hand handling a countable infinity of independent[42] probability distributions does require some special care.

The probability assigned to the perturbation of each player's strategy, s_i, will be denoted by $\epsilon_i \in (0,1)$. Notice that in the case of a countable infinity of players it is essential that this probability be allowed to depend on the identity of the player. This is because we will require that the sequence $\epsilon_1, \epsilon_2, \ldots$ be such that $\prod_{i=1}^{\infty}(1-\epsilon_i) = 1-\epsilon$ with ϵ a real number strictly between 0 and 1.[43] Notice that $1-\epsilon$ therefore represents the probability of the event 'all players play their equilibrium strategy'. To summarize, if we denote by s_i^ϵ the equilibrium strategy of each player i and by P_i^ϵ its perturbation, the 'overall' probability distribution on player i's strategies (the equivalent of (5) for the case of infinitely many players) is given by

$$\overline{P}_i^\epsilon(s_i) = \begin{cases} (1-\epsilon_i) + \epsilon_i P_i^\epsilon(s_i) & \text{if } s_i = s_i^\epsilon \\ \epsilon_i P_i^\epsilon(s_i) & \text{if } s_i \neq s_i^\epsilon \end{cases} \tag{13}$$

Given an infinite array of probability distributions as in Assumption 4, the strategy sets of Assumption 3 and a sequence $\{\epsilon_i\}_{i=1}^{\infty}$, it is clear how (2) can be modified to represent the set of strategies in $\mathcal{S}_i$ which are a best response for i to the

[41] In what follows we will assume that the sets Δ_i and $\mathcal{R}_i$ are chosen in a mutually consistent way in the sense that $\mathcal{P}_i(\mathcal{R}_i) \neq \emptyset$.

[42] As we have pointed out in Section 8 above, correlation across players' types may, by itself, destroy our selection result (Theorem 1) above. The intuition about this point which we developed in Section 8 generalizes to the case of a countable infinity of players. Therefore, since our results of this Section show that the cooperative outcome *cannot* be selected as the unique viable one, using independent perturbations across players strengthens our results.

[43] Notice that (given that each $\epsilon_i > 0$) this will be the case if and only if $\sum_{i=1}^{\infty} \epsilon_i = \xi$ with ξ a number strictly between 0 and 1.

strategy profile s_{-i} and the perturbations P_{-i}, given $\{\epsilon_i\}_{i=1}^{\infty}$. We denote this set by $\mathcal{B}_i(s_{-i}, P_{-i}, \{\epsilon_i\}_{i=1}^{\infty})$.

We are now ready to state our new definition of $(\epsilon, \mathcal{R})$-CTHE for the model with a countable infinity of players.

DEFINITION 15: *An $(\epsilon, \mathcal{R})$-CTHE for the model with an countable infinity of players is an array $\{s_i^\epsilon, P_i^\epsilon, \epsilon_i\}_{i=1}^{\infty}$ with $\prod_{i=1}^{\infty}(1-\epsilon_i) = 1-\epsilon$, $s_i^\epsilon \in \mathcal{S}_i$ and $P_i^\epsilon \in \mathcal{P}_i(\mathcal{R}_i)$ such that for all $i = 1, 2 \ldots$*

$$s_i^\epsilon \in \mathcal{B}_i(s_{-i}^\epsilon, P_{-i}^\epsilon, \{\epsilon\}_{i=1}^{\infty})$$

The set $\Pi^E(\mathcal{R})$ of equilibrium arrays for a given pair $(\epsilon, \mathcal{R})$ is denoted by $E(\epsilon, \mathcal{R})$, and the set of corresponding equilibrium long-run payoff vectors by $\Pi^E(\epsilon, \mathcal{R})$.

The set of $\mathcal{R}$-CTHE is the limit of the set of $(\epsilon, \mathcal{R})$-CTHE as the noise vanishes, exactly as in in Definition 12. We do not repeat it here.

9.3. Results

We are now ready to state our results for the model with a countable infinity of players. It is instructive to start with a Lemma which makes precise the intuition that any history of finite length will not be capable of signalling more than the cooperative intentions of a *finite* subset of players. A history being 'consistent' with a given strategy has the same meaning as in Definition 14.

LEMMA 9: *Let any $(\epsilon, \mathcal{R})$-CTHE, $\{s_i^\epsilon, P_i^\epsilon, \epsilon_i\}_{i=1}^{\infty}$, be given. Then, any history of finite length h_t which takes place with probability strictly greater than zero has the following property. There exists an n (which may depend on h_t) such that for all $i \geq n$, strategy s_i^ϵ is consistent with history h_t.*

PROOF: See Appendix.

In other words, along any history which is generated with positive probability in any $(\epsilon, \mathcal{R})$-CTHE, the posterior probabilities are such that all but a finite set of equilibrium strategies have probability at least as large as $1 - \epsilon_i$.

