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Interaction Games: A Unified Analysis of Incomplete Information, Local Interaction and Random Matching*

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Abstract

Incomplete information, local interaction and random matching games all share a common mathematical structure. A type or player interacts with various subsets of the set of all types/players. A type/player's total payoff is additive in the payoffs from these various interactions. This paper describes a general class of interaction games and shows how each of these three classes of games can be understood as special cases. Techniques and results from the incomplete information literature are translated into this more general framework. A companion paper, Morris [1997], uses these techniques to derive new results concerning contagion in local interaction games.

Contents

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1 Introduction

This paper introduces and analyses a class of *interaction games.* A finite or infinite population interacts strategically. But each player's payoff depends on the population strategy profile in a special way. Each player is involved in a number of *interactions,* consisting of subsets of players. He must choose the same action in each of his interactions. He receives a payoff from each interaction that does not depend on the actions of players not in the interaction. Each interaction has a weight. An equilibrium of an interaction game is a profile of possibly mixed strategies for each player such that each player maximizes the weighted sum of his payoffs from each interaction.

Two restrictions allow an interaction game to have an *incomplete information* interpretation: *N-partite interaction* requires that the players can be partitioned into N groups, such that each interaction consists of exactly one player from each group; *bounded interactions* requires that the weights on the interactions add up to 1. Now consider the N player incomplete information game where each of the N groups represent the set of types of one "big player". Interactions then correspond to type profiles, or states, while the "weight" on an interaction corresponds the probability of the type profile. The definition of equilibrium for general interaction games corresponds to the standard definition of (Bayesian Nash) equilibrium of the incomplete information game. Any incomplete information game can be interpreted as an interaction game in this way.

To interpret an interaction game as a *random matching* game, drop the N-partite interaction assumption but assume bounded interactions and two more restrictions: *player independent payoffs* requires that any player's payoff from an interaction depends only on his action and the actions of others in the group (and not on the identity of the player or interaction); *binary interaction* requires each positive weight interaction consists of exactly two players. Now interpret an interaction as a match of two players and the weight of an interaction as the probability of that match. Players must choose actions without knowing which match is chosen. Again, the definition of equilibrium for general interaction games corresponds to the standard definition of equilibrium for random matching games and any random matching game can be interpreted as an interaction game in this way. Indeed, only the bounded interactions assumption is necessary to interpret an interaction game as a random matching game. One can easily have many player matches where payoffs depend on the identities of all players in the match.

Finally, to interpret an interaction game as a deterministic *local interaction* game, maintain the player independent payoffs and binary interaction assumptions but replace bounded interactions with the following: *constant weights* requires that each interaction receives a weight of either 0 or 1. Now two players are "neighbours" if the interaction consisting of those two players has weight 1; they are not neighbours if that interaction has weight 0. Again, equilibrium notions coincide. Deterministic local interaction is the broadest interpretation of interaction games since no restriction is necessary for the interpretation: one can have an unbounded number of many player interactions where identities matter and different interactions have different deterministic weights.

These equivalences are more than just a curiosity. By understanding the common structure of interaction games, we understand each of these classes of games better. For example, it transpires that what matters in the analysis of incomplete information games is the additive separability of payoffs across interactions; the fact that types of one player do not interact with other types of the same player is irrelevant for most purposes.

Monderer and Samet [1989] introduced a set of techniques ("belief operators") for analysing higher order beliefs (players' beliefs about other players' beliefs, etc...) in incomplete information games. This paper shows how to translate the belief operator techniques, and results proved using them, to general interaction games. In incomplete information games, higher order beliefs are important exactly when players' types are highly *correlated* and belief operators are most useful in such situations. The interaction game viewpoint makes clear that this feature corresponds to highly *local* interaction and highly *non-uniform* random matching. It is thus in these environments that the techniques described are most useful.

The approach described here builds on Mailath, Samuelson and Shaked [1995]. They show that the set of probability distributions over action profiles generated by equilibria of random matching games equals the set of correlated equilibria of the underlying game. This argument (summarized in section 3.3.1) implicitly exploits the equivalence between incomplete information and local interaction / random matching games. The purpose of this paper is to make the equivalence explicit in a more general class of games, but also to develop a unified approach to analysing interaction games. Most results are translations of known results into this general setting. In a companion piece, Morris [1997], these techniques are applied to give new characterizations of which features of a local interaction system allow behaviour to spread contagiously. That paper also contains a discussion of existing results in the local interaction literature and their (sometimes close) connection to the approach described here.

I now provide a more detailed outline of the paper. In section 2, I describe an "investment example" that has incomplete information, local interaction and random matching interpretations. With the incomplete information interpretation, the example is close to the "electronic mail game" of Rubinstein [1989] that is the canonical example illustrating how higher order beliefs can allow small probability events to have high probability impacts in incomplete information games. With the local interaction interpretation, the example is close to the interaction on a line analysis of Ellison [1993] that is the canonical example of how the behaviour of a small number of players can be bootstrapped to influence the behaviour of all players in a local interaction system. This example goes a long way to providing a feel for the equivalence.

The general class of interaction games is described in section 3, together with the restrictions necessary to support the various interpretations. Four examples in section 3.2 serve two purposes. They all clarify the role of the various restrictions introduced. The latter two examples illustrate the usefulness of a more general perspective on incomplete information results. I present a no trade result generalizing the standard incomplete information result. I also present a version of the convention game of Shin and Williamson [1996]. Suppose that each player must choose an action in the interval $[0,1]$ and that each player has an incentive to mimic others' behaviour and, more specifically, choose an action that is a weighted average of all the actions of all players he interacts with. Then all players must choose the same action in any equilibrium if the interaction system is *connected* (each player interacts directly or indirectly via some chain with every other player). This is true even though such equilibria may be strictly Pareto dominated. This negative result is sensitive to the continuum of action assumption. But it is an important benchmark and is quite independent of any structure (beyond connectedness) of the interaction system.

The unified approach is presented in section 4.1. The key tool is the following. Fix a group of players Y. Let $B^p(Y)$ be the group of players x with the property that proportion at least p of interactions including x involve players who are entirely in the original group Y. A group Y is *p-cohesive* if this is true for all members of Y, i.e., $Y \subseteq B^p(Y)$. Cohesion is crucial for the analysis of interaction games: if there exist p-cohesive groups for p close to 1, then it is possible to determine the behaviour of players within those groups independently of what players outside do. On the other hand, it is exactly when p-cohesive groups fail to exist (for large p) that outcomes are highly sensitive to small probability events (in the incomplete information interpretation) or small sets of deviant players (in the random matching/local interaction interpretation). The largest p-cohesive group contained in any group Y can be found by iteratively applying the operator B^p to group Y, i.e., $C^p(Y) = \bigcap_{k \geq 1} [B^p]^k(Y)$ contains every p-cohesive subset of Y.[1]

A few benchmark results employing p-cohesion are presented in section 4.2. When can we characterize equilibrium behaviour of some group of players independently of the behaviour of other players? An action a is said to be *p-dominant* for x if it is best response whenever all other players in at least proportion p of all interactions involving x choose a. If an interaction game includes a p-cohesive group Y for whom action a is p-dominant, then it has an equilibrium where a is played by everyone in Y. But when does there exist such a p-cohesive group? Specifically, which interaction systems have the property that all large groups have p-cohesive sub-groups? A characterization is given using proportion operators in section 4.2.3. This allows a characterization of which equilibria of an N-player game are robust in interaction (Kajii and Morris [1995]). In fact,

[1] The close relation between proportion operators and belief operators is discussed in section 4.1.2.

the general interaction approach developed in this paper allows an extension of the Kajii and Morris results. Finally, in section 4.3, the relation between equilibrium arguments and dynamic arguments is discussed.

2 Leading Example

2.1 Investment Game

Two players (ROW and COL) must choose whether to Invest (I) or Not Invest (D). Each player faces a cost 2 of investing. Each player realizes a gross return of 3 from the investment if both (1) the other player invests and (2) investment conditions are *favorable* for that player. Thus if investment conditions are favorable for both players, then payoffs are given by the following symmetric matrix:

Favorable for ROW Favorable for COL	I	D
I	1,1	-2,0
D	0,-2	0,0

This game has two strict Nash equilibria: both players invest and both players don't invest. On the other hand, if conditions are unfavorable for ROW (but favorable for player COL), payoffs are given by the following matrix:

Favorable for ROW Unfavorable for COL	I	D
I	-2,1	-2,0
D	0,-2	0,0

In this game, ROW has a dominant strategy to not invest, and thus the unique Nash equilibrium has both players not investing.

2.2 Incomplete Information

Now allow a small amount of incomplete information about investment conditions. In particular, investment conditions are always favorable for COL, but not for ROW. ROW knows when investment conditions are favorable for him, but COL does not.

