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Affirmative Action in a Competitive Economy*

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Abstract

This paper analyzes statistical discrimination in a model with endogenous human capital formation and a frictionless labor market. It is shown that in the presence of two distinguishable but *ex ante* identical groups of workers discrimination is sustainable as an equilibrium outcome. This is true irrespective of whether there are multiple equilibria when the groups have no distinguishable characteristics. When an affirmative action policy consisting of an employment quota is introduced in the model it is shown that affirmative action can "fail" in the sense that there may still be equilibria where the groups are treated differently. However, the incentives to invest for agents in the disadvantaged group are better in *any* equilibrium under affirmative action than in the most discriminatory equilibrium without the policy. Thus, the lower bound on the fraction of agents from the disadvantaged group who invest in their human capital is raised by the policy. The welfare effects are ambiguous. It is demonstrated that the policy may increase the incentives to invest and reduce the expected payoffs for all agents in the target group simultaneously. Indeed, the policy may hurt the intended beneficiaries even when the initial equilibrium is the worst equilibrium for the targeted group.

1 Introduction

Since its introduction in the sixties, affirmative action has been and remains one of the most controversial policies to combat discrimination in the labor market. An economist has little to say about issues on fairness and constitutionality, which are extensively discussed in the popular and

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political debate. However, there are important aspects of affirmative action that can be analyzed using economic theory and relatively little has been done.

In particular, the popular debate often focuses on the effects on incentives of the intended beneficiaries. On the one hand side, opponents of affirmative action often argue that affirmative action makes it easier for unqualified members of the target groups to obtain relatively well paid jobs. This, it is argued, reduces the incentives to invest in their skills for members of the target groups, which means that the real problem, namely that skills are unevenly distributed across groups, is only aggravated by affirmative action. On the other side, proponents of affirmative action argue that minorities and in some cases women are at least partially excluded from the more attractive parts of the labor market and that they for this reason simply do not have the same incentives to make human capital investments. Affirmative action policies with numerical goals for hirings of candidates from the discriminated groups helps overcome the situation by forcing employers to hire people from the disadvantaged groups and therefore create incentives for members of these groups to invest in their personal skills.

The purpose of this paper is to analyze what effects affirmative action policies may have on the incentives to invest, in particular for workers from the groups the policy is intended to help. Furthermore, since opponents often claim that the policy only helps already well situated members of the minority groups, we are also interested in identifying winners and losers of affirmative action. However, while our framework in principle allows us to do this, our understanding of the welfare effects of affirmative action is still very incomplete.

In order to study the effects of affirmative action and other anti discriminatory policies we need a model with discrimination as a possible equilibrium outcome. Here there are two main strands in the literature. One approach, pioneered by Becker [3], explains discrimination from preferences. In this class of models employers prefer to hire candidates from the same group, workers prefer to work with coworkers from the same group or consumers are unwilling to buy products produced by firms' employing workers from other groups.

The main alternative to these taste based models is a statistical theory of discrimination, building on work by Arrow [2] and Phelps [12]. Here the main idea is that when worker skills are imperfectly observable discrimination may occur although firms maximize profits and workers have no preferences about their coworkers' group identity: race, sex, religion etc... may serve as a proxy for productivity if the distributions are different across groups. When each worker can affect their own productivity by human capital investments discrimination may occur in equilibrium even if the groups are identical in terms of "intrinsic abilities" or costs of investment in human capital. In this paper we consider the effects of affirmative action within a model of statistical discrimination.

While there is a large theoretical literature on discrimination in general and discrimination on the labor market in particular, surprisingly little attention has been paid to policy analysis. Notable exceptions are Lundberg and Startz [9], Lundberg [8] and Coate and Loury [4][1]. In Lundberg and

[1] All these papers are focusing on statistical discrimination. Welch [15] and Kahn [7] studies employment quotas

Startz [9] it is shown that an equal opportunity policy prohibiting the firms from making wages dependent upon group identity may be an efficiency enhancing policy in a model with statistical discrimination. In Lundberg [8], it is noted that enforcement of this type of policy may be very difficult since there will be incentives for firms to evade the policy by using other variables as proxies for group identity. The main concern of the paper is to find regulatory policies that implement the equal opportunity laws under different informational assumptions.

The paper most closely related to our work is Coate and Loury [4] where the effects of employment quotas are studied in a setup where discrimination is in job assignments rather than in wages. In their model, output can be produced using two different technologies and workers face a costly human capital investment, which if undertaken makes them productive in the more advanced technology. The sole decision made by employers' is how to assign a number of randomly drawn workers in jobs using either of the two technologies based on an imperfect signal of each workers' productivity in the more advanced job. Whenever there are multiple equilibria in the model there will be equilibria where groups are treated differently.

It is shown that there are circumstances under which all equilibria with the affirmative action policy are such that investment behavior is the same in both groups. However, under equally plausible circumstances there are still equilibria where groups behave differently and the employers (rationally) perceive members of one of the groups to be less capable. Indeed, it is shown that group disparity of investment behavior may actually increase as a result of affirmative action.

The intuition for this possible failure of affirmative action is simple. Consider a situation where the fraction of investors is lower in group a than in group b and make the thought experiment that these fractions remains the same even after the introduction of affirmative action. In order to comply with the policy this means that employers must employ agents from group a in the more advanced job who are (rationally) perceived to have a lower probability of being productive than all agents from group b . Hence both agents who have invested and agents who have not invested are more likely to be employed in the skilled job and whether this improves the incentives to invest for agents in group a or not depends on particularities of the probability distributions of the noisy signal.

While the logic may sound compelling the analysis in Coate and Loury [4] raises some questions. Wages as well as the distribution of workers available for any firm are fixed exogenously in their model. In a world where firms are competing with each other to attract workers these assumptions do not make much sense. Rather one would think that equilibrium wages would depend on investment behavior of the workers and policy parameters, which means that the change in the incentives to invest would also depend on how the policy affects wages. In particular, since the expected marginal productivity is increasing in the signal for agents in the complex technology one would think that wages would also be increasing in the signal. But then the expected wage conditional on the agent being employed in the more advanced technology will be higher for agents

in models where discrimination is taste based.

who undertake the investment and it seems that if firms were forced to employ more workers from the disadvantaged group in the advanced job this would indeed create better incentives to invest.

In our paper, human capital accumulation as well as the information technology is modeled as in Coate and Loury [4]. Individual workers have to decide whether to undertake a costly investment in human capital or not. This choice is unobservable to the firms but there is a publicly observable test available that contains information about the likelihood that a particular worker has undertaken the investment.

Instead of randomly assigning workers between firms we assume that the labor market works without frictions. Firms compete in a Bertrand fashion by offering wage schedules, where the wage is a function of the noisy signal. Apart from the fact that wages are endogenized our model departs from that of Coate and Loury in that the production technology exhibits complementaries between tasks. To be specific we assume that production requires input of labor in two tasks, a *complex* task and a *simple* task. It is assumed that only workers who have undertaken the investment are productive in the complex task, whereas all workers can perform the simple task effectively. Output is generated from the two types of labor input according to a standard neoclassical production function.

When we introduce two groups of workers which only differ by some payoff irrelevant but observable characteristic we show that discrimination is possible due to self confirming expectations about differences in behavior between the groups.

The complementarity in the production technology has several interesting consequences. Even if there is a unique equilibrium in the model where there are no observable payoff irrelevant characteristics there will, under mild conditions, be equilibria with discrimination. The intuition is that groups can specialize as high quality and low quality workers respectively. While this hurts the group that specializes as low quality workers and also creates inefficiencies in investment behavior it does reduce the informational problem for the firms[2]. It should be noted in this context that in models where discrimination is explained as different groups coordinating on different equilibria in some "base model", as for example in Spence [14], Akerlof [1] and Coate and Loury [4], there are no conflicts of interests between groups. The discriminated group is discriminated simply because of coordination on a worse equilibrium than the other group and if this coordination failure could be resolved the other group need not be affected at all. In our model on the other hand the group with the higher fraction of investors unambiguously gains from discrimination since the supply of qualified workers is more scarce than otherwise.

The complementarity in production also has the consequence that group size matters in the determination of equilibria with discrimination. We find that the larger the group is, the more stringent are the conditions that must be satisfied in order to support a (particular type of) dis-

[2] It is indeed easy to visualize a version of our model where agents choose different types of human capital investment that enhances the productivity in different types of jobs. In such a model, discrimination may be efficiency enhancing. However, contrary to our framework such a model may also have the property that discrimination is voluntary in the sense that it may be incentive compatible to truthfully announce group identity if this would be unobservable.

crimination and the smaller is the differences in average earnings between groups (given that discrimination is still sustainable). In a loose sense, we interpret this to mean that in our model discrimination of a smaller group is more likely than discrimination of a larger group. To us this seems to conform with the stylized facts about discrimination: to our knowledge there is no other model with this property.

Introducing an affirmative action policy consisting of an employment quota in the model we find that we in general cannot rule out the possibility of discriminatory equilibria. Hence the policy does not guarantee equal treatment across groups in equilibrium. However, this result alone should not be interpreted as a "failure" of affirmative action. While the ultimate goal of equality between groups is not guaranteed by the policy, it may still be that the policy is successful in the sense that the inequality is reduced. Indeed, we get some results in this direction.

In our model, the "direct effect" of affirmative action, i.e. the effect on the benefits of investment in human capital assuming that investment behavior is unchanged, is typically to increase the returns of investment for the discriminated group and to decrease them for the other group. However, we do not have a theory that predicts what particular equilibrium will occur after the introduction of the policy. Due to multiplicity of equilibria with and without the policy we must compare the full set of equilibria with and without affirmative action. The only thing that can be said in general is that the returns to investment and consequently also the fraction of agents who invest in the most discriminatory equilibrium without the policy is lower than in any equilibrium with affirmative action. The welfare effects are inconclusive. Output may decrease or increase as a result of the policy and by example we show that even if the starting point is the most discriminatory equilibrium, it is possible that the discriminated group is worse off with the policy.

The rest of the paper is structured as follows. Section 2 contains the description of the one-group model and section 3 characterizes the equilibria of this model. In section 4 we extend the model by introducing two identical groups of workers and in section 5 we analyze the consequences of affirmative action. The discussion in section 6 concludes the paper.

2 The Model

We assume that firms need to employ workers performing two different tasks to generate output. These tasks will be referred to as the *complex task* and the *simple task* respectively. On the labor market there are workers of two different types. Some workers, called *qualified* workers, are able to perform the complex task and others are not. Let C be the effective input of labor in the *complex task* for the firm, i.e. C equals the number of qualified workers employed in the skilled task. By S we denote the number of workers employed in the *simple task*. The output of the firm is then given by $y(C,S)$ where $y : R_+^2 \to R_+$ satisfies the standard neoclassical assumptions, i.e. it is a twice continuously differentiable function, strictly concave in both arguments, and:

A1 $y(\cdot,\cdot)$ is homogeneous of degree one [constant returns to scale]

A2 $\lim\limits_{C\to 0} y_1(C,S) = \infty$ for any $S > 0$ and $\lim\limits_{S\to 0} y_2(C,S) = \infty$ for any $C > 0$[3] [boundary behavior]

A3 $y(0,S) = y(C,0) = 0$ [both factors essential]

Since we make the extreme assumption that the additional output generated by *unqualified* workers in the *complex task* is zero only *qualified* workers would be hired for this task in a perfect information environment. However, in the model there will be some mismatch due to uncertainty about worker quality.

2.1 The Game

The timing of events is as follows: In **Stage 1** individual workers decide whether to invest or not in their human capital. After the investment decisions (**Stage 2**) each worker is assigned a signal θ by nature. In **Stage 3** firms simultaneously announce wage schedules (i.e. wages as functions of the signal) and in **Stage 4** workers choose which firm to work for. Finally, in **Stage 5** firms decide how to allocate the available workers between the two tasks.

For tractability we do not want the behavior of any individual worker to have any effect on aggregate behavior so we will assume that the population of workers is large, represented by a continuum.

The model will now be described in detail.

Stage 1. There is a continuum of agents with heterogeneous costs of investment. Each agent c has to choose an action $e \in \{e_q, e_u\}$, where $e = e_q$ means that the agent undertakes an investment in his human capital (and becomes a *qualified* worker) and $e = e_u$ that he does not. If agent c undertakes the investment he incurs a cost of c while no cost is incurred if the investment is not undertaken. The agents are distributed on the interval $[\underline{c}, \overline{c}] \subseteq R$ according to the continuous and strictly increasing distribution function G. We assume that $\underline{c} \leq 0$ and $\overline{c} > 0$.

Stage 2. Each worker is assigned a noisy signal $\theta \in [0,1]$. The signal θ is distributed according to density f_q for workers who invested in Stage 1 and f_u for workers who did not invest. It is assumed that f_q and f_u are continuously differentiable, bounded away from zero and satisfies:

A4 $\frac{f_q(\theta)}{f_u(\theta)} > \frac{f_q(\theta')}{f_u(\theta')}$ if $\theta > \theta'$ [strictly monotone likelihood ratio property]

This assumption implies that qualified workers are more likely to get higher values of θ than unqualified workers. We let F_q and F_u denote the associated cumulative distributions.

[3] Subscripts are used to denote partial derivatives.

Stage 3. There are two firms, $i = 1, 2$. The firms simultaneously announce wage schedules. We allow wages to be dependent on the signal so that a (pure) action of firm i in stage 3 is a measurable function $w_i : [0,1] \to R_+$. We assume that the firms cannot observe the distribution of signals when announcing wages[4].

Stage 4. The workers observe w_1 and w_2 and decide which firm to work for.

Stage 5. In the final stage of the game the firms allocate the available workers by using a task assignment rule which is a measurable function $t_i : [0,1] \to \{0,1\}$[5]. The interpretation is that $t_i(\theta) = 1$ (0) means that firm i assigns all workers with signal θ to the complex (simple) task.

We assume that the risk neutral workers' payoffs are additively separable in money income and the cost of investment and that workers do not care directly in which task they are employed. Thus, once the investment cost is sunk, the worker will rationally choose the firm that offers the higher wage for his particular realization of θ. To save on notation we immediately impose optimal behavior by workers in Stage 4 and write payoffs as

$$E_\theta \left[\max\{w_1(\theta), w_2(\theta)\} | e\right] - c(e), \tag{1}$$

where $c(e_q) = c$ and $c(e_u) = 0$.

Next we want to express the firms' profits as a function of the actions and to do this we need frequency distributions over realized values of the signals. Intuitively one would want to appeal to the strong law of large numbers and take these to be given by F_q and F_u, but as noted by Judd [6] and Feldman and Gilles [5] this is problematic with a continuum of random variables. Feldman and Gilles [5] discusses alternative ways to ensure that the individuals' probability distribution and the frequency distribution coincides almost surely. The analysis in this paper relies only on this property and not the particular way we make sure that the property holds. The simplest solution is to use "aggregate shocks" rather than to assume that the signals are i.i.d. draws from F_q and F_u. The investment decisions by the agents induce distributions of qualified workers and unqualified workers on $[\underline{c}, \overline{c}]$. Call these distributions H_q and H_u. Now let the random variable x be uniformly distributed on $[0,1]$ and let $\theta_c(x)$ denote the test-score for a qualified agent c, where

$$\theta_c(x) = \begin{cases} F_q^{-1}(H_q(c) + x) & \text{if } H_q(c) + x \le 1 \\ F_q^{-1}(H_q(c) + x - 1) & \text{if } H_q(c) + x > 1 \end{cases} \tag{2}$$

[4]The only role of this assumption is that it simplifies the description of the strategy sets. See the discussion in the end of Section 3.

[5]This is not the most general way to describe a pure action, but it will be sufficient for our purposes. See footnote 7.

It is straightforward, but somewhat tedious to verify that $\Pr\left[\theta_c(x) \leq \theta \mid e_q\right] = F_q(\theta)$ for all $c \in [\underline{c}, \overline{c}]$ and all $\theta \in [0,1]$ and that $\int\limits_{c \in A(x,\theta)} dH_q(c) = F_q(\theta)$ $x \in [0,1]$ and all $\theta' \in [0,1]$, where $A(x,\theta) = \{c \in [\underline{c}, \overline{c}] \mid \theta_c(x) > \theta\}$. Clearly the construction can be applied to the unqualified agents as well.