Given Lemma 9 it is easy to see how if the equilibrium strategies are all non-cooperative, they will remain optimal along any history of play which takes place with positive probability. This is the intuition behind our next and last result.

THEOREM 2: *Consider the infinite undiscounted repetition of the Common Interest game described in Figure 3. Fix any corresponding strategy sets as in Assumption 3 and any sets of admissible distributions as in Assumption 4. Let also any array of sets $\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2, \ldots)$ be given. Then there exists a $\mathcal{R}$-CTHE equilibrium of the model such that $s_i^\epsilon(h_t) = D$ for all $i = 1, 2, \ldots$ and for all $h_t \in \mathcal{H}$. In other words,*

$$(\beta, \beta, \ldots) \in \Pi^E(\mathcal{R})$$

PROOF: See Appendix.

10. CONCLUDING REMARKS

We have modified the standard model of an infinitely repeated n-player Common Interest game with no discounting modified in two crucial ways. We have restricted the players' to use supergame strategies which are *computable* in the sense of *Church's Thesis.* We have also perturbed the repeated game with computable probability distributions over computable strategies which have sufficiently large support.

In the framework which we have developed the players can use the early stages of the game to signal to the others their intention to play cooperatively in the long-run. This, given the Common Interest structure of the stage game, *selects* the cooperative payoffs of the repeated game as the only possible ones. As the noise becomes negligible, all the Computable Trembling Hand Equilibria of an infinitely repeated undiscounted n-player Common Interest game are cooperative. The results are a generalization of A-S to n-player repeated Common Interest games. In this paper, the signalling technique used in A-S had to be adapted so that it could be applied by 'backwards induction' to all but one players signalling sequentially their cooperative intentions.

We have also demonstrated that our equilibrium selection result fails if there are a countable infinity of players. The backwards induction argument on which our main result relies fails in this case. Intuitively, since the players have to signal their cooperative intentions sequentially, if there are infinitely many players there is 'not enough time' for 'all players but one' to signal that they want to cooperate.

A particular concern regarding the perturbations of a $\mathcal{R}$-CTHE are the *joint* effects of admissibility and large support. It turns out that the interplay between these two requirements is more complex than would seem at first sight. Since admissibility imposes a degree of regularity on the support of the perturbations it is natural to ask to what extent this prevents the support of the perturbations to be larger than the sets $\mathcal{R}_i$ which appear in the statement of Theorem 1. In A-S and in Anderlini (1990) this issue is addressed in some detail. It turns out that, apart from restrictions on the actual support of the perturbations, the joint assumptions of admissibility and large support do not restrict at all how much probability weight is put on what set of strategies.

The basic technique used to prove Theorem 1 is powerful enough to yield mildly weaker cooperation results for the cases of positive but vanishing discounting and finite but large time horizon, keeping the perturbations fixed as discounting vanishes and time horizon increases. For reasons of space we do not present a formal version of these results. It involves a very considerable amount of additional notation and detail. The full-blown formal analysis for the 2-player games is reported as Theorems 2 and 3 in Anderlini and Sabourian (1990). In A-S and in Anderlini and Sabourian (1990), we discuss the problems that might arise in extending the results to the case of vanishing discounting and finite but large horizon if the perturbations are changing along the sequence of equilibria as the discount factor or the time horizon change.

APPENDIX

We start with some Definitions and Theorems which are standard in the computability literature. All the results which are stated without proof can be found in A-S, Cutland (1980) or Rogers (1967).

DEFINITION A.1: *A computable function $f : \mathbb{N}^m \to \mathbb{N}$ is called a total computable function if and only if $f(e_1, \cdots, e_m) \downarrow \ \forall (e_1, \cdots, e_m) \in \mathbb{N}^m$.*

THEOREM A.1 [s-m-n]: *For each $m \geq 0$ and $n \geq 1$ there exists a total computable function of $m+1$ variables f such that $\forall\ e \in \mathbb{N}$ and $\forall\ (h_1, \cdots, h_m, h_{m+1}, \cdots, h_{m+n}) \in \mathbb{N}^{m+n}$ we have*

$$\varphi_e(h_1, \cdots, h_{m+n}) \simeq \varphi_{f(e,h_1,\cdots,h_m)}(h_{m+1}, \cdots, h_{m+n})$$

THEOREM A.2 [Universal Turing Machine]: *Given any $m \geq 1$, there exists a number u, such that*

$$\varphi_u(n, e_1, \cdots, e_m) \simeq \varphi_n(e_1, \cdots, e_m) \ \ \forall\ (n, e_1, \cdots, e_m) \in \mathbb{N}^{m+1}$$

DEFINITION A.2: *A set $S \subseteq \mathbb{N}$ is recursively enumerable (abbreviated r.e.) if and only if it is equal to the domain of a computable function. Formally, $S \subseteq \mathbb{N}$ is r.e. if and only if for some $n \in \mathbb{N}$ we have $\varphi_n(e) \downarrow \Leftrightarrow\ e \in S$. (The empty set is r.e. since the function 'nowhere defined' is computable.)*

THEOREM A.3: *A set $S \subseteq \mathbb{N}$ is r.e. if and only if it is the range of a computable function. Formally $S \subseteq \mathbb{N}$ is r.e. if and only if there exists a Turing machine n such that*

$$e \in S \ \ \Leftrightarrow\ \ \exists v \text{ such that } \varphi_n(v) = e \tag{A.1}$$

Given an r.e. set S, a Turing machine n with the property in (A.1) is said to 'enumerate' S. We refer to $\varphi_n(v)$ as the v-th element in the enumeration of S.

