CARESS Working Paper #97-01

Contagion*

Stephen Morris†
University of Pennsylvania

January 1997

Abstract

Each player in an infinite population interacts strategically with a finite subset of that population. Suppose each player's binary choice in each period is a best response to the population choices of the previous period. When can behaviour that is initially played by only a finite set of players spread to the whole population? This paper characterizes when such contagion is possible for *arbitrary* local interaction systems (represented by general undirected graphs). Maximal contagion occurs when local interaction is sufficiently uniform and there is *low neighbour growth*, i.e., the number of players who can be reached in k steps does *not* grow exponentially in k.

* This paper and a companion paper, Morris [1997], incorporate earlier work on "Local Interaction, Learning and Higher Order Beliefs," prepared for the Second International Conference on Economic Theory (Learning and Games) at the Universidad Carlos III de Madrid, June 1995. This material has been circulated under various titles and presented at the Social Learning Workshop at SUNY Stony Brook and at Chicago, IGIER (Milan), John Hopkins, Penn., Princeton, USC and Yale. I am grateful for comments from seminar participants and valuable discussions with Carlos Alos-Ferrer, Luca Anderlini, Tilman Börgers, Larry Blume, Antonella Ianni, Michael Chwe, Hervé Crès, Atsushi Kajii, Bart Lipman, George Mailath, Hyun Song Shin and Peyton Young. Financial support from the Alfred P. Sloan Foundation is gratefully acknowledged.

† Department of Economics, University of Pennsylvania, 3718 Locust Walk, Philadelphia PA 19104-6297. E-Mail: smorris@econ.sas.upenn.edu.

1. Introduction

When large populations interact strategically, players may be more likely to interact with some players than others. A local interaction system describes a set of players and specifies which players interact with which other players. If in addition, each player at each location has a set of available actions and a payoff function from each of his various interactions, we have a *local interaction game*. The strategic problem becomes interesting when it is assumed that players cannot tailor their behaviour for each neighbour, but must choose a constant action for all neighbours.

A recent literature has examined such local interaction games.[1] A key finding of that analysis is that local interaction may allow some forms of behaviour to spread rapidly in certain dynamic systems.[2] For example, suppose that players are arranged along a line, and each player interacts with his two neighbours. An action is 1/2-dominant if it is a best response when a player has at least one neighbour playing that action.[3] Ellison [1993] showed that if an action was 1/2-dominant action at every location and was played at *any* pair of neighbouring locations, then best response dynamics alone would ensure that it would eventually be played everywhere.[4]

A number of papers have explored how robust this type of phenomenon is to the structure of the local interaction. For example, two-dimensional lattices have been much studied (Anderlini and Ianni [1995], Blume [1995], Ellison [1994]). Blume [1995] considered local interaction systems where locations are on an m-dimensional lattice and there is a translation invariant description of the set of neighbours. Unfortunately, it is hard to know what to make of results which rely

[1]A partial listing includes Anderlini and Ianni [1996], Berninghaus and Schwalbe [1993, 1996], Blume [1993, 1995], Ellison [1993, 1994], Ianni [1996] and Mailath, Samuelson and Shaked [1995b]. This paper follows that literature in treating the local interaction system as exogenous. See Mailath, Samuelson and Shaked [1995a] and Ely [1995] for models with endogenous local interaction. A large literature in economics has examined the role of local interaction in non-strategic settings; see Durlauf [1996] for a survey of this work.

[2]The literature is primarily concerned with stochastic dynamic processes; as I discuss briefly in section 6, many of the conclusions are driven by properties of deterministic best response dynamics.

[3]If there are only two possible actions in a symmetric two player game, both players choosing the 1/2-dominant action is risk dominant in the sense of Harsanyi and Selten [1988].

[4]Ellison used this property of best response dynamics to prove that a stochastic dynamic process converged to 1/2 dominant behaviour very fast.

on a particular geometric structure. It is not clear that the study of lattices will explain which qualitative features of neighbourhood relations determine strategic behaviour.[5]

The primary purpose of this paper is to develop techniques for analyzing *general* local interaction systems. It is useful to focus on one relatively narrow strategic question in order to explore the comparative statics of the local interaction system. In particular, I consider an infinite population of players. Each player interacts with some finite subset of the population and must choose one of two actions (0 and 1) to play against all of them. There exists a critical number q between 0 and 1 such that action 1 is a best response for a player only if proportion q of his neighbours plays 1. Players are assumed to revise their actions according to determinisitic best response dynamics. Contagion is said to occur if one action - say, action 1 - can spread from a finite set of players to the whole population. In particular, for any given local interaction system, there is a critical *contagion threshold* such that contagion occurs if and only if the payoff parameter q is less than the contagion threshold.

Ellison's argument discussed above shows that the contagion threshold for interaction on a line is $1/2$. In fact, the contagion threshold is at most $1/2$ in *all local interaction systems*. A number of characterizations of the contagion threshold are provided. A group of players is said to be *p-cohesive* if every member of that group has at least proportion p of his neighbours within the group. We show that the contagion threshold is the smallest p such that every "large" group (consisting of all but a finite set of players) contains an infinite, $(1-p)$-cohesive, subgroup. We also show that the contagion threshold is the largest p such that it is possible to label players so that, for any player with a sufficiently high label, proportion at least p of his neighbours has a lower label. These characterizations provide simple techniques for calculating the contagion threshold explicitly in examples.

Contagion is most likely to occur if the contagion threshold is close to its upper bound of $1/2$. We show that the contagion threshold will be close to $1/2$ if two properties hold. First, there is *low neighbour growth*: the number of players who can be reached in k steps grows less than exponentially in k. This will occur if there is a tendency for players' neighbours' neighbours to be their own neighbours. Second, the local interaction system must be sufficiently *uniform*, i.e., if there is

[5] Anderlini and Ianni [1995], Berninghaus and Schwalbe [1993] and Ianni [1996] provide some results about general interaction systems.

some number α such that for all players a long way from some core group, roughly proportion α of their neighbours are closer to the core group.

While the focus is on one contagion question, the techniques and critical properties described are important in a range of strategic local interaction problems. For example, when do mixed equilibria exist, i.e., equilibria of the local interaction game where both actions are played? A low contagion threshold implies the existence of mixed equilibria for a wide range of payoff parameters. In section 5, I show (under the low neighbour growth assumption) that mixed equilibria exist whenever the payoff parameter is more than the contagion threshold and less than one minus the contagion threshold. One consequence is that mixed equilibria *always* exist in the (extreme) case of exactly symmetric payoffs. The literature on local interaction games cited above has focussed on *stochastic* revision processes. In section 6, I argue that the key qualitative properties of stochastic processes built around best response dynamics depend on the deterministic process and thus the properties of general local interaction systems studied here.

This paper builds on two literatures. The questions studied and the formal framework used are very close to the earlier literature on local interaction games (see footnote 1). When applied to geometric examples, the contribution of this paper is to provide a useful language for discussing the structure of local interaction that can be used to generalize arguments already used in that literature. More importantly, this approach allows a discussion of the qualitative properties of local interaction systems that is independent of the geometry.

The inspiration for this work is an apparently unrelated literature on the role of higher order beliefs in incomplete information games. It is possible to show a formal equivalence between local interaction games and incomplete information games. The formal techniques in this paper are then analogues of the belief operator techniques, introduced by Monderer and Samet [1989], and used in the higher order beliefs literature.[6] However, this relationship is explored in detail in a companion piece (Morris [1997]), so in this paper, the ideas are developed independently.

The paper is organized as follows. Some geometric examples are discussed in section 2; these illustrate the questions studied but also highlight the risks of taking fixed geometric structures too seriously. The model of local interaction games is introduced in section 3. The contagion question is posed and studied

[6]The papers of Monderer and Samet [1989, 1996], Morris, Rob and Shin [1995] and Kajii and Morris [1995] are especially relevant.

in section 4. Section 5 presents the results on the existence of mixed equilibria. There is a discussion of various ways of adding random elements to this paper's deterministic dynamic in section 6. Section 7 concludes.

2. Examples

I start by discussing examples where players' interaction is described by some geometric relationship on a lattice. The purpose here is twofold. First, the examples can be used to introduce and illustrate the main question addressed in the paper, and provide some intuition for the answers that will eventually be given. But second, the examples suggest that focussing exclusively on regular geometric examples can be somewhat deceptive. Eventually, we want to characterize strategic behaviour in local interaction environments in ways that do not depend on any geometry.

There is an infinite population of players. Each player interacts with a finite subset of the population and chooses one of two actions, 0 or 1. It is assumed that he must play the same action against each neighbour. (We can think of the action representing a norm of behaviour that cannot be altered at will). The payoffs from each interaction are given by the matrix

	0	1
0	q, q	$0, 0$
1	$0, 0$	$1-q, 1-q$

, (2.1)

where $q \in (0, 1)$. We will study what happens under (deterministic) best response dynamics. Thus suppose that in each period, each player chooses an action that maximizes the sum of his payoffs from all his interactions, given his neighbours' actions in the previous period.

This paper is concerned with the following question. Does there exist a *finite* group of players, such that if that group starts out playing some action (say, without loss of generality, action 1), best response dynamics will ensure that that action is eventually played everywhere? If so, we say that action 1 spreads *contagiously.* More specifically, suppose a local interaction system is fixed. For which values of q are contagious dynamics possible? We will show that every local interaction system is characterized by a *contagion threshold* ξ. Contagion occurs if and only if $q \leq \xi$.

Example 1. *(Interaction on a Line)*[7] *The population is arranged on a line and each player interacts with the player to his left and the player to his right. See figure 1.*

If $q \leq 1/2$, action 1 is a best response whenever at least one neighbour chooses action 1. Thus if two neighbours x and $x+1$ initially choose action 1, players $x-1, x, x+1$ and $x+2$ must all choose action 1 in the next period, players $x-2, x-1, x, x+1, x+2$ and $x+3$ must all choose action 1 in the period after that, and so on. So contagion occurs exactly if $q \leq 1/2$.

Thus contagion occurs easily under interaction on a line. But what happens if the interaction structure becomes more complex?

Example 2. *(Nearest Neighbour Interaction in m Dimensions)*[8] *The population is situated on an infinite m dimensional lattice. Each player interacts with all players who are immediate neighbours in the lattice and thus whose coordinates differ in only one dimension. If $m = 1$, then we have the interaction on a line of the previous example. See figure 2 for the case where $m = 2$.*

In this case, each player has $2m$ neighbours. Contagion occurs only if $q \leq 1/2m$.[9] Thus it appears that as interaction becomes "richer" (i.e., as the number of dimensions increases) contagion becomes impossible. However, this example may be somewhat deceptive, as the following example shows.

