

*CARESS Working Paper 96-09*  
An Alternative Approach to Market Frictions:  
An Application to the Market for Taxicab Rides

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**Abstract**

This paper illustrates an alternative approach to modelling frictions. Frictions are not assumed to exist, but are shown to arise endogenously as a distinctive feature of the set of equilibria that correspond to a particular range of parameter values. To avoid building frictions in the environment, the information imperfections typically assumed in search-theoretic models are eliminated. In addition, the model's spacial structure and the agents' moving decisions are explicitly spelled out, allowing the number of contacts that occur to depend on the way agents choose to locate themselves. It is shown that some heterogeneity among locations is necessary although not sufficient for the equilibria of the model to exhibit frictions. An aggregate matching function is shown to exist, and its behavior with respect to changes in parameters such as the distances between locations, the agents' payoffs and the sizes of the populations of searchers on each side of the market is completely characterized. Finally, the model is used to quantify the effect of a recent change in taxicab fares on the process that rules the meetings between passengers and taxicabs in New York City.

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## 1. Introduction

A distinctive feature of the search approach is that trades occur bilaterally between agents rather than between an agent and the market as in the Walrasian model. This feature makes the process that determines how agents meet a key building block of any equilibrium model of search. The literature typically proceeds by assuming agents possess limited information, so time and resources have to be spent seeking trading partners. The information structure adopted prevents some potential traders on one side of the market (say buyers) from contacting potential traders on the other side (say sellers), not allowing the market to clear, in the sense that there are both buyers who want to buy and sellers who want to sell but were unable to meet. In other words, frictions are built in as a feature of the environment<sup>1</sup>.

The matching function approach is another way of introducing frictions that has been widely used in labor market applications<sup>2</sup>. This approach proceeds by directly assuming the existence of an aggregate object the matching function that gives the number of contacts that occur at any moment in time as a function of the number of searchers on both sides of the market. The information imperfections and other features of the environment that must underlie such a function are not made explicit; rather, it is assumed that their interaction gives rise to a well-behaved function of a small number of variables (Pissarides (1990), pp. 3-4).

Well-behaved typically means continuous, differentiable, strictly component-wise increasing, less than the number of searchers on each side of the market, and often also homogeneous of degree one.

Another common assumption formally equivalent to assuming the existence of a matching function, and used in virtually all monetary applications is that agents meet potential partners according to a Poisson process<sup>3</sup>. The fact that the

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<sup>1</sup>At least since Beveridge (1945, p. 409), labor unemployment has been called frictional when it coexists with an unsatisfied demand for labor somewhere.

<sup>2</sup>Blanchard and Diamond (1989), Bowden (1980), Burdett and Smith (1995), Coles and Smith (1995a, 1995b), Lagos (1995), Millard and Mortensen (1994), Mortensen (1992, 1994), Mortensen and Pissarides (1991, 1992), Phelan and Trejos (1996), Pissarides (1990), Ramey and Watson (1996), and Smith (1994) are some examples. For an early application of a job-matching function, see Phelps (1968).

<sup>3</sup>Examples of papers that assume a Poisson meeting process are: Boldrin et al (1993), Burdett and Coles (1994), Diamond (1981, 1982, 1984), Diamond and Yellin (1987, 1990), Kiyotaki and

number of meetings in some interval of time is random and not generated by the agents' behavior is usually motivated by making references to an environment in which the location of any potential trading partner is originally unknown, and can be determined only by a random draw (i.e. by random search) from the group of possible locations. The thing to note, is that either by adopting a Poisson process or a matching function, the bulk of the search and matching literature assumes that meetings are ruled by some *exogenous* process.

Matching functions can be easily derived for environments with information imperfections and agents who search randomly. In the labor literature, for instance, a common story is that workers know where vacancies are, but don't know which particular vacancies other workers will visit, allowing for the possibility that some workers are unable to contact vacancies because they were second in line. Hall (1979), Mortensen (1994) and Pissarides (1979) all derive the number of contacts that will take place in some time interval, as a function of the numbers of vacancies and searching workers that is immediately implied by this information structure.

However, since adopting a matching function amounts to assuming an exogenous aggregate meeting process, it is unclear what kinds of individual search behavior are consistent with the aggregate structure adopted. In particular, could the predictions of a model that uses a matching function characterize the outcome resulting from the interaction of informed agents who are able to direct their search? This question seems relevant since in most situations, people tend to have at least some information that allows them to direct their search in ways that may not be consistent with the random search assumption.

This paper investigates the microeconomic foundations of the matching function approach. Two features of the market for taxicab rides make it an appealing setting to address this question. In first place, it is a market that exhibits meeting frictions: passengers spend time waiting for taxicabs in some parts of the city (typically downtown), while at the same time vacant taxicabs wait for passengers in others (notably the airport). Additionally, the price in this industry is typically regulated, so the analysis highlights the role of meeting probabilities and market tightness.

The treatment of frictions adopted here is in many ways different from that followed by the equilibrium search literature. The main difference is perhaps that frictions are not assumed to exist, but are shown to arise endogenously as a distinctive feature of the set of equilibria that correspond to a particular range

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Wright (1991, 1993), Trejos and Wright (1995), and Wright (1994).

of parameter values. To let agents direct their search and avoid building frictions in the environment, no information imperfections are assumed. In particular, this means that the common nobody-knows-where-anything-is assumption that forces agents to search randomly and guarantees that some potential traders will be unable to meet, is left out of the analysis. In addition, the model's spacial structure and the agents' moving decisions are explicitly spelled out, allowing the number of contacts that occur to depend on the way agents choose to locate themselves.

It is shown that some heterogeneity among locations is necessary although not sufficient for the equilibria of the model to exhibit frictions. In fact, the conditions under which frictions arise depend crucially on the total numbers of searchers on each side of the market as well as on the heterogeneity among locations. From an aggregate perspective, the equilibria that exhibit frictions (in the sense that not all possible bilateral meetings occur) look just like the outcomes obtained from standard equilibrium search models in which meeting frictions result from the fact that agents are assumed uninformed.

A function that expresses the total number of meetings in terms of the aggregate stocks of searchers on both sides of the market is shown to exist. This endogenous matching function is derived, and its behavior with respect to changes in parameters such as distances between locations, the agents' payoffs, and the sizes of the populations of searchers is completely characterized. Since agents can direct their search, changes in parameters affect their search strategies altering the shape of the matching function. This suggests that the results of policy experiments from models that assume an exogenous meeting process are likely to be misleading if the random search assumption is not a good characterization of the agents' underlying search behavior. The reason being that if agents are able to direct their search, then the matching function is an equilibrium object and hence responds to policy changes. Finally, the model is used to quantify the effect of a recent change in taxicab fares on the process that rules the meetings between passengers and taxicabs in New York City.

The rest of the paper is organized as follows. Section 2 describes the environment. Section 3 presents a cab driver's decision problem. Section 4 introduces the notion of equilibrium and presents a formal statement of the main theoretical results. For a particular case, Section 5 characterizes the full set of equilibria, derives the endogenously determined matching function for each relevant spacial arrangement, discusses the nature of the frictions that arise in equilibrium, and analyzes the main properties of the endogenous meeting technology. All the re-

sults in Section 5 are generalized in Appendix A. Section 6 computes the complete set of equilibria as well as the corresponding matching function for Manhattan market for taxicab rides. This matching function is used to answer a particular policy question in Section 7. Sections 6 and 7 are complemented by Appendix B, which proves the main result in Section 6 and provides a map of the geographical abstraction the application of the model is based upon. Section 8 concludes with a summary of the main results.

## 2. Environment

Time is discrete and continues forever. A city consists of  $n \geq 2$  locations across which the populations of people and taxicabs (hereafter cabs) may position themselves<sup>4</sup>. There is a continuum of people with size normalized to unity, and a continuum of cabs of measure  $v$ . The fractions of people and cabs in location  $i$  are denoted  $p_i$  and  $v_i$  respectively.

People's wishes to move between locations are taken to be exogenously given by a Markov chain. Specifically, it is assumed that in each period an agent will wish to remain at the current location with a probability that is constant across locations and denoted by  $(1 - u) \in (0, 1)$ . This means that there are  $u$  movers in the whole city<sup>5</sup>. The probability that a passenger in  $i$  wishes to move to  $j$  is given by  $ua_{ij}$ , with  $\sum_{j \neq i}^n a_{ij} = 1$  and  $a_{ij} \in (0, 1)$ . Therefore there are  $u_i \equiv up_i$  movers in location  $i$ ,  $a_{ij}u_i$  of which want to go to location  $j$ .

People cannot walk to their desired destination: they have to get a cab ride. Cabs cannot drive more than one passenger per trip, and when vacant, are free to choose the location where they will try to find a passenger.

Cabs (passengers) are unable to meet passengers (cabs) in distant locations: contacts only occur among cabs and passengers in the same location. And within each location, the only way in which a cab (person) may not find a passenger (cab)

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<sup>4</sup>It may help to think of the physical environment as a list  $\{\{l^i, \delta_{ij}\}_{i=1}^n\}_{j \neq i}^n$ , with each location  $l^i$  being a point on the plane:  $l^i = (d_1^i, d_2^i) \in \mathfrak{R}^2$ ; and  $\delta_{ij} = \sqrt{\sum_{k=1}^2 (d_k^i - d_k^j)^2}$  the distance between locations  $i$  and  $j$ . Notice that with  $n$  locations, there are  $\binom{n}{2} = \frac{n!}{2(n-2)!}$  pairwise distances.

<sup>5</sup>In equilibrium not everyone who wants to move is able to. The term *mover* refers to a person who *wants* to move, regardless of the ability to do so. Assuming that the probability of wishing to stay at the present location is the same across locations guarantees that the city-wide number of movers is independent of the distribution of agents across locations.

is if there are not enough passengers (cabs) in that location. In other words, letting  $m_i$  be the number of cab-passenger meetings that occur in location  $i$ , we have<sup>6</sup>

$$m_i = \min \{u_i, v_i\}. \quad (2.1)$$

Consequently, assuming the  $m_i$  contacts are random and letting  $\rho_i \equiv u_i/v_i$ , a cab in location  $i$  will find a passenger with probability  $\gamma_i = \min \{\rho_i, 1\}$ , while a mover in  $i$  will find a cab with probability  $\gamma_i \rho_i^{-1}$ .

### 3. How to drive a cab

When moving from  $i$  to  $j$ , a cab incurs a moving cost  $c_{ij}$ , with

$$c_{ij} = \hat{\pi} \delta_{ij},$$

$\delta_{ij}$  being the distance between  $i$  and  $j$ , and  $\hat{\pi} \geq 0$  the (per unit distance) cost of moving. A cab that was unable to find a passenger in a given period can choose to go to a new location where, in the following period, it will try to find a passenger. Hence the value of being unmatched in location  $i$  at the end of a period<sup>7</sup> is just the discounted value of being at the best location at the beginning of the next period, net of moving costs. Using modulo  $n$  arithmetic, this value can be written as follows:

$$U_i = \beta \max \{V_i, V_{i+1} - c_{i,i+1}, \dots, V_{i+n-1} - c_{i,i+n-1}\}, \text{ for } i = 1, \dots, n. \quad (3.1)$$

$\beta \in (0, 1)$  is the discount factor, and  $V_i$  the value of being in  $i$  before contacts take place. When driving a passenger from  $i$  to  $j$ , cabs charge a bag-drop rate  $b \geq 0$  and a rate  $\pi^* > \hat{\pi}$  per unit distance<sup>8</sup>, and hence a cab's profit from driving

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<sup>6</sup>Notice that it is implicitly assumed that contacts only occur between cabs and movers. This can be motivated by giving cabs the ability to identify movers (say because movers always raise one arm until they have found a cab).

<sup>7</sup>The expression "beginning (end) of a period" means before (after) the period's contacts have occurred.

<sup>8</sup>Strictly speaking, fares are calculated as a bag-drop charge and a charge (say \$0.25) per additional unit. A unit can be one of distance and/or of waiting time. In New York City, a unit of distance is 1/5 of a mile, while a unit of waiting time is equal to 75 seconds. As the cab moves at more than 9.6 mph, the meter clocks distance. When the cab is stopped or moving at less than 9.6 mph, the meter clocks time. So for example, traveling 1/10 of a mile (moving faster than 9.6 mph), then waiting at a stop light for  $37\frac{1}{2}$  seconds generates one unit and hence a \$0.25 charge. The rate structure adopted in the model, ignores waiting time, and hence a unit is just a unit of distance.

somebody from  $i$  to  $j$  is

$$\pi_{ij} = b + \pi\delta_{ij}, \quad (3.2)$$

where  $\pi \equiv \pi^* - \hat{\pi}$ . Then the value of giving a ride from  $i$  to  $j$  in any period is given by the profit from the trip between  $i$  and  $j$  plus the value of being located at  $j$  at the beginning of the next period:

$$V_{ij} = \pi_{ij} + \beta V_j. \quad (3.3)$$

Finally, the value of being located in  $i$  (for  $i = 1, \dots, n$ ) before a period's meetings occur is given by:

$$V_i = \gamma_i \left( \sum_{j \neq i}^n a_{ij} \max \{V_{ij}, U_i\} \right) + (1 - \gamma_i) U_i. \quad (3.4)$$

## 4. Steady state equilibrium

A steady state equilibrium is a time-invariant distribution of cabs and movers across locations, such that given this distribution, cabs maximize profits by optimally choosing where to locate themselves. Before formally defining an equilibrium, it is convenient to understand how people (i.e. filled cabs) and vacant cabs move between locations.