THEOREM A.4: *An infinite set $S \subseteq \mathbb{N}$ is r.e. if and only if it is the range of a one-to-one total computable function of one variable. Formally, given an infinite set $S \subseteq \mathbb{N}$, S is r.e. if and only if there exists a Turing machine n computing a total computable function such that $v \neq v' \Rightarrow \varphi_n(v) \neq \varphi_n(v')$ and*

$$e \in S \ \ \Leftrightarrow\ \ \exists v \text{ such that } \varphi_n(v) = e$$

The Turing machine n is said to enumerate S 'without repetitions'.

THEOREM A.5: *Any finite set $S \subset \mathbb{N}$ is r.e. and can be enumerated without repetitions by a Turing machine n as follows. Let $\|S\|$ be the cardinality of S. Then $\varphi_n(v) \downarrow \Leftrightarrow v \in \{0, 1, \cdots, \|S\| - 1\}$, $v \neq v'$ $\varphi_n(v) \downarrow$ and $\varphi_n(v') \downarrow \Rightarrow \varphi_n(v) \neq \varphi_n(v')$, and finally $e \in S \Leftrightarrow \exists v$ such that $\varphi_n(v) = e$.*

THEOREM A.6 [Pseudo-Fixed Point]: *For any computable function f of $m+1$ variables, there exists $\overline{x} \in \mathbb{N}$ such that*

$$\varphi_{\overline{x}}(e_1, \cdots, e_m) \simeq f(\overline{x}, e_1, \cdots, e_m) \ \forall (e_1, \cdots, e_m) \in \mathbb{N}^m$$

LEMMA A.1: *The set $\text{supp}(P_i)$ is r.e. for any $P_i \in \Delta^\infty$ which is computable in the sense of Definition 6. It follows that the same statement is true for any $P_i \in \Delta^\infty$ which is admissible in the sense of Definition 9.*

LEMMA A.2: *If $P_i \in \Delta^\infty$ is admissible according to Definition 9, then the set $\overline{Q}_i^{P} \equiv \text{supp}(P_i) \cap \overline{Q}_i$ is r.e.*

LEMMA A.3: *If $P_i \in \Delta^\infty$ is admissible according to Definition 9, then the set $Q_i^{P_i} \equiv \text{supp}(P_i) \cap Q_i$ is r.e.*

PROOF OF LEMMA 2: A machine d which computes g_i can be constructed as follows. Start by computing the value of $\tilde{t}_i$ as in Lemma 1 for all i. If any of these computations do not halt, leave the output of φ_d undefined. If these computations halt, proceed further as follows.

Given the values of $\tilde{t}_i$, compute the values of v_i for all i as $v_1 = 0$ and $v_i = i - 1 + \sum_{j<i} \tilde{t}_j$ for $i = 2, \ldots, n-1$.

Given the values of v_i, applying Theorem A.2 twice, it is feasible to compute the result of $\varphi_{\varphi_{m_i}(\tau - v_i - 1)}(h_\tau)$, $\forall h_\tau \in \mathcal{H}_\tau$ and $\forall\ v_i < \tau \leq \tilde{t}_i + v_i$ and for all $i = 1, \ldots, n-1$. If any of these computations do not halt, leave the output of φ_d undefined. If all these computations halt proceed as follows.

Applying Theorem A.2 once, it is possible to compute the result of $\varphi_{y_i}(h_\tau)$ $\forall\ h_\tau \in \mathcal{H}_\tau$, $\forall\ \tau < v_i$ and for all $i = 1, \ldots, n-1$. If any of these computations do not halt, leave the output of φ_d undefined. If all all the computations yielding $\varphi_{y_i}(h_\tau)$ halt, then proceed as follows.

Applying Theorem A.2 again, it is possible to compute the result of $\varphi_{y_i}(h_{v_i})$ $\forall\ h_{v_i} \in \mathcal{H}_{v_i}$ and $\forall\ i$. If any of these computations do not halt, leave the output of φ_d undefined. If all all the computations halt, then proceed as follows.

Check whether h_t has $t \geq \tilde{t}_i + v_i + 1$ or not. If this is the case, it is clearly feasible to simply output the cooperative action a_i^e, irrespective of the other inputs.