A single firm does not care directly about the realized frequency distributions for the whole population, but rather about the particular workers the firm has available, which depends on the decisions of the workers in Stage 4. Thus, to evaluate the profits of a single firm we need to aggregate the behavior of the workers in some way. In principle, we could derive the distributions for a firm given by arbitrary actions by workers, but we will immediately impose optimal behavior by workers in Stage 4. To capture this we define

$$I^1_{\langle w_1, w_2 \rangle}(\theta) = \begin{cases} 1 & \text{if } w_1(\theta) > w_2(\theta) \\ \frac{1}{2} & \text{if } w_1(\theta) = w_2(\theta) \\ 0 & \text{if } w_1(\theta) < w_2(\theta) \end{cases} \tag{3}$$

and let $I^2_{\langle w_1, w_2 \rangle}$ be defined symmetrically. The interpretation is that $I^i_{\langle w_1, w_2 \rangle}(\theta) = 1$ means that all workers with signal θ choose to work for firm i. Besides the fact that the tie-breaking rule is arbitrary[6] these functions aggregates the workers optimal responses to $\langle w_1, w_2 \rangle$ in the obvious way.

Given that a fraction π of the workers invests and wage schedules $\langle w_1, w_2 \rangle$ we can now compute the number of *qualified* workers available for firm i with a signal $\theta \leq \overline{\theta}$ as $\int_0^{\overline{\theta}} I^i_{\langle w_1, w_2 \rangle}(\theta) \pi f_q(\theta) d\theta$ and the number of *unqualified* workers can be computed symmetrically. The effective input of labor in the two tasks given a pair of wage schedules $\langle w_1, w_2 \rangle$ and task assignment rule t_i are then given by

$$\begin{aligned} C_i(w_1, w_2, t_i) &= \int I^i_{\langle w_1, w_2 \rangle}(\theta) t_i(\theta) \pi f_q(\theta) d\theta \\ S_i(w_1, w_2, t_i) &= \int I^i_{\langle w_1, w_2 \rangle}(\theta) (1 - t_i(\theta)) (\pi f_q(\theta) + (1-\pi) f_u(\theta)) d\theta \end{aligned} \tag{4}$$

respectively. The profits of firm i can then be expressed as

$$\Pi^i(\cdot) = y(C_i(w_1, w_2, t_i), S_i(w_1, w_2, t_i)) - \int I^i_{\langle w_1, w_2 \rangle}(\theta) w_i(\theta) (\pi f_q(\theta) + (1-\pi) f_u(\theta)) d\theta. \tag{5}$$

After the workers' decisions in Stage 4 have been replaced by the sequentially rational allocation rule (3) a pure strategy for a worker is simply to decide to invest or not. We will summarize the behavior of all workers as a map $i : [\underline{c}, \overline{c}] \to \{e_q, e_u\}$.[7] A pure strategy for a firm is a pair $\langle w_i, \xi_i \rangle$

[6] However, one can show that there are no additional equilibrium outcomes that can be supported by changing the tie-breaking rule.

[7] This assumes that all workers with the same investment costs choose the same strategy. More generally one could model a pure strategy profile in analogy with a "distributional strategy" in the sense of Milgrom and Weber[11], i.e. as a joint distribution over $[\underline{c}, \overline{c}] \times \{e_q, e_u\}$. In our model this generality is not necessary since if the best response of agent c is to invest and $c' < c$, then the unique best response of agent c' is to invest.

where w_i is a measurable function from $[0,1]$ into R_+, $\xi_i : M \times M \to T$, M denotes the set of measurable functions from $[0,1]$ into R_+ and T denotes the set of measurable functions from $[0,1]$ into $\{0,1\}$. The interpretation is that if $\xi_i(w_1, w_2)(\theta) = 1$ then firm i assigns workers with signal θ to the complex task given that the pair of wage schedules offered in Stage 3 is (w_1, w_2) .

3 Characterization of Equilibria

In this section we characterize the set of equilibrium outcomes of the model. Although the model is dynamic, standard refinements such as *perfect Bayesian equilibrium* will not give any sharper predictions than *Nash Equilibrium* in terms of equilibrium *outcomes*[8]. Therefore we take *Nash equilibrium* as our solution concept.

To find the equilibria of the game we will first characterize the firms' equilibrium responses given any investment behavior by the workers. These responses determine a unique wage schedule consistent with any investment behavior by the workers and when we impose optimal behavior by the workers in the initial stage we get a simple fixed point equation that characterizes the set of equilibrium outcomes.

First we will argue that the wage schedules offered by the firms must be identical almost everywhere in any Nash equilibrium of the game. The reason is simple: If one firm would offer a higher wage than the other to a set of workers with positive mass it could decrease the wage bill by lowering wages slightly for all these workers. If the cut in wages is small enough the firm still has the same distribution of workers available and by keeping the task assignment rule on the outcome path as before the deviation profits would increase[9].

Next we consider the decision problem for the firm in the final stage after a history when an arbitrary fraction of agents $\pi \in (0,1]$[10] has chosen to invest and the firms have offered wage schedules $\langle w_1, w_2 \rangle$ with $w_1(\theta) = w_2(\theta)$ for almost all $\theta \in [0,1]$. For an arbitrary θ, the quantity of qualified workers available for firm i with realized signal less than θ is then simply $\frac{1}{2}F_q(\theta)$ and the quantity of unqualified workers with signal less than θ is symmetrically $\frac{1}{2}F_u(\theta)$. By the monotone likelihood ratio assumption any optimal task assignment rule for firm i must be a cutoff-rule[11] of the form

$$t_i(\theta) = \begin{cases} 0 & \text{if } \theta < \widetilde{\theta}_i \\ 1 & \text{if } \theta \geq \widetilde{\theta}_i \end{cases} . \tag{6}$$

Using the identities in (4) and observing that the firms can do nothing about their wage costs in

[8] The reasons why any *Nash equilibrium outcome* can be supported as a *perfect Bayesian equilibrium* will be briefly discussed towards the end of the section.

[9] For a formal proof, see Lemma I in Appendix B.

[10] Given "worker strategy profile" i this is computed as $\pi = \int_{c \in i^{-1}(\{e_q\})} dG(c)$.

[11] For a formal argument, see Lemma II in appendix B.

the final stage of the game the problem to maximize output can be written as

$$\begin{aligned} & \max_{\theta_i, C_i, S_i} y\left(C_i, S_i\right) & (7) \\ \text{subj. to } 2C_i & \leq \pi\left(1 - F_q\left(\theta_i\right)\right) \\ 2S_i & \leq \pi F_q\left(\theta_i\right) + (1-\pi) F_u\left(\theta_i\right). \end{aligned}$$

Eliminating θ_i from the problem it is easy to show that the monotone likelihood ratio implies that the constraint set is (strictly) convex[12] . Furthermore, y is strictly increasing in both arguments

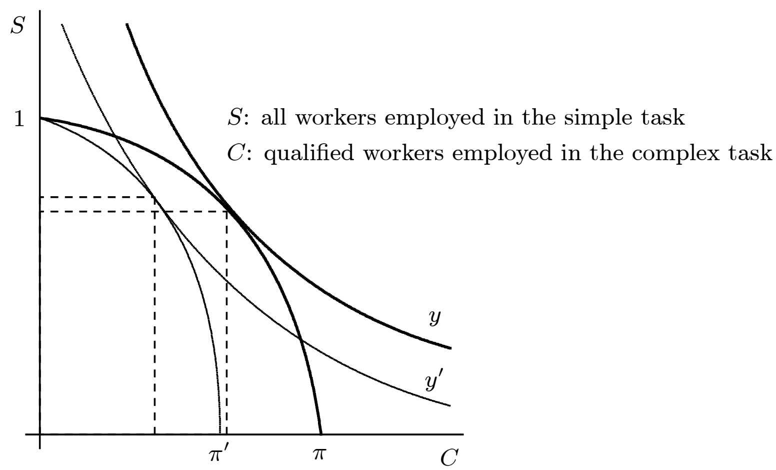

Figure 1: The task-assignment problem

so the constraint must bind with equality and strict quasi-concavity of y guarantees that there is a unique solution to (7). Finally, the boundary condition **A2** guarantees that any solution must be interior, so the problem can be depicted graphically as in Figure 1.

Since both firms face a symmetric problem with a unique solution we drop the indices from now on. Eliminating C and S from the problem and using constant returns to scale we can after some algebra write the first order condition for (7) as

$$p\left(\theta, \pi\right) y_1\left(\pi\left(1 - F_q\left(\theta\right)\right), F_\pi\left(\theta\right)\right) = y_2\left(\pi\left(1 - F_q\left(\theta\right)\right), F_\pi\left(\theta\right)\right), \tag{8}$$

[12]See Lemma III in appendix A.

where $F_\pi(\theta)$ is shorthand notation for $\pi F_q(\theta) + (1-\pi) F_u(\theta)$ and

$$p(\theta, \pi) = \frac{\pi f_q(\theta)}{\pi f_q(\theta) + (1-\pi) f_u(\theta)} \equiv \Pr[e_q | \theta] \tag{9}$$

denotes the posterior probability that a randomly drawn agent with test score θ is *qualified* given prior probability π. The economic interpretation is that an agent with signal equal to the optimal cutoff point determined by (8) has the same expected marginal productivity in both tasks. All agents with a lower realization of the signal are more productive in the simple task and agents with higher signals are more productive in the complex task.

For each $\pi > 0$ we let the unique solution to (8) be denoted by $\widetilde{\theta}(\pi)$. It is shown in Appendix C that $\widetilde{\theta}$ is continuously differentiable and therefore continuous on the open unit interval. To save on notation we also define

$$r(\pi) = \frac{\pi\left(1 - F_q\left(\widetilde{\theta}(\pi)\right)\right)}{\pi F_q\left(\widetilde{\theta}(\pi)\right) + (1-\pi) F_u\left(\widetilde{\theta}(\pi)\right)}, \tag{10}$$

which, for an arbitrary π is the ratio of effective units of complex labor over units of simple labor implied by the equilibrium task assignment rule (6).

Our next task is to determine the equilibrium wage schedules. The most natural guess is that all agents are paid according to their respective expected marginal productivity in the task where they are employed. Thus, given a fraction π of investors we take the candidate "labor market equilibrium" wage function to be given by $w : [0,1] \to R_+$ defined as

$$w(\theta) = \begin{cases} y_2(r(\pi), 1) & \text{for } \theta < \widetilde{\theta}(\pi) \\ p(\theta, \pi) y_1(r(\pi), 1) & \text{for } \theta \geq \widetilde{\theta}(\pi) \end{cases}, \tag{11}$$

where we have used the assumption of constant returns to scale[13]. Observe that the assumption of (strict) monotone likelihood ratio implies that $p(\theta, \pi)$ is strictly increasing in θ given any $\pi > 0$. Hence the proposed wage schedule is strictly increasing on $\left[\widetilde{\theta}(\pi), 1\right]$.

Indeed, it can be shown that the intuition from Bertrand competition with constant returns to scale carries over to our model. Formally:

Proposition 1. *Let the fraction of agents who invest be given by π, let $t : [0,1] \to \{0,1\}$ be a cutoff rule with critical point $\widetilde{\theta}(\pi)$ determined by (8) and let $w : [0,1] \to R$ be given by (11). Furthermore, for any firm strategy profile $\langle w_i, \xi_i \rangle_{i=1,2}$ let $t_i : [0,1] \to \{0,1\}$ be the task assignment rule on the outcome path, i.e. $t_i(\theta) = \xi_i(w_i, w_j)(\theta)$ for all θ. Then both firms are playing best responses if and only if $w_i(\theta) = w(\theta)$ and $t_i(\theta) = t(\theta)$ for $i = 1, 2$ and for all almost all $\theta \in [0,1]$.*

[13] I.e. $y_i\left(\pi\left(1 - F_q\left(\widetilde{\theta}(\pi)\right)\right), F_\pi\left(\widetilde{\theta}(\pi)\right)\right) = y_i(r(\pi), 1)$

The proof is in the appendix. The intuition for the sufficiency part is that any deviating firm has to pay at least the expected marginal productivity (given task assignments according to candidate equilibrium) for all workers and strictly more if it wants to attract any additional workers. The only way this could be a profitable deviation in a constant returns to scale environment is if the deviating firm could allocate the workers more efficiently between tasks, which is impossible since the original task assignment rule maximizes output. Deviations on sets of measure zero will clearly have no effect on profits and it has been argued above that both firms must choose task assignment rules identical to t almost everywhere.

So far we have considered the firms' equilibrium responses for any fixed investment behavior by the workers. In a *Nash Equilibrium* of the full model the additional condition that each worker maximizes (1) given the wage schedules must hold as well. The workers only care about the *maximal* offer for each realization of θ , so we let $w'(\theta) = \max\{w'_1(\theta), w'_2(\theta)\}$ for any pair of wage schedules $\langle w'_1, w'_2 \rangle$ and write the set of pure best responses for agent $c \in [\underline{c}, \overline{c}]$ as

$$\beta_c(w') = \begin{cases} e_q & \text{if } \int w'(\theta) f_q(\theta) d\theta - c > \int w'(\theta) f_u(\theta) d\theta \\ \{e_q, e_u\} & \text{if } \int w'(\theta) f_q(\theta) d\theta - c = \int w'(\theta) f_u(\theta) d\theta \\ e_u & \text{if } \int w'(\theta) f_q(\theta) d\theta - c < \int w'(\theta) f_u(\theta) d\theta \end{cases}. \tag{12}$$

The unique fraction of investors consistent with all workers playing best responses is thus given by

$$\pi = G\left(\int w'(\theta) f_q(\theta) d\theta - \int w'(\theta) f_u(\theta) d\theta\right). \tag{13}$$

Since Proposition 1 guarantees that wages must be given by marginal productivities in any Nash equilibrium of the full model we can substitute the wage schedule (11) into (13) to obtain a fixed point equation in π. We denote by $H(\pi)$ the gross benefits of investment, i.e. the difference between the expected earnings for an agent who invests and the expected earnings for an agent who does not invest. By simple substitution we find that

$$H(\pi) = y_2(r(\pi), 1)\left(F_q\left(\widetilde{\theta}(\pi)\right) - F_u\left(\widetilde{\theta}(\pi)\right)\right) + y_1(r(\pi), 1) \int_{\widetilde{\theta}(\pi)}^{1} p(\theta, \pi)\left(f_q(\theta) - f_u(\theta)\right) d\theta. \tag{14}$$

Since $y_1(r(\pi), 1)\, p\left(\widetilde{\theta}(\pi), \pi\right) = y_2(r(\pi), 1)$ it is easy to see that $H(\pi) > 0$ for all π in the interior of the unit interval. As discussed above optimal behavior of the workers implies that an agent should invest if and only if the cost of doing so is less than the expected benefits. The equilibria of the model are thus fully characterized by the solutions to the equation

$$\pi = G(H(\pi)) \tag{15}$$

Summing up these observations we have:

Proposition 2. *Consider a strategy profile* $\left\{i, \langle w_i, \xi_i \rangle_{i=1,2}\right\}$ *and let* $\pi^* = \int_{c \in i^{-1}(\{e_q\})} dG(c)$ *be the fraction of investors implied by worker strategy profile* i*. Furthermore, let* w*,* t *and* t_i *be defined as in Proposition 1 (with* $\pi = \pi^*$*). Then* $\left\{i, \langle w_i, \xi_i \rangle_{i=1,2}\right\}$ *is a Nash equilibrium if and only if* π^* *solves (15),* $i(c) = e_q$ *for all* $c < G^{-1}(\pi^*)$ *and* $i(c) = e_u$ *for all* $c > G^{-1}(\pi^*)$ *and* $(w_i(\theta), t_i(\theta)) = (w(\theta), t(\theta))$ *for* $i = 1, 2$ *and for almost all* $\theta \in [0, 1]$.

The proof follows from Proposition 1 and the discussion in the preceding paragraph and is omitted.