### 4.1. Flows of filled cabs

Since a person can only move when able to get a cab ride, the number of people who are able to move to their desired destination depends on the number of cabs available at their original location. That is, even though  $u_i$  persons want to move out of  $i$ , (only)  $\gamma_i \rho_i^{-1} u_i$  of them will be able to find a cab to do so. Notice that when  $\rho_i \leq 1$  there are at least as many cabs as movers in  $i$ , and all those who want to leave location  $i$  are able to do so. However, when  $\rho_i > 1$  there are less cabs than movers, and some people wanting to move out of  $i$  will be unable to find a cab. In this case, only a fraction  $\gamma_i \rho_i^{-1}$  of the  $a_{ij} u_i$  people wanting to go to from  $i$  to  $j$  is able to find a cab to go there. Letting  $M_{ij}$  denote the flow of matches from  $i$  to  $j$  (for  $i, j = 1, \dots, n$  and  $i \neq j$ ), we have:

$$M_{ij} = a_{ij} m_i.$$

Stationarity of the distribution of people across locations requires that for each location, the inflow of matches equals the outflow of matches. This will be the case if and only if the following  $n - 1$  conditions are satisfied:

$$\sum_{j \neq i}^n M_{ij} = \sum_{j \neq i}^n M_{ji}, \quad i = 1, \dots, n - 1.$$

## 4.2. Allocation of empty cabs

The analysis will focus on equilibria in which a cab's expected discounted payoff at the beginning of each period is equal across locations, namely equilibria for which

$$V_1 = V_2 = \dots = V_n. \quad (4.1)$$

A feature of the class of equilibria under consideration, is that a cab will never find it optimal to turn a passenger down, regardless of where the passenger wants to go<sup>9</sup>. Equations (3.1), (3.3) and (4.1) can be combined with (3.4), to show that the (flow) value of being in  $i$  (for  $i = 1, \dots, n$ ) at the beginning of a period is given by:

$$(1 - \beta) V_i = \gamma_i \pi_i$$

where

$$\pi_i \equiv \sum_{j \neq i}^n a_{ij} \pi_{ij} \quad (4.2)$$

is a cab's expected profit conditional on having contacted a passenger in location  $i$ .

## 4.3. Definition of equilibrium

A steady state equilibrium is a distribution of movers and cabs across locations,  $\{(u_i, v_i)\}_{i=1}^n$ , such that:

**(E1)** cabs maximize expected profits

$$\gamma_1 \pi_1 = \gamma_i \pi_i, \quad i = 2, \dots, n$$

**(E2)** the distribution is invariant

$$\sum_{j \neq i}^n M_{ij} = \sum_{j \neq i}^n M_{ji}, \quad i = 1, \dots, n - 1.$$

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<sup>9</sup>To see why this is true, let  $V \equiv V_i$  for all  $i$ , and notice that  $V_{ij} = \pi_{ij} + \beta V$  is strictly greater than  $U_i$  since in this class of equilibria  $U_i = U_j = \beta V$  for all  $i, j$ .



**(E3)** the distribution is feasible

$$\sum_{i=1}^n u_i = u \text{ and } \sum_{i=1}^n v_i = v.$$

By ensuring that they have no profitable way to reallocate at the beginning of each period, condition (E1) guarantees that cabs are maximizing expected discounted profits. Condition (E2) assures that the distribution of cabs and people across locations remains constant through time. Condition (E3) requires that the equilibrium distribution is consistent with the total numbers of people and cabs in the city.

#### 4.4. The aggregate matching function

In the search literature frictions are certain features of the environment that prevent some bilateral meetings from taking place. Within the present framework, no feature of the environment rules out the possibility that all possible meetings occur. In particular, (2.1) guarantees that if some cabs and passengers fail to contact each other, it can only be as a result of the way in which cabs chose to locate. Put differently, in this context frictions are a property of the equilibrium allocation, and are not necessarily implied by the type of environment assumed. An equilibrium allocation will be said to exhibit frictions if it simultaneously exhibits vacant cabs and unserved passengers. So letting  $m$  denote the aggregate (i.e. city-wide) number of meetings, an equilibrium exhibits frictions if

$$m < \min \{u, v\},$$

and is frictionless if all possible contacts take place, namely if

$$m = \min \{u, v\}.$$

This is the operational definition of frictions that will be adopted hereafter.

Having introduced the notion of equilibrium and adopted a definition of frictions, we can pose the two main questions addressed by the paper. The first, is trying to understand what conditions cause frictions arise. The second, is asking whether—as often assumed in the equilibrium search literature—it is possible to write down the aggregate number of meetings as a well-behaved function of the numbers searchers on both sides of the market. The following proposition answers both questions.

**Proposition 1.** Let  $\Pi \equiv \max \{\pi_1, \dots, \pi_n\} - \min \{\pi_1, \dots, \pi_n\}$ .

An equilibrium exists for any  $\Pi$ . Furthermore,

(a) If  $\Pi = 0$ , then all equilibria are frictionless, for any aggregate degree of market tightness  $v/u$ .

(b) If  $\Pi > 0$ , then the equilibrium allocations exhibit frictions if and only if the market is tight enough (i.e. if and only if  $v/u$  is small enough).

(c) There always exists a unique aggregate matching function. Moreover, this function is of the Leontief variety.

**Proof.** See Appendix A.

Proposition 1 gives a set of conditions that are necessary and sufficient for frictions to arise in equilibrium. It also establishes the existence of a function that expresses the aggregate number of meetings in terms of the aggregate numbers of agents on each side of the market. The purpose of the following section is to convey the intuition behind these results.

## 5. Understanding the mechanics of the model

This section uses the 3-location as a leading example to illustrate the results stated in the previous section.

### 5.1. Characterization of equilibria

As in Proposition 1, the problem of finding the conditions under which frictions arise will be split in two cases. In the first case, all locations look identical from a cab's perspective, while in the second, some location(s) is (are) better than others. Since all the results in this section are particular cases of Proposition 1, they are summarized in corollaries and their proofs omitted.

#### 5.1.1. When all locations are identical

Suppose that people's wishes to move and the distances between locations are such that<sup>10</sup>

$$\pi_1 = \pi_2 = \pi_3. \tag{5.1}$$

This means that a cab's expected profit conditional on having contacted a passenger is the same across locations. Since all possible trips in this city give the same

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<sup>10</sup>Notice that having  $\pi_{12} = \pi_{13} = \pi_{23}$  is sufficient but not necessary for (5.1) to hold.

profit, cabs maximize expected profits by maximizing the probability of picking up a passenger, so that in equilibrium contact rates must be equalized across locations. Indeed, if (5.1) holds, (E1) becomes:

$$\min \{u_1/v_1, 1\} = \min \{u_2/v_2, 1\} = \min \{u_3/v_3, 1\}. \quad (5.2)$$

Notice that (5.2) implies that in equilibrium, either all locations exhibit excess supply of cabs or none of them do<sup>11</sup>. Hence there are three possible types of equilibria: one with excess supply in all locations, another in which there is market clearing in all locations, and a third in which there is excess demand in at least one location while none of the others exhibit excess supply. These three types of equilibria are characterized in the following result:

**Corollary 1.** Assume (5.1) holds. If:

(a)  $v > u$  then there exists a unique equilibrium: all locations exhibit excess supply of cabs.

(b)  $v = u$  then there exists a unique equilibrium: there is market clearing in all locations.

(c)  $v < u$  then there is a continuum of equilibria in which at least one market exhibits excess demand while none of the others exhibit excess supply.

Whenever there are at least as many cabs as movers in all locations, each period all movers are able to reach their desired destinations. Put differently, in any equilibrium in which no location exhibits excess demand, the steady state distribution of movers across locations is given by the unique invariant distribution of the Markov matrix that rules passengers' wishes to move<sup>12</sup>. In this 3-location case this distribution is  $\mu u$ , with  $\mu = (\mu_1, \mu_2, \mu_3)$  given by

$$\begin{aligned} \mu_1 &= (1 - a_{23}a_{32})/\Delta \\ \mu_2 &= (1 - a_{13}a_{31})/\Delta \\ \mu_3 &= (1 - a_{12}a_{21})/\Delta, \end{aligned}$$

and

$$\Delta \equiv (1 - a_{12}a_{21}) + (1 - a_{13}a_{31}) + (1 - a_{23}a_{32}).$$

Hence  $\mu_i u$  is the (unconstrained) steady state number of movers in location  $i$ . The equilibrium allocations for the equilibria in parts (a) and (b) of Corollary 1 are reported in Table 5.1.

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<sup>11</sup>From now on, excess supply (demand) will mean excess supply (demand) of cabs

<sup>12</sup>For more on this, see Appendix A.

<b>i =</b>	<b>1</b>	<b>2</b>	<b>3</b>
$u_i$	$\mu_1 u$	$\mu_2 u$	$\mu_3 u$
$v_i$	$\mu_1 v$	$\mu_2 v$	$\mu_3 v$
$m_i$	$\mu_1 u$	$\mu_2 u$	$\mu_3 u$

Table 5.1: Case with  $\pi_1 = \pi_2 = \pi_3$ . Equilibrium with excess supply in all locations (or with market clearing in all locations).

In an equilibrium with either excess supply or market clearing in all locations, all movers reach their desired destinations every period and hence are distributed across locations according to  $\mu u$ . Given this distribution, there is a unique way for cabs to position themselves so that (5.2) holds. The equilibrium distribution of cabs is such that the fraction of cabs in location  $i$  is the same as the fraction of movers in that location. Notice that cabs contact rates are the same across locations and equal to  $u/v$ .

The equilibrium allocations for the equilibria in part **(c)** of Corollary 1 are reported in Table 5.2, where  $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$  denotes a vector in the 3-dimensional unit simplex<sup>13</sup>.

<b>i =</b>	<b>1</b>	<b>2</b>	<b>3</b>
$u_i$	$\mu_1 v + \varepsilon_1 (u - v)$	$\mu_2 v + \varepsilon_2 (u - v)$	$\mu_3 v + \varepsilon_3 (u - v)$
$v_i$	$\mu_1 v$	$\mu_2 v$	$\mu_3 v$
$m_i$	$\mu_1 v$	$\mu_2 v$	$\mu_3 v$

Table 5.2: Case with  $\pi_1 = \pi_2 = \pi_3$ . Equilibria with excess demand in at least one location.

In an equilibrium with excess demand in all locations, a cab is indifferent between looking for a passenger in any location because in any of them it gets a passenger with certainty and all trips give the same profits. However, with excess demand everywhere the flows between locations are ruled by the number of cabs in each location (due to (2.1)). This means that the equilibrium distribution of cabs is uniquely determined by (E2), and given by  $\mu v$ . The distribution of movers on the other hand, is indeterminate, as can be seen in Table 5.2. In any case, this indeterminacy has no effect on the number of meetings, since the latter is not affected by the number of unserved passengers (this is obvious from the last row of Table 5.2).

<sup>13</sup>That is,  $\sum_{i=1}^3 \varepsilon_i = 1$ , and  $\varepsilon_i \geq 0$ , for  $i = 1, 2, 3$ .

Finally, notice how the pattern of excess supply changes as the degree of aggregate market tightness  $v/u$  varies. As can be seen from Tables 5.1 and 5.2, the distribution of cabs is  $\mu v$  in all equilibria. So suppose we start with some  $v/u > 1$ . Initially, all markets have (the same) excess supply of cabs, namely  $v - u$ . Hence as  $v/u$  falls, the excess supply shrinks in all markets, until all markets clear at the point when  $v/u = 1$ . As the number of movers comes to exceed the number of cabs, we enter a parameter range with multiple equilibria, in which all equilibria have the property that no market is in excess supply while at least one exhibits excess demand. It is interesting to note how the number of meetings varies continuously from  $u$  to  $v$  as the aggregate degree of market tightness  $v/u$  moves continuously from above 1 to below 1 (see the last rows of Tables 5.1 and 5.2 to verify this fact).

### 5.1.2. The case of heterogeneous locations

Now suppose some location is less attractive from a cab's perspective, in the sense that the conditional expected profit of a trip from that location is strictly less than from the others. Labeling locations so that smaller subindexes correspond to locations with larger conditional expected profits, we now focus on a set of parameters that satisfy either:

$$\pi_1 \geq \pi_2 > \pi_3, \text{ or} \tag{5.3}$$

$$\pi_1 > \pi_2 = \pi_3. \tag{5.4}$$

Notice that conditions (5.1), (5.3) and (5.4) partition the parameter space. If (5.3) holds, then locations 1 and 2 are better than 3 from a cab's perspective because the expected profit of a trip from either 1 or 2 conditional on having contacted a passenger is strictly greater than from 3. If a cab is to be indifferent among locations, it must be that it is less likely to find a passenger in 1 and in 2 than in 3. In fact, under (5.3), (E1) becomes

$$\min \{u_1/v_1, 1\} \leq \min \{u_2/v_2, 1\} < \min \{u_3/v_3, 1\}. \tag{5.5}$$

According to (5.5), there can only be three types of equilibria: location 3 is in excess supply in the first, exhibits market clearing in the second and excess demand in the third. Locations 1 and 2 are in excess supply in all three types. Similarly, if the parameters are such that (5.4) is the case, then (E1) implies

$$\min \{u_1/v_1, 1\} < \min \{u_2/v_2, 1\} = \min \{u_3/v_3, 1\}. \tag{5.6}$$

The following corollary summarizes the full set of equilibria whenever either condition (5.3) or condition (5.4) holds.

**Corollary 2.** Let  $\phi \equiv \pi_3 / (\sum_{i=1}^3 \mu_i \pi_i)$ .

Assume (5.3) holds. If:

(a)  $\phi v > u$  then there is a unique equilibrium: all locations exhibit excess supply.

(b)  $\phi v = u$  then there is a unique equilibrium: locations 1 and 2 exhibit excess supply while the market clears in 3.

(c)  $\phi v < u$  then there is a unique equilibrium: locations 1 and 2 exhibit excess supply and there is excess demand in 3.

Alternatively, suppose (5.4) holds. If:

(d)  $\phi v > u$  then there is a unique equilibrium: all locations exhibit excess supply.

(e)  $\phi v = u$  then there is a unique equilibrium: location 1 exhibits excess supply while the market clears in 2 and 3.

(f)  $\phi v < u$  then there is a continuum of equilibria with excess supply in location 1 only, and excess demand in at least one of the other two locations.

We first comment on parts (a), (b), (d) and (e), whose equilibrium allocations are reported in Table 5.3.