Check whether h_t has $t < v_i$. If this is the case then set the output of $\varphi_d(x_i, x_{-i}, y_i, p, q, m, k, h_t)$ to be equal to $\varphi_{y_i}(h_t)$.

Check whether h_t has $t = v_i$. If this is the case, then set the output of $\varphi_d(x_i, x_{-i}, y_i, p, q, m, k, h_t) \in \mathcal{A}_i$ to be different from *both* a_i^e and $\varphi_{y_i}(h_t)$. The latter step is feasible by Church's thesis and by Assumption 1 which guarantees that $\mathcal{A}_i$ contains at least three distinct elements.

Lastly, if $t \leq \tilde{t}_i + v_i$, and $t > v_i$, then set the output of $\varphi_d(x_i, x_{-i}, y_i, p, q, m, k, h_t) \in \mathcal{A}_i$ to be different from *both* a_i^e and $\varphi_{\varphi_{m_i}(t-v_i-1)}(h_t)$. Again, the last step is feasible by Church's thesis and by Assumption 1 which guarantees that $\mathcal{A}_i$ contains at least three distinct elements.

Clearly, the above algorithm defining d halts on the required set of inputs and is such that for all $(x_i, x_{-i}, y_i, p, q, m, k) \in \mathbb{N}^{4n+2}$, either $\varphi_d(x_i, x_{-i}, y_i, p, q, m, k, h_t) \downarrow \forall h_t$, or $\varphi_d(x_i, x_{-i}, y_i, p, q, m, k, h_t) \uparrow \forall h_t$. ■

PROOF OF LEMMA 3 [COMMUNICATION LEMMA]: As we have outlined above, the proof involves four main manipulations. The first is to use the *s-m-n* Theorem A.1 so as to guarantee that the 'parameterization' of x_i^* is correctly set up; the second is to use Theorem A.6 to carry out the pseudo fixed point step outlined intuitively in Section 1.2. The third is to use the construction in Lemma 1 and Lemma 2 to ensure revelation of the cooperative intentions of strategy x_i^*. The fourth and final step is to see that $\mathcal{R}_i \subset \mathcal{S}_i$ of the statement of the lemma can be put equal to the set of all possible x_i^* yielded by the parameterization of all possible profiles of distributions $P \in \mathcal{P}$, all possible y_i and all possible precision values k.

The *s-m-n* Theorem A.1 guarantees that there exists a total computable function $s : \mathbb{N}^{4n+2} \to \mathbb{N}$ (recall that we are setting $(x, p, q, m) = \{x_i, p_i, q_i, m_i\}_{i=1}^n$) such that

$$\varphi_{s(x_i, x_{-i}, y_i, p, q, m, k)}(h_t) \simeq \varphi_{x_i}(x_{-i}, y_i, p, q, m, k, h_t) \quad \forall\, (x_i, y_i, p, q, m, k, h_t) \in \mathbb{N}^{4n+3} \tag{A.2}$$

By Theorem A.2 and by Church's thesis, f_i from $\mathbb{N}^{4n+3}$ to $\mathbb{N}$ defined by

$$f_i(x_i, x_{-i}, y_i, p, q, m, k, h_t) \equiv g_i(s(x_i, x_{-i}, y_i, p, q, m, k), x_{-i}, y_i, p, q, m, k, h_t) \tag{A.3}$$

where g_i is as in Lemma 2, is a computable function. By the pseudo fixed point Theorem A.6 we then have that $\exists\, \overline{x}_i \in \mathbb{N}$ such that

$$\varphi_{\overline{x}_i}(x_{-i}, y_i, p, q, m, k, h_t) \simeq f_i(\overline{x}_i, x_{-i}, y_i, p, q, m, k, h_t) \tag{A.4}$$

for all $(x_{-i}, y_i, p, q, m, k, h_t) \in \mathbb{N}^{4n+2}$. Substituting (A.2) and (A.3) into (A.4), we finally obtain that for all $(x_{-i}, y_i, p, q, m, k, h_t) \in \mathbb{N}^{4n+2}$ we must have that

$$\varphi_{s(\overline{x}_i, x_{-i}, y_i, p, q, m, k)}(h_t) \simeq g_i(s(\overline{x}_i, x_{-i}, y_i, p, q, m, k), x_{-i}, y_i, p, q, m, k, h_t) \tag{A.5}$$