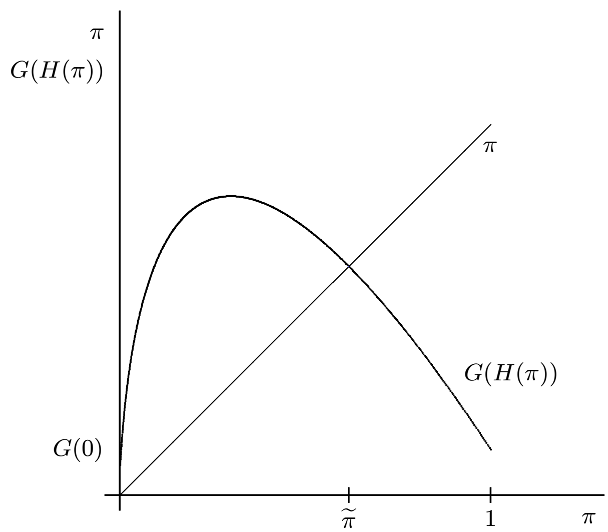

Figure 2: An example with a unique interior equilibrium

Note that Proposition 2 implies that the question of existence of equilibria reduces to the question of existence of a fixed point of the map $G \circ H$. This gives us a relatively easy proof of existence of equilibria, which is the next result:

Proposition 3 *If* $G(0) > 0$[14]*, then there exists a non-trivial equilibrium of the model.*

[14] Under the additional assumption that $\lim_{C \to 0} \frac{y_{11}(C,S)y(C,S)}{(y_1(C,S))^2} > -\infty$ the result holds even when $G(0) = 0$ or, equivalently, when the lower bound on the support of the cost distribution $\underline{c} = 0$. In this case the model also has a trivial equilibrium where nobody invests, which is not the case when $\underline{c} < 0$. While not easy to interpret, the condition holds for several common parametric production functions (for example in the Cobb-Douglas case). The proof, in which it is shown that the slope of $G \circ H$ is unbouded at $\pi = 0$ is available on request from the authors.

The proof is the appendix, but the idea can be understood from Figure 2. While (8) is not defined at $\pi = 0$ it follows directly from the constraint that the only *feasible* input of complex labor is zero when nobody invests. Since by assumption **A3** both tasks are needed in production, output must be zero. Hence $w(\theta) = 0$ for all θ which implies $H(0) = 0$. When $\pi = 1$ we have that $p(\theta, 1) = 1$ for all θ and since this means that the signal does not provide any additional information the wage schedule is a constant function of θ, so $H(1) = 0$. As has already been argued $H(\pi) > 0$ for all intermediate values of π and after verifying that $H(\pi)$ is continuous on $[0, 1]$ existence is established by use of the intermediate value theorem.

3.1 Why Use a Strategic Model?

In our model, it is very important that workers respond to wage schedules when choosing what firm to work for. We capture this by the assumption that workers are allocated between firms in accordance to (3). This makes the model extremely "competitive" in a strategic sense and, as we have seen, the equilibrium conditions have an obvious flavor of competitive equilibrium. One may therefore conjecture that the model could be formalized as a model with price taking agents, but doing this one runs into technical as well as conceptual difficulties[15]. The game-theoretic modelling helps overcome these difficulties and also makes the policy analysis easier to handle.

Note that the "reduced game" obtained by assuming that each firm's distribution of available workers is determined by (3) is still a dynamic game and we have nevertheless been able to ignore unreached information sets: propositions 1 and 2 do not even specify what firms are supposed to do when choosing task assignment rules at information sets where the wage schedules differ on a set of points with positive measure. The reader may therefore be worried that we consider equilibria supported by non-credible threats off the equilibrium path, but this is not the case. The intuitive explanation is that the last stage of the game is non strategic, in the sense that the task assignments by the other firm has no impact on the best responses in Stage 3. Hence it is impossible to enlarge the set of equilibrium *outcomes* compared to the set of *perfect Bayesian equilibrium outcomes* by committing to task assignment rules that are suboptimal off the equilibrium path[16].

In the discussion above we derived the fixed point equation (15) by working from the end of the game as if we were using backwards induction. However, since we are working with a continuum

[15] The problem is that when maximizing over "quantities" the decision variable of the firm is to choose a distribution on the support of the noisy signal. In order to write down sensible market clearing conditions it turns out that a strong law of large numbers is needed. The technical problem is that to guarantee such a strong law of large numbers in an environment with uncountably many independent random variables one has to rely on somewhat arbitrary probability measures (see Judd [6] and Feldman and Gillles [5]). While we also have to deal with this problem in our model it is much easier to circumvent in our framework. The conceptual problem is that even with such a strong law of large numbers it is not clear how the firms should evaluate profits out of equilibrium.

[16] Since it would not affect the set of equilibrium outcomes the reader may wonder why we did not model the wage and task assignment decisions as simultaneous. The reason is that when introducing the affirmative action policy we need task assignments to be done after the firm knows what distribution of workers it has available.

of workers no single worker can affect aggregate variables and the function (14) therefore plays the same role in the analysis as the best response correspondence in a static game.

4 The Model with Two Identifiable Groups of Workers

We now extend the basic model and assume that each worker belongs to one of two identifiable groups, indexed by a and b respectively. The purpose of the section is to demonstrate the existence of equilibria with discrimination. Under the assumption that not too many agents will invest when there are no monetary incentives, we show existence of discriminatory equilibria by construction.

4.1 The Extended Model

We now assume that a fraction λ^a of the workers belongs to group a and a fraction $\lambda^b = 1 - \lambda^a$ to group b. It is assumed that the distribution of investment costs is given by G in each group and that the probability density over signals is given by f_q for any worker (from group a or b) who invests and by f_u for any worker who does not. These two assumptions means that the groups are *ex ante identical* in terms of investment costs and that the signals are *unbiased.*

As in the single group model we assume that the realized frequency distributions of signals coincides with the probability distributions F_q and F_u. This can be *derived* using the obvious generalization of the exact stochastic model described in Section 2.1.

The game is the same as in Section 2.1, except that wage schedules are now allowed to depend on "group identity". Hence, a strategy for firm i is a quadruple $\left\langle w_i^a, w_i^b, \xi_i^a, \xi_i^b \right\rangle$ where $w_i^j : [0,1] \to R_+$ is the wage schedule and ξ_i^j maps pairs of wage schedules to task assignment rules for group j. For the same reason as in the single group model, the specification of the task assignment rule off the equilibrium path will be irrelevant and we can think of the firm as choosing a pair of wage schedules and task assignment rules $t_i^j : [0,1] \to \{0,1\}$.

We maintain the assumption that workers from each group allocate themselves between firms according to (3), now evaluated using the relevant wage schedules for each group. Hence if the fractions of agents who invest are given by $\boldsymbol{\pi} = \left(\pi^a, \pi^b\right)$, the effective input of labor in respective task in firm i is given by

$$\begin{aligned} C_i(\cdot;\boldsymbol{\pi}) &= \sum_{j=a,b} \lambda^j \int I^i_{\left\langle w_1^j, w_2^j \right\rangle}(\theta)\, t_i^j(\theta)\, \pi^j f_q(\theta)\, d\theta \\ S_i(\cdot;\boldsymbol{\pi}) &= \sum_{j=a,b} \lambda^j \int I^i_{\left\langle w_1^j, w_2^j \right\rangle}(\theta) \left(1 - t_i^j(\theta)\right) \left(\pi^j f_q(\theta) + \left(1 - \pi^j\right) f_u(\theta)\right) d\theta \end{aligned} \tag{16}$$

It should be clear how the firms' profit functions should be generalized from (5).

4.2 Equilibrium in the Extended Model

Analogous to the procedure in Section 3 we begin by considering the problem of maximizing output over task-assignment rules with the cutoff-property. This problem can be written as;

$$\max_{(\theta^a,\theta^b)\in[0,1]^2} y\left(\sum_{j=a,b}\lambda^j\pi^j\left(1-F_q\left(\theta^j\right)\right),\sum_{j=a,b}\lambda^j F_{\pi^j}\left(\theta^j\right)\right). \tag{17}$$

The program is qualitatively very similar to (7), the task assignment problem in the basic model, but (partial) corner solutions may now be possible. By similar arguments as in the single group model one shows that there exists a unique solution to (17) for any $\boldsymbol{\pi}\neq(0,0)$ and that the solution satisfies the Kuhn-Tucker conditions. We let γ_j be the multiplier associated with the constraint $\theta^j\geq 0$ and η_j be associated with $1-\theta^j\geq 0$. The Kuhn-Tucker conditions are, after some rearranging, given by

$$-p\left(\theta^j,\pi^j\right)y_1(\cdot)+y_2(\cdot)+\frac{\gamma_j-\eta_j}{f_{\pi^j}\left(\theta^j\right)}=0,\ \text{for } j=a,b \tag{18}$$

together with the complementary slackness conditions. We let the solution be denoted by $\widetilde{\boldsymbol{\theta}}(\boldsymbol{\pi})=\left(\widetilde{\theta}^a(\boldsymbol{\pi}),\widetilde{\theta}^b(\boldsymbol{\pi})\right)$. As in the basic model, continuity of $\widetilde{\boldsymbol{\theta}}$ follows from the implicit function theorem. To economize on notation we will let $\widetilde{r}(\boldsymbol{\pi})$ denote the factor ratio implied by $\widetilde{\boldsymbol{\theta}}(\boldsymbol{\pi})$, that is

$$\widetilde{r}(\boldsymbol{\pi})=\frac{\sum_{j=a,b}\lambda^j\pi^j\left(1-F_q\left(\widetilde{\boldsymbol{\theta}}(\boldsymbol{\pi})\right)\right)}{\sum_{j=a,b}\lambda^j\left(\pi^jF_q\left(\widetilde{\boldsymbol{\theta}}(\boldsymbol{\pi})\right)+\left(1-\pi^j\right)F_u\left(\widetilde{\boldsymbol{\theta}}(\boldsymbol{\pi})\right)\right)} \tag{19}$$

As in the basic model each worker is paid according to his expected marginal productivity in equilibrium, and by the assumption of constant returns the wage schedules can be written as:

$$w^j(\theta)=\begin{cases} y_2\left(\widetilde{r}(\boldsymbol{\pi}),1\right) & \text{for } \theta<\widetilde{\theta}^j(\boldsymbol{\pi}) \\ p\left(\theta,\pi^j\right)y_1\left(\widetilde{r}(\boldsymbol{\pi}),1\right) & \text{for } \theta\geq\widetilde{\theta}^j(\boldsymbol{\pi})\end{cases} \tag{20}$$

Finally, the fraction of agents in group j who invest can be found in the same way as before. Given a wage schedule w^j for group j the fraction of agents who optimally decides to invest is given by (13). The gross benefits of investment given any investment behavior $\boldsymbol{\pi}$ is

$$\begin{aligned} B^j(\boldsymbol{\pi}) &= y_2\left(\widetilde{r}(\boldsymbol{\pi}),1\right)\left(F_q\left(\widetilde{\theta}^j(\boldsymbol{\pi})\right)-F_u\left(\widetilde{\theta}^j(\boldsymbol{\pi})\right)\right)+ \\ &\quad +y_1\left(\widetilde{r}(\boldsymbol{\pi}),1\right)\int_{\widetilde{\theta}^j(\boldsymbol{\pi})}^{1}p\left(\theta,\pi^j\right)\left(f_q(\theta)-f_u(\theta)\right)d\theta \end{aligned} \tag{21}$$

and the relevant system of fixed-point equations is

$$\pi^j = G\left(B^j\left(\boldsymbol{\pi}\right)\right), \text{ for } j = a, b. \tag{22}$$

The characterization results from the single group model generalize in a straightforward way, so the equilibrium set will be fully characterized as the solutions to (22). For expositional convenience we will therefore work directly with the reduced form equations. Given any solution to (22) we can always use (17) and (20) to construct the implied equilibrium wage schedules and task assignment rules.

We will say that an equilibrium is *discriminatory* if $\widetilde{\theta}^a\left(\boldsymbol{\pi}\right) \neq \widetilde{\theta}^b\left(\boldsymbol{\pi}\right)$ or $w^a \neq w^b$ and *non-discriminatory* otherwise. Here it is important to realize that $\widetilde{\theta}^j\left(\boldsymbol{\pi}\right)$ as well as the wage schedule for group j depends on investment behavior in both groups. The reason is that the fraction of investors in the other group affects how scarce a resource qualified workers are and firms will therefore take investment behavior in both groups into consideration when deciding on the task assignments for any of the groups. This implies that the fraction of investors in the other group affects the benefits of investors, both by the effect on the cutoff signal and by affecting the factor ratio. In fact one can show that an increase in the fraction of investors in the other group monotonically decreases the incentives to invest, so investment in the two groups are "aggregate strategic substitutes". As a consequence of these interdependencies the set of equilibria of the extended model will **not** be the set of possible permutations of the equilibria of the single group model.

The set of *non-discriminatory* equilibria corresponds one to one with the set of equilibria of the single group model: the equilibrium conditions of the extended model reduces to the equilibrium conditions of the single group model when it is imposed that both groups are treated symmetrically. Combining this simple observation with Proposition 3, it follows that there exists at least one non-trivial non-discriminatory equilibrium in the extended model.

We are mainly interested in discriminatory equilibria and as the next proposition shows at least one such equilibrium will exist under the assumption that not too many workers derive positive utility from investment in human capital.

Proposition 4 *Let y be a given production function and let f_q, f_u be some fixed densities, where y, f_q and f_u satisfies the assumptions stated in Section 2.1 and let $\left(\lambda^a, \lambda^b\right) \in int\left(\Delta^2\right)$. Then, there exists $\overline{G}_0 > 0$ such that if $G(0) \leq \overline{G}_0$, then there exists an equilibrium where no workers from one of the groups are assigned to the complex task and a positive fraction of the agents from the other group are assigned to the complex task. Moreover, in this equilibrium the wage schedule for the group where all workers are assigned to the simple task is uniformly below the wage schedule for the other group.*

The construction is in the appendix. Assuming that all agents in, say, group a, are assigned to the simple task the equilibrium conditions for the other group are qualitatively as in the single group model. Hence, applying the same steps as in the proof of Proposition 3 we have that there is an equilibrium where a fraction $\pi^b > G(0)$ of the agents in the other group invests and a positive fraction of these are assigned to the complex task, *assuming* that no agents from the discriminated group are assigned to the complex task. Let the implied cutoff for group b be given by $\widetilde{\theta}^b$. In order to check that this is an equilibrium of the model with two groups we just have to check that $\left(1, \widetilde{\theta}^b\right)$satisfies the Kuhn-Tucker conditions for the problem (17) when π^b is given as above and $\pi^a = G(0)$. This is indeed the case given that a sufficiently small fraction of agents have negative costs of investment.

Observe that for $\theta^a = 1$ to satisfy (18) it must be that $p(1, \pi^a) \leq y_2(\cdot)/y_1(\cdot)$. Since an increase in λ^a decreases the factor ratio the right hand of this inequality is decreasing in λ^a and it follows that $\overline{G}_0$ is strictly decreasing in λ^a. Thus, the larger the group is the more difficult it is to sustain this extreme form of discrimination against its members.

An alternative sufficient condition for existence of discriminatory equilibria is existence of multiple equilibria in the single group model. As a general property of discriminatory equilibria, it can be shown that there exists *some* equilibrium in the single group model such that any agent is better off than an agent in the disadvantaged group with the same investment costs c. The discriminated group is thus always better off in *some* "autarchic equilibrium".

5 Affirmative Action

In this section we will use our framework to analyze the effects of affirmative action, which we model as a quota forcing the employers to fulfill certain requirements on the representation of workers from the disadvantaged group in both tasks.

A more natural intervention would perhaps be an equal opportunities law requiring firms to offer wages that do not depend on group identity. In our simple framework this would mean that the firms would be constrained to offer identical wage schedules to both groups. Since the incentives to invest would be the same for both groups this would eliminate discrimination in our model.

The problem with this type of equal opportunity law is that the regulator must observe all information the employer has available in order to implement such a policy. In a more realistic setup where there are other variables than a one-dimensional signal, this type of policy would also be possible to evade by using other variables as proxies for group identity. Moreover, in reality hiring decisions are based on several factors that may or may not be an indicator of the expected productivity of the worker. In particular if there are other variables correlated both with group identity and intrinsic productivity it may be impossible for the regulatory authorities to disentangle what part of the correlation is "real" and what is due to statistical discrimination[17].