<b>i =</b>	<b>1</b>	<b>2</b>	<b>3</b>
$u_i$	$\mu_1 u$	$\mu_2 u$	$\mu_3 u$
$v_i$	$\frac{\mu_1 \pi_1}{\sum_{i=1}^3 \mu_i \pi_i} v$	$\frac{\mu_2 \pi_2}{\sum_{i=1}^3 \mu_i \pi_i} v$	$\mu_3 \phi v$
$m_i$	$\mu_1 u$	$\mu_2 u$	$\mu_3 u$

Table 5.3: Equilibrium allocations corresponding to cases (a), (b), (d) or (e) in Corollary 2.

When all locations are identical, a cab gets the same profit from all trips. In that case, the only reason why there may be more cabs in a location than in another is that there are more passengers in the former than in the latter. In general, there are two features of locations that cabs care about. The first, is the number of potential passengers in each location. All else equal, a location with more passengers will have more cabs. The second, is how profitable rides from each particular location tend to be. This is essentially the notion of conditional (expected) profit introduced earlier. If  $\pi_i > \pi_j$ , then all else equal, there will be

more cabs in  $i$  than in  $j$  because rides originating in  $i$  tend to be longer, and hence (by (3.2)) more profitable than those originating in  $j$ . Notice that  $\pi_i$  is just a weighted average of the profits from selling a trip from  $i$  to each one of the other locations, with the weights given by the fraction of movers in  $i$  wishing to go to each one of these locations (see (4.2)). When locations differ in their conditional expected profit, the equilibrium distribution of cabs will reflect the attractiveness of each location not only in terms of the number of movers, but also in terms of how profitable it is to pick up a random passenger in each one. A good way to illustrate this point is to compare the middle row of Table 5.3 with that of Table 5.2 (or 5.1). The allocation of cabs in Table 5.3 is obtained from the one in Table 5.2 after multiplying the number of cabs in each location by a measure of the relative attractiveness of that location in terms of its conditional expected profit. This measure of the attractiveness of location  $i$ 's conditional expected profit relative to other locations is given by

$$\frac{\pi_i}{\sum_{i=1}^3 \mu_i \pi_i}, \text{ for } i = 1, 2, 3. \quad (5.7)$$

In Corollary 2 the relative attractiveness of the worst location (i.e. of the one with the lowest  $\pi_i$ ) was denoted  $\phi$ . Since in the equilibria described in parts (a), (b), (d) and (e) no location exhibits excess demand, the equilibrium distribution of movers is the unconstrained steady state distribution  $\mu u$ . Finally, a cab's contact rate differs among locations in a way that is consistent with the ranking of the  $\pi_i$ 's, with cabs in more attractive locations facing lower probabilities of meeting passengers<sup>14</sup>.

We now turn to analyze the equilibrium in part (c) of Corollary 2, whose allocations are reported in Table 5.4. The distribution of cabs is the same as in Table 5.3: the fraction of cabs in location  $i$  is still equal to the unconstrained steady state fraction of movers in location  $i$  (namely  $\mu_i$ ) times the relative attractiveness of location  $i$  given in (5.7). However, location 3 is now in excess demand, so each period some of the movers there are unable to find a cab to reach their desired destinations. Consequently, the number of passengers flowing out of 3 is smaller than in the unconstrained steady state, implying a steady state equilibrium distribution of movers with more movers in 3 and less in 1 and 2 relative to the unconstrained steady state  $\mu u$ .

Finally, it should be noted that the fact that cabs are in excess demand in location 3 does not necessarily mean that there are less cabs there than in the lo-

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<sup>14</sup>Hereafter, location  $i$  will be said to be more attractive than  $j$  if  $\pi_i > \pi_j$ .

<b>i =</b>	<b>1</b>	<b>2</b>	<b>3</b>
$u_i$	$\mu_1 \phi v$	$\mu_2 \phi v$	$u - (1 - \mu_3) \phi v$
$v_i$	$\frac{\mu_1 \pi_1}{\sum_{i=1}^3 \mu_i \pi_i} v$	$\frac{\mu_2 \pi_2}{\sum_{i=1}^3 \mu_i \pi_i} v$	$\mu_3 \phi v$
$m_i$	$\mu_1 \phi v$	$\mu_2 \phi v$	$\mu_3 \phi v$

Table 5.4: Case with  $\pi_1 \geq \pi_2 > \pi_3$ . Equilibrium with excess supply in 1 and 2 and excess demand in 3.

cations with excess supply. In particular, if 3 is popular enough as a destination (in the sense that  $a_{13}$  and  $a_{23}$  are relatively big, implying a  $\mu_3$  close enough to 1), there will be more cabs there than in 1 and 2, as can be seen from the middle row in Table 5.4.

The set of equilibria of part (f) is reported in Table 5.5, where  $\epsilon = (\epsilon_2, \epsilon_3)$  denotes a vector in the 2-dimensional unit simplex (i.e.  $\epsilon_2 + \epsilon_3 = 1$  and  $\epsilon_i \geq 0$  for  $i = 2, 3$ ).

<b>i =</b>	<b>1</b>	<b>2</b>	<b>3</b>
$u_i$	$\mu_1 \phi v$	$\mu_2 \phi v + \epsilon_2 (u - \phi v)$	$\mu_3 \phi v + \epsilon_3 (u - \phi v)$
$v_i$	$\frac{\mu_1 \pi_1}{\sum_{i=1}^3 \mu_i \pi_i} v$	$\mu_2 \phi v$	$\mu_3 \phi v$
$m_i$	$\mu_1 \phi v$	$\mu_2 \phi v$	$\mu_3 \phi v$

Table 5.5: Case with  $\pi_1 > \pi_2 = \pi_3$ . Equilibria with excess supply in 1 and no excess supply in 2 and 3.

The only difference between the allocations in Table 5.5 and those in Table 5.4 is that since there are now two locations with excess demand, the distribution of movers between them is indeterminate. Nevertheless, as can be seen in the last row of Table 5.5, the equilibrium number of meetings in each location is uniquely determined.

To conclude the section, notice how starting with an aggregate degree of market tightness  $v/u$  that lies above  $\phi^{-1}$ , the pattern of excess supply in all locations changes as  $v/u$  falls below  $\phi^{-1}$ . To x ideas, suppose condition (5.3) holds. As  $v/u$  decreases, the excess supply in all locations falls (see Figure 2 below). Under (5.3), location 3 exhibits the smallest level of excess supply. In fact, notice that it has excess supply if and only if  $v/u > \phi^{-1}$ . So the excess supply in 3 goes to zero as  $v/u$  approaches  $\phi^{-1}$ , and the market in 3 clears when  $v/u = \phi^{-1}$ . As  $v/u$  falls below  $\phi^{-1}$ , 3 moves in excess demand while 1 and 2 remain with excess supply. As  $v/u$  falls continuously from above  $\phi^{-1}$  to below  $\phi^{-1}$ , the number of meetings



varies in a continuous manner from  $u$  to  $\phi v$  (see the last rows of Tables 5.3, 5.4 and 5.5).

## 5.2. Derivation of the aggregate matching function

The first claim made in Proposition 1 is that if all locations are identical in the conditional expected profit sense, then any equilibrium that may result has the property that all possible bilateral trades occur: the system has no frictions. Part (b) of Proposition 1 claims that although having identical locations is sufficient to guarantee no frictions, it is not necessary. By deriving the aggregate (i.e. city-wide) matching function implied by each possible equilibrium of the model, this section establishes the claims made in Proposition 1 for the 3-location case.

Given any geographic configuration, any set of moving preferences (represented by the  $a_{ij}$ s) and any aggregate populations of cabs and movers, Tables 5.1-5.5 specify the distribution of cabs and movers implied by the equilibrium (or equilibria) of the model. Furthermore, the last row of each table specifies the equilibrium number of meetings that will occur in each location. We show that by aggregating across locations (i.e. by adding up the entries in the last row of the relevant table), an aggregate matching function can be constructed.

Given (5.1) holds, the equilibrium allocations when either  $v \geq u$  or  $v < u$  are given in Tables 5.1 and 5.2 respectively. From the information on their last rows, it follows that when all locations are identical from a cabs perspective, the aggregate number of meetings is

$$m^1(u, v) = \begin{cases} u & \text{if } v \geq u \\ v & \text{if } v < u, \end{cases}$$

which can be re-written as

$$m^1(u, v) = \min\{u, v\}.$$

$m^1(\cdot)$  stands for the aggregate number of meetings under condition (5.1). Therefore there are no frictions in an equilateral city since all possible bilateral meetings take place.

If condition (5.1) does not hold, then the equilibrium distribution will be one of those reported in Tables 5.3-5.5. Notice that the aggregate number of meetings is the same in all equilibria and only depends on the aggregate degree of market tightness. Letting  $m(\cdot)$  denote the aggregate number of meetings under condition

(5.3) or (5.4), we can write:

$$m(u, v; b, \pi, \delta, A) = \begin{cases} u & \text{if } \phi v \geq u \\ \phi v & \text{if } \phi v < u \end{cases} \quad (5.8)$$

where  $b$  is the ag-drop rate,  $\pi$  the per-mileage charge,  $\delta \equiv (\delta_{12}, \delta_{13}, \delta_{23})$  the set of pair-wise distances between locations, and  $A$  the matrix of wishes to move which is explicitly given in Appendix A. The number of meetings in (5.8) implies that the aggregate matching function exists and is given by:

$$m(u, v; b, \pi, \delta, A) = \min \{u, \phi v\}. \quad (5.9)$$

Since

$$m(u, v; b, \pi, \delta, A) < \min \{u, v\}, \text{ if and only if } v/u < \phi^{-1},$$

an equilibrium exhibits friction if and only if the aggregate degree of market tightness  $v/u$  is smaller than  $\phi^{-1}$ .

Two facts about the nature of frictions in this model can be learned from the matching functions just derived, and they are the keys to understanding what exactly is preventing some contacts to occur in some regions of the parameter space. For frictions to arise, first, it is necessary that locations are not all identical (in the sense of having the same  $\pi_i$ s). And second, that the number of cabs be small enough relative to the number of movers. Together, these two conditions are equivalent to the notion of frictions within the present framework. The next section explains why this is so.

### 5.3. Frictions: when and why they arise

The discussion in the previous two sections reveals that for any given degree of heterogeneity among locations (as measured by  $\phi$ , the relative attractiveness of the worst location), there is a level of market tightness such that frictions arise if and only if  $v/u$  is below that level. To provide a simple illustration of this fact, suppose one half of the movers in each location wish to go to each one of the other two locations. This amounts to setting  $a_{ij} = 1/2$  for all  $i, j$  and  $i \neq j$ , which in turn implies  $\mu = (1/3, 1/3, 1/3)$ . That is, if people's wishes to move treat locations identically, then each one ends up with 1/3 of the movers in an unconstrained steady state. This symmetry assumption means that distances between locations are the sole source of heterogeneity among them. Additionally assuming that

$\delta_{12} = \delta_{13}$  implies that  $\pi_1$  is a cab's profit from a trip between location 1 and any other location, as well as that  $\pi_2 = \pi_3$ . It may be useful to picture locations as the vertices of an isosceles triangle with location 1 being the vertex between the two sides of equal length. Define  $x \equiv \pi_1/\pi_{23}$ , and notice that  $x$  is isomorphic to the distance between location 1 and location 2 (or 3) relative to the distance between locations 2 and 3. When  $x = 1$  the city resembles an equilateral triangle, and all locations are identical from a cab's perspective. This is the case analyzed in Corollary 1. When  $x < 1$  the parameters satisfy (5.3) subject to a re-labeling, while when  $x > 1$ , (5.4) holds. According to (5.7), the relative attractiveness of location 1 is given by  $3x/(2x+1)$ , while the relative attractiveness of location 3 is the same as location 2's (because  $\pi_2 = \pi_3$ ), namely  $3(x+1)/2(2x+1)$ . Since 1 is the best location if and only if  $x > 1$ , we know that

$$\phi = \begin{cases} \frac{3x}{2x+1} & \text{if } x \leq 1 \\ \frac{3(x+1)}{2(2x+1)} & \text{if } x > 1 \end{cases} \quad (5.10)$$

since, as before,  $\phi$  is the relative attractiveness of the worst location.

By varying  $x$ , we can get all the possible patterns of heterogeneity among locations. By combining  $x$  with aggregate tightness  $v/u$ , Figure 1 illustrates the areas of the parameter space for which frictions arise in equilibrium.

Figure 1. All equilibria corresponding to parameter values below the line  $v/u = F(x)$  exhibit frictions, except for the equilateral case (i.e. when  $x = 1$ ) whose equilibria are always frictionless.

The two most important curves in Figure 1 are  $x = 1$  and  $F(x)$ . Since all locations are identical when  $x = 1$ , we know (by Corollary 1), that the equilibria that correspond to parameter values on this locus are frictionless. There are also no frictions for all combinations of market tightness  $v/u$  and relative distance  $x$  that lie on or above the no-frictions frontier given by

$$F(x) = \begin{cases} \frac{2x+1}{3x} & \text{if } x \leq 1 \\ \frac{2(2x+1)}{3(x+1)} & \text{if } x > 1. \end{cases} \quad (5.11)$$

Frictions arise everywhere below the frontier, except in the case when the city is equilateral (i.e. when  $x = 1$ ).  $F(x)$  is immediate after putting (5.10) together with the fact that all equilibria corresponding to levels of market tightness that exceed the reciprocal of the relative attractiveness of the worst location are frictionless (Corollary 2). Notice that the line  $v/u = 1$  lies below the no-frictions frontier. Thus even when there are the same number of movers and cabs, so that all agents could potentially meet a partner, some meetings fail to occur in equilibrium, provided all locations are not identical.