Consider now a fixed $P = (P_1, ..P_n) \in \mathcal{P}$ and its 'parameterization' (its basis of Definition 10) $(p, q, m) \in \mathbb{N}^{3n}$. Next, for any i, for any $y_i \in \mathbb{N}$ and for any given precision parameter k, define $x_i^*(y_i, k) \in \mathbb{N}$ and $\tilde{t}_i$ as follows

$$x_i^*(y_i, k) \;=\; s(\overline{x}_i, \overline{x}_{-i}, y_i, p, q, m, k) \tag{A.6}$$

$$\tilde{t}_i \;=\; d_i(\overline{x}_i, p_i, q_i, m_i, k) \tag{A.7}$$

where each $\overline{x}_i$ is the pseudo fixed point of equation (A.4) and d_i is defined as in (3) of Lemma 1. Also let $v_1 = 0$ and $v_i = i - 1 + \sum_{j<i} \tilde{t}_j$ for $i = 2, \ldots, n$ and set

$$t_i = \tilde{t}_i + v_i + 2 = i + 1 + \sum_{j \le i} \tilde{t}_j \quad \forall\, i = 1, \ldots, n \tag{A.8}$$

Suppose that for such given basis, given profile of machines $y = (y_1, .., y_n)$ and a given precision parameter k we have that $\forall\, i$

$$\varphi_{p_i}(x_i^*(y_i, k)) > 0 \tag{A.9}$$

Then it follows from (A.4), (A.6), (A.8), and the construction of g_i in Lemma 2, that

$$x_i^*(y_i, k) \in \mathcal{C}_i^{t_i - 1} \tag{A.10}$$

Again, since g_i is as in Lemma 2, by construction we have that

$$\mathrm{h}_{t+v_i+1}(x_i^*(y_i, k), x_{-i})) \neq \mathrm{h}_{t+v_i+1}(\varphi_{m_i}(t-1), x_{-i})) \quad \forall\, x_{-i} \in \mathcal{S}_{-i} \;\; \forall\, t \le \tilde{t}_i \tag{A.11}$$

Since $\tilde{t}_i$ is computed as in (3) of Lemma 1, (A.9) and (A.11) imply that

$$\begin{aligned} &\frac{1}{k} P_i(x_i^*(y_i, k) | \mathrm{h}_{\tilde{t}_i+v_i+1}(x_i^*(y_i, k), x_{-i})) > \\ &\qquad\qquad P_i(\overline{\mathcal{Q}}_i^P | \mathrm{h}_{\tilde{t}_i+v_i+1}(x_i^*(y_i, k), x_{-i})) \quad \forall\, x_{-i} \in \mathcal{S}_{-i} \end{aligned} \tag{A.12}$$

Note now that it follows from (A.10) that at $t_i - 1$ strategy $x_i^*(y_i, k)$ plays the cooperative action a_i^c. Therefore, since $x_i^*(y_i, k) \in \mathcal{S}_i^H$, by Definition 7 of $\mathcal{Q}_i$, and by the definition of t_i given in (A.8), (A.12) implies that

$$\frac{1}{k}P_i(x_i^*(y_i,k)|\mathrm{h}_{t_i}(x_i^*(y_i,k),x_{-i})) > P_i(\overline{\mathcal{C}}_i^{t_i}|\mathrm{h}_{t_i}(x_i^*(y_i,k),x_{-i})) \quad \forall\, x_{-i} \in \mathcal{S}_{-i} \tag{A.13}$$

Conditions (A.13) and (A.10) in turn imply that

$$P_i(\mathcal{C}_i^{t_i}|\mathrm{h}_{t_i}(x_i^*(y_i,k),x_{-i})) > kP_i(\overline{\mathcal{C}}_i^{t_i}|\mathrm{h}_{t_i}(x_i^*(y_i,k),x_{-i})) \quad \forall\, x_{-i} \in \mathcal{S}_{-i} \tag{A.14}$$

Because $P_i(\mathcal{C}_i^{t_i}|\mathrm{h}_{t_i+1}(x_i^*(y_i,k),x_{-i})) + P_i(\overline{\mathcal{C}}_i^{t_i}|\mathrm{h}_{t_i+1}(x_i^*(y_i,k),x_{-i})) = 1$, setting $k = c$ in (A.14) immediately gives condition *(iii)* of the Communication Lemma. Conditions *(i)* and *(ii)* of the Lemma follow trivially the fact that $v_i = t_{i-1}+1$ and from the definition of g_i in Lemma 2, *(iv)* of the Lemma follows from (A.10) and *(v)* of the Lemma follow from (A.8).

Finally, to close the argument we must now define each $\mathcal{R}_i$ so as to ensure that for all i (A.9) is satisfied for all $P = (P_1, \ldots, P_n)$ such that $P_i \in \mathcal{P}_i(\mathcal{R}_i)$, for all $y = (y_1, \ldots, y_n) \in \mathbb{N}^n$ and for all $k \in \mathbb{N}$. Clearly, by Lemma 2 it is sufficient to set

$$\mathcal{R}_i \equiv \operatorname*{Range}_{(y_i,p,q,m,k)\in\mathbb{N}^{3n+2}} s(\overline{x}_i, \overline{x}_{-i}, y_i, p, q, m, k) \tag{A.15}$$

Finally, notice that by Lemma 2 it must be that $\mathcal{R}_i \subseteq \mathcal{S}_i$. This concludes the proof of the Lemma. ∎