Specifically, suppose that ROW observes a signal $s_R \in \{0, ..., K-1\}$ which is drawn from a uniform distribution. Assume that investment conditions are favorable for ROW *unless* $s_R = 0$. COL observes a noisy version of ROW's signal, $s_C \in \{0, ..., K-1\}$. In particular, assume that

$$s_C = \begin{cases} s_R, \text{ with probability } 1/2 \\ s_R - 1, \text{ with probability } 1/2 \end{cases},$$

with mod K arithmetic, so that $0-1=K-1$. Thus if $s_R=0$, s_C is 0 or $K-1$ with equal likelihood.

The above constitutes a description of an incomplete information game. We can summarize the game in the following diagram:

Type of *COL*

Type of *ROW*

	0	1	2		K-1	
0	$\times$	$\circ$	$\circ$	$\cdot$	$\times$	U
1	$\times$	$\times$	$\circ$	$\cdot$	$\circ$	F
2	$\circ$	$\times$	$\times$	$\cdot$	$\circ$	F
	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
K-1	$\circ$	$\circ$	$\circ$	$\cdot$	$\times$	F
	F	F	F	$\cdot$	F	

Types of *ROW* are represented by rows, types of *COL* by columns. Boxes with a $\times$ correspond to type profiles which occur with positive probability; given the uniform prior assumption, each occurs with ex ante probability $\frac{1}{2K}$. Boxes with a $\circ$ correspond to type profiles that occur with zero ex ante probability. Payoffs are specified by the letter - F for favorable, U for unfavorable - at the end of the row/column corresponding to the type.

The unique equilibrium of this incomplete information game has each investor never investing. To see why, observe first that type 0 of *ROW* will not invest in any equilibrium. But type 0 of *COL* attaches probability 1/2 to *ROW* being of type 0, and therefore not investing. But even if investment conditions are favorable, the best response of a player who believes that his opponent will invest with probability less than or equal to a half is not to invest. Thus type 0 of *COL* will not invest. But now consider type 1 of *ROW*. Although investment conditions are favorable, he attaches probability 1/2 to his opponent not investing; so he will not invest. This argument iterates to ensure that no one will invest.

This example is an elaboration of an example of Rubinstein [1989]; this version follows the leading example of Morris, Rob and Shin [1995]. It illustrates the fact that it is not enough either that investment conditions are favorable for both players with high probability; nor that everyone know that everyone know... up to an arbitrary number of levels... that investment conditions are favorable for both players.

2.3 Local Interaction

Now suppose that there are $2K$ players situated on a circle (see figure 1). Player k interacts with his two neighbours, $k-1$ and $k+1$. We use mod $2K$ arithmetic, so that player $2K$'s neighbours are $2K-1$ and 1. Conditions are favorable for all players except the player at location 1. It is common knowledge for whom investment conditions are favorable.

Each player must decide whether to invest or not. His payoff is the sum of his payoff from his two interactions with each of his two neighbours. A strategy profile specifies which players invest, and which do not. A strategy profile is an equilibrium strategy profile if each player's action is a best response given the behaviour of his two neighbours.

This local interaction game can be summarized by the following table:

	2	4	6		2K	
1	×	○	○	·	×	U
3	×	×	○	·	○	F
5	○	×	×	·	○	F
	·	·	·	·	·	·
2K-1	○	○	○	·	×	F
	F	F	F	·	F	

A cross (×) marks a pair of players who interact with each other. Thus, for example, player 3 interacts with players 2 and 4 and no other player.

The unique equilibrium of this game has all players never investing. The argument is as for the incomplete information game. We know that the player at location 1 will never invest. Consider the player at location 2. Since one of his neighbours is not investing, his best response is not to invest. Similarly, the player at location 3 does not invest, and the argument iterates to ensure the result. This iterated deletion of dominated strategies argument is closely related to the best response dynamics on a line argument of Ellison [1993] (the relation is discussed in section 4.3).

The above table is constructed in such a way as to identify an exact relationship between the incomplete information game and the local interaction game. In particular, the odd numbered players in the local interaction game play the role of ROW's types in the incomplete information game, while the even numbered players play the role of COL's types.

2.4 Random Matching

The local interaction game can be easily interpreted as an environment with non-uniform random matching. Suppose in each period, two players are randomly drawn out of a population of $2K$ to play the investment game. The two players are not randomly chosen: players are labelled 1 through $2K$ and only players with consecutive labels may be chosen. Players must decide on an action before knowing who they are matched against. Investment conditions are favorable for all players except player 1.

3 Interaction Games

Fix a finite or countably infinite population of players, $\mathcal{X}$. A standard strategic form game among these players is described by a set of actions for each player, $\{A_x\}_{x\in\mathcal{X}}$, and payoff functions for each player, $\{v_x\}_{x\in\mathcal{X}}$, where each $v_x : \times_{x\in\mathcal{X}} A_x \to \Re$. Thus the game is described by 3-tuple $\left(\mathcal{X}, \{A_x\}_{x\in\mathcal{X}}, \{v_x\}_{x\in\mathcal{X}}\right)$. A (simple) mixed strategy for player x is a (finite support) probability distribution $\alpha_x \in \Delta(A_x)$. A mixed strategy profile is a vector $\alpha \equiv \{\alpha_x\}_{x\in\mathcal{X}}$. For notational convenience, I want to work with a constant set of actions A (so that $A_x = A$ for all $x \in \mathcal{X}$); we can always re-label actions so that the action set is constant.

This paper is concerned with games with a special form of payoffs. Write $\mathcal{I}$ for the collection of subsets of $\mathcal{X}$ with at least two elements; an element $X \in \mathcal{I}$ will be called an *interaction*. We write $\mathcal{I}(x)$ be the collection of such interactions involving player x, i.e.,

$$\mathcal{I}(x) = \{X \in \mathcal{I} : x \in X\}.$$

Let $P : \mathcal{I} \to \Re_+$, where for all $x \in \mathcal{X}$,

$$0 < \sum_{X\in\mathcal{I}(x)} P(X) < \infty. \tag{1}$$

Write $a_X = (a_x)_{x\in X}$ for a typical element of A^X. Now for each $x \in \mathcal{X}$, let $u_x(a_X, X)$ be the payoff that player x gets from interaction $X \in \mathcal{I}(x)$ if players in X choose according to a_X. Assume that payoffs are bounded, i.e., for each $x \in \mathcal{X}$, there exists M such that $|u_x(a_X, X)| \leq M$ for all $X \in \mathcal{X}$ and $a_X \in A^X$. This assumption ensures that total payoffs are well defined:

$$v_x(a) = \sum_{X\in\mathcal{I}(x)} P(X).u_x(a_X, X).$$

In this paper, we will be studying *interaction games* of the above form, described by the 4-tuple $\left(\mathcal{X}, P, A, \{u_x\}_{x\in\mathcal{X}}\right)$. Payoff functions can be extended to mixed strategies in the usual way; thus for any $\alpha \in [\Delta(A)]^{\mathcal{X}}$:

$$\begin{aligned} u_x(\alpha_X, X) &= \sum_{a_X\in A^X}\left(\prod_{y\in X}\alpha_y(a_y)\right)u_x(a_X, X) \\ \text{and } v_x(\alpha) &= \sum_{X\in\mathcal{I}(x)} P(X).u_x(\alpha_X, X). \end{aligned}$$

Definition 1 *Strategy profile $\alpha^* \in [\Delta(A)]^{\mathcal{X}}$ is a (Nash) equilibrium of $\left(\mathcal{X}, P, A, \{u_x\}_{x\in\mathcal{X}}\right)$ if for all $x \in \mathcal{X}$ and all $\alpha \in \Delta(A)$:*

$$v_x\left(\alpha_x^*, \alpha_{-x}^*\right) \geq v_x\left(\alpha, \alpha_{-x}^*\right).$$

The degenerate interaction game with $P(X) = 0$ for all $X \neq \mathcal{X}$ can capture any form of strategic interaction. But this formulation is of interest when $\mathcal{X}$ is large and $P(X) > 0$ only for small X. I will outline a number of alternative interpretations of interaction games below, each of which relies on extra restrictions on the game $\left(\mathcal{X}, P, A, \{u_x\}_{x\in\mathcal{X}}\right)$.

3.1 Interpretations

3.1.1 Incomplete Information

For an incomplete information interpretation, we require first that for some $N \geq 2$, only interactions with N members have positive weight:

P1 (N-ary Interaction): If $P(X) > 0$, then $\#X = N$.

In the special case where $N = 2$, I refer to *binary interaction*. But we will also require the stronger property that the players can be divided into N groups such that each positive weight interaction involves exactly one player from each of the groups.

P1* (N-partite Interaction): There exists a partition of $\mathcal{X}$ into N disjoint subsets $(\mathcal{X}_1, .., \mathcal{X}_N)$ such that if $P(X) > 0$, X consists of exactly one element of each of $\mathcal{X}_1, .., \mathcal{X}_N$.

In the special case where $N = 2$, I refer to *bipartite interaction*. Note that N-partite interaction (**P1***) implies N-ary interaction (**P1**).