To see how frictions arise in equilibrium, we now turn to the general 3-location case, only requiring from distances and wishes to move that they be such that (5.3) holds. Conditional on having picked up a random passenger, a cab's expected profits from a trip are  $\pi_1$  in location 1,  $\pi_2$  in 2 and  $\pi_3$  in 3. Under (5.3), a cab weakly prefers to pick up a passenger in 1 than in 2, and strictly prefers to get a trip originating in 2 rather than in 3. Since in equilibrium there can be no profitable re-allocation for a cab, (unconditional) expected profits must be equalized across locations and hence a cab's contact rate must be (weakly) smaller in 1 than in 2, and (strictly) smaller in 2 than in 3. Suppose that initially there are more than  $\phi^{-1}$  cabs per mover in the city, where  $\phi$ , the relative attractiveness of the worst location is defined in Corollary 2<sup>15</sup>. The equilibrium allocations for this case are unique and given in Table 5.3. Under (5.3) cabs do not distribute themselves evenly in equilibrium, in the sense that the fraction of cabs in each location differs from the fraction of movers in that location. However, when  $v/u > \phi^{-1}$ , the city-wide number of cabs is so large relative to the city-wide number of movers, that the *number* of cabs that choose to look for a passenger in location 3 exceeds the number of passengers in that location. To verify this, the excess supply functions can be derived from Tables 5.3 and 5.4: letting

$$xs_i \equiv v_i - u_i,$$

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<sup>15</sup>Since  $0 < \pi_3 = \min\{\pi_1, \pi_2, \pi_3\}$ , it is immediate that  $\phi \in (0, 1)$ .

we have:

$$xs_i = \begin{cases} (\phi_i - \phi) \mu_i v & \text{if } v/u < \phi^{-1} \\ (\phi_i v - u) \mu_i & \text{if } v/u \geq \phi^{-1} \end{cases}$$

for  $i = 1, 2$ ; and

$$xs_3 = \begin{cases} \phi v - u & \text{if } v/u < \phi^{-1} \\ (\phi v - u) \mu_3 & \text{if } v/u \geq \phi^{-1}. \end{cases}$$

As can be seen in Figure 2, even though only a fraction  $\mu_3\phi$  of the total number of cabs are in 3, while the fraction of the total number of movers there is  $\mu_3$ , there are so many cabs in the city when  $v/u \geq \phi^{-1}$ , that even the worst location is in excess supply.

Figure 2. Excess supply functions as a function of the total number of cabs when  $\pi_1 \geq \pi_2 > \pi_3$ .

Now consider what happens as  $v/u$  falls. In terms of the parameter space in Figure 1, this experiment amounts to taking a point to the left of  $x = 1$  and above the no-frictions frontier and seeing how the set of equilibria and the implied frictions change as the point is brought down along a vertical line. As can be seen in Table 5.3, initially the distribution of movers remains unchanged as tightness falls since as long as  $v/u > \phi^{-1}$  there is excess supply in all locations. On the other hand, the number of cabs in all locations falls as  $v/u$  falls (see the middle row of Table 5.3). Notice that as a fraction of the total number of cabs, the number of cabs in each location remains constant (and given by  $\mu_j\pi_j / \sum_{i=1}^3 \mu_i\pi_i$

for  $j = 1, 2, 3$ ). In other words each location's share of the total number of cabs remains constant for any level of aggregate market tightness<sup>16</sup>. Being the least attractive of all, location 3 is the one with the smallest number of cabs per mover (namely  $(v/u)\phi$ , as opposed to  $(v/u)\pi_j/\sum_{i=1}^3\mu_i\pi_i$ , in 1 and 2). Thus only in location 3 does the excess supply of cabs vanish as  $v/u$  gets arbitrarily close to  $\phi^{-1}$  (see Figure 2). Indeed, while locations 1 and 2 still exhibit excess supply, market clearing obtains in 3 when  $\phi v = u$ . In fact, notice that the expression for the no-frictions frontier in (5.11) is just the market clearing condition for the least attractive location.

As soon as  $v/u$  falls below  $\phi^{-1}$ , location 3 starts having excess demand while there is still excess supply in the other locations. Figure 2 confirms that below the no-frictions frontier the equilibrium exhibits excess supply of cabs in locations 1 and 2, and excess demand in 3. This equilibrium has cabs (in locations 1 and 2) which are unable to contact passengers, and passengers (in location 3) who are unable to find cabs: the equilibrium exhibits frictions.

Figure 3. Behavior of a cab's contact rates as the city-wide number of cabs per mover increases.

Figure 3 shows the behavior of a cab's contact rate in each of the three locations as the city-wide number of cabs changes. While there is excess demand in location

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<sup>16</sup>This is true both above and below the no-frictions frontier.

3, the contact rates of a cab in locations 1 and 2 remain constant. This is because in equilibrium, profit maximization (refer to equilibrium condition (E1)) requires the ratios of cabs' contact rates across locations to remain constant, and the contact rate of a cab in 3 is constant and equal to one while there is excess demand in that location. Once  $v/u$  reaches  $\phi^{-1}$ , market clearing is attained in location 3 (in Figure 3, notice how  $u_3/v_3$  equals one at that point) and thereafter, further increases in  $v$ , cause a cab's contact rate to deteriorate in each location, preserving their relative sizes. The dotted line in Figure 3 shows how the excess demand in location 3 is reduced and eventually disappears as  $v/u$  rises. The number of cabs per mover in that location rises until the excess demand is eventually worked out and the frictions disappear.

Given all potential meetings always take place within each location, the only way a vacant cab and a mover may fail to contact each other is if they stand in different locations. Hence an equilibrium with frictions must necessarily involve some location(s) with excess supply and some location(s) with excess demand. By increasing  $v/u$ , the location(s) with excess demand move toward market clearing, so that if  $v/u$  gets to be high enough, all movers will be able to contact cabs, and hence all possible bilateral meetings will take place.

#### 5.4. Properties of the aggregate matching function

As mentioned in the Introduction, most equilibrium search models are built around an exogenous aggregate matching function with some convenient properties. In particular, the matching function is often assumed to be strictly component-wise increasing and homogeneous of degree one, and these properties usually play an important role in the analyses. This section presents the properties of the matching function implied by model to see how they compare with those that are typically assumed in the literature.

##### 5.4.1. Meetings and the stocks of searchers: why Leontief?

Recall the matching function derived earlier:

$$\min \{u, \phi v\}.$$

This function responds to changes in the stock of movers only above the no-frictions frontier (i.e. for  $u < \phi v$ ). This is because in that region of the parameter space, any equilibrium involves excess supply of cabs in all locations, so wherever

they may end up located in the steady state equilibrium, any number of extra movers generates the same number of extra meetings. Below the no-frictions frontier (i.e. for  $u > \phi v$ ), increasing the city-wide number of movers just increases the steady state number of movers in the location(s) with excess demand for cabs, and hence has no effect on the number of contacts (for example notice that only  $u_3$  depends on  $u$  in Table 5.4).

Similarly, additional cabs increase the number of contacts only when the meeting process exhibits frictions. When the number of cabs is too big relative to the number of movers (i.e. when  $v > \phi^{-1}u$ ), the additional cabs just increase the excess supply of cabs in each location, and hence don't increase the number of matches. On the other hand, below the no-frictions frontier, additional cabs generate additional meetings, but at a rate smaller than 1. If all the additional cabs placed themselves in the location(s) with excess demand, then each additional cab would generate an extra meeting. In equilibrium, however, the additional cabs spread themselves across all three locations (again, the extra number of cabs that go to each location is proportional to the location's relative attractiveness given in (5.7)), and since some end up in locations with excess supply, the increase in the number of contacts is smaller than the increase in the number of cabs.

It is clear that each additional cab in the location(s) with excess demand generates an extra meeting. However, in an equilibrium with frictions (i.e. with excess supply in some location(s) and excess demand in other(s)) it is *not* true that additional cabs in the location(s) with excess supply generate no extra meetings. This is because the additional cabs in the location(s) with excess demand necessarily mean more movers in the location(s) with excess supply<sup>17</sup>. With more cabs in the location(s) with excess demand, some of the movers who were previously unable to go to locations with excess supply, will now be able to do so. Thus although the aggregate number of movers has not changed, their steady state distribution across locations is changed by the increase in the number of cabs. Relative to the previous one, the new steady state equilibrium will exhibit more movers in the location(s) with excess supply, and less in (some of) the location(s) with excess demand.

#### 5.4.2. Policy experiments and the shape of the matching function

As it can be seen from (5.9), the matching function depends on the ag drop rate  $b$ , the per-mileage charge  $\pi$ , the full set of pair-wise distances  $\delta$ , and the matrix

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<sup>17</sup>To illustrate this, go back to Table 5.5 and verify that  $u_1$  is increasing in  $v$ .



A of wishes to move. In short, it depends on all the variables that determine the profit a cab makes from each possible trip it may sell. Aiming at the policy question addressed in Section 7, the analysis will focus on how changes in fares (i.e. changes in  $b$  and/or  $\pi$ ) affect the endogenous meeting process.

Suppose there are  $n$  locations satisfying

$$\pi_n = \min \{ \pi_1, \dots, \pi_n \} < \pi_1.$$

As shown in Appendix A, under these conditions the matching function is still given by (5.9), but with  $\phi$  given by

$$\frac{\pi_n}{\sum_{i=1}^n \mu_i \pi_1}.$$

Notice that the matching function can be re-written as:

$$m(u, v; \alpha) = \min \{ u, \phi(\alpha) v \},$$

with

$$\begin{aligned} \phi(\alpha) &= \frac{\alpha + \eta}{\alpha + \Sigma} \\ \eta &\equiv \sum_{j=1}^{n-1} a_{nj} \delta_{nj} \\ \Sigma &\equiv \sum_{i=1}^n \mu_i \left( \sum_{j \neq i} a_{ij} \delta_{ij} \right), \end{aligned}$$

and  $\alpha \equiv b/\pi$  being the relevant policy parameter. Differentiating  $m(\cdot)$  with respect to  $\alpha$ , we get:

$$\frac{\partial}{\partial \alpha} m(u, v; \alpha) = \frac{\Sigma - \eta}{(\alpha + \Sigma)^2} > 0,$$

provided  $\phi(\alpha) v < u$ . Thus, in an equilibrium with frictions, the number of meetings increases with  $\alpha$ . An increase in the tag-drop rate relative to the per mileage charge makes shorter trips relatively more attractive than before, inducing some cabs in locations with excess supply to go and look for passengers in locations with excess demand.

The mechanism that brings about this change in the number of meetings differs from the main effects at work in most existing equilibrium search models. The

latter typically rely either on changes in the stocks of searchers or in search intensity for an explanation of changes in the aggregate number of meetings. For example, in an equilibrium search model of the labor market with a free entry condition for firms, a decrease in firms' search costs will increase the equilibrium number of firms, which in turn causes (under the standard assumption that the matching function is strictly increasing in both components) an increase in the number of contacts. On the other hand, within the present framework, even holding the number of cabs and movers fixed, a policy (for instance changing the taxicab-meter rate so that  $\alpha$  rises) will change the equilibrium distribution of cabs across locations, which will in turn affect the *shape* of the matching function. In other words, the number of contacts may react to a policy because cabs change the way in which they look for passengers in response to the policy. When agents are allowed to choose how to conduct their search, the meeting process is endogenous, and hence using a model that *assumes* a meeting technology to predict the effects of a policy may be misleading in some cases. The application worked out in the next section is one such case.

## 6. A matching function for New York City

New York City (NYC) is composed of 5 boroughs: Brooklyn, the Bronx, Queens, Staten Island and Manhattan. This section uses the tools developed in the previous ones to compute an approximation to the matching process according to which cabs and passengers meet in Manhattan, below 79th St.

### 6.1. Why NYC, why Manhattan and why below 79th St.

The main reason why it seems appealing to apply the analysis to NYC's market for taxicab rides is that the city is currently changing the market's regulations. Within the last few months, the New York City Taxi and Limousine Commission (TLC), the regulating entity, has raised the fare, and is considering increasing the number of medallions<sup>18</sup> by 400 over the next 3 years. The remainder of the paper will focus on the effects this recent increase in the fare is likely to have had on the market's meeting process.

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<sup>18</sup>NYC's law currently limits the number of (yellow) taxicabs to 11,787. The term *medallion* refers to the painted aluminum medallion that is affixed to the hood of every yellow cab representing a taxi license.

Choosing to focus on Manhattan, is natural because NYC’s taxi trips *center* on Manhattan: only about 8% of NYC’s taxi trips serve the outerboroughs (i.e. Brooklyn, the Bronx, Queens, Staten Island or northern Manhattan<sup>19</sup>) more than half of which begin in Manhattan south of 96th St. Only 1% of all trips both begin and end in the outerboroughs. Moreover, 70% of all trips transport Manhattan residents. With 78% of its households not owning a car, public transportation is a vital part of life in Manhattan. And transporting 34% of all fare-paying bus, subway, taxi or for-hire vehicle passengers traveling within Manhattan, taxis are a vital part of Manhattan’s transportation network.

Finally, the fact that at least 80% of all NYC’s trips begin and end in Manhattan south of 79th St, suggests that the analysis should specialize in this particular area<sup>20</sup>. In a slight abuse of terminology, the text will refer to this particular area as *Manhattan*.

## 6.2. A simplified map of Manhattan

We proceed to simplify the map of Manhattan by dividing it in 6 locations. This means that in our geographical abstraction, there are only 6 places where people and cabs may be located. All six locations have roughly the same size, and are bounded by the Hudson River on the West and the East River on the East. Table 6.1 reports the North-South boundaries of each location.

<b>i</b>	<b>Name</b> (Areas/well-known spots it includes)	<b>North-South</b> (Boundary)
<b>1</b>	Upper (Metropolitan Museum)	79th St.-59th St.
<b>2</b>	Midtown North (Moma, Rockefeller Ctr.)	59th St.-39th St.
<b>3</b>	Midtown South (Madison Sq. Garden, Penn Station)	39th St.-19th St.
<b>4</b>	Village (Greenwich Village, East Village, Union Sq.)	19th St.-Houston
<b>5</b>	SoHo (SoHo, Chinatown, Little Italy, Lower East Side, Tribeca)	Houston-Chambers
<b>6</b>	Wall St. (Financial District)	Chambers-Battery Park

Table 6.1: North and South boundaries of the six locations.

Distances between locations are measured from a particular point (the central

<sup>19</sup>Northern Manhattan refers to Manhattan above 96th St.

<sup>20</sup>All facts quoted from *The New York City Taxicab Fact Book* (May 1994).

point ) chosen roughly in the geographical center of each location. Table 6.2 reports the street corner chosen to be each location s central point.