[17]See Lundberg [8] for a discussion on these issues and some interesting suggestions on statistical procedures the regulator could use in order to implement equal opportunities laws when there is asymmetric information between the firms and the regulator

Also observe that if the principle of "equal pay for equal work" is interpreted to mean that the average wage for workers *performing a particular task* cannot differ across groups then the discriminatory equilibrium constructed in the proof of Proposition 6 satisfies this principle. All workers in the simple task are paid the same wage and no workers in the discriminated group are assigned to the complex task, so "wage equality" in what seems to us to be the standard operational sense holds.

5.1 The Model with Affirmative Action

For simplicity we will model affirmative action as a requirement for each firm to hire workers for each task in accordance with the population fractions[18]. The quantity of workers (qualified and unqualified) from group j employed in the complex task by firm i given that a fraction π^j has invested and the actions $A = \left\langle w_i^a, w_i^b, t_i^a, t_i^b \right\rangle_{i=1,2}$ is given by

$$\Psi_i^j (A) = \int I^1_{\left\langle w_i^j, w_{i'}^j \right\rangle} (\theta)\, t_i^j (\theta)\, \lambda^j f_{\pi^j} (\theta)\, d\theta \tag{23}$$

There is no distinction between the quantity of $j-$workers and the input of labor in the simple task. This quantity, $S_i^j (A)$, is therefore computed according to equation (16) in Section 4 and the affirmative action requirement is

$$\frac{\Psi_i^a (A)}{\Psi_i^b (A)} = \frac{\lambda^a}{\lambda^b} \text{ and } \frac{S_i^a (A)}{S_i^b (A)} = \frac{\lambda^a}{\lambda^b} \tag{24}$$

The payoffs as functions of actions and the timing of the actions are as before and the only difference compared to the model in Section 4 is that the task assignment rule chosen in the final stage of the game must satisfy the affirmative action constraint[19]. Since we think of the affirmative action policy as a *constraint* on the available actions we should in principle allow the task assignments to be contingent on π^a and π^b and adjust the strategy sets accordingly. However, for the same reasons as earlier we can focus on *Nash equilibria* without any risk of analyzing equilibria supported by non credible threats. In particular, since no worker can affect the fraction of investors it is immaterial if we view the "equilibrium responses" as dynamic reactions to the behavior of the workers or fictitious best responses.

As in the single group model the first step in the equilibrium characterization is to note that both firms must offer wage schedules that are identical almost everywhere in any equilibrium of

[18] We can handle quotas with other numerical goals, but as we will discuss later it is important to have a quota for both tasks.

[19] One could alternatively keep the strategy sets as before and impose affirmative action by *charging penalties* to any firm that violates the numerical goals on employment stipulated by the policymaker. If the penalty for a violation is sufficiently costly the two approaches are equivalent.

the model. By the monotone likelihood ratio property, the task assignment rules must be a pair of cutoff rules. Using this fact we can characterize the optimal task assignment rule after any history where both firms have offered (essentially) the same wage schedules by solving the problem

$$\begin{array}{ll} \max_{\theta^a,\theta^b} & y\left(\sum_{j=a,b}\lambda^j\pi^j\left(1-F_q\left(\theta^j\right)\right),\sum_{j=a,b}\lambda^j\left(\pi^jF_q\left(\theta^j\right)+\left(1-\pi^j\right)F_u\left(\theta^j\right)\right)\right) \\ \text{s.t} & \pi^aF_q\left(\theta^a\right)+\left(1-\pi^a\right)F_u\left(\theta^a\right)=\pi^bF_q\left(\theta^b\right)+\left(1-\pi^b\right)F_u\left(\theta^b\right) \end{array}. \tag{25}$$

This problem is just adding a constraint to the task assignment problem (17) in Section 4. As is easily verified this constraint is the affirmative action requirement (24) for the special case when the wage schedules are the same and the task-assignment rules are taken to have the cutoff property, which as we have argued must be properties of equilibrium.

The first-order conditions[20] for this problem can after some rearranging be written as[21];

$$\begin{array}{l} -y_1\left(\cdot\right)p\left(\theta^a,\pi^a\right)+y_2\left(\cdot\right)-\frac{\mu}{\lambda^a}=0 \\ -y_1\left(\cdot\right)p\left(\theta^b,\pi^b\right)+y_2\left(\cdot\right)+\frac{\mu}{\lambda^b}=0 \end{array} \tag{26}$$

where $\mu>0$ if $\pi^a<\pi^b$ (see Footnote 21). Using the constraint, the multiplier and one of the decision variables can be eliminated and the remaining equation has all the qualitative properties of (8). By arguments more or less identical to the ones used in the single group model one can show that for each $\boldsymbol{\pi}=\left(\pi^a,\pi^b\right)$ such that either π^a or π^b is strictly positive there is a unique $\widehat{\boldsymbol{\theta}}\left(\boldsymbol{\pi}\right)=\left(\widehat{\theta}^a\left(\pi^a,\pi^b\right),\widehat{\theta}^b\left(\pi^a,\pi^b\right)\right)>>0$ that solves (25) and that the implicit function theorem applies[22]. The solution will consequently be a smooth function of $\boldsymbol{\pi}$. To write things down more compactly below will introduce one additional piece of notation. For each $\boldsymbol{\pi}$, we denote by $\widehat{r}\left(\boldsymbol{\pi}\right)$ the unique factor ratio implied by the firms optimal choice of task assignment rules, that is

$$\widehat{r}\left(\boldsymbol{\pi}\right)=\frac{\sum_{j=a,b}\lambda^j\pi^j\left(1-F_q\left(\widehat{\theta}^j\left(\boldsymbol{\pi}\right)\right)\right)}{\sum_{j=a,b}\lambda^j\left(\pi^jF_q\left(\widehat{\theta}^j\left(\boldsymbol{\pi}\right)\right)+\left(1-\pi^j\right)F_u\left(\widehat{\theta}\left(\boldsymbol{\pi}\right)\right)\right)}. \tag{27}$$

While the characterization of equilibrium task-assignment rules is not significantly harder than in the single group model the determination of wages is somewhat counterintuitive. It is tempting

[20]It is straightforward to show that the first-order conditions are sufficient and that there is a unique solution to the program (25).

[21]It is useful to note that the solution(s) to the problem with the affirmative action restriction formulated as an equality constraint must also be solution(s) to the problem with the same objective and the inequality constraint $\pi^aF_q\left(\theta^a\right)+\left(1-\pi^a\right)F_u\left(\theta^a\right)\leq\pi^bF_q\left(\theta^b\right)+\left(1-\pi^b\right)F_u\left(\theta^b\right)$ if $\pi^a<\pi^b$ and the reverse inequality if $\pi^a>\pi^b$. Hence, if the multiplier is taken to be positive it enters with a negative sign for the group with the smaller number of investors and a positive sign for the other group.

[22]The proofs follows the proofs of Lemma I and Lemma II in appendix B step by step.

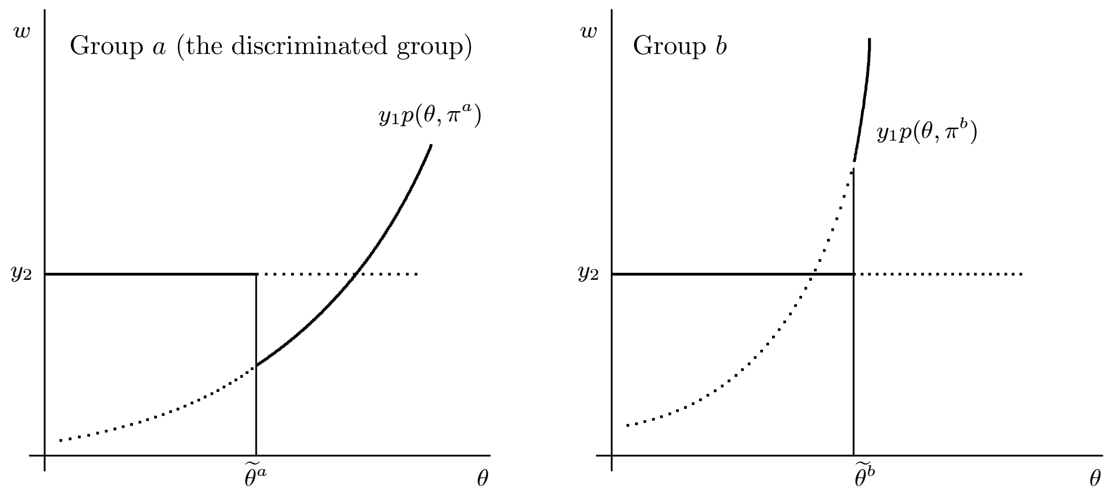

Figure 3: Not an equilibrium

to guess that wages still are given by expected marginal productivities, that is to take (20) as the candidate equilibrium wage function, using the unique cut-off points determined above. This is however not consistent with equilibrium since (assuming $\pi^a \neq \pi^b$) some agents of the discriminated group employed in the complex tasks would be paid less than other agents from the same group who are in the simple task (see Figure 3). Hence a firm could deviate and attract all these workers and replace some of the workers previously in the simple task by the additional workers the firm attracts. If the affirmative action constraint was satisfied prior to the deviation it will be satisfied after the deviation as well and output is unchanged. Since the wage bill has decreased this is a profitable deviation.

The unique wage schedules consistent with equilibrium are depicted in Figure 4, which is drawn under the assumption that $\pi^a < \pi^b$. As can be seen from the graph the wage in the simple task is now determined by the marginal agent's productivity in the complex task rather than the productivity in the simple task. Intuitively, there is no incentive to deviate in order to change the allocation of workers in the complex task for any of the firms: all workers already employed in the complex task are paid their expected marginal productivities and all workers employed in the simple task are paid more than their expected marginal productivity would be if in the complex task.Furthermore, by the affirmative action constraint the firms cannot change the ratio of $a-$workers to $b-$workers in the simple task. This means that what is left to show in order to demonstrate that the proposed

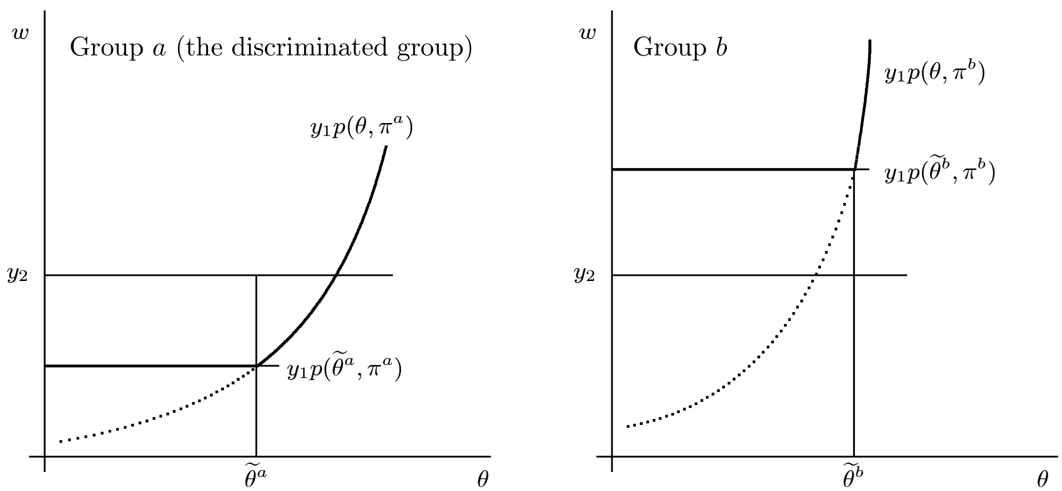

Figure 4: Equilibrium wage schedules under affirmative action

wage schedules are consistent with equilibrium is that the *average wage* in the simple task equals the marginal productivity. To realize this it is useful to note that by combining the two equations in (26) we see that for the optimal choice of $\widehat{\theta}$ the weighted average of the expected marginal productivities in the complex task for the critical agents' in respective group equals the marginal productivity in the simple task. That is, eliminating the multiplier from (26) we have

$$y_1\left(\widehat{r}\left(\boldsymbol{\pi}\right),1\right)\sum_{j=a,b}\lambda^j p\left(\widehat{\theta}^j\left(\boldsymbol{\pi}\right),\pi^j\right)=y_2\left(\widehat{r}\left(\boldsymbol{\pi}\right),1\right) \tag{28}$$

At this point it is simply to note that $F_{\pi^a}\left(\widehat{\theta}^a\left(\boldsymbol{\pi}\right)\right)=F_{\pi^b}\left(\widehat{\theta}^b\left(\boldsymbol{\pi}\right)\right)$ by the affirmative action constraint, so the left hand side of (28) is also the average wage in the simple task, which gives the result.

In terms of the notation introduced above we can write the proposed labor market equilibrium wage schedules depicted in Figure 4 as

$$w^j\left(\theta\right)=\begin{cases} y_1\left(\widehat{r}\left(\boldsymbol{\pi}\right),1\right)p\left(\widehat{\theta}^j\left(\boldsymbol{\pi}\right),\pi^j\right) & \text{for } \theta<\widehat{\theta}^j\left(\boldsymbol{\pi}\right) \\ y_1\left(\widehat{r}\left(\boldsymbol{\pi}\right),1\right)p\left(\theta,\pi^j\right) & \text{for } \theta\geq\widehat{\theta}^j\left(\boldsymbol{\pi}\right)\end{cases}. \tag{29}$$

for $j=a,b$. Note that if $\pi^a<\pi^b$ then $\widehat{\theta}^a\left(\boldsymbol{\pi}\right)<\widehat{\theta}^b\left(\boldsymbol{\pi}\right)$, since otherwise the affirmative action constraint can not be satisfied. Hence $p\left(\widehat{\theta}^a\left(\boldsymbol{\pi}\right),\pi^a\right)<p\left(\widehat{\theta}^b\left(\boldsymbol{\pi}\right),\pi^b\right)$ and by inspection of (26) we

see that $w^a(\theta) < y_2(\widehat{r}(\boldsymbol{\pi}), 1)$ for $\theta < \widehat{\theta}^a(\boldsymbol{\pi})$ and $w^b(\theta) > y_2(\widehat{r}(\boldsymbol{\pi}), 1)$ for $\theta < \widehat{\theta}^b(\boldsymbol{\pi})$. That is, workers in the simple task from group a are paid less than their marginal productivity in the task and workers from group b are paid more.

Summing up the discussion above as a proposition:

Proposition 5. *Let the fractions of agents who invest in each group be given by $\boldsymbol{\pi} = \left(\pi^a, \pi^b\right)$ and let $\widehat{\theta}(\boldsymbol{\pi}) = \left(\widehat{\theta}^a(\boldsymbol{\pi}), \widehat{\theta}^b(\boldsymbol{\pi})\right)$ be the unique solution to (26). Furthermore, for $j = a, b$ let $w^j(\cdot)$ be given by (29) and $t^j(\cdot)$ be the cutoff task assignment rule with critical value $\widehat{\theta}^j(\boldsymbol{\pi})$. Finally, for an arbitrary firm strategy profile $\left\langle w_i^a, w_i^b, \xi_i^a, \xi_i^b \right\rangle_{i=a,b}$ let t_i^a and t_i^b be the task assignment rules on the outcome path for firm $i = 1, 2$. Then, both firms are playing best responses if and only if $w_i^j(\theta) = w^j(\theta)$* and *$t_i^j(\theta) = t^j(\theta)$ for $i = 1, 2$ and $j = a, b$ and for almost all $\theta \in [0, 1]$.*

The proof in the appendix fills in some of the details missing in the paragraphs above

One may believe that when we impose affirmative action as a constraint in both tasks, one constraint is really redundant. The reason for this would be that if the affirmative action constraint is satisfied in one of the tasks and if the market clears, which must be the case in equilibrium, then the affirmative action constraint must be satisfied in the other task as well. This is indeed true and it is also true that in order to characterize the equilibrium task assignment rules we need only one of the constraints. The problem is that if there is affirmative action in the complex task only and if the groups behave differently there is no "labor market equilibrium" in the continuation game. To see this it useful to consider Figure 4, where it is easy to see that any deviation where a firm reduces the number of workers from group b is profitable. Hence, since only non-discriminatory (Nash) equilibria remains in the full game one could in principle interpret this as saying that affirmative action works. However, we think that this is taking the notion of equilibrium too far[23].