PROOF OF LEMMA 4: For each player $i = 1, \ldots, n-1$, simply set $x_i^{c,\epsilon}$ and $t_i^{c,\epsilon}$ to be the machine x_i^* and time period t_i of the Communication Lemma when the admissible probability distribution is P_i^ϵ, the arbitrary machine $y_i = x_i^\epsilon$ and the precision level is c. Notice that (7) follows directly from the fact that the histories $h_{t_1^{c,\epsilon}}^{c,\epsilon}, \ldots, h_{t_{n-1}^{c,\epsilon}}^{c,\epsilon}$ are consecutive given $(x_1^{c,\epsilon}, \ldots, x_{n-1}^{c,\epsilon}, x_n^\epsilon)$ and *(i)* of the Communication Lemma. From (7) and from the fact that $x_i^{c,\epsilon} \in \mathcal{R}_i \subseteq \operatorname{supp}(\overline{P}_i^\epsilon)$ for all $i = 1, \ldots, n-1$, we immediately get that each $h_{t_i^{c,\epsilon}}^{c,\epsilon}$ occurs with positive probability as required.

Now condition (8) follows directly from *(ii)* and *(iii)* of the Communication Lemma. Finally, condition (9) follows from *(iv)* of the Communication Lemma. ∎

LEMMA A.4: *Let any $(\epsilon, \mathcal{R})$-CTHE be given as in Lemma 4. For each $i = 1, \ldots, n-1$ let $\overline{P}_i^\epsilon$, $h_{t_i^{c,\epsilon}}^{c,\epsilon}$, $t_i^{c,\epsilon}$ and $x_i^{c,\epsilon}$ be the total probabilities, consecutive histories, dates and signalling machines yielded by Lemma 4.*

For each possible values of c and ϵ let also an arbitrary profile of probability distributions over $\mathcal{S}$ be given. Denote this array by $\tilde{P}^{c,\epsilon} = (\tilde{P}_1^{c,\epsilon}, \ldots, \tilde{P}_n^{c,\epsilon})$. Then

(a) For any $j, i = 1, \ldots, n-1$ and $k = 1, \ldots, n$ if

$$\overline{P}_j^\epsilon(\mathcal{C}_j^{t_i^{c,\epsilon}}|h_{t_i^{c,\epsilon}}^{c,\epsilon}) > \frac{c}{c+1} \tag{A.16}$$

then

$$\lim_{c\to\infty,\,\epsilon\to 0}\left|\mathcal{E}_k(x_j^{c,\epsilon},\widetilde{P}_{-j}^{c,\epsilon},h_{t_i^{c,\epsilon}}^{c,\epsilon})-\mathcal{E}_k(\overline{P}_j^{\epsilon}|h_{t_i^{c,\epsilon}}^{c,\epsilon},\widetilde{P}_{-j}^{c,\epsilon},h_{t_i^{c,\epsilon}}^{c,\epsilon})\right|=0 \tag{A.17}$$

(b) For any $k,j=1,\ldots,n$, and for any history h_t which takes place with positive probability, whenever x_j^ϵ is consistent with h_t we have that

$$\lim_{c\to\infty,\,\epsilon\to 0}\left|\mathcal{E}_k(x_j^{\epsilon},\widetilde{P}_{-j}^{c,\epsilon},h_t)-\mathcal{E}_k(\overline{P}_j^{\epsilon}|h_t,\widetilde{P}_{-j}^{c,\epsilon},h_t)\right|=0 \tag{A.18}$$

PROOF: (a) It follows from (A.16) and (9) that for any ϵ

$$\left|\mathcal{E}_k(x_j^{c,\epsilon},\tilde{P}_{-j}^{c,\epsilon},h_{t_i^{c,\epsilon}}^{c,\epsilon})-\mathcal{E}_k(\overline{P}_j^{\epsilon},\tilde{P}_{-j}^{c,\epsilon},h_{t_i^{c,\epsilon}}^{c,\epsilon})\right|\le\theta_k(1-\frac{c}{c+1}) \tag{A.19}$$

where $\theta_k = \max_{a\in\mathcal{A}}\pi_k(a) - \min_a \pi_k(a)$. Since θ_k does not depend on ϵ or c, (A.17) follows from taking the limits of both sides of (A.19) as $c\to\infty$ and $\epsilon\to 0$.