Example 3. *(n-Max Distance Interaction in m Dimensions) The population is again situated on an infinite m dimensional lattice. Each player interacts with all players who are less than n steps away in each of the m dimensions. See figure 3 for the case where $m = 2$ and $n = 1$.*

In this case, there is contagion whenever $q \leq \frac{n(2n+1)^{m-1}}{(2n+1)^m - 1}$. The following table gives the values of this expression for different values of m and n:

[7] This case has been analysed by Ellison [1993] and others.

[8] This case has been analysed by Blume [1995] and others.

[9] Proofs for this and other results given in this section are reported in the appendix. They use the general results developed in the text.

	1	2	3	·	n	·	$n \to \infty$
1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	·	$\frac{1}{2}$	·	$\frac{1}{2}$
2	$\frac{3}{8}$	$\frac{5}{12}$	$\frac{7}{16}$	·	$\frac{n(2n+1)}{(2n+1)^2-1}$	·	$\frac{1}{2}$
3	$\frac{9}{26}$	$\frac{25}{62}$	$\frac{147}{342}$	·	$\frac{n(2n+1)^2}{(2n+1)^3-1}$	·	$\frac{1}{2}$
·	·	·	·	·	·	·	·
m	$\frac{3^{m-1}}{3^m-1}$	$\frac{2.5^{m-1}}{5^m-1}$	$\frac{3.7^{m-1}}{7^m-1}$	·	$\frac{n(2n+1)^{m-1}}{(2n+1)^m-1}$	·	$\frac{1}{2}$
·	·	·	·	·	·	·	·
$m \to \infty$	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{3}{7}$	·	$\frac{n}{2n+1}$	·	$\frac{1}{2}$

This example illustrates the lack of robustness of the nearest neighbour analysis. If we simply add players at one diagonal remove, then increasing the number of dimensions never eliminates contagion if $q \leq 1/3$.

So restricting attention to interaction on a lattice, higher dimensions may be associated with decreasing contagion, but there is no simple relationship. We can use the lattice examples and one more (non-lattice) example to illustrate features that we will show are important for contagion.

- *"Cohesive" Groups limit Contagion.*

Example 4. *(4-Families) The population is divided into an infinite number of "families" of four players. Each player in a family interacts with every other player in that family. Each player also interacts with exactly one player outside his family. Figure 4 illustrates this structure.*

Suppose $q > 1/4$. If many families start out with all their members playing action 0, they will never stop playing action 0. Thus no contagion can occur if $q > 1/4$. Each family is a *cohesive* group with the property that each member has a high proportion (three out of four) of his interactions within the group.

The lack of contagion (for $q > 1/4$) in the nearest neighbour interaction in two dimensions (example 2 and figure 2) occurs for essentially the same reason. Consider groups of players made up of two adjacent rows of players in the lattice. Each player in the pair of rows has three of his four neighbours within the pair of rows. Thus we have an equally cohesive group that will prevent contagion.

- *Low Neighbour Growth allows Contagion.*

The n-max distance case (example 3) illustrates that as the radius of interaction becomes large, the contagion threshold approaches 1/2. What drives this result? Intuitively, as the radius of interaction becomes large, irregularities, or lumpiness, in the lattice structure disappears. But this is not sufficient to ensure contagion. We also require one element that corresponds to "localness" in the lattice examples. It must be the case there is neighbour correlation, so that a player's neighbours' neighbours are more likely to be the original player's neighbour than some player picked at random. This property is guaranteed to be true if there is not too much *neighbour growth*: the number of players reached in k steps grows less than exponentially in k.

3. Local Interaction Games

This section introduces general local interaction systems and the binary action local interaction games studied in this paper, as well as introducing the critical cohesion properties that we will be exploiting later in the paper.

3.1. Local Interaction Systems

Fix a countably infinite set of players $\mathcal{X}$ and let $\sim$ be a binary relation on $\mathcal{X}$. If $x \sim y$, we say that "y is a neighbour of x." Write $\Gamma(x)$ for the set of neighbours of x, i.e., $\Gamma(x) \equiv \{y : y \sim x\}$. We will assume, for all $x \in \mathcal{X}$,

1. *Irreflexivity*: $x \notin \Gamma(x)$. No player is his own neighbour.
2. *Symmetry*: $x \sim y \Rightarrow y \sim x$. If y is a neighbour of x, then x is a neighbour of y.
3. *Bounded Neighbours*: there exists M such that $1 \leq \#\Gamma(x) \leq M$. Each player has at least 1 and at most M neighbours.

A *local interaction system* is a pair $(\mathcal{X}, \sim)$, where $\sim$ satisfies properties [1] through [3]. A *group* of players, X, is an arbitrary subset of $\mathcal{X}$. The complementary group in $\mathcal{X}$ is written as $\overline{X}$, i.e., $\overline{X} = \{x \in \mathcal{X} : x \notin X\}$.

3.2. Cohesion

For any given group of players X, do players in the group mostly interact with players within the group or with players outside the group? In this section, critical tools for analyzing this question are introduced. Let $\pi[X|x]$ be the proportion of x's neighbours who are in group X, i.e.,

$$\pi\left(X|x\right) = \frac{\#\left(X \cap \Gamma\left(x\right)\right)}{\#\Gamma\left(x\right)}.$$

Write $\Pi^p(X)$ for the players for whom at least proportion p of their interactions are with players in X, i.e.,

$$\Pi^p(X) = \left\{x \in \mathcal{X} : \pi\left(X|x\right) \geq p\right\}.$$

Let the *cohesion* of group X be the smallest p such that each player in X has proportion p of his interactions within X, i.e.,

$$c(X) = \min_{x \in X} \pi\left(X|x\right) = \max\left\{p : X \subseteq \Pi^p(X)\right\}.$$

The minimum and maximum exist since, for all players x and groups X, $\pi\left(X|x\right)$ is a rational number with denominator less than or equal to M. Say that group X is *p-cohesive* if $c(X) \geq p$.[10]

3.3. Local Interaction Games

We will focus on the case where each player has two possible actions, $\{0, 1\}$. Write $u(a, a')$ for the payoff of a player from a particular interaction if he chooses a and his neighbour chooses a'. This payoff function corresponds to symmetric payoff matrix:

	0	1
0	$u(0,0), u(0,0)$	$u(0,1), u(1,0)$
1	$u(1,0), u(0,1)$	$u(1,1), u(1,1)$

We assume that this game has two strict Nash equilibria, so that $u(0,0) > u(1,0)$ and $u(1,1) > u(0,1)$. However, for the analysis of this paper all we care about

[10]Sociologists have introduced and applied various related measures of the cohesion of groups; see Wasserman and Faust [1994, chapter 7].

is the best response correspondence of this game. In particular, observe that action 1 is best response for some player exactly if he assigns probability at least $q = \frac{(u(0,0)-u(1,0))}{(u(0,0)-u(1,0))+(u(1,1)-u(0,1))}$ to the other player choosing action 1. Thus payoffs are parameterized by the critical probability $q \in (0,1)$.[11] Now a *local interaction game* is a 3-tuple $(\mathcal{X}, \sim, q)$.

A conventional description of best responses and equilibrium would proceed as follows. A (pure) *configuration* is a function $s : \mathcal{X} \to \{0,1\}$. Given configuration s, player x's best response is to choose an action which maximizes the sum of his payoffs from his interactions with each of his neighbours. Thus action a is a best response to configuration s for player x, i.e. $a \in b(s,x)$ if

$$\sum_{y \in \Gamma(x)} u(a, s(y)) \geq \sum_{y \in \Gamma(x)} u(1-a, s(y)).$$

Configuration s' is best response to configuration s if $s'(x)$ is a best response to s for each x, i.e., if $s'(x) \in b(s,x)$ for all $x \in \mathcal{X}$; and configuration s is an *equilibrium* if it is a best response to itself.

However, it is useful for us to identify a configuration with the group of players who choose action 1 in that configuration. Thus configuration s is identified with the group $X = \{x : s(x) = 1\}$; group X is identified with configuration s where

$$s(x) = \begin{cases} 1, \text{ if } x \in X \\ 0, \text{ if } x \notin X \end{cases}.$$

Now we have:

Definition 1. *X is a best response to Y if $X \subseteq \Pi^q(Y)$ and $\overline{X} \subseteq \Pi^{1-q}\left(\overline{Y}\right)$.*

There exist multiple best responses to Y only if there are players who have exactly proportion q of their neighbours within Y and thus exactly proportion $1-q$ outside Y. For a generic choice of q, this will not occur. So for generic q, $\Pi^q(X)$ is the unique best response to X. Thus throughout the paper, $\Pi^q(X)$ will be referred to as *the* best response to X.

[11] Thus in particular, we can restrict attention to payoff matrices of the form given in equation 2.1 in section 2. Note that $(1,1)$ is a *risk dominant equilibrium* (in the sense of Harsanyi and Selten [1988]) exactly if $q < 1/2$.

Definition 2. *X is an equilibrium of $(\mathcal{X}, \sim, q)$ if X is a best response to X, i.e., if $X \subseteq \Pi^q(X)$ and $\overline{X} \subseteq \Pi^{1-q}\left(\overline{X}\right)$.*

Thus X is an equilibrium if and only if X is q-cohesive and $\overline{X}$ is $(1-q)$-cohesive. Again, for generic q, X is an equilibrium if and only if $X = \Pi^q(X)$.

4. Deterministic Contagion

If X is the set of players who initially choose action 1, then under determinisitic best response dynamics, $[\Pi^q]^k(X)$ will be the set of players choosing action 1 after k periods.

4.1. The Contagion Threshold

The *contagion threshold* is the largest q such that action 1 can spread from a finite group of players to a co-finite group of players,[12] i.e.,

$$\xi = \max\left\{q : \bigcup_{k\geq 1} [\Pi^q]^k(X) \text{ is co-finite for some finite } X\right\}.$$

The maximum can be shown to always exist, using properties of the operator Π^p descrbied below. Operator Π^p is non-monotonic: X may contain $\Pi^p(X)$, X may be contained in $\Pi^p(X)$, or neither might be true. To characterize the contagion threshold, it will be useful to study an always increasing version of the operator

$$\Pi^p_+(X) \equiv X \cup \Pi^p(X).$$

The following properties of Π^p and Π^p_+ will be used extensively in the following analysis (the elementary proofs appear in Appendix C). For a sequence of groups X_k, we write $X_k \uparrow X$ if $X = \bigcup_{k\geq 1} X_k$ and $X_k \subseteq X_{k+1}$ for each k; and $X_k \downarrow X$ if $X = \bigcap_{k\geq 1} X_k$ and $X_{k+1} \subseteq X_k$ for each k. The properties hold for all $X \subseteq \mathcal{X}$.