<b>i</b>	<b>Location i s Central Point</b>
<b>1</b>	69th St. & 3rd Av.
<b>2</b>	49th St. bet. 7th Av. & Av. of the Americas
<b>3</b>	29th St. bet. 7th Av. & Av. of the Americas
<b>4</b>	8th St. and University Pl.
<b>5</b>	Canal and Lafayette
<b>6</b>	Wall St. and Broadway

Table 6.2: Street corners chosen as locations central points.

Given that distances between locations are symmetric<sup>21</sup>, Table 6.3 reports the distances between each pair of locations, measured in miles.

<b>i \ j</b>	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>	$\sum_{j=1}^6 \delta_{ij}$
<b>1</b>	0	1.7	3	4.3	5.5	6.6	21.1
<b>2</b>	1.7	0	1.3	2.6	3.8	4.9	14.3
<b>3</b>	3	1.3	0	1.3	2.5	3.6	11.7
<b>4</b>	4.3	2.6	1.3	0	1.2	2.3	11.7
<b>5</b>	5.5	3.8	2.5	1.2	0	1.1	14.1
<b>6</b>	6.6	4.9	3.6	2.3	1.1	0	18.5

Table 6.3: Pairwise distances between locations.

Having worked out a geographical model for the area of interest, the next section will compute all equilibria and derive the corresponding matching function for Manhattan s market for cab rides.

### 6.3. Manhattan in equilibrium

The notation used in this section follows the one adopted for the case with 3 locations. Unfortunately, the information needed to make the Markov matrix of people s wishes to move consistent with the actual observed flows of cabs between locations is unavailable<sup>22</sup>. Given this limitation, the analysis will proceed under

<sup>21</sup>This assumption seems innocuous given the level of geographical abstraction.

<sup>22</sup>The information on what fractions of those who get cabs at each location want to go to any other location (i.e. what fraction of those who hail a cab in SoHo go to the Village, to Wall St.,

the assumption that people's wishes to move are ruled by a completely symmetric Markov chain. This is equivalent to making the following two assumptions on people's moving behavior: (a) when a cab driver spots a person in some location, the probability this person wants a cab is the same (i.e.  $u$ ) in any location, and (b) a passenger in any location is equally likely to want to go to any one of the other five locations. It should be noted that adopting this kind of moving behavior will magnify the effect distances have on the cabs' locating decisions.

The definition of equilibrium for this application is obtained from the general definition in Section 4.3 by letting  $n = 6$  and  $a_{ij} = 1/5$  for  $i = 1, \dots, 6$  and  $i \neq j$ . The following section characterizes the set of all the possible equilibria for Manhattan, as a function of the bag-drop charge ( $b$ ), the per-mile profit rate ( $\pi$ ), the city-wide number of cabs ( $v$ ) and the fraction of people wishing to find a cab ( $u$ ).

### 6.3.1. Equilibrium allocations: on where cabs are when you need one

When the number of cabs is very large relative to the number of potential passengers, there will be more cabs than people needing cabs in every location throughout the city: everybody is able to find a cab within the first period. Conditional on having found a passenger, the expected profit for a cab in Midtown South (location 3) and the Village (location 4) is lower than elsewhere in the city, and hence when the number of cabs is small enough relative to the number of people wishing to find a cab, there will be a shortage of cabs in at least one of those areas. The following proposition formalizes the notions of very large and small enough.

**Proposition 2.** If

(a)  $v/u > \frac{15b+45.7\pi}{15b+35.1\pi}$ , then there exists a unique equilibrium, and it exhibits excess supply of cabs in all locations.

(b)  $v/u = \frac{15b+45.7\pi}{15b+35.1\pi}$ , then there exists a unique equilibrium with market clearing in Midtown South and the Village and excess supply elsewhere.

(c)  $v/u < \frac{15b+45.7\pi}{15b+35.1\pi}$ , then there is an equilibrium with excess demand in Midtown South, market clearing in the Village and excess supply elsewhere, another one in which the market clears in Midtown South, exhibits excess demand in the Village and excess supply elsewhere, and a continuum with excess demand in both

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etc.) can be obtained from the trip sheet data collected periodically from cab drivers by the New York City Taxi and Limousine Commission. I have been unable to access this data so far.

Midtown South and the Village and excess supply elsewhere.

**Proof.** See the Appendix B.

Once the equilibria have been computed, the city-wide matching function can be derived simply by adding up the number of contacts that occur in equilibrium in each location, which are reported in the last row of Tables B.1-B.5. Letting  $m(\cdot)$  denote Manhattan's matching function, and  $\alpha$  be the ratio of the bag-drop charge to the per-mile profit rate (i.e.  $\alpha \equiv b/\pi$ ), the model predicts that

$$m(u, v, \alpha) = \min\{u, \phi(\alpha)v\} \tag{6.1}$$

where

$$\phi(\alpha) \equiv \frac{15\alpha + 35.1}{15\alpha + 45.7}.$$

Notice that

$$m(u, v, \alpha) < \min\{u, v\} \text{ iff } v/u < \phi(\alpha)^{-1},$$

and hence the market exhibits frictions if and only if  $v/u < \phi(\alpha)^{-1}$ . Figure 4 shows a plot of  $v/u = \phi(\alpha)^{-1}$ , namely Manhattan's no-frictions frontier.

Figure 4. Manhattan's no-frictions frontier.

## 7. The recent increase in taxicab fares: some predictions

Taxi fares in NYC where increased earlier this year. The following table reports the old and the new fares<sup>23</sup>.

	<b>b</b>	<b><math>\pi</math></b>	<b><math>\alpha</math></b>
old fare (1990-1996)	1.50	1.25	$\frac{6}{5}$
new fare	2	1.50	$\frac{4}{3}$

Table 7.1: Flag-drop and per mile charges: new and old.

The last column of Table 7.1 shows that the flag-drop charge ( $b$ ) was increased proportionately more than the per-mile profit rate ( $\pi$ ), causing their ratio ( $\alpha$ ) to rise. The effect of this policy on the city-wide matching function (given in (6.1)) depends on the value of the parameters  $v$  and  $u$ . To see this, notice that if  $u/v \geq \phi(6/5)^{-1} = \frac{637}{531}$ , then (given that  $v$  and  $u$  are unchanged by the policy) the fare increase has no effect on the matching function and consequently the number of meetings between cabs and passengers in Manhattan remains unchanged. Conversely, if  $u/v < \frac{637}{531}$ , because this new rate structure increases the profits from short trips relatively more than it does for long ones, cabs and locations such as Midtown South and the Village now more attractive than before the change in fares. This reallocation<sup>24</sup> increases the number of contacts by shifting<sup>25</sup> the matching function.

In order to assess the effects of the fare increase, the values of  $v$  and  $u$  will be chosen so that the model is consistent with two documented features of NYC's market for taxicab rides before the policy took place, namely the average waiting times faced by a cab driver looking for a passenger and by a passenger looking for a cab.

The length of a model period will be chosen so that the model reproduces the average number of meetings observed in Manhattan in a typical weekday. There are approximately 490,000 trips on an average weekday in NYC. Given that 80%

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<sup>23</sup>The figures for  $\pi$  ignore the per-mile cost of operation, namely the cost of gas. Being only about 7 cents per mile, introducing this cost will have no significant effect on the conclusions.

<sup>24</sup>The policy also induces a reallocation of cabs when  $v/u \geq \frac{637}{531}$ , but in that case the reallocation has no effect on the number of meetings because -since the total number of cabs in the city was large enough relative to the total number of movers- there were already enough cabs to serve all passengers in all locations before the policy took place.

<sup>25</sup>Recall that  $\phi(\alpha)$  increases with policies that raise  $\alpha$  (see Section 5.4.2).

of the trips begin and end in Manhattan, this means that there are roughly 272 contacts per minute in Manhattan during an average weekday. Hence, for given values of  $u$ ,  $v$ , and  $\alpha$ , the length of a model period measured in minutes is denoted  $\lambda(\cdot)$  and given by

$$\lambda(u, v, \alpha) = \frac{1}{272} m(u, v, \alpha).$$

### 7.1. A cab's average waiting time and the number of cabs

The expected number of model periods a cab in location  $i$  waits before meeting a passenger is<sup>26</sup>

$$\max\{1, v_i/u_i\}.$$

The average (across locations) expected number of periods a cab waits for a passenger is  $v/u$  for values of  $v$  and  $u$  such that  $v/u$  exceeds  $\frac{637}{531}$ ; and  $\frac{637}{531}$  ( $\cong 1.2$ ) for values of  $v/u$  that are smaller than or equal to  $\frac{637}{531}$ . This means that the average expected waiting time for a cab trying to find a passenger in the model is

$$\frac{637}{531} \min\{u, \phi(6/5)v\} \frac{1}{272} = \frac{637}{531} \phi(6/5) \frac{v}{272}$$

minutes when below or on the no-frictions frontier, and

$$\frac{v}{u} \min\{u, \phi(6/5)v\} \frac{1}{272} = \frac{v}{272}$$

minutes when above it. Notice that since  $\phi(6/5) = \frac{531}{637}$ , it follows that the average waiting time faced by a cab is  $\frac{v}{272}$  regardless of whether  $v/u$  lies above or below the no-frictions frontier.

According to The New York City Taxicab Fact Book (NYC Taxi and Limousine Commission (1994b) p.33), on average, cab drivers spend 6.5 minutes waiting to

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<sup>26</sup>While in location  $i$ , the probability a cab finds a passenger is  $\gamma_i$  in any given period. Hence the probability of finding a passenger in the  $k$ th period is

$$g_i(k) = (1 - \gamma_i)^{k-1} \gamma_i,$$

and thus the expected number of periods until a cab finds a passenger in location  $i$  is given by

$$\sum_{k=0}^{\infty} g_i(k) k = \gamma_i^{-1} = \min\{\rho_i, 1\}^{-1}.$$



and a fare<sup>27</sup>. Hence, for the model to be consistent with this fact,  $v$  must solve

$$\frac{v}{272} = 6.5,$$

which implies that

$$v = 1768.$$

## 7.2. A passenger's average waiting time and the number of movers

The expected number of model periods a passenger in location  $i$  spends waiting for a cab is

$$\max \left\{ 1, \frac{u_i}{v_i} \right\}.$$

If  $v/u > \frac{637}{531}$ , then the model predicts excess supply of cabs in all locations, and hence a passenger's waiting time is 1 model period in any location. Conversely, if  $v/u \leq \frac{637}{531}$ , then the average (across locations) expected number of periods a passenger waits for a cab is

$$\frac{637 u}{531 v}.$$

Hence above the no-frictions frontier a passenger's average expected waiting time is

$$\min \{u, \phi(6/5) v\} \frac{1}{272} = \frac{u}{272}$$

minutes, and below it, it is

$$\frac{637 u}{531 v} \min \{u, \phi(6/5) v\} \frac{1}{272} = \frac{637}{531} \phi(6/5) \frac{u}{272} = \frac{u}{272}$$

minutes as well.

Then if  $z$  is the (observed) average number of minutes a passenger has to wait for a cab in Manhattan it follows that the number of movers  $u$  has to be given by

$$u = 272 \cdot z \tag{7.1}$$

for the model to be consistent with this observation. Notice that

$$\frac{v}{u} = \frac{1768}{272 \cdot z} > \frac{637}{531} \text{ iff } z < 5.4; \tag{7.2}$$

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<sup>27</sup>This figure excludes time spent waiting in airport hold areas and dead-heading after airport drop-offs, as well as breaks of over 20 minutes taken by drivers.

that is,  $v/u$  lies above the no frictions frontier if and only if on average, a passenger has to wait for less than 5.4 minutes to find a cab in Manhattan.

In 1989, a study prepared for the City of New York (Parsons et al (1989), pp. III14-III18 and B1-B38) estimated the average (across locations and time of the day) waiting time for passengers trying to hail a cab in the relevant area of Manhattan to be slightly above 1 minute. Unfortunately, the result of this study can only be taken seriously as a lower bound to hailers' waiting time<sup>28</sup>. However, notice that according to (7.2),  $v/u$  would lie above the no-frictions frontier even if the true average waiting time of a hailer was *five times* the one obtained from the report by Parsons et al. Hence it seems safe to presume that Manhattan's market for taxicab rides was above the no-frictions frontier before the fare was increased. Nevertheless, the following section predicts the effects of the fare increase on Manhattan's meeting process without making any assumptions on the value of  $u$ .

### 7.3. Effects of the fare increase

The available evidence on the effects that the last two fare increases (namely May 1987 and January 1990) had on  $u$  and  $v$  suggests that both variables would most likely remain unchanged by the recent increase.

Two independent studies argue that the demand for taxicab rides in NYC is perfectly inelastic. Parsons et al (1989) (p. VIII12) report that the 22% fare increase of May 1987 brought about nearly 22% in extra revenue. Similarly, according to The New York City Taxicab Fact Book (NYC Taxi and Limousine Commission (1994b) p.16), the 12% fare increase of January 1990 induced no ridership loss.

Since the number of medallions is fixed by law, the number of cabs cannot

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<sup>28</sup>The study was carried out by a group of workers who, in several locations throughout NYC tried to spot people hailing cabs and measured their hailing times. The timing of a hailer began when the person started hailing and ended either when he/she entered a cab or walked away from the observers' range of view. There are at least two reasons why the hailing times measured in this way will turn out to be biased downward. First, most hailers were probably identified as such only after they had been waiting for a cab for some time (there is no point in hailing if there are no cabs coming down the street). Second, some of the people who walked away from the observers' range of view may have not given up the idea of finding a cab altogether (as assumed by the observers who just stopped timing people walking out of their sights), but most likely moved on to try finding a cab elsewhere, resulting in their waiting time being underestimated as well. The results might have been more accurate using alternative data-collecting techniques such as making the workers involved in the study hail cabs themselves or conducting an on-the-cab survey of passengers' hailing times.

increase. Nevertheless, a fare increase could, in principle, induce the existing number of licensed cabs to be operated more hours, with effects similar to those of an increase in the actual number of cabs. However, there are good reasons why a significant increase in the number of operating hours due to rising fares would be an unlikely outcome. Of the 11,787 outstanding licenses, 6,818 are corporate licenses which are required by TLC regulation to be run for two shifts of nine to twelve hours duration each day. Furthermore, around 33% of the 4,969 individual taxis are double-shifted (Parsons et al (1989), p.VIII16). This means that 70% of the existing cabs are already running practically full time. Additionally there is also some evidence suggesting that drivers preferences would work against the increase in operating hours: according to taxi industry sources, some drivers will work fewer hours if they make more per hour, preferring the leisure time (Parsons et al (1989), p.VIII13). Finally, the study by Parsons et al (1989) (pp. VIII12-VIII16), concludes that most likely, a fare increase of 23% would not appreciably increase the number of taxis double-shifting, nor cause a change in the basic operating pattern of the existing taxis. In accordance with these facts and arguments, the effects of the recent fare increase should be analyzed keeping  $v/u$  constant.