Second, the sum of the interaction weights over the whole system is bounded. Without loss of generality, we can assume the sum is equal to one.

P2 (Bounded Interactions): $\sum_{X\in\mathcal{I}} P(X) = 1$.

Now $\left(\mathcal{X}, A, P, \{u_x\}_{x\in\mathcal{X}}\right)$ can be interpreted as an incomplete information game, where there are N "big players", $\{1, ..., N\}$, A is the action set of each player n and $\mathcal{X}_n$ is the set of types of big player n; writing $\mathcal{I}_N$ for the set of interactions consisting of exactly N players, each element of $\mathcal{I}_N$ corresponds to a type profile in $\mathcal{X}_1 \times .. \times \mathcal{X}_N$, i.e., the cross product of player types, or state space; P is the probability distribution over type profiles. Note that payoffs depend on the type profile (state) X. Now a strategy profile α can be thought of as a collection of mixed strategies for each big player, $\alpha \equiv \{\alpha_n\}_{n=1}^N$, where each $\alpha_n \equiv \{\alpha_x\}_{x\in\mathcal{X}_n}$ describes the behaviour of each type of big player n. The definition of Nash equilibrium for general interaction games given above corresponds to an *interim* definition of Bayesian Nash equilibrium. But this is equivalent to the standard ex ante definition.[2] N-partite interaction (**P1**)

[2] Harsanyi [1967, page 177] described an interim interpretation of incomplete information games (he attributes it to Selten) where each type is treated as a separate player.

and bounded interactions (**P2**) are both necessary for the interpretation of the interaction game as a standard game of incomplete information.[3]

3.1.2 Random Matching

If $\widehat{a} \in A^K$ is a K-vector of actions, write $\widetilde{\pi}[\widehat{a}]$ for the frequencies of actions in that action profile, i.e.,

$$\widetilde{\pi}[\widehat{a}](a) = \frac{\#\{k \in \{1, .., K\} : \widehat{a}_k = a\}}{K}$$

for each $a \in A$. Now suppose that N players are matched together to play a game. Each player cares only about the frequency of actions of his $N-1$ opponents (not who takes which action). Thus if a player chooses action $a \in A$ and his $N-1$ opponents choose action profile $\widehat{a} \in A^{N-1}$, his payoff is $g(a, \widetilde{\pi}[\widehat{a}])$. A function $g : A \times \Delta(A) \to \Re$ is a *symmetric payoff function.* For any $N \geq 2$, write $g[N]$ for the symmetric N-player game where the nth player's payoff from action profile $\{a_m\}_{m=1}^N$ is $g\left(a_n, \widetilde{\pi}\left[\{a_m\}_{m \neq n}\right]\right)$.

Write X/x for the group consisting of all members of X except x. Requiring that each player's payoff from each interaction is given by some symmetric payoff function gives us:

P3 (Symmetric Payoffs): For each $x \in \mathcal{X}$ and $X \in \mathcal{I}(x)$, there is a symmetric payoff function g such that $u_x(a_X, X) = g\left(a_x, \widetilde{\pi}\left[a_{X \backslash x}\right]\right)$ for all $a_X \in A^X$.

Note that the symmetric payoffs assumption is empty for those X with $\#X = 2$. Requiring in addition that each player's payoff function does not depend on which interaction he is involved in gives us:

P3*(Interaction Ind't Payoffs): For each $x \in \mathcal{X}$, there is a symmetric payoff function g such that $u_x(a_X, X) = g\left(a_x, \widetilde{\pi}\left[a_{X \backslash x}\right]\right)$ for all $X \in \mathcal{I}(x)$ and $a_X \in A^X$.

Finally, requiring also that payoff functions do not depend on the identity of the player gives us:

P3 (Player Ind't Payoffs):** There is a symmetric payoff function g such that $u_x(a_X, X) = g\left(a_x, \widetilde{\pi}\left[a_{X \backslash x}\right]\right)$ for all $x \in \mathcal{X}$, $X \subseteq \mathcal{I}(x)$ and $a_X \in A^X$.

[3] Bounded interactions (**P2**) is necessary for a standard *ex ante* interpretation of incomplete information games. But with an interim interpretation, no inconsistency arises if we allow for improper priors (Hartigan [1983]) with infinite mass. Note that equation (1) is a maintained restriction on P that implies that conditional probabilities are always well defined.

The most standard one population model of random matching assumes binary interaction (**P1**, with $N = 2$), bounded interactions (**P2**) and player ind't payoffs (**P3****). Now $\mathcal{X}$ is a collection of players. Each (positive probability) match consists of two players. Thus $\mathcal{I}$ is the set of possible matches and P is a probability distributions over matches. Payoffs are independent of all features of the match. An equilibrium has the following interpretation. Each player picks a possibly mixed strategy. He does not know with whom he will interact. His mixed strategy is a best response to the expected distribution over actions.

Only the bounded interactions assumption (**P2**) is *necessary* for this interpretation. Matches may consist of more than two players. Payoffs may be different for each player and may depend on who they interact with.

3.1.3 Local Interaction

A standard model of local interaction considers a *graph* $(\mathcal{X}, \sim)$, where $\mathcal{X}$ is the set of players (or "locations") and $\sim$ is an irreflexive symmetric relation; player x is a "neighbour" of player y if $x \sim y$. Players must choose the same action against each neighbour, all players have the same payoff function from all interactions, and their total payoff is the sum of their payoffs from each neighbour.

This model corresponds in this framework to assuming binary interaction (**P1**, with $N = 2$), player ind't payoffs (**P3****) and

P4 (Constant Weights): $P(X) \in \{0, c\}$ for all $X \in \mathcal{I}$ for some $c > 0$.

Now x and y are neighbours exactly if $P(\{x, y\}) = c$. An equilibrium has the following interpretation. Each player picks a possibly mixed strategy. His mixed strategy maximizes the sum of his payoffs from all interactions, given the strategies of others.

The local interaction interpretation is the most general, in the sense that no restriction is necessary for the interpretation. We can allow an unbounded quantity of interactions involving many players with varying payoffs that depend on the interactions and the opponents' identities. We can drop the constant weights assumption. If $P(X) > 0$, we would say that the group X interacts and $P(X)$ measures the importance of that interaction.

3.2 Examples

Four examples will illustrate the general structure of interaction games. The investment example (section 3.2.1) and co-ordination on a lattice example (section 3.2.2) illustrate the various properties that we have introduced in the alternative interpretations. No trade (3.2.3) and convention (3.2.4) examples illustrate how results that hold for incomplete information game generalize to interaction games.

3.2.1 Investment Game

The following is a formal description of the example of section 2. Let $\mathcal{X} = \{1,..,2K\}$; $A = \{I, D\}$;

$$P(X) = \begin{cases} \frac{1}{2K}, \text{ if } X = \{x,y\} \text{ and either } |x-y| = 1 \text{ or } \{x,y\} = \{1,2K\} \\ 0, \text{ otherwise} \end{cases};$$

$$u_1(a_X, X) = \begin{cases} -2, \text{ if } a_1 = I \\ 0, \text{ if } a_1 = D \end{cases}$$

and if $x \neq 1$, then

$$u_x(a_X, X) = \begin{cases} 1, \text{ if } a_y = I \text{ for all } y \in X \\ -2, \text{ if } a_x = I \text{ and } a_y = D \text{ for some } y \in X \\ 0, \text{ if } a_x = D \end{cases}.$$

- This game satisfies bipartite interaction (**P1***, with $N = 2$), bounded interactions (**P2**), interaction ind't payoffs (**P3***), constant weights (**P4**), but not player ind't payoffs (**P3****). To check for bipartite interaction, let $\mathcal{X}_1 = \{x : x \text{ is odd}\}$ and $\mathcal{X}_2 = \{x : x \text{ is even}\}$.

- The argument given in section 2 showed unique equilibrium α^* has $\alpha^*_x(D) = 1$ for all $x \in \mathcal{X}$. This is also the unique strategy profile satisfying iterated deletion of strictly dominated strategies (we provide a formal definition for this in the next section).

3.2.2 Co-ordination on a Lattice

Versions of this example have been studied in the local interaction literature (Blume [1995], Ellison [1994], Anderlini and Ianni [1996]). Suppose that the set of players consists of all points on a two dimensional lattice, each player interacts with his nearest neighbours and each player's payoffs from each interaction are given by the symmetric matrix:

	I	D
I	$1,1$	$0,0$
D	$0,0$	$2,2$

This game may be formally represented at follows. Writing $\mathcal{Z}$ for the set of integers, $\mathcal{X} = \mathcal{Z}^2$; $A = \{I, D\}$;

$$P(X) = \begin{cases} 1, \text{ if } X = \{x,y\} \text{ and } |x_1 - y_1| + |x_2 - y_2| = 1 \\ 0, \text{ otherwise} \end{cases}$$

$$u_x(a_X, X) = \begin{cases} 1, \text{ if } a_y = I \text{ for all } y \in X \\ 2, \text{ if } a_y = D \text{ for all } y \in X \\ 0, \text{ otherwise} \end{cases}.$$

- This game satisfies bipartite interaction (**P1***, with $N = 2$), player ind't payoffs (**P3****), constant weights (**P4**), but not bounded interactions (**P2**). To check for bipartite interaction, let $\mathcal{X}_1 = \{x : x_1 + x_2 \text{ is odd}\}$ and $\mathcal{X}_2 = \{x : x_1 + x_2 \text{ is even}\}$.