B1 (*Operator Monotonicity*). $\Pi^p(X) \subseteq \Pi^p_+(X)$.

B2 (*Group Continuity*). If $X_k \uparrow X$, then $\Pi^p(X) = \bigcup_{k\geq 1} \Pi^p(X_k)$ and $\Pi^p_+(X) = \bigcup_{k\geq 1} \Pi^p_+(X_k)$.

[12] A group X is *co-finite* if the complementary group $\overline{X}$ is finite.

B2 implies:

B2* (*Group Monotonicity*). If $X \subseteq Y$, then $\Pi^p(X) \subseteq \Pi^p(Y)$ and $\Pi^p_+(X) \subseteq \Pi^p_+(Y)$.

B3 (*Probability Continuity*). If $p_k \uparrow p$, then $\Pi^{p_k}(X) \downarrow \Pi^p(X)$ and $\Pi^{p_k}_+(X) \downarrow \Pi^p_+(X)$.

B3 implies:

B3* (*Probability Monotonicity*). If $p < r$, then $\Pi^r(X) \subseteq \Pi^p(X)$ and $\Pi^r_+(X) \subseteq \Pi^p_+(X)$.

B4 (*Inverse Operator*). If $p + r > 1$, then $\Pi^p(X) \subseteq \overline{\Pi^r\left(\overline{X}\right)}$.

Now we can show the following equivalences.

Lemma 1. *If ξ be the contagion threshold of local interaction system $(\mathcal{X}, \sim)$, the following properties are equivalent:*

[0] $p \leq \xi$;
[1] $\underset{k\geq 1}{\cup} [\Pi^p]^k (X)$ is co-finite, for some finite X;
[2] $\underset{k\geq 1}{\cup} [\Pi^p_+]^k (X)$ is co-finite, for some finite X;
[3] $[\Pi^p_+]^k (X) \uparrow \mathcal{X}$, for some finite X;
[4] $[\Pi^p]^k (X) \uparrow \mathcal{X}$, for some finite X.

It is straightforward to construct examples where X is finite, $\underset{k\geq 1}{\cup} [\Pi^p]^k (X)$ is finite, but $\underset{k\geq 1}{\cup} [\Pi^p_+]^k (X) = \mathcal{X}$. But the lemma implies that it must then be possible to find another, possibly larger but still finite, group Y with $\underset{k\geq 1}{\cup} [\Pi^p]^k (Y) = \mathcal{X}$.

Proof. By definition of ξ and property B3*, [0] is equivalent to [1].

If $X \subseteq Y$ and X is co-finite, then Y is co-finite. With property B1, this gives [1] $\Rightarrow$ [2].

To show [2] $\Rightarrow$ [3], suppose $\underset{k\geq 1}{\cup} [\Pi^p_+]^k (X)$ is co-finite, for some finite X; let $Y = X \cup \left(\overline{\underset{k\geq 1}{\cup} [\Pi^p_+]^k (X)}\right)$; Y is finite by construction. But by property B2*, $\underset{k\geq 1}{\cup} [\Pi^p_+]^k (X) \subseteq \underset{k\geq 1}{\cup} [\Pi^p_+]^k (Y)$; and

$$\overline{\underset{k\geq 1}{\cup} [\Pi^p_+]^k (X)} \subseteq Y \subseteq \underset{k\geq 1}{\cup} [\Pi^p_+]^k (Y),$$

so $\underset{k\geq 1}{\cup}\left[\Pi_{+}^{p}\right]^{k}(Y)=\mathcal{X}$. But $\left[\Pi_{+}^{p}\right]^{k}(Y)$ is increasing by construction, so $\left[\Pi_{+}^{p}\right]^{k}(X)\uparrow\mathcal{X}$.

To show [3] $\Rightarrow$ [4], we first show by induction that for all groups X and $k\geq 1$,

$$\left[\Pi_{+}^{p}\right]^{k}(X)=X\cup\Pi^{p}\left(\left[\Pi_{+}^{p}\right]^{k-1}(X)\right). \tag{4.1}$$

This is true by definition for $k=1$. Suppose it is true for arbitrary k. Now

$$\begin{aligned}\left[\Pi_{+}^{p}\right]^{k+1}(X) &= \Pi_{+}^{p}\left(\left[\Pi_{+}^{p}\right]^{k}(X)\right)\\ &= \left[\Pi_{+}^{p}\right]^{k}(X)\cup\Pi^{p}\left(\left[\Pi_{+}^{p}\right]^{k}(X)\right),\text{ by definition of }\Pi_{+}^{p}\\ &= X\cup\Pi^{p}\left(\left[\Pi_{+}^{p}\right]^{k-1}(X)\right)\cup\Pi^{p}\left(\left[\Pi_{+}^{p}\right]^{k}(X)\right),\text{ by inductive hypothesis}\\ &= X\cup\Pi^{p}\left(\left[\Pi_{+}^{p}\right]^{k}(X)\right),\text{ by B2*, since }\left[\Pi_{+}^{p}\right]^{k-1}(X)\subseteq\left[\Pi_{+}^{p}\right]^{k}(X).\end{aligned}$$

Now suppose that X is finite and $\underset{k\geq 1}{\cup}\left[\Pi_{+}^{p}\right]^{k}(X)=\mathcal{X}$. Let $Y=X\cup\{x:\Gamma(x)\cap X\neq\emptyset\}$; since X is finite, Y is finite, and we can choose K such that $Y\subseteq\left[\Pi_{+}^{p}\right]^{K}(X)$ and thus $Y\subseteq\left[\Pi_{+}^{p}\right]^{k}(X)$ for all $k\geq K$. Now $x\in X\Rightarrow\Gamma(x)\subseteq Y\Rightarrow\Gamma(x)\subseteq\left[\Pi_{+}^{p}\right]^{k}(X)$ for all $k\geq K\Rightarrow x\in\Pi^{p}\left(\left[\Pi_{+}^{p}\right]^{k}(X)\right)$ for all $k\geq K$. Thus $X\subseteq\Pi^{p}\left(\left[\Pi_{+}^{p}\right]^{k}(X)\right)$ for all $k\geq K$. Now by (4.1), $\left[\Pi_{+}^{p}\right]^{k+1}(X)=X\cup\Pi^{p}\left(\left[\Pi_{+}^{p}\right]^{k}(X)\right)=\Pi^{p}\left(\left[\Pi_{+}^{p}\right]^{k}(X)\right)$ for all $k\geq K$. So $\left[\Pi^{p}\right]^{k}\left(\left[\Pi_{+}^{p}\right]^{K}(X)\right)=\left[\Pi_{+}^{p}\right]^{K+k}(X)$ for all $k\geq 0$. Thus $\left[\Pi^{p}\right]^{k}\left(\left[\Pi_{+}^{p}\right]^{K}(X)\right)$ is increasing and $\underset{k\geq 1}{\cup}\left[\Pi^{p}\right]^{k}\left(\left[\Pi_{+}^{p}\right]^{K}(X)\right)=\mathcal{X}$. Thus $\left[\Pi_{+}^{p}\right]^{K}(X)$ is a finite group satisfying property [4].

Finally, since $\mathcal{X}$ is co-finite, [4] $\Rightarrow$ [1]. $\square$

We will use lemma 1 in constructing some useful characterizations of the contagion threshold.

4.2. Upper and Lower Bounds on the Contagion Threshold

This section provides upper and lower bounds on the contagion threshold. These bounds are used in appendix A to analyse geometric examples (and thus prove the results reported in section 2).

Proposition 1. *The contagion threshold is the smallest p such that every co-finite group contains an infinite, $(1-p)$-cohesive, subgroup.*

The proposition gives a characterization of the contagion threshold. The proposition is most useful in one direction, giving a constructive way of providing an upper bound on the contagion threshold:

Corollary 1. *[Upper Bound] Suppose every co-finite group contains an infinite, $(1-p)$-cohesive, subgroup. Then $\xi \leq p$.*

The proof of proposition 1 will exploit the following lemma (proved in appendix B).

Lemma 2. *For any local interaction system $(\mathcal{X}, \sim)$ and probability $p \in (0,1)$, there exists $\varepsilon > 0$ such that $\overline{\underset{k\geq 1}{\cup} [\Pi_+^r]^k (X)}$ is $(1-p)$-cohesive for all $X \subseteq \mathcal{X}$ and $r \leq p + \varepsilon$.*

Proof. (of proposition 1). The proposition can be re-stated as: "every co-finite group contains an infinite, $(1-p)$-cohesive, subgroup if and only if $\xi \leq p$." Suppose every co-finite group contains an infinite, $(1-p)$-cohesive, subgroup. Let X be any finite group. Let Y be any infinite, $(1-p)$-cohesive, subgroup of co-finite group $\overline{X}$. Fix $r > p$. We will show by induction that $Y \subseteq \overline{[\Pi_+^r]^k (X)}$. True for $k = 0$ (since $Y \subseteq \overline{X}$). Suppose true for k. Now:

$$\begin{aligned}
Y &\subseteq \Pi^{1-p}(Y), \text{ since } Y \text{ is } (1-p)\text{-cohesive} \\
&\subseteq \Pi^{1-p}\left(\overline{[\Pi_+^r]^k (X)}\right), \text{ by inductive hypothesis and B2*} \\
&\subseteq \overline{\Pi^r\left([\Pi_+^r]^k (X)\right)}, \text{ by B4.} \\
\text{Thus } Y &\subseteq \overline{[\Pi_+^r]^k (X)} \cap \overline{\Pi^r\left([\Pi_+^r]^k (X)\right)} \\
&= \overline{[\Pi_+^r]^k (X) \cup \Pi^r\left([\Pi_+^r]^k (X)\right)} \\
&= \overline{[\Pi_+^r]^{k+1} (X)}.
\end{aligned}$$

So $\underset{k\geq 1}{\cup} \left[\Pi_+^r\right]^k (X)$ is not co-finite for all $X \subseteq \mathcal{X}$ and thus, by lemma 1, $\xi < r$. But $\xi < r$ for all $r > p$ implies $\xi \leq p$.