All that is needed to characterize the market for taxicab rides is a pair  $(v/u, \alpha)$ . Since due to lack of good information on the average waiting time of a passenger (denoted  $z$ ), we cannot come up with a number for  $v/u$ , all we can say (see Figure 4) is that the policy caused the point  $(v/u, 6/5)$  to shift to  $(v/u, 4/3)$ . That is, whatever  $v/u$  was initially, it was lying on the  $\alpha = 6/5$  line before, and is lying on  $\alpha = 4/3$  locus after the fare increase. However, it turns out that even without assigning a value to  $u$ , the effects of the policy on the aggregate meeting process can be bounded quite tightly, and in fact, shown to be very small.

Let us first consider what the results look like if the hailers average waiting time  $z$  is very low. Let  $\hat{z} \equiv \phi(6/5) \frac{1768}{272}$ , and assume that  $z \leq \hat{z}$ . Then (7.1) implies that  $u \leq \phi(6/5) \cdot 1768$ , and hence  $\frac{1768}{u} \geq \phi(6/5)^{-1}$ , indicating that the city was above the no-frictions frontier before the policy. Thus the city will remain above the frontier after the policy (see Figure 4), and consequently (according to (6.1)) the fare increase has no effect on the matching function.

On the other hand, if the real waiting time  $z$  is high enough, it can be immediately seen (from Figure 4 and (6.1)) that the market will start below the frontier, and hence that the number of matches will rise as a result of the policy. Let  $\bar{z} \equiv \phi(4/3) \frac{1768}{272}$ , and assume  $z > \bar{z}$ . Then the market starts and ends below the frontier (again see Figure 4 and notice that  $z > \bar{z}$  implies that  $1768/u <$

$\phi(4/3)^{-1}$ ), and hence the number of matches increases from  $\phi(6/5) \cdot 1768$  to  $\phi(4/3) \cdot 1768$ . Alternatively, if  $z \in (\hat{z}, \bar{z}]$ , the market will start below the frontier and end up above it as a result of the fare increase. The increase in the number of meetings is given by

$$[\phi(\alpha^*) - \phi(6/5)] \cdot 1768,$$

where  $\alpha^*$  is characterized by

$$\frac{1768}{272 \cdot z} = \phi(\alpha^*).$$

Letting  $\Delta(z)$  denote the percentage change in the number of meetings induced by the fare increase, and defining it as

$$\Delta(z) \equiv [m(u, v, 4/3) - m(u, v, 6/5)] m(u, v, 6/5)^{-1} \times 100,$$

the above discussion implies that

$$\Delta(z) = \begin{cases} 0 & \text{if } z \leq \hat{z} \\ \frac{\phi\left(\frac{272 \cdot 45.7 \cdot z - 1768 \cdot 35.1}{15 \cdot (1768 - 272 \cdot z)}\right) - \phi(6/5)}{\phi(6/5)} \times 100 & \text{if } \hat{z} < z \leq \bar{z} \\ \frac{\phi(4/3) - \phi(6/5)}{\phi(6/5)} \times 100 \cong 0.6\% & \text{if } \bar{z} < z \end{cases}.$$

The function  $\Delta(z)$  is plotted in Figure 5.

Figure 5. Percentage change in the number of meetings induced by the fare increase as a function of a passenger's average waiting time.

Notice that independently of the real value of  $z$  (and thus regardless of what the implied value for  $u$  might be), the meeting process barely responds at all to the fare increase: the 11% increase in  $\alpha$  will *at most* induce a 0.6% increase in the number of meetings.

On a closing note, the fact that the effect that a change in fares has on the matching process depends on what happens to  $\alpha$ , and *not* on what happens to the overall cost of a trip should be stressed. For example, if fares are increased by raising  $\pi$  proportionately more than  $b$ , then *less* matches will result in equilibrium (provided we're below  $\phi(\alpha)^{-1}$ ). This result stands in contrast with what one would find by attempting to predict the effects of a rate change by using any model that does not explicitly account for the endogeneity of cabs' locating decisions. In particular, a model that *assumes* a matching process and possibly allows cabs to affect it by letting them choose their search intensity will reach the conclusion that the number of contacts will increase whenever the *cost of a trip* rises. The answer differs because the class of models that *assume* a matching process cannot account for the fact that in environments in which agents can direct their search, the *shape* of the matching function is affected by the change in the rate structure.

## 8. Concluding remarks

The model constructed shows how, when distance and the choice of a location are relevant for the agents (in the sense that they enter their payoffs), and their searching strategies are explicitly modeled, frictions may arise endogenously as a feature of a set of equilibria, even though agents are perfectly informed (they know the distribution of cabs and passengers across locations), and face no coordination problems (i.e. in equilibrium no cab wishes to change locations under the notion of equilibrium adopted). Within this framework, it was shown that some heterogeneity among locations is a necessary condition for frictions to exist.

The environment laid out is such that locations are characterized by two features: how far away they stand from each of the other two locations, and the moving wishes of the people in it. For those combinations of these features that deliver environments with identical locations (in the sense that conditional expected profits are equal in all of them), frictions do not arise since cabs spread themselves evenly in order to maximize the probability of contacting a passenger.

When at least one location is *better* (in the sense that the expected profit of having contacted a passenger there is higher) than another, the possibility that cabs may *overcrowd* that location leaving another location with less cabs

than people needing a ride, arises. So although all possible contacts occur within each location, cabs may distribute themselves in such a way that some of them are unable to find passengers, while some passengers are unable to get a cab. From an aggregate perspective (i.e. by looking at the total numbers of movers and cabs disregarding their distribution across locations), this situation looks just like the environments with meeting frictions typically assumed in search-theoretic models, although the nature of these frictions is very different.

The models that assume an exogenously specified matching function (i.e. one that is not an equilibrium object, derived from the agents' optimal search behavior) that generates the number of meetings, are implicitly assuming that agents can only engage in random search. That is, they do not choose how or where to look for a partner, and hence they cannot affect the meeting *process*. In other words, assuming an exogenous matching function is equivalent to assuming that nobody knows where anything is. This must be so, since in a model in which agents choose how to search (for instance, they may choose *where* to go look for a partner, as was the case above), the number of resulting meetings will necessarily be an equilibrium outcome. Put differently, doing policy experiments is meaningless in a model that assumes a matching function but in which agents supposedly engage in something other than random search, because any results will ignore the fact that the *shape* of the matching function is changing in some way in response to the policy.

Sometimes, as in the taxicab model presented here, the nobody knows where anything is assumption may not be appropriate, and hence tacitly adopting it by studying the problem with an exogenous matching function may be misleading.

## A. Appendix

Before proceeding with the proof of Proposition 1, it is convenient to write down the equilibrium conditions explicitly:

$$\left. \begin{aligned} \min \left\{ \frac{u_1}{v_1}, 1 \right\} \pi_1 &= \min \left\{ \frac{u_i}{v_i}, 1 \right\} \pi_i, \quad i = 2, \dots, n \\ \min \{u_i, v_i\} &= \sum_{j \neq i}^n a_{ji} \min \{u_j, v_j\}, \quad i = 1, \dots, n-1 \\ \sum_{i=1}^n u_i &= u, \text{ and } v = \sum_{i=1}^n v_i = v. \end{aligned} \right\} \quad (\text{A.1})$$

Notice that these are  $2n$  equations in the  $2n$  unknowns  $\{(u_i, v_i)\}_{i=1}^n$ . The first and second sets of  $n-1$  equations are the equal profit and steady state conditions respectively. The last two equations are the adding up conditions.

### Proof of Proposition 1.

(a). Suppose the parameters are such that  $\Pi = 0$ , or equivalently, such that

$$\pi_1 = \pi_2 = \dots = \pi_n.$$

That is to say, all locations are identical from a cabs perspective in the sense that the conditional expected profit  $\pi_i$  is the same in all of them. Under this condition the first  $n-1$  equations in (A.1) become

$$\min \left\{ \frac{u_1}{v_1}, 1 \right\} = \min \left\{ \frac{u_2}{v_2}, 1 \right\} = \dots = \min \left\{ \frac{u_n}{v_n}, 1 \right\}.$$

So in equilibrium, either *all locations* have excess supply of cabs, or *no location* exhibits excess supply. Therefore there are three types of equilibria. In the first, all locations exhibit excess supply of cabs. We deal with this case in Step 1a. In the second type, which is analyzed in Step 2a, there is market clearing in all locations. Finally, Step 3a characterizes the set of equilibria for the case in which at least one location exhibits excess demand while none of the others exhibit excess supply.

**Step 1a.** [Existence and uniqueness of an equilibrium with excess supply in all locations]. If there is excess supply of cabs in all locations, then (A.1)

becomes:

$$\frac{u_1}{v_1} = \frac{u_2}{v_2} = \dots = \frac{u_n}{v_n} \quad (\text{A.2})$$

$$v = \sum_{i=1}^n v_i \quad (\text{A.3})$$

$$u_i = \sum_{j \neq i}^n a_{ji} u_j, \quad i = 1, \dots, n-1 \quad (\text{A.4})$$

$$u = \sum_{i=1}^n u_i. \quad (\text{A.5})$$

Notice that since there is excess supply of cabs in all locations, the flows of movers between locations are driven by the Markov process that rules people's wishes to move. In fact,  $\bar{u} = (\bar{u}_1, \dots, \bar{u}_n)$  solves (A.5) and the  $n-1$  row equations labeled (A.4) if and only if it solves

$$\bar{u} \cdot A = \bar{u} \quad (\text{A.6})$$

and  $\bar{u}_n = u - \sum_{i=1}^{n-1} \bar{u}_i$ , where

$$A = \begin{bmatrix} 1 - u & ua_{12} & \cdots & ua_{1n} \\ ua_{21} & 1 - u & \cdots & ua_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ua_{n1} & ua_{n2} & \cdots & 1 - u \end{bmatrix}$$

is the Markov matrix of people's wishes to move. Although there are  $n$  equations in the system given by (A.6) and only  $n-1$  in (A.4), both systems are identical since the first  $n-1$  equations are the same, while the  $n$ th equation in (A.6) is just a linear combination of the previous  $n-1$ . To see this, rearrange the first  $n-1$  equations in (A.4) or in (A.6) to get

$$\begin{aligned} a_{n1} u_n &= u_1 - \sum_{i=2}^{n-1} a_{i1} u_i \\ a_{n2} u_n &= u_2 - \sum_{i \neq 2}^{n-1} a_{i2} u_i \\ &\vdots \\ a_{n,n-1} u_n &= u_{n-1} - \sum_{i=1}^{n-2} a_{i,n-1} u_i. \end{aligned}$$

Adding up these  $n-1$  conditions implies:

$$u_n \sum_{i=1}^{n-1} a_{ni} = \sum_{j=1}^{n-1} \left( 1 - \sum_{i \neq j}^{n-1} a_{ji} \right) u_j$$



which is the same as the last equation in (A.6), namely

$$u_n = \sum_{j=1}^{n-1} a_{jn} u_j,$$

since  $\sum_{i=1}^{n-1} a_{ni} = 1$  and  $1 - \sum_{i \neq j}^{n-1} a_{ji} = a_{jn}$ . The matrix  $A$  is a strictly positive Markov matrix, so it has a unique stationary distribution, namely there is a unique vector  $\mu = (\mu_1, \mu_2, \dots, 1 - \sum_{i=1}^{n-1} \mu_i)$  such that

$$\mu A = \mu. \quad (\text{A.7})$$

Furthermore,  $\mu > 0$  because  $A > 0$ <sup>29</sup>. Multiplying both sides of (A.7) through by  $u$ , we see that  $\bar{u} = \mu u$  is the unique solution to system of equations given by (A.4) and (A.5). Since  $u$  can be divided out of (A.7),  $\mu$  does not depend on  $u$  and hence  $\bar{u}$  is linear in  $u$ . So we can write:

$$\bar{u}_i = \mu_i u, \quad \text{for } i = 1, \dots, n, \quad (\text{A.8})$$

where  $\mu_i > 0$  is a function of the elements of the Markov matrix  $A$  only. Having solved for  $\bar{u}$ , the  $n$  equations given by (A.2) and (A.3) can be solved for the equilibrium allocation of taxicabs (denoted  $\bar{v}_i$ ):

$$\bar{v}_i = \nu_i v$$

where

$$\begin{aligned} \nu_1 &\equiv \left(1 + (1/\mu_1) \sum_{i=2}^n \mu_i\right)^{-1}, \text{ and} \\ \nu_i &\equiv \mu_i \nu_1 / \mu_1, \quad i = 2, \dots, n. \end{aligned}$$

Since  $\sum_{i=1}^n \mu_i = 1$ , it follows that

$$\bar{v}_i = \mu_i v, \quad \text{for } i = 1, \dots, n. \quad (\text{A.9})$$

Finally, verify that

$$\bar{v}_i > \bar{u}_i, \quad \text{for } i = 1, \dots, n,$$

---

<sup>29</sup>If  $z$  is a vector,  $z > 0$  means  $z_i > 0$  for all  $i$ , while  $z \geq 0$  means  $z_i \geq 0$  for all  $i$  and  $z_i > 0$  for some  $i$  (i.e.  $z \neq 0$ ). Similarly, if  $Q$  is a matrix,  $Q > 0$  means  $q_{ij} > 0$  for all  $i$  and  $j$ . The fact that  $A$  is a Markov matrix, implies that 1 is an eigenvalue (hence we know that a  $\mu$  satisfying (A.7) exists). Additionally,  $A > 0$  implies that 1 is  $A$ 's largest eigenvalue. Since  $\mu$  is the eigenvector associated with the largest non-negative eigenvalue of a non-negative matrix, by Frobenius theorem we know that  $\mu \geq 0$  (see Nikaido (1970) or Takayama (1985)). Finally, since  $A > 0$ , it is obvious that (A.7) cannot hold if  $\mu_i = 0$  for some  $i$ , so  $\mu > 0$  must be the case.

holds if and only if

$$v > u. \quad (\text{A.10})$$

Hence the unique equilibrium described in (A.8) and (A.9) exists if and only if (A.10) holds.