- There are many equilibria (see Blume [1995] for a characterization). For example, α^* is an equilibrium where $\alpha^*_x(I) = 1$ if $x_1 \geq 0$ and $\alpha^*_x(D) = 1$ if $x_1 < 0$.

3.2.3 No Trade Theorem

The standard no trade theorem for incomplete information games states that if there are no ex ante gains from trade, no trade will take place in any trading game where players always have the option of not trading. As a number of researchers have noted, this result remains true if there are no *interim* gains from trade (a weaker assumption, and thus a stronger result). One special case where there are no interim gains from trade is when (i) there are no *ex post* gains from trade; (ii) players are risk neutral; and (iii) players share a common prior. This result has a natural analogue in all interaction games.

Let $\mathcal{X}$ be finite and $A = \{I, D\}$. For each $X \in \mathcal{I}$, let $f_X : X \to \Re$ satisfy $\sum_{x \in X} f_X(x) \leq 0$. Let

$$u_x(a_X, X) = \begin{cases} f_X(x) - \varepsilon, \text{ if } a_y = I \text{ for } \textit{all } y \in X \\ 0, \text{ if } a_y = D \text{ for } \textit{some } y \in X \end{cases}$$

where $\varepsilon > 0$. The interpretation is that player x must decide whether to participate (I) or not (D). If he participates, he pays a transaction cost ε. Each interaction in which he participates is zero sum.

- This game satisfies bounded interactions (**P2**) and symmetric payoffs (**P3**) but for non-trivial functions f_X will fail interaction ind't payoffs (**P3***). It may or may not satisfy N-ary interaction (**P1**) or constant weights (**P4**).

Let α^* be any equilibrium and let $\beta^*(X)$ be the corresponding probability that all players participate in interaction X, i.e., $\beta^*(X) = \prod_{x \in X} \alpha^*_x(I)$. Now player x's payoff is $u^*_x = \sum_{X \in \mathcal{I}(x)} P(X) \beta^*(X)(f_X(x) - \varepsilon) \geq 0$ (since he can guarantee himself 0 by choosing D). So

$$\begin{aligned} 0 &\leq \sum_{x \in \mathcal{X}} u^*_x \\ &= \sum_{x \in \mathcal{X}} \sum_{X \in \mathcal{I}(x)} P(X) \beta^*(X)(f_X(x) - \varepsilon) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{X\in\mathcal{I}}\sum_{x\in X} P(X)\,\beta^{*}(X)\,(f_X(x)-\varepsilon) \\
&\leq -\varepsilon \sum_{X\in\mathcal{I}} \#X.P(X).\beta^{*}(X).
\end{aligned}
$$

Thus $P(X) > 0 \Rightarrow \beta^{*}(X) = 0 \Rightarrow \alpha_x^{*}(D) = 1$ for some $x \in X$. In other words:

- In every positive probability interaction, at least one player chooses D.

In the incomplete information interpretation, the common prior assumption plays a crucial role in this result (it ensures that the ex post zero sum property implies no interim gains from trade). The analogous property in interaction games (built into this formulation) is that each player uses the same interaction weights.

3.2.4 Conventions

Shin and Williamson [1996] described and analysed (a more general version of) the following game (with an incomplete information interpretation). Let $\mathcal{X}$ be finite, $A = [0,1]$ and

$$
u_x(a_X, X) = \widetilde{u}_x\left(a_{X\backslash x}, X\right) - \varepsilon\left(a_x - \frac{1}{\#X-1}\sum_{y\in X\backslash x} a_y\right)^2
$$

for some $\varepsilon > 0$. Thus player x's payoff from interaction X is additively separable in two components. The first component, $\widetilde{u}_x\left(a_{X\backslash x}, X\right)$, does not depend on player x's action. The second component is a quadratic loss function proportional to the squared distance between player x's action and the weighted average of the actions of others in the interaction.

- This game satisfies bounded interactions (**P2**); it may or may not satisfy N-ary interaction (**P1**), symmetric payoffs (**P3**) and constant weights (**P4**).

Each player's best response is always to choose an action that is a weighted average of actions chosen by the other players in the interactions he is a member of. Thus this is a convention game where each player wants to mimic those he interacts with. Thus for any $\psi \in [0,1]$, there is an equilibrium where $\alpha_x(\psi) = 1$ for all $x \in \mathcal{X}$. More surprisingly, if every player is linked, directly or indirectly, to every other player, *all* equilibria take this form. More precisely, this is true if the following property is satisfied.

P5 (Connectedness): For all $x, y \in \mathcal{X}$, there exists a sequence of interactions $X_1, ..., X_K$ such that $x \in X_1$; $y \in X_K$; $P(X_k) > 0$ for all $k = 1, .., K$; and $X_k \cap X_{k+1} \neq \emptyset$ for all $k = 1, .., K-1$.

The argument is straightforward. Let $\overline{\psi}$ be largest action played with positive probability by any player (say it is player x). Since each player's action is a strictly convex combination of the actions played by all players he interacts with, we have $\alpha_y\left(\overline{\psi}\right) = 1$ for all $y \in \bigcup_{\{X \in \mathcal{I}\{x\}: P(X)>0\}} X$. Iterating this argument, connectedness ensures that $\alpha_y\left(\overline{\psi}\right) = 1$ for all $y \in \mathcal{X}$.

It might be highly inefficient to have all players choose a constant action, i.e., if ε is very small and $\widetilde{u}_x$ depends non-trivially on $a_{X \backslash x}$.[4]

3.3 Related Literature and Further Solution Concepts

3.3.1 Role Dependent Payoffs, Player Independent Payoffs and Correlated Equilibria

N-partite interaction (**P1***) was the defining characteristic of an incomplete information game. But it also has a natural interpretation under a local interaction / random matching interpretation: each player has a role and each interaction consists of exactly one player in each of N roles. Under this interpretation, it is natural to consider settings where a player's payoff depends on his role, but nothing else. This restriction can be described formally as follows. Write $\mathcal{I}_N$ for the set of interactions with $\#X = N$, and $\mathcal{I}_N(x) \equiv \mathcal{I}(x) \cap \mathcal{I}_N$. Any $X \in \mathcal{I}_N$ can be written as $X = \{\nu(n, X)\}_{n=1}^{N}$, where $\nu(n, X)$ is the unique (by N-partite interaction) element of $X \cap \mathcal{X}_n$. An N-player game (not necessarily symmetric) is parameterized by payoff functions $\{g_n\}_{n=1}^{N}$, with each $g_n : A^N \to \Re$.

P3a (Role Dependent Payoffs): There is an N-player game $\{g_n\}_{n=1}^{N}$, such that $u_x(a_X, X) = g_n\left(\left(a_{\nu(m,X)}\right)_{m=1}^{N}\right)$ for all $x \in \mathcal{X}_n$, $X \in \mathcal{I}_N(x)$ and $a_X \in A^X$.[5]

Mailath, Samuelson and Shaked [1995] studied interaction games (with a random matching interpretation) satisfying N-partite interaction (**P1**), bounded interactions (**P2**) and role dependent payoffs (**P3a**). They showed the following. Let $\mu \in \Delta\left(A^N\right)$ be the probability distribution over action profiles generated by some equilibrium α of an interaction game, i.e.,

$$\mu(\widehat{a}) = \sum_{X \in \mathcal{I}} P(X) \left(\prod_{n=1}^{N} \alpha_{\nu(n,X)}\left(\widehat{a}_{\nu(n,X)}\right) \right).$$

This probability distribution μ is a correlated equilibrium of the N-player game parameterized by $\{g_n\}_{n=1}^{N}$. Under the incomplete information interpretation,

---

[4] Morris [1997] contains positive results on the co-existence of conventions with *discrete* actions. See also Sugden [1995] and Young [1996].