Now suppose $\xi \leq p$. Let X be any co-finite group. By lemma 2, there exists $\varepsilon > 0$ such that $\overline{\underset{k\geq 1}{\cup} \left[\Pi_+^{p+\varepsilon}\right]^k \left(\overline{X}\right)}$ is $(1-p)$-cohesive. Since $p + \varepsilon > \xi$, we have by lemma 1 that $\overline{\underset{k\geq 1}{\cup}\left[\Pi_+^{p+\varepsilon}\right]^k\left(\overline{X}\right)}$ is infinite. Since $\overline{\underset{k\geq 1}{\cup}\left[\Pi_+^{p+\varepsilon}\right]^k\left(\overline{X}\right)} \subseteq X$, every co-finite group contains an infinite $(1-p)$-cohesive subgroup. □

To develop *lower* bounds on the contagion threshold, some additional notation is useful. A labelling of locations $\mathcal{X}$ is a bijection $l : \mathcal{Z}_{++} \to \mathcal{X}$. Write $\mathcal{L}$ for the set of labellings and $\alpha_l(k)$ for the proportion of neighbours of player $l(k)$ who have a lower label under labelling l, i.e.,

$$\alpha_l(k) = \frac{\#\{j : l(j) \sim l(k) \text{ and } j < k\}}{\#\{j : l(j) \sim l(k)\}}. \tag{4.2}$$

Proposition 2. *The contagion threshold is the largest p such thus under some labelling l, $\alpha_l(k) \geq p$ for all sufficiently large k. Formally:*

$$\xi = \max_{l\in\mathcal{L}}\left(\mathit{lim}_{K\to\infty}\left(\min_{k\geq K}\alpha_l(k)\right)\right). \tag{4.3}$$

Again, this characterization is especially useful because it can be used to provide a constructive bound, in this case a lower bound:

Corollary 2. *[Lower Bound I]. Suppose there exists a labelling l with $\alpha_l(k) \geq p$ for all sufficiently large k. Then $\xi \geq p$.*

Proof. (of proposition 2). The proposition can be re-stated as: "There exists a labelling l with $\alpha_l(k) \geq p$ for all sufficiently large k if and only if $\xi \geq p$". Suppose $\alpha_l(k) \geq p$ for all $k > K$. Now let X be the finite group $\{l(j) : j \leq K\}$. Now by induction $\{l(j) : j \leq K + k\} \subseteq [\Pi_+^p]^k(X)$, so $\underset{k\geq 1}{\cup}[\Pi_+^p]^k(X) = \mathcal{X} \Longrightarrow p \leq \xi$ (by lemma 1).

Conversely, suppose $p \leq \xi$. By lemma 1, there exists finite group X_0 such that $\underset{n\geq 1}{\cup}[\Pi_+^p]^n(X_0) = \mathcal{X}$. Let $X_n = [\Pi_+^p]^n(X_0) \cap \overline{[\Pi_+^p]^{n-1}(X_0)}$ for $n = 1, 2....$ Consider any labelling with $j > k$ whenever $l(j) \in X_m$, $l(k) \in X_n$ and $m > n$. Now $\alpha_l(k) \geq p$ for all $k > \#X_0$. □

A second lower bound holds under an additional restriction on the local interaction system:

Definition 3. *Local Interaction System $(\mathcal{X}, \sim)$ is connected if, for all $\underline{x}, \overline{x} \in \mathcal{X}$, there exist $\{x_1, x_2, ..., x_K\} \subseteq \mathcal{X}$ such that $x_1 = \underline{x}$, $x_K = \overline{x}$ and $x_k \sim x_{k+1}$ for each $k = 1, ..., K-1$.*

Recall that M was an upper bound on the number of possible neighbours.

Corollary 3. *[Lower Bound II]. If Local Interaction System* $(X, \sim)$ *is connected, then the contagion threshold* ξ *is at least* $1/M$.

Proof. By connectedness, there exists a labelling such that for all $k \geq 2$, there exists $j < k$ with $l(j) \sim l(k)$. Since $\#\Gamma(x) \leq M$ for all $x \in \mathcal{X}$, $\alpha_l(k) \geq 1/M$ for all k. By corollary 2, $\xi \geq 1/M$. □

4.3. Contagion Thresholds close to 1/2

Proposition 3. *Every local interaction system* $(\mathcal{X}, \sim)$ *has a contagion threshold less than or equal to* $1/2$.

This can proved from a result of Kajii and Morris [1995], via the incomplete information game/local interaction game equivalence discussed in Morris [1997]. For completeness, however, a direct proof is reported in appendix B.

The contagion threshold cannot exceed 1/2. The examples discussed in section 2 suggested that if there were sufficiently low neighbour growth and sufficient uniformity in the local interaction system, then the contagion threshold would be close to 1/2. We will now introduce two formal properties which capture the intuition of low neighbour growth and uniformity.

Low Neighbour Growth Say that player y is within Erdös distance k of group X if it takes at most k steps to reach y from X. Formally, let $\Gamma^0(X) = X$ and $\Gamma^{k+1}(X) = \Gamma^k(X) \cup \left\{y : y \sim x \text{ for some } x \in \Gamma^k(X)\right\}$; $\Gamma^k(X)$ is the set of players within Erdös distance k of X.

We will be interested in the behaviour of $\#\Gamma^k(X)$ as a function of k. Consider the following example.

Example 5. *(m-Hierarchy) The population is arranged in a hierarchy. Each player has* $m-1$ *subordinates. Each player, except the root player, has a single superior. Thus let* $\mathcal{X} = \bigcup_{k=0}^{\infty} X_k$, *where* $X_0 = \{\emptyset\}$ *and* $X_k = \{1, .., m-1\}^k$ *for all* $k \geq 1$, *with* $m \geq 3$; $x \sim y$ *if and only if* $x = (y, n)$ *or* $y = (x, n)$ *for some* $n = 1, .., m-1$. *See figure 5 for the case where* $m = 3$.

We can calculate how fast $\Gamma^k(X_0)$ grows as a function of k: $\#\Gamma^1(X_0) = 1 + (m-1) = m$, $\#\Gamma^2(X_0) = 1 + (m-1) + (m-1)^2$, etc.., so that

$$\#\Gamma^k(X_0) = \left(1 + (m-1) + .. + (m-1)^k\right) = \left((m-1)^{k+1} - 1\right) \Big/ (m-2)$$

Thus $\#\Gamma^k(X_0)$ grows exponentially in k. Clearly this is the worst case scenario: as long as each player has at most m neighbours, this is the fastest rate at which $\#\Gamma^k(X)$ can grow. If a significant proportion of players' neighbours' neighbours are their own neighbours, then this will tend to slow down the exponential growth. We will be interested in the case where there is enough neighbour correlation to prevent exponential growth of $\#\Gamma^k(X)$.

Definition 4. *Local Interaction System $(\mathcal{X}, \sim)$ satisfies low neighbour growth if $\gamma^{-k}\#\Gamma^k(X) \to 0$ as $k \to \infty$, for all finite groups X and $\gamma > 1$.*

In the all the geometric examples considered in section 2, $\#\Gamma^k(X)$ is a polynomial function of k, and thus low neighbour growth is satisfied. In fact, as long as the local interaction system is connected, requiring the definition to hold for all finite X is redundant since we have:

Lemma 3. *If $(\mathcal{X}, \sim)$ is connected and $\gamma^{-k}\#\Gamma^k(X) \to 0$ as $k \to \infty$, for all $\gamma > 1$, for some finite X, then $\gamma^{-k}\#\Gamma^k(X) \to 0$ as $k \to \infty$, for all $\gamma > 1$ and all finite X.*

Proof. Suppose X is finite and $\gamma^{-k}\#\Gamma^k(X) > \varepsilon > 0$ for infinitely many k. Fix any finite group Y. By connectedness, $X \subseteq \Gamma^n(Y)$ for some n. Now $\Gamma^k(X) \subseteq \Gamma^{n+k}(Y)$, so $\gamma^{-(n+k)}\#\Gamma^{n+k}(Y) > \varepsilon\gamma^{-n} > 0$ for infinitely many k, i.e., $\gamma^{-k}\#\Gamma^k(Y)$ does not tend to zero. $\square$

Researchers in the sociology literature have compared the growth of $\#\Gamma^k(X)$ for graphs describing different relationships; $\#\Gamma^k(X)$ grows more slowly when the graph describes an important relationship than when it describes a more peripheral relationship. For example, Rapoport and Horvarth [1961] examined levels of friendship among junior high school students. In a graph based on seventh and eighth best friends, $\#\Gamma^k(X)$ grows fast. In a graph based on best and second best friends, $\#\Gamma^k(X)$ grows much more slowly.[13]

[13]See also Granovetter [1973]. I'm grateful to Michael Chwe for bringing this literature to my attention. See Chwe [1996] for more on the strategic implications of different kinds of social links.

To get a feel for the growth of $\#\Gamma^k(X)$, consider the experimental finding of Milgram [1967] that the median Erdös distance between two randomly chosen individuals in the U.S. was five. To interpret this finding, consider two extreme cases. The US population at the time was around 200 million and Milgram estimated that each individual had approximately 500 acquaintances. Suppose that one individual has no overlap between his acquaintances, his acquaintances' acquaintances and his acquaintances' acquaintances' acquaintances. Then over half the population would be within Erdös distance 3 of this individual ($500^3 = 125,000,000$). On the other hand, suppose the population of 200 million was arranged in a circle and each individual knew 250 people on either side, the median Erdös distance would be 200,000.

$\delta-$**Uniformity** To discuss the uniformity property, we need some additional notation.

Definition 5. *Labelling l is an Erdös labelling if there exists a finite group X such that $l(i) \in \Gamma^k(X)$ and $l(j) \notin \Gamma^k(X) \Rightarrow i < j$.*

Definition 6. *Local Interaction System $(\mathcal{X}, \sim)$ satisfies δ-uniformity if there exists an Erdös labelling l such that for all sufficiently large K,*

$$\max_{k',k \geq K} \quad |\alpha_l(k') - \alpha_l(k)| \leq \delta \tag{4.4}$$

The hierarchy case discussed above (example 5 and figure 5) satisfies 0-smoothness. Consider any Erdös labelling with initial (singleton) group X_0. Now $\Gamma^k(X_0) = \bigcup_{j=0}^{k} X_j$. For any K, player $l(K)$ has exactly one neighbour with a lower label. Thus $\alpha_l(K) = 1/m$ for all K.[14]

Proposition 4. *If a Local Interaction System satisfies low neighbour growth and δ-uniformity, then the contagion threshold $\xi \geq 1/2 - \delta$.*

Proof. Suppose Erdös labelling l satisfies (4.4). Then there exist α and K with

$$\alpha - \delta \leq \frac{\#\{j : l(j) \sim l(k) \text{ and } j < k\}}{\#\{j : l(j) \sim l(k)\}} \leq \alpha$$

[14]The contagion threshold ξ for this example is thus $1/m$: by corollary 1, $\xi \leq 1/m$; by corollary 3, $\xi \geq 1/m$.