Since for each $\boldsymbol{\pi}$ there is a unique wage function consistent with the firms playing mutual best responses we can proceed as in the single group model and characterize the equilibrium set as fixed points of a function from $[0,1]^2$ to $[0,1]^2$. The interpretation of this function is as simple as in the single group model. The function computes the fractions of agents in each group who invests as a best response to the wage function implied by any given investment behavior.

Using the wage schedules (29) we can express the expected gross benefits from undertaking the investment for an agent in group j when $\boldsymbol{\pi} = \left(\pi^a, \pi^b\right)$ as

$$\begin{aligned} H^j(\boldsymbol{\pi}) &= y_1(\widehat{r}(\boldsymbol{\pi}), 1)\, p\left(\widehat{\theta}^j(\boldsymbol{\pi}), \pi^j\right)\left(F_q\left(\widehat{\theta}^j(\boldsymbol{\pi})\right) - F_u\left(\widehat{\theta}^j(\boldsymbol{\pi})\right)\right) \\ &\quad + y_1(\widehat{r}(\boldsymbol{\pi}), 1)\int_{\widehat{\theta}^j(\boldsymbol{\pi})}^1 p\left(\theta, \pi^j\right)\left(f_q(\theta) - f_u(\theta)\right) d\theta. \end{aligned} \tag{30}$$

[23] It has been suggested to us to consider a quota in the complex task together with a "civil rights law" prohibiting wage discrimination in the simple task. However, the same nonexistence problem remains under this policy.

Arguing as in the single group model we see easily that $\boldsymbol{\pi} = \left(\pi^a, \pi^b\right)$ is an equilibrium if and only if

$$\pi^j = G\left(H^j\left(\boldsymbol{\pi}\right)\right) \text{ for } j = 1, 2 \tag{31}$$

From these expressions it is easily seen that *any non-discriminatory equilibrium in the extended model is an equilibrium under affirmative action.* This should be fairly obvious since if the groups behave the same way then the employers voluntarily treat the groups identically. To see it formally we observe that the multiplier in the conditions (26) is zero when $\pi^a = \pi^b$, so both equations in (31) reduces to the fixed point equation for the single group model.

If there are asymmetric equilibria under affirmative action, inspection of (29) reveals that the wage schedule for the group with the lower fraction of agents who invests will be uniformly below the wage schedule for the other group. Hence, wage discrimination persists in our model unless the policy forces the economy to an equilibrium where the fractions of agents who invest are the same in both groups. For this reason we will say that an equilibrium is *discriminatory* unless $\pi^a = \pi^b$ and although one could potentially think of alternatives, we say that the *most discriminatory equilibrium* is the equilibrium for which the difference $\left|\pi^a - \pi^b\right|$ is the largest. We observe:

Observation *If $0 < G(0) < \overline{G}_0$ (with $\overline{G}_0$ defined as in Proposition 4) so that the most discriminatory equilibrium $\left(\pi^a, \pi^b\right)$ is such that $\pi^a = G(0)$ and if $\left(\widehat{\pi}^a, \widehat{\pi}^b\right)$ is the most discriminatory equilibrium under affirmative action with $\widehat{\pi}^a < \widehat{\pi}^b$. Then $\widehat{\pi}^a > \pi^a$.*

To see this, observe that if $G(0) > 0$ there must clearly be a positive fraction of agents from both groups who invest in any equilibrium. Hence the wage schedule for both groups will be *strictly* increasing and there will consequently be some (possibly very small) monetary incentives to invest for agents in both groups in any equilibrium under affirmative action.

The fact that affirmative action provides a lower bound on the fraction of investors in the discriminated group that is higher than in the "worst" equilibrium without affirmative action does not help us if we want to analyze the consequences of introducing affirmative action in general. In particular, it provides no guidance at all if we want to say something about the effects of affirmative action starting from a situation where there is discrimination, but where some agents from both groups are employed in both tasks.

If all equilibria under affirmative action would be non-discriminatory the situation would be different. Hence, it would be desirable to have some sufficient conditions under which affirmative action always eliminates the possibility of discriminatory equilibria. Intuitively, affirmative action makes it harder to sustain discrimination since it pushes up wages in the simple task for workers from the group with the higher fraction of investors and pushes down wages in the simple task for the discriminated group. However, as we show next, any such sufficient conditions must involve stringent restrictions on the distribution of investment costs.

Proposition 6 *Let y be a given production function and let f_q, f_u be some fixed densities, where y, f_q and f_u satisfies the assumptions stated in Section 2.1. Furthermore fix any $\left(\lambda^a, \lambda^b\right) \in int\left(\Delta^2\right)$. Then there exists some strictly increasing distribution function G with $G(0) > 0$ such that the model with affirmative action has an equilibrium $\left(\widehat{\pi}^a, \widehat{\pi}^b\right)$ with $\widehat{\pi}^a < \widehat{\pi}^b$.*

Proof. Suppose $\pi^a = 0$ and $0 < \pi^b < 1$. Then optimality conditions for the problem (25) can in this case be written as

$$\begin{array}{c} y_2(\cdot) - \frac{\mu}{\lambda^a} = 0 \\ -y_1(\cdot)\, p\left(\theta^b, \pi^b\right) + y_2(\cdot) + \frac{\mu}{\lambda^b} = 0 \end{array} \tag{32}$$

we observe that the unique solution $\left(\widehat{\theta}^a, \widehat{\theta}^b\right)$ must still be interior. Using (30) we also see that $H^a\left(0, \pi^b\right) = 0 < H^b\left(0, \pi^b\right)$. It is straightforward to verify that H^j is continuous at $\left(0, \pi^b\right)$ for $j = a, b$ and it follows that there must exist some $\left(\widehat{\pi}^a, \widehat{\pi}^b\right)$ where $0 < \widehat{\pi}^a < \widehat{\pi}^b$ and, since H^a is initially increasing, $0 < H^a\left(\widehat{\pi}^a, \widehat{\pi}^b\right) < H^b\left(\widehat{\pi}^a, \widehat{\pi}^b\right)$. There must therefore exist some strictly increasing function G such that $G(0) > 0$, $G\left(H^a\left(\widehat{\pi}^a, \widehat{\pi}^b\right)\right) = \widehat{\pi}^a$ and $G\left(H^b\left(\widehat{\pi}^a, \widehat{\pi}^b\right)\right) = \widehat{\pi}^b$, i.e. $\left(\widehat{\pi}^a, \widehat{\pi}^b\right)$ is an equilibrium in the economy with fundamentals $\left\{y, f_q, f_u, \left(\lambda^a, \lambda^b\right), G\right\}$. ■

The result can be strengthened in several directions. First, it should be clear that we get multiplicity for a generic set of distribution functions. To see this one notes that we can always find an open set U containing $\left(\widehat{\pi}^a, \widehat{\pi}^b\right)$ such that the expected benefits of investment for members in group b exceeds the benefits for members in group a for all $\boldsymbol{\pi} \in U$. Also, the argument only relies on existence of a function G that takes on particular values at a few points which means that assumptions about its curvature will not be enough to get any sufficient conditions for ruling out discriminatory equilibria. For example, one can show that there always exist uniform distributions such that there are discriminatory equilibria in the model. The idea should be clear from the proof above, but a slightly more complicated argument is needed to assure that $G(0) \geq 0$.

5.2 Welfare Effects of Affirmative Action

The purpose of this section is to illustrate that even if starting from the most extreme form of discrimination, the disadvantaged group may or may not be hurt by affirmative action. To show this we will consider simple distribution functions of the form:

$$G(c) = \begin{cases} \pi^a & \text{if } c \leq \overline{c} \\ \pi^b & \text{if } c > \overline{c} \end{cases} \tag{33}$$

It is not difficult to extend the examples to strictly increasing distribution functions, but for analytical simplicity we will not do so.

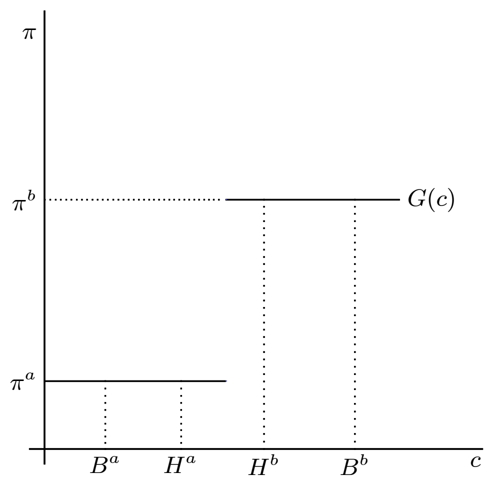

Figure 5:

In the first example it is demonstrated that even if we start from the most extreme form of discrimination introduction of affirmative action may make the disadvantaged group may be worse off. The construction is illustrated in Figure 5.

The idea is to choose $\left(\pi^a, \pi^b\right)$ so that an there is an equilibrium with nobody in group a employed in the complex task. Using the notation from Section 4 we let $B^j\left(\boldsymbol{\pi}\right)$ denote the expected gross benefits of investment in the model without affirmative action. Now, if we fix $\pi^b > 0$ and π^a is sufficiently small the solution to the task assignment problem for the firm is to assign all workers from group a to the simple task, while some workers from group b will be assigned to the complex task. Hence $B^a\left(\pi^a, \pi^b\right) = 0 < B^b\left(\pi^a, \pi^b\right)$ for π^a sufficiently small. Thus, if $\overline{c}$ in (33) is in between 0 and $B^b\left(\pi^a, \pi^b\right)$ we have that $\left(\pi^a, \pi^b\right)$ is an equilibrium in the model without affirmative action. Next we proceed in the spirit of the argument of the proof of Proposition 6 and argue that for π^a small enough and for the right choice of $\overline{c}$ this will also be an equilibrium with affirmative action. To see this we note that $H^a\left(\pi^a, \pi^b\right) \to 0$ as $\pi^a \to 0$ while $H^b\left(\pi^a, \pi^b\right) \to H^b\left(0, \pi^b\right) > 0$ as $\pi^a \to 0$. Hence there exists some $\pi^a > 0$ such that

$$\min\left(B^b\left(\pi^a, \pi^b\right), H^b\left(\pi^a, \pi^b\right)\right) > H^a\left(\pi^a, \pi^b\right) > B^a\left(\pi^a, \pi^b\right) = 0. \tag{34}$$

The desired result follows: choosing $\overline{c}$ in (33) between $H^a\left(\pi^a, \pi^b\right)$ and the minimum of $B^b\left(\pi^a, \pi^b\right)$ and $H^b\left(\pi^a, \pi^b\right)$ we have that $\left(\pi^a, \pi^b\right)$ is an equilibrium both with and without the policy. It is now obvious that *output decreases when affirmative action is introduced* in this case. This follows since the set of feasible production plans with the policy is a strict subset of the feasible plans without the policy and the unique solution without the restriction is not in this subset.

The change in expected utility for an agent of group j who invests[24] is given by the difference in expectation of the wage schedules with respect to f_q:

$$\begin{aligned} \Delta W^j_{INV} \;=\;& y_2\left(\widetilde{r}\left(\boldsymbol{\pi}\right),1\right)F_q\left(\widetilde{\theta}^j\left(\boldsymbol{\pi}\right)\right)+\int_{\widetilde{\theta}^j\left(\boldsymbol{\pi}\right)}^1 y_1\left(\widetilde{r}\left(\boldsymbol{\pi}\right),1\right)p\left(\theta,\pi^j\right)f_q\left(\theta\right)d\theta - \\ & -\left[p\left(\widehat{\theta}^j\left(\boldsymbol{\pi}\right),\pi^j\right)y_1\left(\widehat{r}\left(\boldsymbol{\pi}\right),1\right)F_q\left(\widehat{\theta}\left(\boldsymbol{\pi}\right)\right)+\int_{\widehat{\theta}^j\left(\boldsymbol{\pi}\right)}^1 y_1\left(\widehat{r}\left(\boldsymbol{\pi}\right),1\right)p\left(\theta,\pi^j\right)f_q\left(\theta\right)d\theta\right] \end{aligned} \tag{35}$$

The change in expected utility for agents who do not invest is derived symmetrically. Note that the factor ratio in general changes when the policy is introduced. This effect may go either way depending on the choice of production function and of distribution functions for θ. However, given the way this example is constructed this creates no difficulties. If the factor ratio increases when the policy is introduced we can use (35) directly to show that the expected benefits for *any* agent in group a decreases with the policy. On the other hand, if the factor ratio decreases we cannot say anything in general since the wage under affirmative action for high realizations of θ now may be higher than the (constant) wage in the original discriminatory equilibrium. However, no matter what happens to the factor ratio we can always rely on the fact that under affirmative action $w^a\left(\theta\right)\leq y_1\left(\widehat{r}\left(\boldsymbol{\pi}\right),1\right)p\left(1,\pi^a\right)$ for all θ, so for π^a small enough the policy must decrease the expected utility for all agents in group a. The welfare effect for the other group is ambiguous.

It is also possible to construct examples where the discriminated group gains even if an equilibrium occurs that leaves the group discriminated. To illustrate the point we want to make it is however more straightforward to show that introduction of affirmative action may imply that a symmetric equilibrium is the only possibility.

Consider the benefits of investment if a fraction π^b would invest in *both groups,* $B^j\left(\pi^b,\pi^b\right)$ in the model without the quota and $H^j\left(\pi^b,\pi^b\right)$ in the model with, both which equals the benefits of investment in the single group model if a fraction π^b invests, $H\left(\pi^b\right)$. It is straightforward to show that the benefits of investment for agents in one of the groups is monotonically decreasing in the fraction of investors in the other group, so

$$B^a\left(\pi^a,\pi^b\right)<B^a\left(\pi^b,\pi^b\right)=H\left(\pi^b\right)<B^b\left(\pi^a,\pi^b\right)$$

As we argued above $B^a\left(\pi^a,\pi^b\right)=0$ for some small enough $\pi^a>0$, so for $\overline{c}$ chosen in between 0 and $B^b\left(\pi^a,\pi^b\right)$ and π^a small enough $\left(\pi^a,\pi^b\right)$ is an equilibrium. But if $\overline{c}$ is chosen in the interval

[24] Since we constructed the equilibria so that the fraction of investors remains the same we do not need to worry about agents who change their behaviour when the policy is introduced.

$\left(0, H^a\left(\pi^a, \pi^b\right)\right)$ then $\left(\pi^a, \pi^b\right)$ is not an equilibrium when affirmative action is introduced. Thus (assuming a is the smaller group we cannot switch to discrimination of the other group) the only remaining equilibrium candidate is one where a fraction of π^b invests in both groups. Since $\overline{c}$ can be chosen so that it is smaller than $H\left(\pi^b\right)$ there is a range of parameter values so that this is indeed an equilibrium. One can show that the wage for agents employed in the simple task will be higher in the symmetric equilibrium and this means that all agents in group a benefits from the policy. Since production increases the other group may or may not be made worse off.

By relatively standard continuity arguments both examples can be extended to some strictly increasing distribution functions as well.

We also conjecture that there are circumstances where affirmative action is necessarily a Pareto improvement and other circumstances where removing affirmative action is a Pareto improvement, but we have not been able to show this yet.

6 Discussion

We believe that our model captures important aspects of how discrimination may be sustained in the real world: when few workers of a particular group invest in their skills, the firms will tend not to promote these workers to higher paid more qualified jobs. This in turn suggests that the incentives to invest in human capital should be lower for agents from a group where few workers invest in their skills than for agents from a group where more workers invest. Hence discrimination as a consequence of self-fulfilling expectations seems like a plausible explanation for differences in labor market performance between groups.

Multiple equilibrium explanations of discrimination (as well as of other economic phenomena) are often criticized on the grounds that the model gives no prediction. Our model is also vulnerable to this type of criticism since it does not give a unique prediction for any fixed fundamentals. However, when we combine the logic of self-confirming expectations with factor complementarity this problem becomes less severe since the model has some implications about the relation between relative group size and possibilities for discrimination. In this context we again want to stress that since group size matters in the determination of discriminatory equilibria we cannot take an arbitrary discriminatory equilibrium and construct a new equilibrium by reversing the roles of the groups.