[5] Player ind't payoffs (**P3****) implies role dependent payoffs (**P3a**), but role dependent payoffs need not imply even symmetric payoffs (**P3**).

the role dependent payoffs assumption (**P3a**) is equivalent to assuming that each big player's payoffs are independent of his type. Thus the above result is equivalent to Aumann's [1987] classic characterization of correlated equilibrium.[6]

A related result holds in the case where N-partite interaction (**P1***) is weakened to N-ary interaction (**P1**), although in this case it is necessary to have player ind't payoffs (**P3****) (see Mailath, Samuelson and Shaked [1995] and Ianni [1996]). Since player ind't payoffs is satisfied, assume that payoffs of all players are given by symmetric payoff function g. For some equilibrium α of an interaction game, we can calculate the probability distribution over *unordered* profiles of actions. We can then construct a probability distribution $\mu \in \Delta\left(A^N\right)$ over *ordered* action profiles by assuming that any ordering is equally likely. This probability distribution is a symmetric correlated equilibrium of the symmetric N-player game $g[N]$. The formal construction is

$$\mu(\widehat{a}) = \frac{1}{\#\left\{\widehat{a}' \in A^N : \widetilde{\pi}(\widehat{a}') = \widetilde{\pi}(\widehat{a})\right\}} \sum_{X \in \mathcal{I}} \sum_{\left\{a_X : \widetilde{\pi}(a_X) = \widetilde{\pi}(\widehat{a})\right\}} P(X) \left(\prod_{x \in X} \alpha_x(a_x) \right)$$

(note that $\widetilde{\pi}(\widehat{a}') = \widetilde{\pi}(\widehat{a})$ exactly if $\widehat{a}'$ and $\widehat{a}$ represent the same collection of actions - possibly in a different order).

3.3.2 Iterated Deletion of Strictly Dominated Strategies

The natural definitions of equilibrium in incomplete information games, random matching games and local interaction games all correspond to the natural definition of equilibrium in the general interaction games. However, other solution concepts do not translate quite as straightfowardly. Consider the following definition of iterated deletion of strictly dominated strategies for interaction games.

Definition 2 *Define* $\left\{\mathcal{U}_x^k\right\}_{x \in \mathcal{X}}$*, each* $\mathcal{U}_x^k \subseteq A$*, iteratively as follows:* $\mathcal{U}_x^0 = A$*;*

$$\mathcal{A}^k = \left\{\alpha \in [\Delta(A)]^{\mathcal{X}} : \alpha_x(a) = 0 \text{ if } a \notin \mathcal{U}_x^k\right\};$$

$$\mathcal{U}_x^{k+1} = \left\{a \in \mathcal{U}_x^k : v_x(a, \alpha_{-x}) \geq v_x(a', \alpha_{-x}) \text{ for all } a' \in A_x, \text{ for some } \alpha \in \mathcal{A}^k\right\}.$$

Action a survives iterated deletion of strictly dominated strategies for player x if $a \in \mathcal{U}_x^\infty \equiv \bigcap_{k \geq 1} \mathcal{U}_x^k$.

This definition corresponds to iterated deletion of strictly *interim* dominated strategies in an incomplete information game [Fudenberg and Tirole 1991, p. 226].

[6]The common prior assumption was necessary for Aumann's [1987] characterization. Dropping the common prior assumption, his assumptions imply only that each player x (with $x \in \mathcal{X}_n$) chooses an action that survives iterated deletion of strictly dominated strategies (for player n in the N-player game $\{g_n\}_{n=1}^N$). The same conclusion would follow if we relaxed the assumption (in interaction games) that players use the same weights in calculating payoffs.

4 A Unified Analysis of Interaction Games

Some tools for analyzing interaction systems $(\mathcal{X}, P)$ are introduced in section 4.1; these tools are applied to characterizing equilibrium behaviour and dynamics in interaction games in sections 4.2 and 4.3 respectively.

4.1 The Structure of Interaction

4.1.1 Proportion Operators

Let $Y \subseteq \mathcal{X}$ be a group of players. We are interested in the set of players who have most of their interactions within group Y. For any $p \in (0,1]$, let $B^p(Y)$ be the set of players for whom proportion at least p of their interactions involve exclusively players within group Y, i.e.,

$$B^p(Y) \equiv \left\{ x \in \mathcal{X} : \frac{\sum\limits_{\{X \in \mathcal{I}(x): X \subseteq Y\}} P(X)}{\sum\limits_{X \in \mathcal{I}(x)} P(X)} \geq p \right\}.$$

Observe that $B^p(Y) \subseteq Y$ for all $Y \subseteq \mathcal{X}$. Group Y is *p-cohesive* if each member of Y has proportion p of his interactions within Y, i.e., $Y \subseteq B^p(Y)$. Iterating the operator, we have:

$$C^p(Y) = \underset{k \geq 1}{\cap} [B^p]^k (Y).$$

Thus $C^p(Y)$ is the collection of players for whom all the following statements are true. At least proportion p of their interactions are within Y. At least proportion p of their interactions involve players who have at least proportion p of their interactions within Y. And so on.

It is straightforward to show that B^p satisfies the following two properties:

$$\text{if } Y \subseteq Y', \text{ then } B^p(Y) \subseteq B^p(Y'); \qquad \text{(monotonicity)}$$

$$\text{if } Y_{k+1} \subseteq Y_k \text{ for all } k, \text{ then } \underset{k \geq 1}{\cap} B^p(Y_k) \subseteq B^p\left(\underset{k \geq 1}{\cap} Y_k\right). \qquad \text{(continuity)}$$

The following result is a consequence of these two properties.

Proposition 1 *For all groups Y: (1) $C^p(Y)$ is the largest p-cohesive group contained in Y. (2) $x \in C^p(Y)$ if and only if there exists a p-cohesive group Y' such that (i) $x \in Y'$ and (ii) $Y' \subseteq Y$.*

Proof. $C^p(Y) \equiv \underset{k \geq 1}{\cap} [B^p]^k (Y) \subseteq \underset{k \geq 2}{\cap} [B^p]^k (Y) \subseteq B^p\left(\underset{k \geq 1}{\cap} [B^p]^k (Y)\right) = B^p(C^p(Y))$, by continuity, so $C^p(Y)$ is p-cohesive. Now for all $Y' \subseteq Y$, $[B^p]^k (Y') \subseteq [B^p]^k (Y)$ for all $k \geq 1$, by iterated application of monotonicity; thus $C^p(Y') \subseteq C^p(Y)$. If in addition Y' in p-cohesive, then $Y' = C^p(Y') \subseteq$

$C^p(Y)$, proving (1). For the "only if" part of (2), set $Y' = C^p(Y)$. For the "if" part of (2), we have $x \in Y'$ by (i), $Y' = C^p(Y')$ by assumption that Y' is p-cohesive and $C^p(Y') \subseteq C^p(Y)$ by (ii) and part (1). So $x \in Y' = C^p(Y') \subseteq C^p(Y)$. □

4.1.2 Interpretation of Proportion Operators and Cohesion

Incomplete Information Assume N-partite interaction (**P1***) and bounded interactions (**P2**). Thus under the incomplete information interpretation, $\mathcal{X}_n$ is the set of types of big player n. For any $X \in \mathcal{I}_N$, write $\nu(n, X)$ for the unique element of $X \cap \mathcal{X}_n$: $\nu(n, X)$ is the type of big player n if the state (i.e., the interaction) is X. For arbitrary *events* $E \subseteq \mathcal{I}$, Monderer and Samet [1989] defined $\widetilde{B}_n^p(E)$ to be the set of states where player n believes event E with probability at least p. Thus $\widetilde{B}_n^p : 2^{\mathcal{I}} \to 2^{\mathcal{I}}$ is defined by

$$\widetilde{B}_n^p(E) \equiv \left\{ X \in \mathcal{I} : \frac{\sum\limits_{\{X' \in \mathcal{I}(\nu(n,X)) : X' \in E\}} P(X')}{\sum\limits_{X' \in \mathcal{I}(\nu(n,X))} P(X')} \right\}.$$

Now let $\widetilde{B}_*^p(E)$ be the set of states where *all* big players believe event E with probability at least p, i.e., $\widetilde{B}_*^p(E) \equiv \bigcap\limits_{n=1}^{N} \widetilde{B}_n^p(E)$. How is this belief operator $\widetilde{B}_*^p$ related to the proportion operators defined above? Under the incomplete information interpretation, a group Y is a collection of types. We can associate with each collection of types an event $\widetilde{E}(Y) \equiv \{X \in \mathcal{I} : X \subseteq Y\}$. Now it is true by definition that

$$\begin{aligned}
\widetilde{E}(B^p(Y)) &= \{X \in \mathcal{I} : X \subseteq B^p(Y)\} \\
&= \left\{ X \in \mathcal{I} : \frac{\sum\limits_{\{X' \in \mathcal{I}(x) : X' \subseteq Y\}} P(X')}{\sum\limits_{X' \in \mathcal{I}(x)} P(X')} \geq p \text{ for all } x \in X \right\} \\
&= \bigcap_{n=1}^{N} \left\{ X \in \mathcal{I} : \frac{\sum\limits_{\{X' \in \mathcal{I}(\nu(n,X)) : X' \subseteq Y\}} P(X')}{\sum\limits_{X' \in \mathcal{I}(\nu(n,X))} P(X')} \geq p \right\} \\
&= \bigcap_{n=1}^{N} \left\{ X \in \mathcal{I} : \frac{\sum\limits_{\left\{X' \in \mathcal{I}(\nu(n,X)) : X' \in \widetilde{E}(Y)\right\}} P(X')}{\sum\limits_{X' \in \mathcal{I}(\nu(n,X))} P(X')} \geq p \right\} \\
&= \bigcap_{n=1}^{N} \widetilde{B}_n^p\left(\widetilde{E}(Y)\right)
\end{aligned}$$

$$= \widetilde{B}^p_* \left(\widetilde{E}(Y)\right).$$

Thus proportion operators can be thought of as belief operators restricted to *simple* events that have the form $\widetilde{E}(Y)$ (for the game theory applications that we will discuss in the next section, these are *exactly* the events we are interested in). Proposition 1 is thus a simple corollary of proposition 3 of Monderer and Samet [1989].[7] In the language of Monderer and Samet [1989], an event $\widetilde{E}(Y)$ is "evident p-belief" if and only if the group Y is p-cohesive; and $\widetilde{E}(C^p(Y))$ is the set of states where the event $\widetilde{E}(Y)$ is "common p-belief."