for all $k \geq K$. By corollary 2, $\xi \geq \alpha - \delta$. So if $\alpha \geq 1/2$, we are done. Suppose then that $\alpha < 1/2$. Now:

$$\#\{j : l(j) \sim l(k) \text{ and } j < k\} \leq \left(\frac{\alpha}{1-\alpha}\right) \#\{j : l(j) \sim l(k) \text{ and } j > k\} \quad (4.5)$$

for all $k \geq K$. Since l is an Erdös labelling, there exists a finite group X such that $l(i) \in \Gamma^k(X)$ and $l(j) \notin \Gamma^k(X) \Rightarrow i < j$. Let $X_0 = X$ and $X_k = \Gamma^k(X) \cap \overline{\Gamma^{k-1}(X)}$ for $k = 1, 2, ...$ Choose N such that $l(K) \in X_{N-1}$. Now if $n \geq N$, summing equation (4.5) across all k with $l(k) \in X_n$ implies

$$\begin{aligned} &\left\{ \begin{array}{c} \#\{(j,k) : l(j) \sim l(k), l(j) \in X_{n-1} \text{ and } l(k) \in X_n\} \\ +\#\{(j,k) : l(j) \sim l(k), \ \{l(j), l(k)\} \subseteq X_n \text{ and } j < k\} \end{array} \right\} \\ \leq \ &\left(\frac{\alpha}{1-\alpha}\right) \left\{ \begin{array}{c} \#\{(j,k) : l(j) \sim l(k), l(j) \in X_{n+1} \text{ and } l(k) \in X_n\} \\ +\#\{(j,k) : l(j) \sim l(k), \ \{l(j), l(k)\} \subseteq X_n \text{ and } j > k\} \end{array} \right\}. \end{aligned}$$

$$\begin{aligned} \text{Writing } d^n &= \#\{\{x,y\} : x \sim y, \, x \in X_{n-1} \text{ and } y \in X_n\} \\ \text{and } e^n &= \#\{\{x,y\} : x \sim y, \, x \in X_n \text{ and } y \in X_n\}, \end{aligned}$$

the above expression can be re-written as

$$d^n + e^n \leq \left(\frac{\alpha}{1-\alpha}\right) \left(d^{n+1} + e^n\right).$$

Since $\alpha < 1/2$, this implies $d^{n+1} \geq \left(\frac{1-\alpha}{\alpha}\right)^n d^n$ for all $n \geq N$ so $d^n \geq \left(\frac{1-\alpha}{\alpha}\right)^{N-n} d^N$ for all $n \geq N$. But $\#X_n \geq d^n / M \geq \left(\frac{1-\alpha}{\alpha}\right)^{N-n} d^N / M$. Thus $\gamma^{-n} \#\Gamma^n(X) \to \infty$ if $\gamma < \left(\frac{1-\alpha}{\alpha}\right)$, contradicting the assumption of low neighbour growth. Thus our hypothesis that $\alpha < 1/2$ is false and the lemma is proved. □

Two earlier examples illustrate why both conditions are required. The hierarchy case (example 5 and figure 5) satisfied 0-uniformity but failed low neighbour growth. The contagion threshold was $1/m$ and thus not close to $1/2$.

On the other hand, consider two dimensional nearest neighbour interaction on a lattice (example 2 and figure 2). For any x, we have $\#\Gamma^0(\{x\}) = 1$ and $\#\Gamma^{k+1}(\{x\}) = \#\Gamma^k(\{x\}) + 4(k+1)$ for all $k \geq 1$. Thus $\#\Gamma^k(\{x\}) = 1 + 2k(k+1)$ and the local interaction system satisfies low neighbour growth. On the other hand, δ-smoothness fails, for any $\delta < 1/4$. Consider any Erdös labelling l. We noted above that, for any k, there are $4(k+1)$ players who are contained in $\Gamma^{k+1}(\{x\})$ but not $\Gamma^{k}(\{x\})$. Those locations form an empty square with $k+2$ players on each side (see figure 6); now $4k$ of those locations (those *not* on the corners) have $\alpha_l(k) = 1/2$. But the four corners have $\alpha_l(k) = 1/4$. Intuitively, the irregularity arises because of the lumpiness of the lattice.

The intuition for proposition 4 is that behaviour must always spread slowly when contagion occurs: if it spreads fast initially, it must spread to players who do not interact much with each other, and therefore it will not spread further. Given the uniformity condition, low neighbour growth ensures that it spreads slowly. The uniformity condition is quite necessary for this result, as the following example shows.

Example 6. *Let $\mathcal{X} = \mathcal{Z}$. Let $\sim^*$ be a symmetric, irreflexive relation such that $(\mathcal{X}, \sim^*)$ is an m-hierarchy (example 5). Now suppose that*

$$\Gamma(x) = \{y : y \sim^* x \, or \; |y - x| \leq n\}.$$

So y is a neighbour of x if either *y is no more than n distance from x on the line or they are related by the hierarchy.*

In this example, $\#\Gamma^k(X)$ grows at exponential rate $m-1$. But the contagion threshold $\xi \geq \frac{n}{2n+m}$, by corollary 2 (consider the labelling l with $l(k) = \frac{k}{2}$ if k is even, $l(k) = -\frac{k+1}{2}$ if k is odd). By choosing m large but n larger, it is possible to get arbitrarily large growth of $\#\Gamma^k(X)$ with a contagion threshold arbitrarily close to 1/2. Thus it is possible to have high neighbour growth (as the evidence of Milgram [1967] suggests) but still have high contagion if, as in this example, most neighbours are "local" but a few relations generate most of the growth.

5. The Existence of Mixed Equilibria

When do there exist equilibria of local interaction games where different players take different actions? How does the answer depend on the structure of interaction? This question has been studied by researchers under the rubric of the co-existence of conventions; Sugden [1995] and Young [1996] both note the importance of asymmetries in interaction in allowing co-existence.[15] The contagion

[15] See also Shin and Williamson [1996] for an analysis of conventions with a continuum of actions (Morris [1997] shows how their incomplete informations result translates to a local interaction setting).

threshold provides a relevant measure of the degree of asymmetry.

An equilibrium X is said to be *mixed* if $X \notin \{\emptyset, \mathcal{X}\}$.

Proposition 5. *Suppose local interaction system* $(\mathcal{X}, \sim)$ *satisfies low neighbour growth and has contagion threshold* ξ. *Then for all* $q \in [\xi, 1-\xi]$, *local interaction game* $(\mathcal{X}, \sim, q)$ *has a mixed equilibrium.*

Since we know that $\xi \leq 1/2$, we have:

Corollary 4. *Local interaction game* $(\mathcal{X}, \sim, 1/2)$ *has a mixed equilibrium whenever* $(\mathcal{X}, \sim)$ *satisfies low neighbour growth.*

Thus we know that at least in the degenerate case of exactly symmetric payoffs, a mixed equilibrium *always* exists. The proposition will be proved via a pair of lemmas.

Lemma 4. $(\mathcal{X}, \sim, q)$ *has a mixed equilibrium if and only if there exist disjoint non-empty* q*-cohesive and* $(1-q)$*-cohesive groups in* $\mathcal{X}$.

Lemma 5. *If* $(\mathcal{X}, \sim)$ *satisfies low neighbour growth, then there exists a non-empty, finite,* $(1/2)$*-cohesive, group.*

Both lemmas are proved in appendix B.

Proof. (of proposition 5). Suppose that $\xi \leq q \leq 1/2$ (a symmetric argument applies if $1/2 \leq q \leq 1-\xi$). By lemma 5, there exists a finite, $(1/2)$-cohesive group X. Thus X is q-cohesive. By proposition 1, $q \geq \xi \Rightarrow \overline{X}$ contains an (infinite) $(1-q)$-cohesive subgroup. Thus by lemma 4, $(\mathcal{X}, \sim, q)$ has a mixed equilibrium. □

6. Adding Randomness

Deterministic best response dynamics need not converge to an equilibrium. For example, if players are arranged in a line and odd players choose action 1 and even players choose action 0, then best responses will lead odd players to switch to 0 and even players to switch to action 1. Best response dynamics, then, will lead to a two cycle as every player alternates between actions.[16] Partly to rule out such cycles, a number of researchers have considered adding stochastic elements to the best response dynamics. This section contains an informal discussion of alternative ways of adding random elements to the dynamic process considered in this paper. We can use this discussion to describe the connection to some of the related literature.

6.1. Random Initial Conditions

Say that the local interaction system is *isoregular* if that there exist an infinite number of isomorphisms between players that preserve the neighbourhood structure. This property is satisfied by all the geometric examples discussed in this paper. The definition of contagion given in this paper requires the existence of *one* finite group from which action 1 will spread. But if the interaction system is isoregular, then there will always exist an infinite number of disjoint groups from which action 1 will spread (if there exists one such group). Suppose then that the initial configuration is chosen randomly, with the independent probability that each player chooses action 1 equal to $\alpha \in (0, 1)$. By the law of large numbers, we will have action 1 initially played by all players in some contagion triggering group, and thus probability one convergence to action 1 being played everywhere if the payoff parameter q is less than the contagion threshold. Note that this is true even though there exist initial conditions which lead to cycling.

6.2. Random Order of Revisions

Instead of having all players best responding simultaneously, suppose they revised their actions *sequentially*, according to some randomly chosen rule (suppose also that they randomized when indifferent). Thus in each period, the player called upon to revise would choose a best response to other players' actions, while other players' actions would remain constant. Suppose X_0 was the initial configuration and X_t described the sequence of configurations under such a rule. It is straightforward to check that with probability one, we must have:

$$\overline{\bigcup_{k\geq 1} \left[\Pi_{+}^{1-q}\right]^{k} \left(\overline{X_0}\right)} \subseteq X_t \subseteq \bigcup_{k\geq 1} \left[\Pi_{+}^{q}\right]^{k} (X_0) \text{ for all } t = 1, 2, ... \tag{6.1}$$

[16] If $\mathcal{X}$ is finite and $(\mathcal{X}, \sim)$ is bipartite, deterministic best response dynamics must always converge to a two-cycle.

Furthermore, there will exist two critical realizations of the revision process such that X_t converges towards the upper bound and lower bound respectively. Thus there exists a finite initial group X_0 such that action 1 spreads contagiously, with positive probability, under a random revision rule if and only if there exists a finite initial group X_0 such that action 1 spreads contagiously under determinisitic best response dynamics.

Blume [1995] considers a model with random order of moves (he also shows results for random initial conditions). He focuses on interaction on a two dimensional lattice, but considers both nearest neighbour interaction in n dimensions and more general interaction with translation invariant neighbourhoods. In particular, he shows that on an infinite two dimensional lattice, as neighbourhoods of a fixed shape are expanded, a variable analogous to the contagion threshold tends to 1/2.

6.3. Random Responses

Suppose now that players do not always choose a best response. Consider two cases.