Since we are explicitly taking competitive forces on the labor market into consideration our model is a natural framework to analyze the consequences of anti-discriminatory policies. In this paper, we focus on affirmative action. The specific way we model it is subject to criticism since one would more naturally require quotas only on the skilled job, rather than in both jobs. Unfortunately, a quota in the skilled job only implies non-existence of "labor market equilibria" in continuation games where the two groups behave differently. Besides this technical aspect, we do not think that modelling affirmative action as a quota is particularly problematic. We already pointed out that we could as well impose a penalty on employers not conforming to a specific requirement and that a sufficiently stiff penalty would give the same results as our policy gives. The quotas we are considering also have the attractive feature that they are possible to implement under rather weak assumptions about what is observable to the policymaker.

One can think of alternative anti-discriminatory policies, such as different kinds of subsidies, that are feasible under the same informational assumptions that are needed in order to implement employment quotas. In a sequel to this paper we intend to compare the effects of quotas of the type considered in this paper and different subsidies and to analyze optimal policies under different informational assumptions.

A Appendix: Proof of Proposition 1

(sufficiency) Suppose that one of the firms would deviate from the proposed equilibrium strategies and play some arbitrary strategy $\langle w_i', \xi_i' \rangle$ so that the actions on the implied *outcome path* $\langle w_i', t_i' \rangle$ is different from $\langle w, t \rangle$ given by (11) and (6) on a set of positive measure (in principle both firms could be playing according to the characterization and one firm could deviate by offering a wage schedule different only on a finite set of points and this way trigger the other firm to react by changing the task assignment rule. However, such a deviation would not change profits for the deviator, which is why we without loss of generality can assume that the *actions* by the deviator is changed on a set of positive measure). Define the following sets: $\Theta^h = \{\theta : w'(\theta) > w(\theta)\}$, $\Theta^l = \{\theta : w'(\theta) < w(\theta)\}$, $\Theta^e = \{\theta : w'(\theta) = w(\theta)\}$. For ease of notation let C' and S' denote the implied factor inputs for the deviating firm given that the other firm plays according to the proposed equilibrium strategies. Using (4) we see that these quantities can be expressed as:

$$\begin{aligned} C' &= \int_{\theta\in\Theta^h} t'(\theta)\,\pi f_q(\theta)\,d\theta + \frac{1}{2}\int_{\theta\in\Theta^e} t'(\theta)\,\pi f_q(\theta)\,d\theta \\ S' &= \int_{\theta\in\Theta^h} \left(1 - t'(\theta)\right) f_\pi(\theta)\,d\theta + \frac{1}{2}\int_{\theta\in\Theta^e} \left(1 - t'(\theta)\right) f_\pi(\theta)\,d\theta \end{aligned} \tag{36}$$

where $f_\pi(\cdot)$ denotes the density where $f_\pi(\theta) = \pi f_q(\theta) + (1-\pi) f_u(\theta)$ for each $\theta \in [0,1]$. Using the definition of the profit function (5) and the allocation rule (3) we can express the profits for the deviating firm as;

$$\Pi^i_{dev} = y\left(C', S'\right) - \int_{\theta\in\Theta^h} w'(\theta) f_\pi(\theta)\,d\theta - \frac{1}{2}\int_{\theta\in\Theta^e} w(\theta) f_\pi(\theta)\,d\theta \tag{37}$$

Let $C, S > 0$ be the implied factor inputs if both firms are playing according to the equilibrium strategies. By concavity of y and Euler's theorem it follows that

$$\Pi^i_{dev} \leq y_1(C,S)\,C' + y_2(C,S)\,S' - \int_{\theta\in\Theta^h} w'(\theta) f_\pi(\theta)\,d\theta - \frac{1}{2}\int_{\theta\in\Theta^e} w(\theta) f_\pi(\theta)\,d\theta \tag{38}$$

From the definition of $p(\theta,\pi)$ we have that $\pi f_q(\theta) = p(\theta,\pi) f_\pi(\theta)$. Furthermore we have from the definition of the proposed equilibrium wage schedule (11) that $w(\theta) = \max\{y_1(C,S)\,p(\theta,\pi), y_2(C,S)\}$. Some algebraic manipulations using these equalities gives;

$$y_1(C,S)\,C' \leq \int_{\theta\in\Theta^h} t'(\theta)\,y_1(C,S)\,p(\theta,\pi) f_\pi(\theta)\,d\theta + \frac{1}{2}\int_{\theta\in\Theta^e} t'(\theta)\,y_1(C,S)\,p(\theta,\pi) f_\pi(\theta)\,d\theta \tag{39}$$

$$y_2(C,S)\,S' \quad \leq \quad \int\limits_{\theta\in\Theta^h} \left(1-t'(\theta)\right) w(\theta)\, f_\pi(\theta)\, d\theta + \frac{1}{2} \int\limits_{\theta\in\Theta^e} \left(1-t'(\theta)\right) w(\theta)\, f_\pi(\theta)\, d\theta \tag{40}$$

Summing over these inequalities we see that

$$y_1(C,S)\,C' + y_2(C,S)\,S' \leq \int\limits_{\theta\in\Theta^h} w(\theta)\, f_\pi(\theta)\, d\theta + \frac{1}{2} \int\limits_{\theta\in\Theta^e} w(\theta)\, f_\pi(\theta)\, d\theta, \tag{41}$$

and by substituting this into (38) we get

$$\Pi^i_{dev} \leq \int_{\theta\in\Theta^h} \left(w(\theta) - w'(\theta)\right) f_\pi(\theta)\, d\theta, \tag{42}$$

and since $w'(\theta) > w(\theta)$ for all $\theta \in \Theta^h$ this means that no deviation earns positive profits (we can not conclude that a deviation leaves the deviator strictly worse off since there are deviations that "scale down" production that also gives zero profits). Since the deviation was arbitrary this completes the proof of the sufficiency part of Proposition 1. ■

The necessity part of Proposition 1 will be proved by using a sequence of intermediate results:

Lemma I *Suppose $\langle w_i, \xi_i\rangle_{i=1,2}$ is a pair of best responses. Then $w_1(\theta) = w_2(\theta)$ for almost all $\theta \in [0,1]$*

Proof. Suppose $\langle w_i, \xi_i\rangle_{i=1,2}$ are best responses and that there exists a set $\Theta \subseteq [0,1]$ positive measure such that $w_i(\theta) - w_j(\theta) > 0$ for all $\theta \in \Theta$. Consider a deviation $\langle w_i', \xi_i'\rangle$ such that $\xi_i'(w_i', w_j)(\theta) = \xi_i(w_i, w_j)(\theta)$ for all θ, $w_i'(\theta) = w_i(\theta)$ for $\theta \in [0,1]\setminus\Theta$ and $w_i'(\theta) = (w_i(\theta) + w_j(\theta))/2$ for $\theta \in \Theta$. We notice that the deviation leaves the distribution of available workers unchanged and since the task assignment rule on the outcome path also is unchanged this means that output is unchanged. The difference in expected profits is then simply the difference in the total wage bill, i.e.

$$\Delta\Pi^i_{dev} = \int\limits_{\theta\in\Theta} \frac{w_i(\theta) - w_j(\theta)}{2} \left(\pi f_q(\theta) + (1-\pi)\, f_u(\theta)\right) d\theta > 0 \tag{43}$$

which contradicts the hypothesis that $\langle w_i, \xi_i\rangle_{i=1,2}$ is a pair of best responses. ■

Lemma II *Let t_i denote the implied task assignment rule on the equilibrium path for firms $i = 1,2$. Then there exists some $\widetilde{\theta}^i \in (0,1)$ such that $t_i(\theta) = 1$ for almost all $\theta > \widetilde{\theta}^i$ and $t_i(\theta) = 0$ for almost all $\theta < \widetilde{\theta}^i$ and for $i = 1,2$.*

Proof. By Lemma I, $w_i(\theta) = w_j(\theta)$ for almost all θ, so $I^i_{\langle w_1,w_2\rangle}(\theta) = \frac{1}{2}$ for almost all θ. It follows that if t_i must satisfy:

$$\begin{array}{rl} t_i(\cdot) \in & \arg\max\limits_{t(\cdot)} y(c_i, s_i) \\ \text{subj. to} \quad C_i = & \int t(\theta)\, \pi f_q(\theta)\, d\theta \\ S_i = & \int (1-t(\theta))\, f_\pi(\theta)\, d\theta \end{array} . \tag{44}$$

For a contradiction, suppose that the claim is false. Then there are sets $\Theta^h, \Theta^l \subseteq [0,1]$ with positive measure such that $\theta^h > \theta^l$ for all $\theta^h, \theta^l \in \Theta^h \times \Theta^l$, but $t_i\left(\theta^h\right) = 0$ for all $\theta^h \in \Theta^h$ and $t_i\left(\theta^l\right) = 1$ for all $\theta^l \in \Theta^l$. Since f_q and f_u are continuous the mixture f_π is continuous as well and we may therefore without loss of generality assume that $\int_{\theta\in\Theta^h} f_\pi(\theta)\, d\theta = \int_{\theta\in\Theta^l} f_\pi(\theta)\, d\theta > 0$. Consider the alternative task-assignment rule,

$$t_i^a(\theta) = \begin{cases} 1 & \text{if } \theta \in \Theta^h \\ 0 & \text{if } \theta \in \Theta^l \\ t_i(\theta) & \text{otherwise} \end{cases} \tag{45}$$

Let S_i^a and C_i^a be the factor inputs implied by t_i^a and let S_i and C_i be the inputs given the rule t_i. Since $\int_{\theta \in \Theta^h} f_\pi(\theta)\, d\theta = \int_{\theta \in \Theta^l} f_\pi(\theta)\, d\theta$ it follows that $S_i^a = S_i$, the input of simple labor is unchanged. Since the deviation assigns to the complex task workers who are productive with a higher probability it is rather obvious that $C_i^a > C_i$. To see this formally we note that

$$C_i^a = \frac{\pi}{2}\left[\int_{\theta \in \Theta^h} f_q(\theta)\, d\theta - \int_{\theta \in \Theta^l} f_q(\theta)\, d\theta\right] + C_i. \tag{46}$$

Suppose $C_i^a \leq C_i$, which by (46) implies $\int_{\theta \in \Theta^h} f_q(\theta)\, d\theta \leq \int_{\theta \in \Theta^l} f_q(\theta)\, d\theta$. Let $l(\theta)$ denote the likelihood ratio $f_q(\theta)/f_u(\theta)$ and rewrite this inequality as

$$\int_{\theta \in \Theta^h} l(\theta)\, f_u(\theta)\, d\theta \leq \int_{\theta \in \Theta^l} l(\theta)\, f_u(\theta)\, d\theta, \tag{47}$$

which since $l\left(\theta^h\right) > l\left(\theta^l\right)$ for all θ^h, θ^l implies that $\int_{\theta \in \Theta^h} f_u(\theta)\, d\theta \leq \int_{\theta \in \Theta^l} f_u(\theta)\, d\theta$. But then

$$\begin{aligned} \int_{\theta \in \Theta^h} f_\pi(\theta)\, d\theta &= \pi \int_{\theta \in \Theta^h} f_q(\theta)\, d\theta + (1-\pi) \int_{\theta \in \Theta^h} f_u(\theta)\, d\theta < \\ &< \pi \int_{\theta \in \Theta^l} f_q(\theta)\, d\theta + (1-\pi) \int_{\theta \in \Theta^l} f_u(\theta)\, d\theta = \int_{\theta \in \Theta^l} f_\pi(\theta)\, d\theta \end{aligned} \tag{48}$$

which contradicts our original assumption. Hence $C_i^a > C_i$ and since $S_i^a = S_i$ this means that output is higher under t_i^a, so t_i could not solve (44). ■

Lemma III *Let t_i denote the implied task assignment rule on the equilibrium path for firms $i = 1, 2$ and let t be defined by (6). Then $t_i(\theta) = t(\theta)$ almost everywhere.*

Proof. By Lemma II the problem of finding an optimal task assignment rule reduces to finding an optimal solution to the programing problem (7) in the main text. Since firms are facing symmetric problems we drop indices and perform a change in variables by defining $C = \pi(1 - F_q(\theta))$ and $S = \pi F_q(\theta) + (1-\pi) F_u(\theta)$. The problem can then be restated as

$$\begin{aligned} &\max_{c,s}\, y(C, S) \\ &\text{subj. to } g(C, S) \equiv \pi - C - S + (1-\pi) F_u\left(F_q^{-1}\left(\tfrac{\pi - C}{\pi}\right)\right) \geq 0 \end{aligned} \tag{49}$$

It is verified that

$$\frac{\partial g(C,S)}{\partial C} = -1 - \frac{1-\pi}{\pi} \frac{f_u\left(F_q^{-1}\left(\frac{\pi-C}{\pi}\right)\right)}{f_q\left(F_q^{-1}\left(\frac{\pi-C}{\pi}\right)\right)} = -1 - \frac{1-\pi}{\pi} \frac{1}{\psi\left(F_q^{-1}\left(\frac{\pi-C}{\pi}\right)\right)} \tag{50}$$

and taking second derivatives we find that $\partial^2 g/\partial C^2 < 0$ while all other elements of the Hessian matrix is zero by the linearity in S. Hence g is concave (one can actually see that $\partial^2 g/\partial C^2 < 0$ without explicitly performing the differentiation since ψ and F_q^{-1} are both strictly increasing). Since y is concave the Kuhn-Tucker conditions are sufficient conditions for a solution to (49) and necessity follows since concavity of g is sufficient for constraint qualification. Invoking the boundary conditions we easily see that any solution to the Kuhn-Tucker conditions must be interior. Since the programs (8) and (49) are equivalent this completes the proof. ■

Lemma IV *Suppose $\langle w_1, w_2 \rangle$ is a pair of equilibrium wage schedules and let $\widetilde{\theta}(\pi)$ be the solution to (6). Then there is a pair (k_s, k_c) such that $w_i(\theta) = k_s$ for $i = 1, 2$ and for almost all $\theta < \widetilde{\theta}(\pi)$ and $w_i(\theta) = p(\theta, \pi) k_c$ for $i = 1, 2$ and for almost all $\theta > \widetilde{\theta}(\pi)$.*

Proof. We will begin by showing that $w_i(\theta) = k_s$ for almost all $\theta < \widetilde{\theta}$. For contradiction assume that there exists sets $\Theta^a, \Theta^b \subseteq \left[0, \widetilde{\theta}(\pi)\right]$ with strictly positive measure such that $w_i(\theta) < k$ for all $\theta \in \Theta^a$ and $w_i(\theta) \geq k$ for all $\theta \in \Theta^b$. By continuity of f_π we may without loss of generality assume that $\int_{\theta \in \Theta^a} f_\pi(\theta) d\theta = \int_{\theta \in \Theta^b} f_\pi(\theta) d\theta > 0$. To show that this is inconsistent with equilibrium we will construct a deviation where the firm replace get rid of some workers that are paid above k and attracts some workers that are being paid a lower wage. Intuitively it is rather clear that this deviation will be profitable as long as the total input of workers in both tasks constant. To show this formally consider the following deviation by firm i

$$w_i'(\theta) = \begin{pmatrix} w_i(\theta) + \epsilon & \text{for } \theta \in \Theta^a \\ 0 & \text{for } \theta \in \Theta^b \\ w_i(\theta) & \text{otherwise} \end{pmatrix} \tag{51}$$

Since input of both factors remains constant under the deviation (given that the task assignment rule is unchanged, which we assume) the difference in payoffs for the deviating firm is just the difference in wage payments, i.e.

$$\Delta(\epsilon) = \frac{1}{2}\left[\int_{\theta \in \Theta^b} w_i(\theta) f_\pi(\theta) d\theta - \int_{\theta \in \Theta^a} (w_i(\theta) + \epsilon) f_\pi(\theta) d\theta\right] \tag{52}$$

Since $\lim_{\epsilon \to 0} \Delta(\epsilon) > 0$ there exists $\epsilon > 0$ such that $\Delta(\epsilon) > 0$. Hence, for ϵ small enough the deviation is profitable.