Local Interaction Under the local interaction interpretation, a group Y is p-cohesive if at least proportion p of the interactions of each member involve only members of that group. The local interaction interpretation of cohesion is discussed extensively in a companion piece, Morris [1997]. That paper explores a simple form of local interaction described by a graph $(\mathcal{X}, \sim)$, where $\sim$ is a symmetric and irreflexive relation. Two players x and y are said to be neighbours if $x \sim y$. This corresponds (in the language of this paper) to the case of binary interaction (**P1**, with $N = 2$) and constant weights (**P4**), i.e., $P(X) = \begin{cases} 1, \text{ if } X = \{x,y\} \text{ and } x \sim y \\ 0, \text{ otherwise} \end{cases}$. In this simple setting, it was natural to consider an operator defined by:

$$\Pi^p(Y) = \left\{ x \in \mathcal{X} : \frac{\#\{y \in Y : y \sim x\}}{\#\{y \in \mathcal{X} : y \sim x\}} \geq p \right\}.$$

This operator is related to the proportion operator of this paper as follows:

$$\begin{aligned} B^p(Y) &= \left\{ x \in \mathcal{X} : \frac{\sum\limits_{\{\{x,y\} \in \mathcal{I}(x) : \{x,y\} \subseteq Y\}} P(\{x,y\})}{\sum\limits_{\{x,y\} \in \mathcal{I}(x)} P(\{x,y\})} \geq p \right\} \\ &= \begin{cases} \left\{ x \in \mathcal{X} : \frac{\#\{y \in Y : y \sim x\}}{\#\{y \in \mathcal{X} : y \sim x\}} \right\}, \text{ if } x \in Y \\ \emptyset, \text{ if } x \notin Y \end{cases} \\ &= Y \cap \Pi^p(Y). \end{aligned}$$

Random Matching Under the random matching interpretation, group Y is p-cohesive if each member of Y attaches probability at least p to any interaction he is in involving only members of group Y.

[7] In fact, the restriction to simple events simplifies the argument: $\widetilde{B}^p_*$ is monotonic when restricted to simple events, but not otherwise.

4.1.3 The Size of p-Cohesive Groups

It will be useful to know something about the relation between the size of group Y and the size of the group $C^p(Y)$. Write $\overline{Y}$ for the complement of Y in $\mathcal{X}$. Proposition 4.2 of Kajii and Morris [1995] can be modified to show:

Corollary 1 *If N-ary interaction (**P1**) holds*[8] *and $p < \frac{1}{N}$, then for all $Y \subseteq \mathcal{X}$:*

$$\sum_{\left\{X \in \mathcal{I} : X \cap \overline{C^p\left(\overline{Y}\right)} \neq \emptyset\right\}} P(X) \leq \left(\frac{(N-1)\,p}{1-Np}\right) \sum_{\{X \in \mathcal{I} : X \cap Y \neq \emptyset\}} P(X).$$

Note that this result only has content if $\sum\limits_{\{X \in \mathcal{I} : X \cap Y \neq \emptyset\}} P(X)$ is finite. The right hand expression is a positive constant (that depends only on N and p) times the sum of the weights of all interactions involving members of Y. The left hand expression is the sum of the weights of all interactions that involve players who are not in $C^p\left(\overline{Y}\right)$.

To translate this into a restriction on the number of players, we require some relationship between numbers of players and the sum of the weights of the interactions that they are involved in. Thus we have:

P6 (Bounded Player Weights): There exists $\kappa > 0$ and $\nu \geq 1$ such that for all $x \in \mathcal{X}$,

$$\kappa \leq \sum_{X \in \mathcal{I}(x)} P(X) \leq \kappa\nu.$$

Note that under this restriction $\mathcal{X}$ is finite if and only if bounded interactions (**P2**) holds. Now we have:

Lemma 1 *If N-ary interaction (**P1**) and bounded player weights (**P7**) hold, and $p < \frac{1}{N}$, then for all finite $Y \subseteq \mathcal{X}$:*

$$\#\overline{C^p\left(\overline{Y}\right)} \leq \left(\frac{1-p}{\frac{1}{N}-p}\right) \nu \# Y.$$

So under the premises of the lemma, if Y contains most players, $C^p(Y)$ must contain most players.

Proof. By **P6**, $\sum\limits_{X \in \mathcal{I}(x)} P(X) \leq \kappa\nu$ for all $x \in Y$; thus

$$\sum_{\{X \in \mathcal{I} : X \cap Y \neq \emptyset\}} P(X) \leq \sum_{x \in Y} \sum_{X \in \mathcal{I}(x)} P(X) \leq \kappa\nu\#Y. \tag{2}$$

[8] N-ary interaction could be weakened to the assumption that all interactions involve at *most* N players, i.e., $P(X) > 0 \Rightarrow \#X \leq N$.

By **P6**, $\sum_{X \in \mathcal{I}(x)} P(X) \geq \kappa$ for all $x \in \overline{C^p\left(\overline{Y}\right)}$; thus

$$\begin{aligned} N\left(\sum_{\left\{X \in \mathcal{I}: X \cap \overline{C^p\left(\overline{Y}\right)} \neq \emptyset\right\}} P(X)\right) &= \sum_{x \in \mathcal{X}}\left(\sum_{\left\{X \in \mathcal{I}(x):\ X \cap \overline{C^p\left(\overline{Y}\right)} \neq \emptyset\right\}} P(X)\right) \\ &\geq \sum_{x \in \overline{C^p\left(\overline{Y}\right)}} \sum_{X \in \mathcal{I}(x)} P(X) \\ &\geq \kappa \# \overline{C^p\left(\overline{Y}\right)}. \end{aligned} \tag{3}$$

Now equation (3), corollary 1 and equation (2) imply the result. □

4.2 Equilibrium

This section reports versions of (incomplete information) results in Morris, Rob and Shin [1995] and Kajii and Morris [1995] applied to general interaction games. Most proofs are omitted, as the arguments are essentially unchanged.

4.2.1 Existence

Througout this section, the following pair of assumptions sufficient for equilibrium existence are assumed.[9]

P7 (Finite Action Set): A is a finite set.

P8 (Finite Interactions): $P(X) > 0 \Rightarrow X$ is finite.

Remark 1 *If interaction game* $\left(\mathcal{X}, P, A, \{u_x\}_{x \in \mathcal{X}}\right)$ *satisfies* **P7** *and* **P8**, *then there exists an equilibrium.*

I also assume interaction ind't payoffs (**P3***) throughout this section. This assumption is inessential: slightly more complicated results could be proved without it. Under assumption (**P3***), we can write $u_x(a, \pi)$ for player x's payoff from any interaction in which he chooses action a, his opponents choose $a_{X \backslash x}$ and $\pi = \widehat{\pi}\left(a_{X \backslash x}\right)$.

[9] Existence fails in the following example satisfying **P7** but not **P8**. Let $\mathcal{X} = \mathcal{Z}$; $A = \{I, D\}$; $P(\mathcal{X}) = 1$ and $P(X) = 0$ for all $X \neq \mathcal{X}$; and

$$u_x(a_{\mathcal{X}}, \mathcal{X}) = \begin{cases} 1, \text{ if } a_x = I \text{ and } a_y = D \text{ for all } y > x \\ -1, \text{ if } a_x = I \text{ and } a_y = I \text{ for some } y > x \\ 0, \text{ if } a_x = D \end{cases}.$$

4.2.2 The Basic Lemma

The first question we want to address is: when is it possible to characterize equilibrium behaviour for some group of players independently of what other players do? We provide one set of sufficient conditions, combining the cohesion properties of the interaction system with the following property of payoffs, adapted from Morris, Rob and Shin [1995]. Write π_a for the probability distribution putting probability 1 on action a, i.e., $\pi_a(a') = \begin{cases} 1, \text{ if } a' = a \\ 0, \text{ otherwise} \end{cases}$.