(b) If x_j^ϵ is consistent with history h_t,it follows that $\overline{P}_j^\epsilon(x_j^\epsilon|h_t)\ge(1-\epsilon)$. Therefore

$$\left|\mathcal{E}_k(x_j^{\epsilon},\widetilde{P}_{-j}^{c,\epsilon},h_t)-\mathcal{E}_k(\overline{P}_j^{\epsilon}|h_t,\widetilde{P}_{-j}^{c,\epsilon},h_t)\right|\le\theta_k\epsilon \tag{A.20}$$

Since θ does not depend on ϵ or c, (A.18) follows from taking the limits of both sides of (A.20) as $c\to\infty$ and $\epsilon\to 0$. ∎

PROOF OF LEMMA 5: From (7) and (8) of Lemma 4 we know that a strategy for player n that always cooperates after history $h_{t_{n-1}^{c,\epsilon}}^{c,\epsilon}$ would have an expected continuation pay-off not smaller than

$$\left(\frac{c}{c+1}\right)^{n-1}\pi_n^e+\left[1-\left(\frac{c}{c+1}\right)^{n-1}\right]b_n \tag{A.21}$$

where b_n is the worst payoff player n can achieve in any outcome of the stage game. Recall that x_n^ϵ must be optimal after any history which takes place with positive probability and that by Lemma 4 we know that $h_{t_{n-1}^{c,\epsilon}}^{c,\epsilon}$ does take place with positive probability. Therefore, taking the limit as c tends to infinity in (A.21) is clearly enough to prove the claim. ∎

LEMMA A.5: *Recall that $V\subset\mathbb{R}^n$ is the payoff space of the stage game G. Let V^* be its convex hull. Then*

$$\forall\,\alpha>0\;\;\exists\beta>0\;\;\textit{such that}\;\;\pi\in V^*\;\;\textit{and}\;\;|\pi_i^e-\pi_i|<\beta\;\;\Rightarrow\;\;|\pi_j^e-\pi_j|<\alpha\;\;\forall j\neq i$$

PROOF: The claim is obvious from Definition 1 of a Common Interest game. ■

PROOF OF LEMMA 6: By Lemma 4, for all $j \leq i$ and for all $\epsilon > 0$

$$\overline{P}_j^{\epsilon}(\mathcal{C}_j^{t_i^{c,\epsilon}} | h_{t_i^{c,\epsilon}}^{c,\epsilon}) > \frac{c}{c+1}$$

Therefore, it follows from (10) and part (a) of Lemma A.4 that

$$\lim_{c\to\infty,\epsilon\to 0} \mathcal{E}_{i+1}(x_{i+1}^{\epsilon}, x_1^{c,\epsilon}, \ldots, x_i^{c,\epsilon}, \overline{P}_{i+2}^{\epsilon} | h_{t_i^{c,\epsilon}}^{c,\epsilon}, \ldots, \overline{P}_n^{\epsilon} | h_{t_i^{c,\epsilon}}^{c,\epsilon}, h_{t_i^{c,\epsilon}}^{c,\epsilon}) = \pi_{i+1}^{e} \tag{A.22}$$

Notice now that by (7), for any $j > i$, machine x_j^{ϵ} is consistent with history $h_{t_i^{c,\epsilon}}^{c,\epsilon}$. Therefore, we can then conclude from part *(b)* of Lemma A.4 that (A.22) implies that

$$\lim_{c\to\infty,\epsilon\to 0} \mathcal{E}_{i+1}(x_{i+1}^{\epsilon}, x_1^{c,\epsilon}, \ldots, x_i^{c,\epsilon}, x_{i+2}^{\epsilon}, \ldots, x_n^{\epsilon}, h_{t_i^{c,\epsilon}}^{c,\epsilon}) = \pi_{i+1}^{e} \tag{A.23}$$

Together with Lemma A.5, (A.23) implies that

$$\lim_{c\to\infty,\epsilon\to 0} \mathcal{E}_{i}(x_{i}^{c,\epsilon}, x_1^{c,\epsilon}, \ldots, x_{i-1}^{c,\epsilon}, x_{i+1}^{\epsilon}, \ldots, x_n^{\epsilon}, h_{t_i^{c,\epsilon}}^{c,\epsilon}) = \pi_{i}^{e} \tag{A.24}$$

Since players do not discount the future, and because $h_{t_{i-1}^{c,\epsilon}}^{c,\epsilon}$ and $h_{t_i^{c,\epsilon}}^{c,\epsilon}$ are consecutive histories when the players choose the machine profile $(x_1^{c,\epsilon}, \ldots, x_i^{c,\epsilon}, x_{i+1}^{\epsilon}, \ldots, x_n^{\epsilon})$, (A.24) implies that

$$\lim_{c\to\infty,\epsilon\to 0} \mathcal{E}_{i}(x_{i}^{c,\epsilon}, x_1^{c,\epsilon}, \ldots, x_{i-1}^{c,\epsilon}, x_{i+1}^{\epsilon}, \ldots, x_n^{\epsilon}, h_{t_{i-1}^{c,\epsilon}}^{c,\epsilon}) = \pi_{i}^{e} \tag{A.25}$$

Again, by Lemma 4, for all $j < i$ and for all $\epsilon > 0$

$$\overline{P}_j^{\epsilon}(\mathcal{C}_j^{t_{i-1}^{c,\epsilon}} | h_{t_{i-1}^{c,\epsilon}}^{c,\epsilon}) > \frac{c}{c+1}$$