Noise at the Margin. Suppose first that players always stay with their current action if it is a best response, but they randomize if their current action is not a best response. Now if X_0 was the initial configuration and X_t described the sequence of configurations chosen under the process, we must again have equation (6.1) holding with probability one, and thus contagion occurring with positive probability under the noise at the margin process if and only if it occurs under deterministic best response dynamics. Anderlini and Ianni [1995] show a convergence result for arbitrary local interaction systems of a finite population version of the noise at the margin process.

Ergodic Noise. Suppose that there is a positive probability of any configuration in any period, but the probability of a player not choosing a best response is small. In the environment of this paper, this would give rise to an infinite state Markov process. Ellison [1993, 1994] considers finite population versions of such a process. In the long run, action 1 will be played most of the time whenever $q < \frac{1}{2}$. But a finite version of the contagion discussed in this paper is used to show that convergence to action 1 being played occurs very fast. Blume [1993] considers a random revision rule where the log odds ratio of choosing alternative actions is linear in payoff differences (see also Ianni [1996]). This process can be shown to be equivalent to stochastic Ising models. But as the parameter on payoff differences becomes large, the process converges to best response dynamics.

7. Conclusion

This paper focussed on one narrow question: when do we get contagion under determinisitic best response dynamics in binary action games? This narrow focus allowed a detailed analysis of the comparative statics of the local interaction system. However, the techniques and some of the results presented here are relevant to a broader range of questions: for example, the existence of mixed equilibria and stochastic dynamics.

Many of the results extend straightforwardly to more general interaction structures (for example, allowing different interactions to have different weights). A companion paper, Morris [1997], considers a very general class of interaction games and it is straightforward to extend many of the results in this paper.

The contribution of the paper is to characterize contagion in terms of qualitative properties of the interaction system, such as cohesion, neighbour growth and uniformity (rather than in terms of, say, the dimension of lattices). But one would like to go one step further and understand how likely these critical properties are to emerge.

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Appendix A: Interaction on a Lattice

This appendix shows how the upper and lower bounds on the contagion threshold developed in section 4 can be used to identify the contagion threshold in the type of geometric examples studied in the literature.

Consider then local interaction systems on an m-dimensional lattice, i.e. $\mathcal{X} = \mathcal{Z}^m$. Nearest neighbour interaction (example 2) is now formally described by adding the neighbourhood relation $\sim$ defined by

$$x \sim y \Leftrightarrow \sum_{i=1}^{m} |x_i - y_i| = 1.$$

See figures 1 and 2 for the cases where m equals 1 and 2 respectively. The contagion threshold $\xi = 1/2m$.

Since each player has exactly $2m$ neighbours, we have $\xi \geq 1/2m$ by corollary 3. But now consider any group of the form $\Lambda(K) = \{x : |x_i| > K \text{ for all } i = 1, m\}$. This is a co-finite group that includes all players except an m-dimensional cube with side length $(2K+1)$; $\Lambda(K)$ is $\left(\frac{2m-1}{2m}\right)$-cohesive for all K; but for all co-finite groups X, there exists positive integer K such that $\Lambda(K) \subseteq X$. So by corollary 1, $\xi \leq 1/2m$.

In order to analyze n-max distance interaction on an m-dimensional lattice, it is useful to prove a general result about translation invariant neighbourhoods in $\mathcal{Z}^m$ (the two dimensional version was studied by Blume [1995]). Write $\|x\|$ for the length of vector x in $\Re^m$, i.e., $\|x\| = \sqrt{\sum_{i=1}^{m} x_i^2}$; $\mathbf{0}$ for the origin in $\Re^m$; and S^{m-1} for the surface of a sphere of radius 1 in m dimensions, i.e., $S^{m-1} = \{x \in \Re^m : \|x\| = 1\}$.

Example 7. *(Arbitrary translation invariant neighbourhoods in m dimensions). Let $\mathcal{X} = \mathcal{Z}^m$ and define the neighbourhood relation $\sim$ by*

$$x \sim y \Leftrightarrow y - x \in W,$$

where W is some finite subset of $\mathcal{Z}^m \backslash \{\mathbf{0}\}$. The maintained symmetry assumption on $(\mathcal{X}, \sim)$ implies that W is radially symmetric ($x \in W \Longrightarrow -x \in W$).

Let $\zeta(W)$ be the maximum proportion of points in W that an $(m-1)$-dimensional plane through the origin can go through, i.e.,

$$\zeta(W) = \frac{\max\limits_{\lambda \in S^{m-1}} \#\{x \in W : \lambda.x = 0\}}{\#W}.$$

It will be shown that the local interaction system $(\mathcal{X}, \sim)$ has contagion threshold $\xi \in \left[\frac{1}{2}(1 - \zeta(W)), \frac{1}{2}\right]$. By radial symmetry,

$$\#\{x \in W : \lambda.x > 0\} = \#\{x \in W : \lambda.x < 0\}$$

for all $\lambda \in S^{m-1}$. Thus

$$\begin{aligned} \#\{x \in W : \lambda.x < 0\} &= \tfrac{1}{2}(\#W - \#\{x \in W : \lambda.x = 0\}) \\ &\geq \tfrac{1}{2}(1 - \zeta(W))\#W \end{aligned}$$

for all $\lambda \in S^{m-1}$. Since W is a finite set, there exists $\varepsilon > 0$ such that

$$\#\{x \in W : \lambda.x \leq -\varepsilon\} = \#\{x \in W : \lambda.x < 0\} \geq \frac{1}{2}(1 - \zeta(W))\#W.$$

for all $\lambda \in S^{m-1}$. Let $w = \max_{x \in W} \|x\|$. Suppose $p < \frac{1}{2}(1 - \zeta(W))$. We will construct a labelling satisfying $\alpha_l(k) \geq p$ for all k sufficiently large. Let labelling l satisfy $\|x\| \geq \|y\| \Rightarrow l(x) \geq l(y)$. Fix K such that

$$2\varepsilon \|l(K)\| \geq w^2 \tag{7.1}$$

Let $\lambda = \frac{1}{\|l(K)\|}.l(K)$. Now suppose (i) $y \in \Gamma(l(K))$, so $(y - l(K)) \in W$; and (ii) $\lambda.(y - l(K)) \leq -\varepsilon$, so $\|l(K) - \lambda.y\| \geq \varepsilon$. Now

$$\|l(K) - y\| = \sqrt{\|y - \lambda.y\|^2 + \|l(K) - \lambda.y\|^2} \leq w,$$

so $\|y - \lambda.y\|^2 \leq w^2 - \|l(K) - \lambda.y\|^2 \leq w^2 - \varepsilon^2$. Also $\|\lambda.y\| \leq \|l(K)\| - \varepsilon$, so

$$\begin{aligned}
\|y\| &= \sqrt{\|y - \lambda.y\|^2 + \|\lambda.y\|^2} \\
&\leq \sqrt{(\|l(K)\| - \varepsilon)^2 + (w^2 - \varepsilon^2)} \\
&= \sqrt{\|l(K)\|^2 - 2\|l(K)\|\varepsilon + \varepsilon^2 + w^2 - \varepsilon^2} \\
&< \|l(K)\| \text{ by (7.1).}
\end{aligned}$$

(see figure 7 for the geometry of this argument[17]). Thus

$$\{k : l(k) \sim l(K) \text{ and } \lambda.(y - l(K)) \leq -\varepsilon\} \subseteq \{k : l(k) \sim l(K) \text{ and } k < K\}$$

$$\begin{aligned}
\text{So } \frac{\#\{k : l(k) \sim l(K) \text{ and } k < K\}}{\#W} &\geq \frac{\#\{k : l(k) \sim l(K) \text{ and } \lambda.(y - l(K)) \leq -\varepsilon\}}{\#W} \\
&= \frac{\#\{x \in W : \lambda.x \leq -\varepsilon\}}{\#W} \\
&\geq \frac{1}{2}(1 - \zeta(W)) \\
&> p.
\end{aligned}$$

Now the result follows by corollary 2.

[17] A similar diagram appears in the appendix of Carlsson and van Damme [1993]. The similarity of the geometric arguments illustrates the incomplete information / local interaction equivalence described in Morris [1997].

This result can be applied to give an exact characterization of the contagion threshold in n-max distance interaction in m dimensions (example 3 and figure 3). This example is formally described by letting $\mathcal{X} = \mathcal{Z}^m$ and defining neighbourhood relation $\sim$ by

$$x \sim y \Leftrightarrow 1 \leq \max_{i=1,m} |x_i - y_i| \leq n,$$

for some integer $n \geq 1$. The contagion threshold ξ is

$$\psi(n,m) = \frac{n(2n+1)^{m-1}}{(2n+1)^m - 1}.$$

First note that this example fits the structure of example 7 with

$$W = \{x \in \mathcal{Z}^m \backslash \{\mathbf{0}\} : |x_i| \leq n, \text{ for all } i = 1, m\}.$$

Observe that for each $x \in \mathcal{X}$ and any $i = 1, m$,

$$\#\{y : y \sim x \text{ and } y_i > x_i\} = \#\{y : y \sim x \text{ and } y_i < x_i\} = n(2n+1)^{m-1},$$

$$\#\{y : y \sim x \text{ and } y_i = x_i\} = (2n+1)^{m-1} - 1,$$

and thus $\#\Gamma(x) = (2n+1)^m - 1$ for all $x \in \mathcal{X}$. Now $\zeta(W) = \frac{(2n+1)^{m-1}-1}{(2n+1)^m-1}$. By the argument of the example 7, the contagion threshold $\xi \geq \frac{1}{2}(1 - \zeta(W)) = \psi(n,m)$. But consider the group $\Lambda(K) = \{x : |x_k| > K\}$. For any $x \in \Lambda(K)$, $\#(\Gamma(x) \cap \Lambda(K)) \geq n(2n+1)^{m-1} + (2n+1)^{m-1} - 1$, so for all $x \in \Lambda(K)$,

$$\frac{\#(\Gamma(x) \cap \Lambda(K))}{\#\Gamma(x)} \geq \frac{n(2n+1)^{m-1} + (2n+1)^{m-1} - 1}{((2n+1)^m - 1)} = 1 - \psi(n,m);$$

so $\Lambda(K)$ is p-cohesive for $p \leq 1 - \psi(n,m)$. But every co-finite group X contains the group $\Lambda(K)$ if K is chosen sufficiently large. Thus every co-finite group has an infinite, $(1 - \psi(n,m))$-cohesive, subgroup. So by corollary 1, $\xi \leq \psi(n,m)$. □

Appendix B: Proofs

Proposition 3 and lemmas 2, 4 and 5 are proved in this appendix. Two additional lemmas are used and will be stated and proved first.