**Step 2a.** [Existence and uniqueness of an equilibrium with market clearing in all locations]. With market clearing in all locations the equilibrium is characterized by (A.3)-(A.5) and:

$$v_i = u_i, \text{ for } i = 1, \dots, n. \quad (\text{A.11})$$

As in Step 1a, (A.4) and (A.5) can be solved for (A.8). The distribution of cabs is then obtained from (A.11), and it is seen to satisfy (A.3) if and only if

$$v = u. \quad (\text{A.12})$$

Thus the equilibrium with market clearing in all locations is unique and exists if and only if (A.12) holds. The equilibrium allocations derived in Step 1a and Step 2a are summarized in the following table.

<b>i =</b>	<b>1, ..., n</b>
$u_i$	$\mu_i u$
$v_i$	$\mu_i v$
$m_i$	$\mu_i u$

Table A.1:  $\Pi=0$ . Equilibrium of Step 1a (or 2a).

**Step 3a.** [Characterization of equilibria with excess demand in at least one location]. Suppose there are  $k$  locations with excess demand and  $n - k$  that clear, with  $1 \leq k \leq n$ . If  $k \leq n - 1$ , label locations so that,  $i = 1, \dots, n - k$  correspond to those with market clearing.

$$\underbrace{1, \dots, n - k}_{\text{market clearing}}, \underbrace{n - k + 1, \dots, n}_k$$

In this case, the equilibrium conditions are:

$$v_i = u_i, \text{ for } i = 1, \dots, n - k, \text{ and} \quad (\text{A.13})$$

$$v_i = \sum_{j \neq i}^n a_{ji} v_j, \text{ } i = 1, \dots, n - 1 \quad (\text{A.14})$$

together with (A.3) and (A.5). If  $k = n$ , then the equilibrium is characterized by (A.14), (A.3) and (A.5) only. Notice that the system given by (A.14) is the same as that in (A.4) but with the  $v_i$  s replacing the  $u_i$  s. Therefore the equilibrium distribution of cabs is  $\bar{v} = \mu v$  (as given in (A.9)), with  $\mu$  defined by (A.7). So in this class of equilibria, the distribution of cabs across locations is the same as in the equilibrium with excess supply of cabs in all locations analyzed in Step 1a, and is given by (A.9). Having derived the distribution of cabs, the distribution of movers across the location(s) with market clearing follows immediately from (A.13). Since there are  $2n - k + 1$  independent equilibrium conditions and  $2n$  unknowns, the system will be under-determined if there is more than one market with excess demand (i.e. if  $k \geq 2$ ). In this case there will be a continuum of equilibria since the distribution of movers across the locations with excess demand is indeterminate<sup>30</sup>. Any distribution  $\{\hat{u}_i, \hat{v}_i\}_{i=1}^n$  with  $\hat{v}_i = \bar{v}_i$  for  $i = 1, \dots, n$ ;  $\hat{u}_i = \hat{v}_i$ , for  $i = 1, \dots, n - k$ , and  $\{\hat{u}_j\}_{j=n-k+1}^n$  satisfying

$$\hat{u}_j > \hat{v}_j, \text{ and} \tag{A.15}$$

$$\sum_{j=n-k+1}^n \hat{u}_j = u - v \sum_{i=1}^{n-k} \mu_i \tag{A.16}$$

constitutes an equilibrium. So equilibria of this kind exist if and only if<sup>31</sup>

$$v < u. \tag{A.17}$$

If the equilibrium is not unique, then there is a continuum. Uniqueness obtains if and only if  $k = 1$ . The following table summarizes the allocations corresponding to the equilibria of Step 3a.

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<sup>30</sup>This severe multiplicity is irrelevant for the purposes of characterizing the aggregate matching function.

<sup>31</sup>To show  $(\Rightarrow)$ , notice that (A.15) and (A.16) imply

$$\sum_{i=n-k+1}^n \hat{v}_i < u - \sum_{i=1}^{n-k} \hat{v}_i,$$

so  $v < u$  is necessary for both conditions to hold. For  $(\Leftarrow)$ , assume that  $v < u$ , and construct equilibria as follows. Let  $\hat{v}_i = \mu_i v$  for  $i = 1, \dots, n$ , and let  $\hat{u}_i = \hat{v}_i$ , for  $i = 1, \dots, n - k$ . For  $j = n - k + 1, \dots, n$ ; let

$$\hat{u}_j = \hat{v}_j + \varepsilon_j (u - v)$$

with  $\varepsilon = (\varepsilon_{n-k+1}, \dots, \varepsilon_n)$  being a vector in the  $k$ -dimensional unit simplex (i.e.  $\varepsilon_j \geq 0$  and  $\sum_{j=n-k+1}^n \varepsilon_j = 1$ ).

$\mathbf{i} =$	$\mathbf{1}, \dots, \mathbf{n} - \mathbf{k}$	$\mathbf{n} - \mathbf{k} + \mathbf{1}, \dots, \mathbf{n}$
$u_i$	$\mu_i v$	$\mu_i v + \varepsilon_j (u - v)$
$v_i$	$\mu_i v$	$\mu_i v$
$m_i$	$\mu_i v$	$\mu_i v$

Table A.2:  $\Pi=0$ . Equilibria of Step 3a.

This concludes Step 3a.

To show that all equilibria are always (i.e. for any  $u$  and  $v$ ) frictionless when  $\Pi = 0$ , it is sufficient to verify that the aggregate number of meetings  $m$  satisfies

$$m = \min \{u, v\}$$

in all the equilibria characterized in Steps 1a, 2a and 3a. Let  $m_{1,2}^a$  and  $m_3^a$  denote the aggregate number of meetings corresponding to any equilibrium characterized in Steps 1a (or 2a) and 3a respectively. By virtue of (A.8), (A.9), (A.10) and (A.12), in an equilibrium with either excess supply in all locations (Step 1a) or market clearing in all locations (Step 2a), the aggregate number of meetings is

$$m_{1,2}^a = \sum_{i=1}^n \min \{\bar{u}_i, \bar{v}_i\} = u.$$

Notice that since the equilibria in Steps 1a and 2a exist if and only if  $v \geq u$ , it follows that

$$m_{1,2}^a = \min \{u, v\}. \quad (\text{A.18})$$

Using (A.8), (A.9), (A.13), (A.15) and (A.17), it is immediate that in any of the (possibly multiple) equilibria with excess demand in at least one location and market clearing in the rest (Step 3a), the aggregate number of meetings is

$$m_3^a = \sum_{i=1}^n \min \{\hat{u}_i, \hat{v}_i\} = v.$$

Since the equilibria in Step 3a exist if and only if  $v < u$  (see (A.17)), we also have

$$m_3^a = \min \{u, v\}. \quad (\text{A.19})$$

This concludes part **(a)**.

(b). Suppose the parameters are such that  $\Pi > 0$ . This means that the conditional expected profits (i.e. the  $\pi_i$ s) are not all the same across locations. Label locations so that bigger subindexes correspond to locations with smaller conditional expected profit. This labeling implies

$$\pi_1 \geq \cdots \geq \pi_{k-1} > \pi_k = \cdots = \pi_n. \quad (\text{A.20})$$

Since there could be more than one location with the smallest level of conditional expected profit in the city, (A.20) allows for  $n - k + 1$  such locations, with  $2 \leq k \leq n$ . The ranking in (A.20) together with the first  $n - 1$  equations in (A.1) imply

$$\min \left\{ \frac{u_1}{v_1}, 1 \right\} \leq \cdots \leq \min \left\{ \frac{u_{k-1}}{v_{k-1}}, 1 \right\} < \min \left\{ \frac{u_k}{v_k}, 1 \right\} = \cdots = \min \left\{ \frac{u_n}{v_n}, 1 \right\}.$$

These conditions mean there must be excess supply in locations 1 through  $k - 1$ , while for locations  $k$  through  $n$  it must be the case that either they all have excess supply, or none of them does. Hence there are three possible types of equilibria. Once again, we proceed in three steps, one for each type.

**Step 1b.** [Existence and uniqueness of an equilibrium with excess supply in all locations]. With excess supply in all locations, the equilibrium is characterized by  $2n$  equations, namely

$$\frac{u_1}{v_1} \pi_1 = \frac{u_2}{v_2} \pi_2 = \cdots = \frac{u_n}{v_n} \pi_n, \quad (\text{A.21})$$

(A.3), (A.4) and (A.5). As in Step 1a, the  $n$  equations labeled (A.4) and (A.5) can be solved for the unique distribution of movers across locations given in (A.8). Knowing the distribution of movers, the  $n$  equations labeled (A.3) and (A.21) can be solved for the distribution of taxicabs, which is given by

$$v_i = \frac{\mu_i \pi_i}{\sum_{i=1}^n \mu_i \pi_i} v, \quad (\text{A.22})$$

for  $i = 1, \dots, n$ . The distributions of movers and cabs (given in (A.8) and (A.22) respectively) constitute an equilibrium if and only if

$$v/u > (1/\pi_i) \sum_{i=1}^n \mu_i \pi_i, \text{ for } i = 1, \dots, n. \quad (\text{A.23})$$

The set of conditions in (A.23) can be rewritten as

$$v/u > \max \{1/\pi_1, \dots, 1/\pi_n\} \sum_{i=1}^n \mu_i \pi_i.$$

Since  $\pi_n = \min \{\pi_1, \dots, \pi_n\}$ , the equilibrium with excess supply in all locations exists if and only if

$$v/u > \phi^{-1} \quad (\text{A.24})$$

where

$$\phi \equiv \frac{\pi_n}{\sum_{i=1}^n \mu_i \pi_i}. \quad (\text{A.25})$$

The following table summarizes the equilibrium allocations characterized in Step 1b.

$\mathbf{i} =$	$\mathbf{1}, \dots, \mathbf{n}$
$u_i$	$\mu_i u$
$v_i$	$\frac{\mu_i \pi_i}{\sum_{i=1}^n \mu_i \pi_i} v$
$m_i$	$\mu_i u$

Table A.3:  $\Pi > 0$ . Equilibrium of Step 1b.

**Step 2b.** [Existence and uniqueness of an equilibrium with excess supply in locations 1 through  $k - 1$  and market clearing in all others].

$$\underbrace{1, \dots, k-1}_{\substack{\text{excess} \\ \text{supply}}}, \underbrace{k, \dots, n}_{\substack{\text{market} \\ \text{clearing}}}$$

The conditions that characterize an equilibrium with excess supply in the first  $k - 1$  locations and market clearing in the remaining  $n - k + 1$  are:

$$\frac{u_1}{v_1} \pi_1 = \frac{u_2}{v_2} \pi_2 = \dots = \frac{u_{k-1}}{v_{k-1}} \pi_{k-1} = \pi_n \quad (\text{A.26})$$

$$v_i = u_i, \text{ for } i = k, \dots, n, \quad (\text{A.27})$$

together with (A.3), (A.4) and (A.5). As in Step 1b, (A.4) and (A.5) can be solved for the unique distribution of movers across locations given in (A.8). Then, the  $n$  equations labeled (A.26) and (A.27) can be solved for the unique distribution of cabs:

$$v_i = (1/\pi_n) \mu_i \pi_i u, \text{ for } i = 1, \dots, k - 1 \quad (\text{A.28})$$

$$v_j = \mu_j u, \text{ for } j = k, \dots, n. \quad (\text{A.29})$$

Finally, for the unique distributions of movers and cabs given in (A.8), (A.28) and (A.29) to be an equilibrium, condition (A.3) must hold, so we must verify that

$$\sum_{i=1}^{k-1} (1/\pi_n) \mu_i \pi_i u + \sum_{i=k}^n \mu_i u = v. \quad (\text{A.30})$$

Because  $\pi_i = \pi_n$  for  $i = k, \dots, n$ , the left-hand side of (A.30) can be written as

$$u (1/\pi_n) \sum_{i=1}^n \mu_i \pi_i,$$

and therefore this equilibrium exists if and only if

$$v/u = \phi^{-1}. \quad (\text{A.31})$$

**Step 3b.** [Characterization of equilibria with excess supply only in locations 1 through  $k-1$ , and excess demand in at least one of the remaining  $n-k+1$  locations]. We focus on an equilibrium with excess supply in locations 1 through  $k-1$  only, market clearing in the first  $h$  of the remaining  $n-k+1$  locations (with  $0 \leq h \leq n-k$ ), and excess demand in the rest.

$$\underbrace{1, \dots, k-1}_{\text{excess supply}}, \underbrace{k, \dots, k+h-1}_{\text{market clearing}}, \underbrace{k+h, \dots, n}_{\text{excess demand}}$$