Symmetrically, suppose there are sets $\Theta^a, \Theta^b \subseteq \left[\widetilde{\theta}(\pi), 1\right]$ with strictly positive measure (where we again w.l.o.g. may assume $\int_{\theta \in \Theta^a} f_q(\theta) d\theta = \int_{\theta \in \Theta^b} f_q(\theta) d\theta$) such that $(w_i(\theta)/p(\theta, \pi)) < k$ for all $\theta \in \Theta^a$ and $(w_i(\theta)/p(\theta, \pi)) \geq k$ for all $\theta \in \Theta^b$. Again we consider a deviation according to (51). Noting that $p(\theta, \pi) = (\pi f_q(\theta)/f_\pi(\theta))$ we find that output is unchanged in this case as well and it is easy to verify by a similar argument as above that the deviation is profitable for ϵ small enough. ■

We now collect the pieces together and prove the necessity part of Proposition 1:

Proof. (*necessity*)**.** By Lemma III it follows that the task assignment rule on any equilibrium path must satisfy $t_i(\theta) = t(\theta)$ for $i = 1, 2$ and almost all θ. Using the notation from the main text and constant returns to scale we have from Lemma IV that in any equilibrium of the model both firms must offer almost identical wage schedules (disregarding sets of measure zero) of the form:

$$w(\theta) = \begin{cases} k_s \\ p(\theta, \pi) k_c \end{cases} \tag{53}$$

It remains to be shown that $k_s = y_2(C, S)$ and $k_c = y_1(C, S)$. By straightforward calculations it can be shown that if $k_s < y_2(C, S)$ and $k_c < y_1(C, S)$ then both firms are making positive profits and a uniform deviation where firm i offers $w_i'(\theta) = w_i(\theta) + \epsilon$ for all θ would be profitable for ϵ small enough. Also, if both inequalities would go the other way and the wages would be uniformly above the candidate equilibrium wage schedule both firms would make strictly negative profits and a deviation to $w_i(\theta) = 0$ for all θ would be profitable. The cases that requires a little work are when the inequalities work in opposite directions.

The two cases can be taken care of with perfectly symmetric arguments we will only consider the case with $k_s > y_2(C, S)$ and $k_c < y_1(C, S)$. The idea behind the construction is illustrated in Figure 6.

Recall that $y_1(C, S)\, p\left(\widetilde{\theta}(\pi), \pi\right) = y_2(C, S)$ by (8). Hence if $k_s > y_2(C, S)$ and $k_c < y_1(C, S)$ then $k_c p\left(\widetilde{\theta}(\pi), \pi\right) < k_s$ and there is an interval $\left(\widetilde{\theta}(\pi), \theta^*\right)$ such that $w_i(\theta) = p(\theta, \pi) k_c < k_s$ for all θ in this interval. The idea behind the construction (see Figure 6) is now simply to demonstrate that it is better to dispose of some of the workers being paid k_s and replace them by cheaper workers from $\left(\widetilde{\theta}(\pi), \theta^*\right)$. While this logic is perfectly simple the formal argument

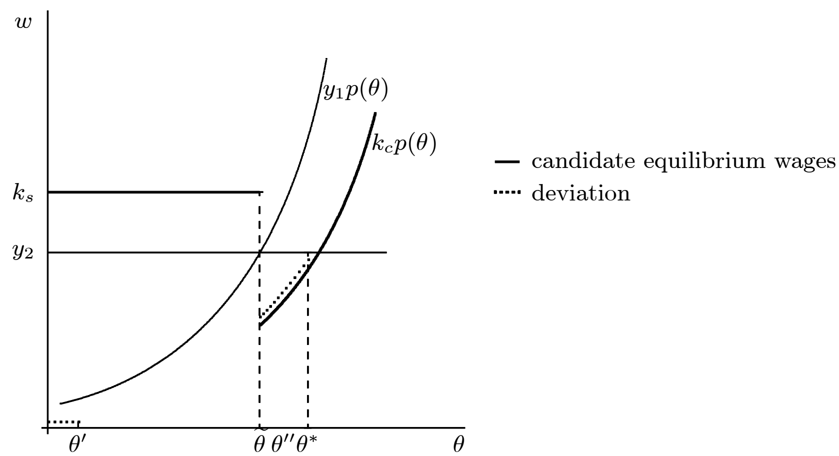

Figure 6: A profitable deviation

below is rather messy. The reason for this being that we need to keep track of the changes in the effective factor inputs as well as changes in the wage bill.

Let θ' solve $F_\pi(\theta') = F_\pi(\theta^*) - F_\pi\left(\widetilde{\theta}(\pi)\right)$ and define θ'' as the solution to $F_\pi(\theta'') - F_\pi\left(\widetilde{\theta}(\pi)\right) = (F_\pi(\theta^*) - F_\pi(\widetilde{\theta}(\pi)))/2$. Consider the following deviation for firm i :

$$w_i'(\theta) = \begin{cases} 0 & \text{for } \theta \in [0, \theta') \\ w_i(\theta) + \epsilon & \text{for } \theta \in [\widetilde{\theta}(\pi), \theta^*) \\ w_i(\theta) & \text{for } \theta \in [\theta^*, 1) \cup [\theta', \widetilde{\theta}) \end{cases} \tag{54}$$

$$t_i'(\theta) = \begin{cases} 0 & \text{for } \theta \in [0, \theta'') \\ 1 & \text{for } \theta \in [\theta'', 1) \end{cases} \tag{55}$$

By construction, the input of simple labor is unchanged (i.e. $1/2F_\pi\left(\widetilde{\theta}(\pi)\right)$ is the input of simple labor before the deviation and since the workers on $[0, \theta')$ will go to the other firm and since all workers on $[\widetilde{\theta}(\pi), \theta'')$ will be in thee firm after the deviation a quantity of $F_\pi(\theta'') - F_\pi\left(\widetilde{\theta}(\pi)\right) + 1/2\left(F_\pi\left(\widetilde{\theta}(\pi)\right) - F_\pi(\theta')\right)$ will be in the simple task after the deviation). The change in effective units of complex labor is given by $C' - C = \frac{\pi}{2}\left(F_q(\theta^*) - 2F_q(\theta'') + F_q\left(\widetilde{\theta}\right)\right)$and using that $F_\pi(\theta^*) - 2F_\pi(\theta'') + F_\pi\left(\widetilde{\theta}\right) = 0$ it is not hard to show that $C' - C > 0$. This should be fairly obvious since the mass of workers assigned to the complex task is unchanged but the average value of θ has increased (the formal argument will be similar to the one used in Lemma II). Thus, output increases under the deviation so the difference in payoffs must be larger than the difference in the wage bill, that is:

$$\begin{aligned}\Delta(\epsilon) &> \tfrac{1}{2}\int_0^1 w_i(\theta) f_\pi(\theta)\, d\theta - \tfrac{1}{2}\int_{\theta'}^{\widetilde{\theta}} w_i(\theta) f_\pi(\theta)\, d\theta - \int_{\widetilde{\theta}}^{\theta^*} (w_i(\theta)-\epsilon) f_\pi(\theta)\, d\theta - \tfrac{1}{2}\int_{\theta^*}^1 w_i(\theta) f_\pi(\theta)\, d\theta = \\ &= \tfrac{1}{2} k_s F_\pi(\theta') - \tfrac{1}{2}\int_{\widetilde{\theta}}^{\theta^*} p(\theta,\pi) k_c f_\pi(\theta)\, d\theta - \int_{\widetilde{\theta}}^{\theta^*} \epsilon f_\pi(\theta)\, d\theta\end{aligned} \tag{56}$$

Recall that $p(\theta,\pi)\, k_c < k_s$ for $\theta \in \left(\widetilde{\theta}, \theta^*\right)$ and that $F_\pi(\theta') = F_\pi(\theta^*) - F_\pi\left(\widetilde{\theta}\right)$ so that:

$$\frac{1}{2} k_s F_\pi\left(\theta'\right) = \frac{1}{2}\int_{\widetilde{\theta}}^{\theta^*} k_s f_\pi(\theta)\, d\theta > \frac{1}{2}\int_{\widetilde{\theta}}^{\theta^*} p(\theta,\pi) k_c f_\pi(\theta)\, d\theta. \tag{57}$$

Hence, $\lim_{\epsilon\to 0}\Delta(\epsilon) > 0$ and there exists $\epsilon > 0$ such that the deviation is profitable. The case with $k_s < y_2(C,S)$ and $k_c > y_1(C,S)$ can be treated symmetrically and Proposition 2 follows. ■

B Appendix: Proof of Proposition 3

Lemma I *Suppose that $y : R_+^2 \to R$ is strictly concave in both arguments and homogenous of degree 1. Then for each $\pi \in (0,1]$ there exists a unique $\widetilde{\theta}(\pi) \in (0,1)$ such that (8) is satisfied.*

Proof. Define $\rho : (0,1)\times(0,1] \to R_+$ by $\rho(\theta,\pi) = \frac{\pi\left(1-F_q(\theta)\right)}{\pi F_q(\theta)+(1-\pi)F_u(\theta)}$ for all $(\theta,\pi) \in (0,1)\times(0,1]$ and let the function $D : (0,1)\times(0,1] \to R$ be defined as

$$D(\theta,\pi) = p(\theta,\pi) - \frac{y_2(\rho(\theta,\pi),1)}{y_1(\rho(\theta,\pi),1)}. \tag{58}$$

Since y is homogenous of degree one $\widetilde{\theta}(\pi)$ solves the first order condition for the task assignment problem (equation (8) in Section 3) if and only if $D(\widetilde{\theta}(\pi),\pi) = 0$. It follows the (strict) monotone likelihood ratio property that $p(\theta,\pi)$ is strictly increasing in θ for any $\pi > 0$. Fixing S , $y_1(C,S)$ is strictly decreasing in C while $y_2(C,S)$ is strictly increasing in C. Since F_q and F_u are strictly increasing $\rho(\theta,\pi)$ is strictly decreasing in θ. Consequently $y_1(\rho(\theta,\pi),1)$ is strictly increasing and $y_2(\rho(\theta,\pi),1)$ is strictly decreasing in θ. Hence, the ratio $y_2(\rho(\theta,\pi),1)/y_1(\rho(\theta,\pi),1)$ is strictly decreasing which implies that $D(\theta,\pi)$ is strictly increasing in θ. Thus, there can be at most one solution $D(\widetilde{\theta}(\pi),\pi) = 0$ and the next task is to show that a solution exists for any $\pi > 0$. We note that $0 < p(0,\pi) < p(1,\pi) < 1$ for any $\pi > 0$. Since F_q and F_u are cdfs it is easy to check that $\lim_{\pi\to 0}\rho(\theta,\pi) = \infty$ and $\lim_{\pi\to 1}\rho(\theta,\pi) = 0$. Using the boundary conditions A2, constant returns to scale and standard limit laws

$$\lim_{\theta\to 0}\frac{y_2(\rho(\theta,\pi),1)}{y_1(\rho(\theta,\pi),1)} = \lim_{\theta\to 0}\frac{y_2\left(1,\frac{1}{\rho(\theta,\pi)}\right)}{y_1(\rho(\theta,\pi),1)} = \frac{\lim_{x\to 0} y_2(1,x)}{\lim_{z\to\infty} y_1(z,1)} = \infty. \tag{59}$$

where the first equality follows from constant returns. Symmetrically we have that $\lim_{\theta\to 1} y_2(\rho(\theta,\pi),1)/y_1(\rho(\theta,\pi),1) = 0$. Combining these facts we see that $\lim_{\theta\to 0} D(\theta,\pi) = -\infty$ and that $\lim_{\theta\to 1} D(\theta,\pi) = p(1,\pi) > 0$. Hence there exists a unique $\widetilde{\theta}(\pi) \in (0,1)$ satisfying $D(\widetilde{\theta}(\pi),\pi) = 0$ for each $\pi > 0$ ■

Lemma II *$\widetilde{\theta}$ and r satisfy the following properties:*

1. $\widetilde{\theta}$ *is continuously differentiable on* $(0,1)$.
2. $\lim_{\pi\to 0+} r(\pi) = 0$
3. r *is monotonically increasing in* π

Proof. 1) It is easy to check that $D_1(\theta,\pi) > 0$ for each $\pi > 0$ which means that the hypotheses of the implicit function theorem is satisfied for each $\widetilde{\theta}(\pi)$.

2) The equation $D\left(\widetilde{\theta}(\pi),\pi\right) = p\left(\widetilde{\theta}(\pi),\pi\right) - \frac{y_2(r(\pi),1)}{y_1(r(\pi),1)} = 0$ must be satisfied for each $\pi > 0$. But $\lim_{\pi\to 0}$ $p\left(\widetilde{\theta}(\pi),\pi\right) \leq \lim_{\pi\to 0} p(1,\pi) = 0$ so $\lim_{\pi\to 0} \frac{y_2(r(\pi),1)}{y_1(r(\pi),1)} = 0$, which implies that $\lim_{\pi\to 0} r(\pi) = 0$.

3) For a contradiction suppose $r(\pi) < r(\pi')$ for $\pi > \pi'$. Since (8) must be satisfied for both π and π' and since the first derivatives of y are decreasing in its own argument it follows that $p\left(\widetilde{\theta}(\pi),\pi\right) < p\left(\widetilde{\theta}(\pi'),\pi'\right)$. But since $p(\theta,\pi)$ is increasing in the second argument (that is, the posterior is increasing in the prior) this means that $\widetilde{\theta}(\pi) < \widetilde{\theta}(\pi')$. Plugging this into the definition of r it follows that $r(\pi) > r(\pi')$ which is a contradiction. ■

Proof. (Proposition 3) By Proposition 2 the set of all equilibria of the model are fully characterized as fixed points of the map $G \circ H : [0,1] \to [0,1]$, where H is defined by (14) in Section 3 (Proposition 2 says how equilibrium *strategies* consistent with a particular fixed point can be constructed).

By Lemma II, $\widetilde{\theta}$ is continuously differentiable on $(0,1)$ and since it follows that r and H are compositions of continuously differentiable functions these are also continuously differentiable on $(0,1)$. This implies that H is a continuous function of π on $(0,1)$. By constant returns to scale the first derivatives of y are homogenous of degree 0 and since $\widetilde{\theta}(\pi)$ must satisfy the first order condition (8) we have that $\widetilde{\theta}(\pi), r(\pi)$ must satisfy

$$p\left(\widetilde{\theta}(\pi),\pi\right) y_1(r(\pi),1) = y_2(r(\pi),1) \tag{60}$$

for every $\pi \in (0,1)$. For $\pi = 1$ we have that $p(\theta,1) = 1$ for all θ, so it does not really matter what workers are assigned to the respective tasks. However, $r(1)$ must nevertheless satisfy $y_1(r(1),1) = y_2(r(\pi),1)$ (one particular way of achieving this is by a cutoff rule). Since the workers are all equally productive in both tasks we get that $w(\theta) = y_1(r(\pi),1) = y_2(r(\pi),1)$ for all θ and it follows that the benefits of investment is given by $H(1) = 0$. It is easy to verify that $\lim_{\pi\to 1} r(\pi) = r(1)$ by use of (60) and using (14) it follows that $\lim_{\pi\to 1} H(\pi) = H(1)$. The case with $\pi = 0$ is taken care of in the same way. No matter how workers are allocated between tasks output is zero, which implies that $w(\theta) = 0$ for all θ. Hence $H(0) = 0$. Furthermore from (60) we have that

$$0 = \lim_{\pi\to 0} p\left(\widetilde{\theta}(\pi),\pi\right) = \lim_{\pi\to 0} \frac{y_2(r(\pi),1)}{y_1(r(\pi),1)}. \tag{61}$$

and using the boundary conditions we see that the only possibility for this to be satisfied is if $\lim_{\pi\to 0} r(\pi) = 0$, $\lim_{\pi\to 0} y_2(r(\pi),1) = 0$ and $\lim_{\pi\to 0} y_1(r(\pi),1)\, p(\widetilde{\theta}(\pi),\pi) = 0$. Since $F_q(\theta) - F_u(\theta)$ and $\int_{\widetilde{\theta}(\pi)}^{1} p(\theta,\pi)(f_q(\theta) - f_u(\theta))\, d\theta$ are bounded below and above it follows from (14)that $\lim_{\pi\to 0} H(\pi) = 0 = H(0)$ establishing continuity of $G \circ H$ on the whole interval $[0,1]$.