Lemma 6. *Suppose X is finite and $p > 1/2$. Then $\#\left([\Pi_+^p]^k(X)\right) \leq \left(1 + \frac{M}{2p-1}\right)\#X$ for all $k \geq 1$.*

Proof. Let X be a finite group and let $p > 1/2$. Let $\rho = \frac{1-p}{p} < 1$. Consider the increasing sequence of groups $\left[\Pi_+^p\right]^k (X)$, for $k \geq 0$. Fix $K \geq 1$. Let $Y_0 = X$; $Y_k = \left[\Pi_+^p\right]^k (X) \cap \overline{\left[\Pi_+^p\right]^{k-1} (X)}$ for $k = 1, .., K$; and $Y_{K+1} = \overline{\left[\Pi_+^p\right]^K (X)}$. By construction, the collection of groups $\{Y_0, Y_1, .., Y_{K+1}\}$ partition $\mathcal{X}$.

Now let $h(i,j) = \#\{(x,y) : x \sim y,\ x \in Y_i \text{ and } y \in Y_j\}$. Note that $h(x,y)$ is the number of *ordered* pairs of players. Thus, in the language of graph theory, $h(i,i)$ is twice the number of edges connecting vertices in Y_i; but if $j \neq i$, $h(i,j)$ is exactly the number of edges joining vertices in Y_i to vertices in Y_j. Observe that $h(i,j) = h(j,i)$.

Now suppose there exists $x \in Y_k = \left[\Pi_+^p\right]^k (X) \cap \overline{\left[\Pi_+^p\right]^{k-1} (X)}$, for some $k = 1, .., K$. Since $\left[\Pi_+^p\right]^k (X) = \left[\Pi_+^p\right]^{k-1} (X) \cup \Pi^p \left(\left[\Pi_+^p\right]^{k-1} (X)\right)$, we must have $x \in \Pi^p \left(\left[\Pi_+^p\right]^{k-1} (X)\right)$. This in turn implies that

$$\frac{\#\left(\left[\Pi_+^p\right]^{k-1} (X) \cap \Gamma(x)\right)}{\#\Gamma(x)} \geq p.$$

Re-arranging gives

$$\begin{aligned}
\#\left\{y : y \in \bigcup_{i=0}^{k-1} Y_i \text{ and } y \sim x\right\} &\geq p\#\{y :\ y \sim x\}, \text{ so} \\
(1-p)\#\left\{y : y \in \bigcup_{i=0}^{k-1} Y_i \text{ and } y \sim x\right\} &\geq p\#\left\{y : y \in \bigcup_{i=k}^{K+1} Y_i \text{ and } y \sim x\right\}, \text{ so} \\
\rho\#\left\{y : y \in \bigcup_{i=0}^{k-1} Y_i \text{ and } y \sim x\right\} &\geq \#\left\{y : y \in \bigcup_{i=k}^{K+1} Y_i \text{ and } y \sim x\right\}.
\end{aligned}$$

Summing the above expression across $x \in Y_k$ gives (for each $k = 1, ..., K$)

$$\rho \sum_{i=0}^{k-1} h(i,k) \geq \sum_{i=k}^{K+1} h(k,i). \tag{7.2}$$

We have just shown that (7.2) holds if Y_k is non-empty. It trivially holds if Y_k is empty (since each expression equals zero). Now multiplying the kth equation of (7.2) by $1 + \rho + .. + \rho^{K-k}$ and summing gives

$$\sum_{k=1}^{K} \sum_{i=0}^{k-1} \left(1 + \rho + .. + \rho^{K-k}\right) \rho h(i,k) \geq \sum_{k=1}^{K} \sum_{i=k}^{K+1} \left(1 + \rho + .. + \rho^{K-k}\right) h(k,i). \tag{7.3}$$

But

$$\sum_{k=1}^{K}\sum_{i=0}^{k-1}\left(1+\rho+..+\rho^{K-k}\right)\rho h(i,k) = \left(\begin{array}{c}\sum_{k=1}^{K}\left(1+\rho+..+\rho^{K-k}\right)\rho h(0,k) \\ +\sum_{k=1}^{K}\sum_{i=1}^{k-1}\left(1+\rho+..+\rho^{K-k}\right)\rho h(i,k)\end{array}\right)$$

$$= \left(\begin{array}{c}\sum_{k=1}^{K}\left(1+\rho+..+\rho^{K-k}\right)\rho h(0,k) \\ +\sum_{k=1}^{K-1}\sum_{i=k+1}^{K}\left(1+\rho+..+\rho^{K-i}\right)\rho h(k,i)\end{array}\right)$$

$$= \left(\begin{array}{c}\sum_{k=1}^{K}\left(1+\rho+..+\rho^{K-k}\right)\rho h(0,k) \\ +\sum_{k=1}^{K-1}\sum_{i=k+1}^{K}\left(\rho+\rho^2+..+\rho^{K-i+1}\right)h(k,i)\end{array}\right) \tag{7.4}$$

$$\text{and } \sum_{k=1}^{K}\sum_{i=k}^{K+1}\left(1+\rho+..+\rho^{K-k}\right)h(k,i) = \left(\begin{array}{c}\sum_{k=1}^{K}\left(1+\rho+..+\rho^{K-k}\right)h(k,k) \\ +\sum_{k=1}^{K-1}\sum_{i=k+1}^{K}\left(1+\rho+..+\rho^{K-k}\right)h(k,i) \\ +\sum_{k=1}^{K}\left(1+\rho+..+\rho^{K-k}\right)h(k,K+1)\end{array}\right) \tag{7.5}$$

Substituting (7.4) and (7.5) in (7.3) gives

$$\sum_{k=1}^{K}\left(1+\rho+..+\rho^{K-k}\right)\rho h(0,k) \geq \left(\begin{array}{c}\sum_{k=1}^{K}\left(1+\rho+..+\rho^{K-k}\right)h(k,k) \\ +\sum_{k=1}^{K-1}\sum_{i=k+1}^{K}\left(\begin{array}{c}1+\rho+..+\rho^{K-k} \\ -\rho-\rho^2-..-\rho^{K-i+1}\end{array}\right)h(k,i) \\ +\sum_{k=1}^{K}\left(1+\rho+..+\rho^{K-k}\right)h(k,K+1)\end{array}\right).$$

Each $h(k,i)$ on the right hand side is multiplied by a number which is at least 1, while each $h(k,0)$ on the left hand side is multiplied by a number less than or equal to $\frac{1-p}{2p-1}$, since $\left(1+\rho+..+\rho^{K-k}\right)\rho = \frac{\rho}{1-\rho}\left(1-\rho^{K-k+1}\right) < \frac{\rho}{1-\rho} = \frac{1-p}{2p-1}$. Thus subtracting positive terms from the right hand side and adding positive terms to the left hand side, we have

$$\frac{1-p}{2p-1}\sum_{k=1}^{K}h(0,k) \geq \sum_{k=1}^{K}\sum_{i=k}^{K+1}h(k,i). \tag{7.6}$$

But $1 \leq \Gamma(x) \leq M$ for all $x \in \mathcal{X}$ implies that for all $k = 0, .., K+1$,

$$M.\#Y_k \geq \sum_{i=0}^{K+1} h(i,k) \geq \#Y_k. \tag{7.7}$$

Now we have

$$\begin{aligned}
\sum_{k=1}^{K} \#Y_k &\leq \sum_{k=1}^{K} \sum_{i=0}^{K+1} h(i,k), \text{ by (7.7)} \\
&= \sum_{k=1}^{K} h(0,k) + 2\sum_{k=1}^{K}\sum_{i=k}^{K} h(k,i) + \sum_{k=1}^{K} h(k,K+1) \\
&\leq \sum_{k=1}^{K} h(0,k) + 2\sum_{k=1}^{K}\sum_{i=k}^{K+1} h(k,i) \\
&\leq \left(1 + \frac{2(1-p)}{2p-1}\right) \sum_{k=1}^{K} h(0,k), \text{ by (7.6)} \\
&\leq \left(\frac{1}{2p-1}\right) \sum_{k=0}^{K+1} h(0,k) \\
&\leq \frac{M}{2p-1} \#Y_0.
\end{aligned}$$

Thus $\# \left[\Pi_+^p\right]^K (X) = \sum_{k=0}^{K} \#Y_k \leq \left(1 + \frac{M}{2p-1}\right) \#Y_0 = \left(1 + \frac{M}{2p-1}\right) \#X$. □

Lemma 7. *Suppose $(\mathcal{X}, \sim)$ satisfies low neighbour growth. Then for all $\varepsilon > 0$ and all finite groups $X \subseteq \mathcal{X}$, there exists n such that $\frac{\#\left(\Gamma^{n+1}(X) \cap \overline{\Gamma^n(X)}\right)}{\#\Gamma^{n+1}(X)} < \varepsilon$.*

Proof. Suppose $\frac{\#\left(\Gamma^{n+1}(X) \cap \overline{\Gamma^n(X)}\right)}{\#\Gamma^{n+1}(X)} \geq \varepsilon$ for all n. Then, for all n, $\#\left(\Gamma^{n+1}(X) \cap \overline{\Gamma^n(X)}\right) \geq \left(\frac{\varepsilon}{1-\varepsilon}\right) \#\Gamma^n(X)$, so $\#\Gamma^{n+1}(X) \geq \left(1 + \frac{\varepsilon}{1-\varepsilon}\right) \#\Gamma^n(X) = \left(\frac{1}{1-\varepsilon}\right) \#\Gamma^n(X)$ and $\#\Gamma^n(X) \geq \left(\frac{1}{1-\varepsilon}\right)^n \#X$. This contradicts low neighbour growth. □

PROOF OF LEMMA 2. Consider the following *finite* subset of $[0,1]$:

$$F(M) = \left\{ \alpha \in (0,1) : \alpha = \frac{n}{m}, \quad \begin{array}{l} \text{for some integers } m,\, n \\ \text{with } 0 < m \leq M \text{ and } 0 \leq n \leq m \end{array} \right\}.$$

Given p, choose $\varepsilon > 0$ such that $F(M) \cap (p, p+\varepsilon)$ is empty. Since $\frac{\#(X \cap \Gamma(x))}{\#\Gamma(x)} \in F(M)$ for all $x \in \mathcal{X}$ and $X \subseteq \mathcal{X}$, we have $\Pi^r(X) = \Pi^{r'}(X)$ for all $X \subseteq \mathcal{X}$ and $r, r' \in (p, p+\varepsilon)$. So for any $X \subseteq \mathcal{X}$, there exists a group Y with

$$Y = \overline{\underset{k \geq 1}{\cup} \left[\Pi_+^r\right]^k (X)} \text{ for } all\ r \in (p, p+\varepsilon).$$