Therefore, it follows from part *(a)* of Lemma A.4 and (A.25) that

$$\lim_{c\to\infty,\epsilon\to 0} \mathcal{E}_{i}(x_{i}^{c,\epsilon}, \overline{P}_1^{\epsilon} | h_{t_{i-1}^{c,\epsilon}}^{c,\epsilon}, \ldots, \overline{P}_{i-1}^{\epsilon} | h_{t_{i-1}^{c,\epsilon}}^{c,\epsilon}, x_{i+1}^{\epsilon}, \ldots, x_n^{\epsilon}, h_{t_{i-1}^{c,\epsilon}}^{c,\epsilon}) = \pi_{i}^{e} \tag{A.26}$$

Similarly it is also the case that, by (7), for any $j > i$, machine x_j^{ϵ} is consistent with history $h_{t_{i-1}^{c,\epsilon}}^{c,\epsilon}$. Therefore, we can conclude that (A.26) and part *(b)* of Lemma A.4 imply that

$$\lim_{c\to\infty,\epsilon\to 0} \mathcal{E}_{i}(x_{i}^{c,\epsilon}, \overline{P}_{-i}^{\epsilon} | h_{t_{i-1}^{c,\epsilon}}^{c,\epsilon}, h_{t_{i-1}^{c,\epsilon}}^{c,\epsilon}) = \pi_{i}^{e} \tag{A.27}$$

Finally, recall that by Lemma 4 we know that $h^{c,\epsilon}_{t^{c,\epsilon}_{i-1}}$ takes place with positive probability. Therefore, we can now see that (11) follows directly from (A.27) and from the requirement that x^ϵ_i must be optimal in expected terms after any history which takes place with positive probability. ■

PROOF OF LEMMA 7: Using Lemmas 5 and 6, by backwards induction on the set of players, we obtain that

$$\lim_{c\to\infty,\epsilon\to 0} \mathcal{E}_1(x^\epsilon_1, \overline{P}^\epsilon_{-1}|\emptyset, \emptyset) = \pi^e_1 \tag{A.28}$$

The claim then follows directly from (A.28) and from Lemma A.5. ■

PROOF OF LEMMA 9: We proceed by contradiction. Let an $(\epsilon, \mathcal{R})$-CTHE be given, and let h_t be a history of length t which takes place with positive probability in this equilibrium. Let this probability be denoted by $P^*(h_t)$.

If the claim is false we must be able to find an infinite sequence of players $i_1, i_2, \ldots, i_m, \ldots$ with the property that for all $m = 1, 2, \ldots$, the strategy $s^\epsilon_{i_m}$ is *not* consistent with h_t. This immediately implies that

$$P^*(h_t) \leq \prod_{m=1}^{\infty} \epsilon_{i_m} \tag{A.29}$$

But since $\prod_{i=1}^{\infty}(1-\epsilon_i) = (1-\epsilon) \in (0,1)$, we must have that $\prod_{m=1}^{\infty} \epsilon_{i_m} = 0$. Therefore (A.29) implies that $P^*(h_t) = 0$. This contradiction is sufficient to establish the claim. ■

PROOF OF THEOREM 2: We proceed by construction. Let a sequence $\{\epsilon_i\}_{i=1}^{\infty}$, satisfying $\prod_{i=1}^{\infty}(1-\epsilon_i) = (1-\epsilon) \in (0,1)$, and any array of perturbations P be given. We can now construct an $(\epsilon, \mathcal{R})$-CTHE as follows.

Because of Assumption 2 (dominance), in any $(\epsilon, \mathcal{R})$-CTHE, each s^ϵ_i must be a halting strategy. For each i let s^ϵ_i be such that $s^\epsilon_i(h_t) = D$ for any finite length h_t. Notice that such s^ϵ_i is guaranteed to belong to $\mathcal{S}_i$ by Assumption 3. Now let h_t any history of finite length which takes place with positive probability in this equilibrium. Using Lemma 9 we have that for all but finitely many players $P_i(s_i|h_t) \leq \epsilon_i$ for any $s_i \neq s^\epsilon_i$. Recall now that, for every i any strategy s_i which satisfies $s_i(h_t) = C$ must be different from s^ϵ_i. Therefore, this implies that the probability of the event 'at t all players play C' conditional on h_t is zero.

Therefore, using the payoffs described in Figure 3 and Assumption 2 (best response), a strategy s^ϵ_i as described is optimal in expected terms after any history of finite length which takes place with positive probability. It follows that the array $\{s^\epsilon_i, P^\epsilon_i, \epsilon_i\}_{i=1}^{\infty}$ which we have just described is indeed an $(\epsilon, \mathcal{R})$-CTHE. Since the value of ϵ in this construction is arbitrary, the claim now follows trivially from the definition of a $\mathcal{R}$-CTHE. ■

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