In this case, the equilibrium conditions are given by a set of  $n+k+h$  equations; namely

$$u_i \pi_i = v_i \pi_n, \text{ for } i = 1, \dots, k-1 \quad (\text{A.32})$$

$$u_i = v_i, \text{ for } i = k, \dots, k+h-1 \quad (\text{A.33})$$

$$u_i = \sum_{j \neq i}^{k-1} a_{ji} u_j + \sum_{j=k}^n a_{ji} v_j, \text{ for } i = 1, \dots, k-1 \quad (\text{A.34})$$

$$v_i = \sum_{j=1}^{k-1} a_{ji} u_j + \sum_{j=k, j \neq i}^n a_{ji} v_j, \text{ for } i = k, \dots, n-1, \quad (\text{A.35})$$

together with (A.3) and (A.5). Let

$$\sigma \equiv \sum_{i=1}^{k-1} u_i + \sum_{i=k}^n v_i,$$

and notice that the system of  $n - 1$  equations labeled (A.34) and (A.35) can be written as<sup>32</sup>

$$\bar{s}A = \bar{s} \quad (\text{A.36})$$

with  $\bar{s} = (u_1/\sigma, \dots, u_{k-1}/\sigma, v_k/\sigma, \dots, 1 - \sum_{i=1}^{k-1} u_i/\sigma - \sum_{i=k}^{n-1} v_i/\sigma)$ . Notice that (A.36), the system obtained by dividing  $\{u_i\}_{i=1}^{k-1}$  and  $\{v_j\}_{j=k}^n$  by  $\sigma$ , is identical to (A.7), and hence  $\bar{s} = \mu = (\mu_1, \dots, \mu_{n-1}, 1 - \sum_{i=1}^{n-1} \mu_i)$ . Therefore the distributions of movers across the first  $k - 1$  locations, and of cabs across locations  $k$  through  $n$  that satisfy (A.34) and (A.35) are

$$u_i = \mu_i \sigma, \text{ for } i = 1, \dots, k - 1 \quad (\text{A.37})$$

$$v_j = \mu_j \sigma, \text{ for } j = k, \dots, n. \quad (\text{A.38})$$

(A.37) and (A.32) imply that

$$v_i = (1/\pi_n) \mu_i \pi_i \sigma, \text{ for } i = 1, \dots, k - 1. \quad (\text{A.39})$$

The distribution of cabs in (A.39) and (A.38) satisfies (A.3), if and only if

$$\sigma \left( \sum_{i=1}^{k-1} (1/\pi_n) \mu_i \pi_i + \sum_{i=k}^n \mu_i \right) = v,$$

or equivalently, since  $\pi_k = \dots = \pi_n$ , if and only if

$$\sigma = \phi v.$$

This allows us to rewrite the distribution of movers in (A.37) and (A.33), and the distribution of cabs in (A.39) and (A.38) as:

$$u_i = \mu_i \phi v, \text{ for } i = 1, \dots, k + h - 1, \quad (\text{A.40})$$

$$v_i = \frac{\mu_i \pi_i}{\sum_{i=1}^n \mu_i \pi_i} v, \text{ for } i = 1, \dots, k - 1, \text{ and} \quad (\text{A.41})$$

$$v_j = \mu_j \phi v, \text{ for } j = k, \dots, n. \quad (\text{A.42})$$

Since there are  $n + k + h$  equilibrium conditions (namely  $n + k + h - 2$  equations in (A.32)-(A.35) plus the two adding up conditions) and  $2n$  unknowns, there are  $n - k - h$  undetermined variables. This indeterminacy arises because the distribution of movers across the locations with excess

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<sup>32</sup>The  $n$ th equation in (A.36) is implied by the  $n - 1$  equations in (A.34) and (A.35).



demand is not pinned down by the equilibrium conditions. If  $h = n - k$ , then there is only one location in excess demand, and the system is exactly determined. On the other hand, if  $h \leq n - k - 1$ , then there are at least two locations with excess demand. In this case, the distribution described in (A.40), (A.41) and (A.42) together with any distribution  $\{u_j\}_{j=k+h}^n$  satisfying

$$u_j > v_j, \text{ and} \tag{A.43}$$

$$\sum_{j=k+h}^n u_j = u - \phi v \sum_{i=1}^{k+h-1} \mu_i \tag{A.44}$$

constitutes an equilibrium. Consequently, an equilibrium of the type described in this Step exists if and only if

$$v/u < \phi^{-1}. \tag{A.45}$$

To show (A.45) is necessary, notice that (A.43) and (A.44) imply

$$\sum_{j=k+h}^n v_j < u - \phi v \sum_{i=1}^{k+h-1} \mu_i$$

or equivalently (using (A.42))

$$\phi v < u.$$

To show it is sufficient for existence, assume (A.45) holds, and construct equilibria as follows. Let  $v_i$ , for  $i = 1, \dots, n$  be given by (A.41) and (A.42), and let  $u_i$ , for  $i = 1, \dots, k + h - 1$  be given by (A.40). For  $j = k + h, \dots, n$ ; let

$$u_j = v_j + \epsilon_j (u - \phi v) \tag{A.46}$$

with  $\epsilon = (\epsilon_{k+h}, \dots, \epsilon_n)$  being a vector in the  $(n - h - k + 1)$ -dimensional unit simplex (i.e.  $\epsilon_j \geq 0$  and  $\sum_{j=k+h}^n \epsilon_j = 1$ ).

The following table reports the allocations for the equilibria characterized in Step 3b.

$\mathbf{i} =$	$\mathbf{1}, \dots, \mathbf{k} - \mathbf{1}$	$\mathbf{k}, \dots, \mathbf{k} + \mathbf{h} - \mathbf{1}$	$\mathbf{k} + \mathbf{h}, \dots, \mathbf{n}$
$u_i$	$\mu_i \phi v$	$\mu_i \phi v$	$\mu_i \phi v + \epsilon_j (u - \phi v)$
$v_i$	$\frac{\mu_i \pi_i}{\sum_{i=1}^n \mu_i \pi_i} v$	$\mu_i \phi v$	$\mu_i \phi v$
$m_i$	$\mu_i \phi v$	$\mu_i \phi v$	$\mu_i \phi v$

Table A.4:  $\Pi > 0$ . Equilibria of Step 3b.

This concludes Step 3b.

To determine the conditions under which the equilibria described in the previous steps exhibit frictions, we characterize the aggregate number of meetings implied by each possible type of equilibrium. Let  $m_{1,2}^b$  and  $m_3^b$  denote the aggregate number of meetings corresponding to any equilibrium characterized in Steps 1b (or 2b) and 3b respectively. In an equilibrium with either excess supply in all locations (Step 1b) as well as in one with market clearing in all locations that do not exhibit excess supply (Step 2b), the number of meetings is

$$m_{1,2}^b = \sum_{i=1}^n \min \{u_i, v_i\} = u,$$

with  $\{u_i, v_i\}_{i=1}^n$  being the equilibrium allocation derived in Step 1b (or 2b). Since the equilibria in Steps 1b and 2b exist if and only if either condition (A.24) or (A.31) hold, it follows that

$$m_{1,2}^b = \min \{u, \phi v\}. \quad (\text{A.47})$$

In any of the (possibly multiple) equilibria with excess supply in some locations and excess demand others (Step 3b), the aggregate number of meetings is

$$m_3^b = \sum_{i=1}^n \min \{u_i, v_i\} = \phi v,$$

with  $\{u_i, v_i\}_{i=1}^n$  given by (A.40)-(A.42) and (A.46). Since this type of equilibria exist if and only if (A.45) is satisfied, we can write

$$m_3^b = \min \{u, \phi v\}. \quad (\text{A.48})$$

Notice that since  $\phi < 1$ , it follows that

$$\min \{u, \phi v\} \leq \min \{u, v\}$$

if and only if

$$v/u \geq \phi^{-1}.$$

So when  $\Pi > 0$ , the model delivers no frictions if and only if the number of cabs is large enough relative to the number of movers. In particular, notice that if  $v = u$  (i.e. when everyone could potentially find a match), unserved passengers and vacant cabs coexist in equilibrium. This concludes part **(b)**.

**(c)**. As shown in parts **(a)** and **(b)**, there always exists at least one equilibrium for any value of  $\Pi$ . If  $\Pi = 0$ , the aggregate number of meetings is given by

$$\min \{u, v\}$$

in any equilibrium, as can be seen from (A.18) and (A.19).

According to (A.47) and (A.48), the aggregate number of meetings is

$$\min \{u, \phi v\}$$

in any possible equilibrium when  $\Pi > 0$ .

This concludes the proof of Proposition 1. ■

## B. Appendix

This appendix provides a proof to Proposition 2, and describes all the possible equilibrium allocations for Manhattan.

### Proof of Proposition 2.

(a). Solving the system given by (E1)-(E3) for  $n = 6$  and  $a_{ij} = 1/5$  for  $i \neq j$  under the assumption that  $\rho_i < 1$  for  $i = 1, \dots, 6$ , one obtains the unique solution reported in Table B.1. It is then verified that  $\rho_i < 1$  for  $i = 1, \dots, 6$  holds if and only if  $v/u$  satisfies the restriction stated in part (a) of the proposition. The last row of Table B.1 shows the equilibrium number of matches that take place in each location.

$\mathbf{i} =$	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>
$u_i$	$\frac{1}{6}u$	$\frac{1}{6}u$	$\frac{1}{6}u$	$\frac{1}{6}u$	$\frac{1}{6}u$	$\frac{1}{6}u$
$v_i$	$\frac{1}{6} \frac{15b+63.3\pi}{15b+45.7\pi} v$	$\frac{1}{6} \frac{15b+42.9\pi}{15b+45.7\pi} v$	$\frac{1}{6} \frac{15b+35.1\pi}{15b+45.7\pi} v$	$\frac{1}{6} \frac{15b+35.1\pi}{15b+45.7\pi} v$	$\frac{1}{6} \frac{15b+42.3\pi}{15b+45.7\pi} v$	$\frac{1}{6} \frac{15b+55.5\pi}{15b+45.7\pi} v$
$m_i$	$\frac{1}{6}u$	$\frac{1}{6}u$	$\frac{1}{6}u$	$\frac{1}{6}u$	$\frac{1}{6}u$	$\frac{1}{6}u$

Table B.1: Equilibrium with excess supply in all locations.

(b). Solving the system given by (E1)-(E3) for  $n = 6$  and  $a_{ij} = 1/5$  for  $i \neq j$  under the assumption that  $\rho_i < 1$  for  $i = 1, 2, 5, 6$  and  $\rho_3 = \rho_4 = 1$ , the unique solution is the one reported in Table B.2. It is then verified that  $\rho_i < 1$  for  $i = 1, 2, 5, 6$  holds if and only if  $v/u$  satisfies the restriction stated in part (b) of the proposition. The last row of Table B.2 shows the equilibrium number of matches that take place in each location.

$\mathbf{i} =$	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>
$u_i$	$\frac{1}{6}u$	$\frac{1}{6}u$	$\frac{1}{6}u$	$\frac{1}{6}u$	$\frac{1}{6}u$	$\frac{1}{6}u$
$v_i$	$\frac{1}{6} \frac{15b+63.3\pi}{15b+35.1\pi} u$	$\frac{1}{6} \frac{15b+42.9\pi}{15b+35.1\pi} u$	$\frac{1}{6}u$	$\frac{1}{6}u$	$\frac{1}{6} \frac{15b+42.3\pi}{15b+35.1\pi} u$	$\frac{1}{6} \frac{15b+55.5\pi}{15b+35.1\pi} u$
$m_i$	$\frac{1}{6}u$	$\frac{1}{6}u$	$\frac{1}{6}u$	$\frac{1}{6}u$	$\frac{1}{6}u$	$\frac{1}{6}u$

Table B.2: Equilibrium with market clearing in locations 3 and 4 and excess supply elsewhere.

(c). The solutions to the system given by (E1)-(E3) (with  $n = 6$  and  $a_{ij} = 1/5$  for  $i \neq j$ ) under the assumption that  $\rho_i < 1$  for  $i = 1, 2, 5, 6$ , and  $\rho_j \geq 1$  for

$j = 3, 4$ , are reported in Tables B.3 and B.4. These allocations are verified to satisfy the equilibrium conditions and the pattern of excess supply/demand imposed on the system if and only if the condition stated in part (c) of the proposition holds. As always,  $\tilde{\varepsilon} \equiv (\tilde{\varepsilon}_3, \tilde{\varepsilon}_4)$  denotes a vector in the 2-dimensional unit simplex (i.e.  $\tilde{\varepsilon}_3 + \tilde{\varepsilon}_4 = 1$  and  $\tilde{\varepsilon}_i \geq 0$  for  $i = 3, 4$ ).

<b>i =</b>	<b>1</b>	<b>2</b>	<b>3</b>
$u_i$	$\frac{1}{6} \frac{15b+35.1\pi}{15b+45.7\pi} v$	$\frac{1}{6} \frac{15b+35.1\pi}{15b+45.7\pi} v$	$\frac{1}{6} \frac{15b+35.1\pi}{15b+45.7\pi} v + \tilde{\varepsilon}_3 \left( u - \frac{15b+35.1\pi}{15b+45.7\pi} v \right)$
$v_i$	$\frac{1}{6} \frac{15b+63.3\pi}{15b+45.7\pi} v$	$\frac{1}{6} \frac{15b+42.9\pi}{15b+45.7\pi} v$	$\frac{1}{6} \frac{15b+35.1\pi}{15b+45.7\pi} v$
$m_i$	$\frac{1}{6} \frac{15b+35.1\pi}{15b+45.7\pi} v$	$\frac{1}{6} \frac{15b+35.1\pi}{15b+45.7\pi} v$	$\frac{1}{6} \frac{15b+35.1\pi}{15b+45.7\pi} v$

Table B.3: Equilibria for Proposition 2, part (c). (allocations for i=1,2,3).

<b>i =</b>	<b>4</b>	<b>5</b>	<b>6</b>
$u_i$	$\frac{1}{6} \frac{15b+35.1\pi}{15b+45.7\pi} v + \tilde{\varepsilon}_4 \left( u - \frac{15b+35.1\pi}{15b+45.7\pi} v \right)$	$\frac{1}{6} \frac{15b+35.1\pi}{15b+45.7\pi} v$	$\frac{1}{6} \frac{15b+35.1\pi}{15b+45.7\pi} v$
$v_i$	$\frac{1}{6} \frac{15b+35.1\pi}{15b+45.7\pi} v$	$\frac{1}{6} \frac{15b+42.3\pi}{15b+45.7\pi} v$	$\frac{1}{6} \frac{15b+55.5\pi}{15b+45.7\pi} v$
$m_i$	$\frac{1}{6} \frac{15b+35.1\pi}{15b+45.7\pi} v$	$\frac{1}{6} \frac{15b+35.1\pi}{15b+45.7\pi} v$	$\frac{1}{6} \frac{15b+35.1\pi}{15b+45.7\pi} v$

Table B.4: Equilibria for Proposition 2, part (c). (allocations for i=4,5,6).

This completes the proof of Proposition 2. ■

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