Definition 3 *Action a is p-dominant for x if, for all $a' \in A$ and $\pi \in \Delta(A)$,*

$$pu_x(a, \pi_a) + (1-p)\, u_x(a, \pi) \geq pu_x(a', \pi_a) + (1-p)\, u_x(a', \pi).$$

Thus action a is p-dominant if it a best response whenever proportion p of interactions involve all other players choosing a also. If a is 1-dominant at x, then everyone playing a is a symmetric Nash equilibrium of $u_x[N]$, the N-player game where each player's payoffs are given by u_x. If a is 0-dominant, then action a is a (weakly) dominant action.

Fix an interaction game and write $\Psi(a,p)$ for the set of players for whom action a is p-dominant.

Lemma 2 *Consider a disjoint collection of groups $\{Y_1, ..., Y_K\}$ with $Y_k \subseteq \Psi(a_k, p_k)$ and Y_k p_k-cohesive for each $k = 1, .., K$; there is an equilibrium α of the interaction game with $\alpha_x(a_k) = 1$ for all $x \in Y_k$, $k = 1, .., K$.*

Proof. Consider the modified interaction game where all players in Y_k are required to play action a_k with probability one. Let α be an equilibrium of the modified game (an equilibrium exists by remark 1). I will show that α is an equilibrium of the original game. By construction, α_x is a best response at all $x \notin \bigcup_{k=1}^{K} Y_k$. But if $x \in Y_k$, then, since Y_k is p_k-cohesive, proportion at least p_k of x's neighbours are in Y_k. Thus proportion at least p_k are playing a_k. Since $x \in Y_k \subseteq \Psi(a_k, p_k)$, a_k is a best response. □

This result is an extension (to interaction games) of lemma 5.2 of Kajii and Morris [1995] which in turn builds on theorem B of Monderer and Samet [1989]. By proposition 1, the largest p-cohesive group contained in $\Psi(a,p)$ is $C^p(\Psi(a,p))$. Thus the following proposition follows from lemma 2.

Proposition 2 *Interaction game $\left(\mathcal{X}, P, A, \{u_x\}_{x\in\mathcal{X}}\right)$ has an equilibrium α with $\alpha_x(a) = 1$ for all $x \in C^p(\Psi(a,p))$.*

The following example (which is a generalization of the investment game of section 2) illustrates the sense in which this result is tight.

Unanimity Game: Consider interaction games $\left(\mathcal{X}, P, A, \{u_x\}_{x\in\mathcal{X}}\right)$ of the following form. Let $A = \{I, D\}$; let $\mathcal{X}$ be partitioned into two sets F (favorable) and U (unfavorable); let

$$u_x(a,\pi) = \begin{cases} \frac{q}{1-q}, \text{ if } a = I,\ \pi = \pi_a \text{ and } x \in F \\ -1, \text{ if } a = I \text{ and } \textit{either } x \in U \textit{ or } \pi \neq \pi_a \\ 0, \text{ if } a = D \end{cases}$$

for some $q \in (0,1)$.[10]

Action I is $(1-q)$-dominant at all $x \in F$. Thus by proposition 2, there exists an equilibrium where all players in $C^{1-q}(F)$ choose action I. But we can also show that action D is the only action that survives iterative deletion of strictly dominated strategies for all players not in $C^{1-q}(F)$. To see why, observe that $\mathcal{U}_x^1 = \{D\}$ for all $x \in U$. Now if $x \in \overline{B^{1-q}(F)}$, x has more than proportion q of his interactions not contained in F. Thus his payoff to action I is strictly less than $(1-q)\left(\frac{q}{1-q}\right) + q(-1) = 0$. Thus $\mathcal{U}_x^2 = \{D\}$ for all $x \in \overline{B^{1-q}(F)}$; iterating this argument shows that $\mathcal{U}_x^{k+1} = \{D\}$ for all $x \in \overline{[B^{1-q}]^k(F)}$ and so $\mathcal{U}_x^{\infty} = \{D\}$ for all $x \in \overline{C^{1-q}(F)}$.[11]

4.2.3 Contagion Threshold

Suppose that action a is p-dominant for almost all players. Is this enough to ensure that action a is played in some equilibrium? The investment game of section 2 suggests not. The action I was $\frac{2}{3}$-dominant at almost every location (for large K) but nonetheless was never played. The problem was that although $\Psi(a,p)$ contained almost all players, $C^{2/3}(\Psi(a,p))$ was empty. In order to exploit proposition 2, it is necessary to find conditions when $\Psi(a,p)$ large implies that $C^p(\Psi(a,p))$ is large.

Because I want to make statements concerning "almost all" players, it is convenient to work with interaction games where there is an infinite mass of interactions. Thus in the remainder of this sub-section, it is assumed that bounded interactions (**P2**) is *not* satisfied but bounded player weights (**P6**) is satisfied (assumptions **P3***, **P7** and **P8** are maintained). Analogous results hold when **P2** is satisfied (and without **P6**); but more complicated "ε, δ" characterizations of large and small groups are required.

Fix an interaction system $(\mathcal{X}, P)$. A group is *co-finite* if it contains all but a finite number of players. A property is said to be true for "almost all" players if it is true for any co-finite group of players. Define a *contagion threshold* for an interaction system as follows:

[10] This game satisfies interaction ind't payoffs (**P3***) but - for non-trivial F - not player ind't payoffs (**P3****). It may or may not satisfy N-partite interaction (**P1***), bounded interactions (**P2**) or constant weights (**P4**).

[11] This argument is a many person version of the *infection* argument of Morris, Rob and Shin [1995].

$$\xi = \inf \left\{ p \in [0,1] : \begin{array}{l} \text{every co-finite group } Y \text{ contains an} \\ \text{infinite } (1-p)\text{-cohesive subgroup} \end{array} \right\}.$$

Lemma 3 *Suppose $(\mathcal{X}, P)$ has contagion threshold ξ; then every interaction game $(\mathcal{X}, P, A, \{u_x\}_{x\in\mathcal{X}})$ where a is $(1-\xi)$-dominant for almost all players has an equilibrium where an infinite number of players choose a.*

This follows immediately from the definition of ξ and proposition 2. Conversely, for any $q < \xi$, we can construct a unanimity game where the interaction system $(\mathcal{X}, P)$ has contagion threshold ξ and F is co-finite, but action D is the only action surviving iterated deletion of strictly dominated strategies for almost all players.

Morris [1997] provides a number of alternative characterizations of this contagion threshold and shows how to calculate it for geometric binary interaction systems. For example, if players are distributed on an infinite m-dimensional lattice and have weight 1 on all their nearest neighbours (and weight 0 on all others), the contagion threshold is $\frac{1}{2m}$. If players are distributed on an infinite m-dimensional lattice and interact with all players within Euclidean distance r, the contagion threshold is close to $\frac{1}{2}$ for large r (independent of m).[12]

4.2.4 Robust Equilibria

We can provide a bound on the contagion threshold that depends on the size of positive weight interactions. By lemma 1, we have that if N-ary interaction (**P1**) holds and $p < \frac{1}{N}$, $\#\overline{C^p\left(\overline{Y}\right)} \leq \left(\frac{1-p}{\frac{1}{N}-p}\right)\nu\#Y$ for all finite Y. It follows that $\overline{C^p\left(Y\right)} \leq \left(\frac{1-p}{\frac{1}{N}-p}\right)\nu\#\overline{Y}$ for all co-finite Y, so $C^p(Y)$ is co-finite if Y is co-finite. So $1-p < \frac{1}{N}$ (i.e., $p > \frac{N-1}{N}$) implies that every co-finite group contains a co-finite $(1-p)$-cohesive subgroup. Thus:

Lemma 4 *If bounded interactions (**P2**) does not hold but bounded player weights (**P6**) and N-ary interaction (**P1**) hold, then the contagion threshold $\xi \leq \frac{N-1}{N}$.*

This bound is tight: there exist interaction systems satisfying the two premises with contagion threshold $\frac{N-1}{N}$:

Example 1 *Let $\mathcal{X} = \{1, ..., N\} \times \mathcal{Z}_+$. Let*

$$P(X) = \begin{cases} 1, \textit{ if} & \begin{array}{c} X = (n,k), (m,k+1)_{m\in\{1,..,N\}\setminus n} \\ \textit{for some } n = 1,..,N \textit{ and } k = 0,1... \end{array} \\ 0, \textit{ otherwise} \end{cases}.$$

[12] The latter result is closely related to the incomplete information analysis of Carlsson and van Damme [1993]. Indeed, the proof appeals to essentially the same geometric argument used by Carlsson and van Damme [1993] in their appendix on multi-dimensional signals (see appendix A of Morris [1997] and the related argument of Blume [1995]).

See figure 2 for the case where $N = 3$.