Now if $x \in Y$, $\pi\left(\underset{k\geq 1}{\cup}\left[\Pi_+^r\right]^k(X)\middle| x\right) < r$ and so $\pi(Y|x) > 1-r$ for all $r \in (p, p+\varepsilon)$. Then $\pi(Y|x) > 1-p$ for all $x \in Y$. □

PROOF OF PROPOSITION 3. Fix any $p > 1/2$ and any finite group X. By lemma 6, $\#\left(\left[\Pi_+^p\right]^k(X)\right) \leq \left(1 + \frac{M}{2p-1}\right)\#X$ for all $k \geq 1$. Thus $\underset{k\geq 1}{\cup}\left(\left[\Pi_+^p\right]^k(X)\right)$ is finite. By the definition of the contagion threshold and part [3] of lemma 1, $\xi < p$. Thus $\xi < p$ for all $p > 1/2$, so $\xi \leq 1/2$. □

PROOF OF LEMMA 4. [only if] follows from the definition of equilibrium. For [if], let X_0 and Y_0 be disjoint q-cohesive and $(1-q)$-cohesive groups in $\mathcal{X}$. Define X_k inductively as follows: $X_{k+1} = \Pi_+^q(X_k) \cap \overline{Y_0}$. Let $X_* = \underset{k\geq 1}{\cup} X_k$. Now suppose $x \in X_*$. If $x \in X_0$, then $x \in \Pi^q(X_0) \subseteq \Pi^q(X_*)$. If $x \notin X_0$, then $x \in X_{k+1} \backslash X_k$ for some $k \geq 0$, so (by definition of X_{k+1}), $x \in \Pi^q(X_k) \subseteq \Pi^q(X_*)$. Thus X_* is q-cohesive. But

$$\begin{aligned} X_* &= \underset{k\geq 1}{\cup} X_k \\ &= \underset{k\geq 1}{\cup} X_{k+1}, \text{ since } X_1 \subseteq X_2 \\ &= \underset{k\geq 1}{\cup}\left(\left(X_k \cup \Pi^q(X_k)\right) \cap \overline{Y_0}\right) \\ &= \left(\left(\underset{k\geq 1}{\cup} X_k\right) \cup \left(\underset{k\geq 1}{\cup} \Pi^q(X_k)\right)\right) \cap \overline{Y_0} \\ &= \left(X_* \cup \Pi^q(X_*)\right) \cap \overline{Y_0}, \text{ by B2.} \qquad (7.8) \\ &= \Pi^q(X_*) \cap \overline{Y_0} \qquad (7.9) \end{aligned}$$

Now suppose $x \in \overline{X_*}$. If $x \in Y_0$, then $x \in \Pi^{1-q}(Y_0) \subseteq \Pi^{1-q}\left(\overline{X_*}\right)$, since $Y_0 \subseteq \overline{X_*}$. If $x \notin Y_0$, then by (7.8) $x \notin \Pi^q(X_*)$, so $x \in \Pi^{1-q}\left(\overline{X_*}\right)$. Thus $\overline{X_*}$ is $(1-q)$-cohesive. So X_* is an equilibrium. □

PROOF OF LEMMA 5. By lemma 2, there exists $\varepsilon > 0$ such that $\overline{\underset{k\geq 1}{\cup}\left[\Pi_+^{\left(\frac{1}{2}+\varepsilon\right)}\right]^k(X)}$ is (1/2)-cohesive for all $X \subseteq \mathcal{X}$. Fix any finite group Y. Let

$$Z_n = \Gamma^{n+1}(Y) \cap \overline{\bigcup_{k\geq 0}\left[\Pi_+^{\left(\frac{1}{2}+\varepsilon\right)}\right]^k \left(\Gamma^{n+1}(Y) \cap \overline{\Gamma^n(Y)}\right)} \text{ for all } n = 0, 1, ..$$

If $x \in Z_n$, then $x \in \Gamma^{n+1}(Y)$. But $x \in \overline{\bigcup_{k\geq 0}\left[\Pi_+^{\left(\frac{1}{2}+\varepsilon\right)}\right]^k \left(\Gamma^{n+1}(Y) \cap \overline{\Gamma^n(Y)}\right)}$ implies $x \notin \Gamma^{n+1}(Y) \cap \overline{\Gamma^n(Y)}$. Thus $x \in \Gamma^n(Y)$. By construction of $\Gamma^{n+1}(Y)$, this implies that $\pi\left(\Gamma^{n+1}(Y) \middle| x\right) = 1$. Now since $\overline{\bigcup_{k\geq 0}\left[\Pi_+^{\left(\frac{1}{2}+\varepsilon\right)}\right]^k \left(\Gamma^{n+1}(Y) \cap \overline{\Gamma^n(Y)}\right)}$ is (1/2)-cohesive, we have that Z_n is (1/2)-cohesive. Also, Z_n is finite since $\Gamma^{n+1}(Y)$ is finite. Now observe that by lemma 6,

$$\begin{aligned}
\#\left(\left[\Pi_+^{\frac{1}{2}+\varepsilon}\right]^k \left(\Gamma^{n+1}(Y) \cap \overline{\Gamma^n(Y)}\right)\right) &\leq \left(1 + \frac{M}{2\left(\frac{1}{2}+\varepsilon\right)-1}\right) \#\left(\Gamma^{n+1}(Y) \cap \overline{\Gamma^n(Y)}\right) \\
&= \left(1 + \frac{M}{2\varepsilon}\right) \#\left(\Gamma^{n+1}(Y) \cap \overline{\Gamma^n(Y)}\right), \text{ for all } k \geq 1.
\end{aligned}$$

$$\begin{aligned}
\text{Thus } \#Z_n &= \#\Gamma^{n+1}(Y) - \#\left(\overline{\bigcup_{j\geq 0}\left[\Pi_+^{\left(\frac{1}{2}+\varepsilon\right)}\right]^j \left(\Gamma^{n+1}(Y) \cap \overline{\Gamma^n(Y)}\right)}\right) \\
&\geq \#\Gamma^{n+1}(Y) - \#\left(\overline{\left[\Pi_+^{\left(\frac{1}{2}+\varepsilon\right)}\right]^k \left(\Gamma^{n+1}(Y) \cap \overline{\Gamma^n(Y)}\right)}\right) \\
&\geq \#\Gamma^{n+1}(Y) - \left(1 + \frac{M}{2\varepsilon}\right) \#\left(\Gamma^{n+1}(Y) \cap \overline{\Gamma^n(Y)}\right) \\
&= \#\Gamma^{n+1}(Y) \left(1 - \left(1 + \frac{M}{2\varepsilon}\right) \frac{\#\left(\Gamma^{n+1}(Y) \cap \overline{\Gamma^n(Y)}\right)}{\#\Gamma^{n+1}(Y)}\right)
\end{aligned}$$

By lemma 7, $\frac{\#\left(\Gamma^{n+1}(Y) \cap \overline{\Gamma^n(Y)}\right)}{\#\Gamma^{n+1}(Y)} < \frac{1}{1+\left(\frac{M}{2\varepsilon}\right)}$ for *some* n. Thus $\#Z_n > 0$ and thus Z_n is non-empty for that n. Thus Z_n is a finite (1/2)-cohesive group. □

Appendix C: Proportion Operator Properties

B1: $\Pi^p(X) \subseteq X \cup \Pi^p(X) = \Pi_+^p(X)$

B2*: If $X \subseteq Y$, then $\pi(X|x) \leq \pi(Y|x)$ for all x; so $\pi(X|x) \geq p$ implies $\pi(Y|x) \geq p$, and thus $\Pi^p(X) \subseteq \Pi^p(Y)$. Now $X \subseteq Y$ and $\Pi^p(X) \subseteq \Pi^p(Y)$ imply that $\Pi^p_+(X) = X \cup \Pi^p(X) \subseteq Y \cup \Pi^p(Y) = \Pi^p(Y)$.

B2: Suppose $X_k \uparrow X$. First, observe that $x \in \bigcup_{k\geq 1} \Pi^p(X_k) \Rightarrow x \in \Pi^p(X_k)$ for some $X_k \Rightarrow x \in \Pi^p(X)$ (by B2*). Now if $\pi[X|\,x] \geq p$, there exists k such that $\Gamma(x) \cap X \subseteq X_k$. Thus $x \in \Pi^p(X) \Rightarrow x \in \Pi^p(X_k)$ for some $k \Rightarrow x \in \bigcup_{k\geq 1} \Pi^p(X_k)$. Thus $\Pi^p(X) = \bigcup_{k\geq 1} \Pi^p(X_k)$. Now $\Pi^p_+(X) = X \cup \Pi^p(X) = \left[\bigcup_{k\geq 1} X_k\right] \cup \left[\bigcup_{k\geq 1} \Pi^p(X_k)\right] = \bigcup_{k\geq 1}[X_k \cup \Pi^p(X_k)] = \bigcup_{k\geq 1} \Pi^p_+(X_k)$.

B3*: If $\pi(X|x) \geq r$ and $r > p$, then $\pi(X|x) \geq p$. Thus $r > p$ implies $\Pi^r(X) \subseteq \Pi^p(X)$. Now $\Pi^r(X) \subseteq X \cup \Pi^r(X) \subseteq X \cup \Pi^p(X) = \Pi^p_+(X)$.

B3: Suppose $p_k \uparrow p$. By B3*, $\Pi^{p_k}(X)$ is a decreasing sequence of sets and $\Pi^p(X) \subseteq \Pi^{p_k}(X)$ for all k. But now if $x \in \bigcap_{k\geq 1} \Pi^{p_k}(X)$, $\pi(X|x) \geq p_k$ for all k, so $\pi(X|x) \geq p$, so $x \in \Pi^p(X)$. Thus $\Pi^{p_k}(X) \downarrow \Pi^p(X)$. Now $\Pi^{p_k}_+(X) = [X \cup \Pi^{p_k}(X)] \downarrow [X \cup \Pi^p(X)] = \Pi^p_+(X)$.

B4: Suppose $p + r > 1$; $x \in \Pi^p(X) \Rightarrow \pi(X|x) \geq p \Rightarrow \pi\left(\overline{X}|x\right) \leq 1 - p < r \Rightarrow x \notin \Pi^r\left(\overline{X}\right) \Rightarrow x \in \overline{\Pi^r\left(\overline{X}\right)}$.

Thing 1. *Interaction on a Line*

Thing 2. *Nearest Neighbour Interaction*

Thing 3. *n-Max distance*

Thing 4. *Families*

Thing 5. *Hierarchies*

Thing 6. *a*

Thing 7. *s*