Let $Y = \mathcal{X} \Big\backslash \left\{ (n,0)_{n=1}^{N} \right\}$. Choose any $q > \frac{1}{N}$. By induction, $[B^q]^k(Y) = \mathcal{X} \Big\backslash \left\{ (n,j)_{n \in \{1,..,N\}, j \leq k} \right\}$; thus $C^q(Y) = \emptyset$ and for all $p < \frac{N-1}{N}$, there exists a co-finite group Y such that $C^{1-p}(Y)$ is empty. Thus $\xi \geq \frac{N-1}{N}$.

This implies that we can prove results about the equilibria of interaction games that depend only on the size of interactions. Fix a symmetric payoff function g.

Definition 4 *Action a is p-dominant in g if, for all $a' \in A$ and $\pi \in \Delta(A)$,*

$$pg(a, \pi_a) + (1-p) g(a, \pi) \geq pg(a', \pi_a) + (1-p) g(a', \pi).$$

Definition 5 *Action a is robust in $g[N]$ if every N-ary interaction game where almost all players' payoffs are given by g has an equilibrium where a is played by almost all players.*

Lemma 5 *If action a is $\frac{1}{N}$-dominant in g, then a is robust in $g[N]$.*

This is a special case of proposition 5.3 in Kajii and Morris [1995] (which had an incomplete information interpretation). This result becomes weak for large N, as requiring a to be $\frac{1}{N}$-dominant is close to requiring that it be a dominant strategy. But we can use the general structure of interaction games to provide an easy extension that works for games with large N.

Consider any interaction game $(\mathcal{X}, P, A, \{u_x\}_{x \in \mathcal{X}})$ with

$$u_x(a_X, X) = \sum_{y \in X \backslash x} \widetilde{u}_x(\{a_x, a_y\}, \{x, y\})$$

for all $x \in \mathcal{X}$, $X \in \mathcal{I}(x)$ and $a_X \in A^X$, so that payoffs are additively separable in the actions of the different opponents. Now consider instead the *binary* interaction game $\left(\mathcal{X}, \widetilde{P}, A, \{\widetilde{u}_x\}_{x \in \mathcal{X}}\right)$ where $\widetilde{P}(\{x,y\}) = \sum_{\{X \in \mathcal{I} : \{x,y\} \subseteq X\}} P(X)$. This latter game is identical (in terms of best responses) to the former game.

But now let g be an additively separable symmetric payoff function, i.e., with $g(a, \pi) = \sum_{a' \in A} \pi(a') \overline{g}(a, \pi_{a'})$, where $\overline{g}$ is a symmetric payoff function. By the transformation described above, any N-ary interaction game where almost all players' payoffs are given by g is equivalent to a binary interaction game where almost all players' payoffs are given by $\overline{g}$. Now if a is $\frac{1}{2}$-dominant in $\overline{g}$, a is robust in $g[N]$.[13]

[13] This argument can be extended to games without additively separable payoffs as follows: action a is robust in $g[N]$ (for all $N \geq 2$) if for all $a' \in A$ and $\lambda \in \Delta(\Delta(A))$ with

4.3 Dynamics

Much of the literature on local interaction and random matching is concerned with dynamics. A companion paper, Morris [1997], uses the techniques described in this paper to provide new results about best response dynamics in local interaction systems represented by a simple graph (i.e., satisfying binary interaction (**P1**, with $N=2$) and constant weights (**P4**)). Here I simply want to point out in this more general context the relation between best response dynamics and the equilibrium arguments that I have been developing here.

Suppose that player ind't payoffs (**P3****) holds and that payoffs are given by the symmetric payoff function of players in F in the unanimity game above,

$$g^U(a,\pi)=\begin{cases}\frac{q}{1-q}, \text{ if } a=I \text{ and } \pi=\pi_I\\ -1, \text{ if } a=I \text{ and } \pi\neq\pi_I\\ 0, \text{ if } a=D\end{cases}.$$

Consider the following deterministic dynamic process. At time 0, players in Y_0 invest (i.e., choose action I), while players in $\overline{Y_0}$ do not invest (i.e., choose action D). At each time $t+1$, a player that did not invest in period t continues to not invest. A player who did invest in period t invests in period $t+1$ only if it is a best response to the previous period strategies to do so, i.e., if all other players invested in at least proportion p of his period t interactions. Thus $Y_t=B^{1-q}(Y_{t-1})$ and so $Y_t=\left[B^{1-q}\right]^t(Y_0)$ and $Y_t\downarrow C^{1-q}(Y_0)$ as $t\to\infty$.[14]

$\sum\limits_{\pi\in\Delta(A)}\lambda(\pi)\pi(a)\geq\frac{1}{2}$:

$$\sum_{\pi\in\Delta(A)}\lambda(\pi)g(a,\pi)\geq\sum_{\pi\in\Delta(A)}\lambda(\pi)g\left(a',\pi\right)$$

It is also possible to extend the Kajii and Morris [1995] results for asymmetric games. Say that $\{a_n^*\}_{n=1}^N$ is a *robust equilibrium* of N-player game $\{g_n\}_{n=1}^N$ if every N-partite interaction game, where (for each n) almost every player in $\mathcal{X}_n$ has payoffs given by g_n, has an equilibrium where $\{a_n^*\}_{n=1}^N$ is played in almost all interactions. Say that $\{a_n^*\}_{n=1}^N$ is a *p-dominant equilibrium* of $\{g_n\}_{n=1}^N$ if for all $n=1,..,N$, $a_n\in A$ and $a_{-n}\in A^{N-1}$,

$$pg_n\left(a_n^*,a_{-n}^*\right)+(1-p)g_n\left(a_n^*,a_{-n}\right)\geq pg_n\left(a_n,a_{-n}^*\right)+(1-p)g_n\left(a_n,a_{-n}\right).$$

KM showed that $\{a_n^*\}_{n=1}^N$ is robust if $\{a_n^*\}_{n=1}^N$ is $\frac{1}{N}$-dominant. But the argument in the text can be adapted to show that $\{a_n^*\}_{n=1}^N$ is also robust if for all $n=1,..,N$, $a_n\in A$ and $\lambda\in\Delta\left(A^{N-1}\right)$ satisfying $\sum\limits_{a_{-n}\in A^{N-1}}\lambda(a_{-n})\#\{m\neq n:a_m=a_m^*\}\geq\frac{1}{2}$,

$$\sum_{a_{-n}\in A^{N-1}}\lambda(a_{-n})g_n(a_n^*,a_{-n})\geq\sum_{a_{-n}\in A^{N-1}}\lambda(a_{-n})g_n(a_n,a_{-n}).$$

[14] $Y_t\downarrow Y$ if $Y_{t+1}\subseteq Y_t$ for all $t=0,1,...$ and $Y=\bigcap\limits_{t\geq0}Y_t$.

Thus the contagion threshold described above tells us exactly whether it is possible for behaviour initially played almost everywhere to eventually be played almost nowhere. Specifically, there exists a co-finite group of players Y_0 such that action I is eventually played almost nowhere exactly if $q < \xi$.

This dynamic process was unidirectional best response dynamics: players were only allowed to switch from I to D and not vice versa. If players switched from D back to I when I was a best response, it would in principle be harder to get action I to disappear. But Morris [1997] shows that the *same* contagion threshold is critical even when looking at *two sided* best response dynamics. More precisely, if there exists a co-finite Y_0 such that $C^{1-q}(Y_0)$ is finite then there exists another co-finite group Z_0 (typically strictly contained in Y_0) such that two sided best response dynamics applied to Z_0 is decreasing and converges to the empty set.

5 Conclusion

Incomplete information, local interaction and random matching games can all be understood as special cases of a general class of interaction games. The distinguishing features of particular classes of games - for example, N-partite interaction for incomplete information games - are in many cases a distraction. A more abstract approach may both allow productive arbitrages across the different research areas and provide a better understanding of what is driving results. Future work will show whether this is in fact the case. Morris [1997] represents one attempt to exploit the equivalence.

One can think of further games that can be embedded in this class. Dynamic games, where each player gets to make many choices, are routinely interpreted as games between "agents" of those players, where each agent gets to make only one choice. If payoffs are additively separable through time, each agent's payoff depends only on interactions with a small subset of all agents (i.e., those acting in the same time period). But the characteristic feature of dynamic games - that players must anticipate the impact of their actions on others' actions - is not naturally embedded in the class of games described in this paper. However, there are two special cases where the analysis translates. First, there is the case where players make a sequence of choices at different points in time, without observing others' choices until the end of the game. In this case, Morris [1995] shows that the incomplete information argument of Carlsson and van Damme [1993] translates to show that if players' clocks are not perfectly co-ordinated, they must play the risk dominant equilibrium in any two player two action co-ordination game. It was noted in footnote 12 that the Carlsson and van Damme argument is closely related to crucial local interaction results. Second, there is the continuum of players case. In this case, again, individual players cannot influence others' actions. Burdzy, Frankel and Pauzner [1996] show that if there is symmetric noise concerning how payoffs evolve through time, the risk dominant equilibrium must be played always. They note the connection with the incomplete information argument of Carlsson and van Damme.

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