Consider any $\pi \in (0,1)$. Inspection of (60) shows that $0 < r(\pi) < \overline{r}$ where $\overline{r}$ is the unique value satisfying $y_1(\overline{r},1) = y_2(\overline{r},1)$. Hence $0 < \widetilde{\theta}(\pi) < 1$ and since $p(\theta,\pi)$ is strictly increasing in θ for any $0 < \pi < 1$ by the assumption of strictly monotone likelihood ratio it follows that

$$\int_{\widetilde{\theta}(\pi)}^{1} p(\theta,\pi)(f_q(\theta) - f_u(\theta))\, d\theta > p\left(\widetilde{\theta}(\pi),\pi\right)\left[F_u\left(\widetilde{\theta}(\pi)\right) - F_q\left(\widetilde{\theta}(\pi)\right)\right]. \tag{62}$$

Hence $H(\pi) > 0$ for all $\pi \in (0,1)$. Since $G(0) > 0$ it follows as a simple application of the intermediate value theorem that there exists at least one fixed point of $G \circ H$. It follows directly from the assumption that $G(0) > 0$ that any fixed point must be in the open interval $(0,1)$. Hence there exists at least one non trivial equilibrium ■

C Appendix: Proof of Proposition 4

Assume that there is an equilibrium such that all agents in group a are assigned to the complex task. Such an equilibrium exist if and only if there is a $\left(\pi^a, \pi^b\right)$ solving $\pi^j = G\left(H^j\left(\pi^a, \pi^b\right)\right)$ for $j = a, b$ such that $\widetilde{\theta}^a(\pi) = \widetilde{\theta}^a\left(\pi^a, \pi^b\right) = 1$. Note that in any such equilibrium we have that $w(\theta) = y_2(r(\pi), 1)$ for all θ, implying that $\pi^a = G(0)$. The cutoff rules for the task assignments must satisfy the Kuhn-Tucker conditions for the problem (17), in particular

$$p\left(\widetilde{\theta}^b(\pi), \pi^b\right) y_1(r(\pi), 1) = y_2(r(\pi), 1) + (\gamma_b - \eta_b) / f_{\pi^b}\left(\widetilde{\theta}^b(\pi)\right), \tag{63}$$

where γ_b is the multiplier associated with the constraint that $\theta^b \geq 0$ and η_b with the constraint that $1 - \theta^b \geq 0$. Note that $\widetilde{\theta}^b(\pi) < 1$ if $\widetilde{\theta}^a(\pi)$ since it follows from the definition of r that $r(\pi) = 0$ if $\left(\widetilde{\theta}^a(\pi), \widetilde{\theta}^b(\pi)\right) = (1,1)$. But then the left hand side of (63) is less than or equal to zero and the right hand side is unbounded, so this could not be the case in equilibrium.

Imposing $\widetilde{\theta}^a(\pi) = 1$ the condition (63) together with the complementary slackness conditions is indeed necessary and sufficient conditions for optimality. Proceeding step by step as in the proof of Lemma I in appendix B one shows that there is a unique solution to these conditions (which may or may not involve assigning all workers in group b to the complex task) for any $\pi^b \in (0,1]$. Let this solution be given by $\theta^b\left(\pi^b\right)$ (observe that so for all we know so far this need not coincide with $\widetilde{\theta}^b\left(G(0), \pi^b\right)$). Fixing exogenously both the investment behavior and the task assignments for group a the model is qualitatively the same as the single group model (with some unexplained input of labor in the simple task) and we can establish that $\theta^b\left(\pi^b\right)$ is continuous in π^b by use of the implicit function theorem, exactly as in the single group model (the possibility of corner solutions where all b workers are assigned to the complex task does not create any discontinuities). Define $r^d : (0,1] \to R_+$ as the ratio of factor inputs function of π^b, assuming that all agents from group a are assigned to the simple task, i.e.:

$$r^d\left(\pi^b\right) = \frac{\lambda^b \pi^b \left(1 - F_q\left(\widetilde{\theta}^b\left(\pi^b\right)\right)\right)}{\lambda^a + \lambda^b \pi^b F_{\pi^b}\left(\widetilde{\theta}^b\right)} \tag{64}$$

In order for π^b to be consistent with equilibrium we have to have that $\pi^b = G\left(H^b\left(G(0), \pi^b\right)\right)$. But this is equivalent to finding a fixed point of (15) with $H(\pi)$ defined as in 14 but with $\widetilde{\theta}$ replaced by θ^b and r replaced by r^d. It is easily checked that θ^b and r^d has all the properties of $\widetilde{\theta}$ and r that were used in the proof of Proposition 5 (i.e. Lemma I and Lemma II still holds). It then remains to check that it is optimal for the firms to assign all workers in group a to the simple task. From the (full) Kuhn-Tucker conditions it follows that this is the case if and only if

$$p(1, G(0)) \leq \frac{y_2\left(r^d\left(\pi^b\right), 1\right)}{y_1\left(r^d\left(\pi^b\right), 1\right)} = p\left(\theta^b\left(\pi^b\right), \pi^b\right) \equiv p(1, \overline{G}_0) \tag{65}$$

$\lim_{G(0)\to 0} p(1, G(0)) = 0$ and $y_2\left(r^d\left(\pi^b\right), 1\right) / y_1\left(r^d\left(\pi^b\right), 1\right) > 0$ this shows that if $G(0)$ is small enough the solution to the task assignment problem is indeed $\left(\widetilde{\theta}^a(\pi), \widetilde{\theta}^b(\pi)\right) = \left(1, \theta^b\left(\pi^b\right)\right)$ if $\pi = \left(G(0), \pi^b\right)$. Hence $r^d(\pi) = r(\pi)$ and $\left(G(0), \pi^b\right)$ is an equilibrium of the model. The fact that all agents in group b are paid a higher wage than any agent from group a follows directly from the wage schedules. ■

D Appendix: Proof of Proposition 5

The proof of the sufficiency part of the proposition uses the following intermediate result.

Lemma *Suppose both firms choose the proposed equilibrium strategy* $\left\langle w^a, w^b, \xi^a, \xi^b \right\rangle$. *Then both firms are earning zero profits.*

Proof. Using the proposed wage schedules (29) the total wage bill can be expressed as:

$$\begin{aligned} W &= \sum_{j=a,b} \lambda^j \int w^j(\theta) f_{\pi^j}(\theta) d\theta = \\ &= y_1(\cdot) \sum_{j=a,b} \lambda^j p\left(\widehat{\theta}^j, \pi^j\right) F_{\pi^j}\left(\widehat{\theta}^j\right) + y_1(\cdot) \sum_{j=a,b} \lambda^j \pi^j \left[1 - F_q\left(\widehat{\theta}^j\right)\right] \end{aligned} \tag{66}$$

where the omitted arguments of y_1 are the implied factor inputs, C and S. But $F_{\pi^a}\left(\widehat{\theta}^a\right) = F_{\pi^b}\left(\widehat{\theta}^b\right)$ due to the affirmative action constraint and using the fact that the weighted average of the marginal productivities in the complex task for the critical workers equals the marginal productivity in the simple task (equation 28) we get

$$W = y_2(\cdot) F_{\pi^a}\left(\widehat{\theta}^a\right) + \sum_{j=a,b} \lambda^j \pi^j \left[1 - F_q\left(\widehat{\theta}^j\right)\right] y_1(\cdot) \tag{67}$$

Since $\sum_{j=a,b} \lambda^j \pi^j \left[1 - F_q\left(\widehat{\theta}^j\right)\right] = C$ and, by the affirmative action constraint, $F_{\pi^a}\left(\widehat{\theta}^a\right) = \lambda^a F_{\pi^a}\left(\widehat{\theta}^a\right) + \lambda^b F_{\pi^b}\left(\widehat{\theta}^b\right) = S$ the result follows from (67) and Euler's theorem. ■

Proof of proposition 5 (*sufficiency*). The proof parallels the proof of Proposition 1, but since the affirmative action constraint has to be used in a non-obvious way we give a rather detailed version of the proof. Suppose one firm should deviate from the candidate equilibrium strategies and play $\left\{w^a_{dev}, w^b_{dev}, \xi^a_{dev}, \xi^b_{dev}\right\}$ so that the actions implied on the outcome path are $\left\langle w^a_{dev}, w^b_{dev}, t^a_{dev}, t^b_{dev}\right\rangle$. Define the following sets: $\Theta^h_j = \{\theta : \widehat{w}^a(\cdot) > w^a(\cdot)\}, \Theta^l_j = \{\theta : \widehat{w}^a(\cdot) < w^a(\cdot)\}, \Theta^e_j = \{\theta : \widehat{w}^a(\cdot) = w^a(\cdot)\}$ for $j = a, b$. Let C and S the implied factor inputs employed in the candidate equilibrium and C_{dev}, S_{dev} be the implied factor inputs for the deviating firm i given that the other firm still plays according to the proposed equilibrium strategies (all these quantities are computed in analogy to (36). The profits for the deviating firm, Π^i_{dev}, can be expressed as ;

$$\Pi^i_{dev} = y(C_{dev}, S_{dev}) - \sum_{j=a,b} \lambda^j \left[\int\limits_{\theta \in \Theta^h_j} w^j_{dev}(\theta) f_\pi(\theta) d\theta + \frac{1}{2} \int\limits_{\theta \in \Theta^e_j} w^j(\theta) f_\pi(\theta) d\theta \right] \tag{68}$$

Using concavity and constant returns to scale as in the derivation of the inequality (38) in the proof of Proposition 1:

$$\Pi^i_{dev} \leq y_1(C, S) C_{dev} + y_2(C, S) S_{dev} - \sum_{j=a,b} \left[\int\limits_{\theta \in \Theta^h_j} w^j_{dev}(\theta) f_\pi(\theta) d\theta + \frac{1}{2} \int\limits_{\theta \in \Theta^e_j} w^j(\theta) f_\pi(\theta) d\theta \right] \tag{69}$$

Using $\pi^j f_q(\theta) = p(\theta, \pi^j) f_{\pi^j}(\theta)$, and manipulating, we have:

$$y_1(C,S)\,C_{dev} = \sum_{j=a,b} \lambda^j \left[\int\limits_{\theta\in\Theta_j^h} t_{dev}^j(\theta)\, y_1(C,S)\, p\left(\theta,\pi^j\right) f_{\pi^j}(\theta)\, d\theta + \frac{1}{2} \int\limits_{\theta\in\Theta_j^e} t_{dev}^j(\theta)\, y_1(C,S)\, p\left(\theta,\pi^j\right) f_{\pi^j}(\theta)\, d\theta \right] \tag{70}$$

But by definition of (29) we have that $w^j(\theta) = y_1(C,S)\,p\left(\theta,\pi^j\right)$ for $\theta \geq \widehat{\theta}^j$ and $w^j(\theta) > y_1(C,S)\,p\left(\theta,\pi^j\right)$ for $\theta < \widehat{\theta}^j$.Hence

$$y_1(C,S)\,C_{dev} \leq \sum_{j=a,b} \lambda^j \left[\int\limits_{\theta\in\Theta_j^h} t_{dev}^j(\theta)\, w^j(\theta)\, f_{\pi^j}(\theta)\, d\theta + \frac{1}{2} \int\limits_{\theta\in\Theta_j^e} t_{dev}^j(\theta)\, w^j(\theta)\, f_{\pi^j}(\theta)\, d\theta \right] \tag{71}$$

Symmetrically, note that $w^j(\theta) = y_1(C,S)\,p\left(\widehat{\theta}^j,\pi^j\right)$ for $\theta \leq \widehat{\theta}^j$ and $w^j(\theta) > y_1(C,S)\,p\left(\widehat{\theta}^j,\pi^j\right)$ for $\theta > \widehat{\theta}^j$. Note that

$$y_1(C,S)\,p\left(\widehat{\theta}^j,\pi^j\right) \frac{S_{dev}^j}{\lambda^j} \leq \int\limits_{\theta\in\Theta_j^h} w^j(\theta)\left(1 - t_{dev}^j(\theta)\right) f_{\pi^j}(\theta)\, d\theta + \frac{1}{2} \int\limits_{\theta\in\Theta_j^e} w^j(\theta)\left(1 - t_{dev}^j(\theta)\right) f_{\pi^j}(\theta)\, d\theta \tag{72}$$

Making use of affirmative action constraint it follows that:

$$y_2(C,S)\,S_{dev} = y_2(C,S)\left(\frac{\lambda^a S_{dev}^a}{\lambda^a} + \frac{\lambda^b S_{dev}^b}{\lambda^b}\right) = y_2(C,S)\left(\lambda^a + \lambda^b\right)\frac{S_{dev}^b}{\lambda^b} \tag{73}$$

and combining with (72) and (28) we get:

$$y_2(C,S)\,S_{dev} \leq \sum_{j=a,b} \lambda^j \left[\int\limits_{\theta\in\Theta_j^h} w^j(\theta)\left(1 - t_{dev}^j(\theta)\right) f_{\pi^j}(\theta)\, d\theta + \frac{1}{2} \int\limits_{\theta\in\Theta_j^e} w^j(\theta)\left(1 - t_{dev}^j(\theta)\right) f_{\pi^j}(\theta)\, d\theta \right] \tag{74}$$

Summing over (71) and (74) we get:

$$y_1(C,S)\,C_{dev} + y_2(C,S)\,S_{dev} \leq \sum_{j=a,b} \lambda^j \left[\int\limits_{\theta\in\Theta_j^h} w^j(\theta)\, f_{\pi^j}(\theta)\, d\theta + \frac{1}{2} \int\limits_{\theta\in\Theta_j^e} w^j(\theta)\, f_{\pi^j}(\theta)\, d\theta \right] \tag{75}$$

The last steps of the argument is exactly as in the proof of Proposition 1. Substituting (75) into the expression for the profits and noting that the deviator must pay higher wages than the candidate equilibrium wages over the relevant ranges gives the result. ■

The necessity part is proved using the following steps.

Lemma I *Suppose* $\left. w_i^a, w_i^b, \xi_i^a, \xi_i^b \right\rangle_{i=1,2}$ *is a pair of best responses. Then,* (1) $w_1^j(\theta) = w_2^j(\theta)$ *for almost all* $\theta \in [0,1]$, $j = a, b$. (2) *Firms earn zero profits. (3)* $\xi_1^j(w_1^j, w_1^j) = \xi_2^j(w_2^j, w_2^j) = t^j(\theta)$, $j = a, b$, *for almost all* $\theta \in [0,1]$, *where* $t^j(\cdot)$ *is the cutoff task assignment rule with critical value* $\widehat{\theta}^j$. (3) *Let* t_i^j *denote the task assignment rule on the equilibrium path for firm group* $j = a, b$ *and firm* $i = 1, 2$. *Then, there exists* $\widehat{\theta}^a$ *and* $\widehat{\theta}^b$ *such that the optimal task assignment rule for group* i *has the cutoff property with critical value* $\widehat{\theta}^i$.

Proof. (1) Equality of wages follows easily observing that if one firm offer higher wages to a positive mass of workers, it could profitably deviate by reducing it. (2) Given constant returns to scale, if firms earned positive profits, one could profitably deviate by reducing the entire wage schedule by a small amount so as to capture the entire labor supply and double profits. (3) Finally, given that firms offer the same wage schedule, an argument similar to the one used in the proof of Proposition 1, Lemma II and III shows that (26) has a unique solution $\widehat{\theta} = (\widehat{\theta}^a, \widehat{\theta}^b)$, and the optimal task assignment rule is the cutoff rule $t^j(\cdot)$ with critical value $\widehat{\theta}^j$, $j = a, b$. ■

Lemma II *Suppose* $\left\langle w_1^j, w_2^j \right\rangle_{j=a,b}$ *is a pair of equilibrium wage schedules and* $\widehat{\theta} = (\widehat{\theta}^a, \widehat{\theta}^b)$ *is the solution to (26). Then there is a pair* $\left\langle k_c^j, k_s^j \right\rangle$ *for each group* $j = a, b$ *such that* $w_i^j(\theta) = k_s^j$ *for almost all* $\theta \leq \widehat{\theta}^j$ *and* $w_i^j(\theta) = k_c^j p(\theta, \pi^j)$ *for almost all* $\theta > \widehat{\theta}^j$, $j = a, b$.

Proof. $w_i^j(\theta) = k_s^j$ *for almost all* $\theta \leq \widehat{\theta}^j$, $j = a, b$ follows from the same argument used in the proof of Proposition 1, Lemma IV (see Appendix A). To prove that $w_i^j(\theta) = k_c^j p(\theta, \pi^j)$ *for almost all* $\theta > \widehat{\theta}^j$ the argument is slightly more cumbersome, since now we have to make sure that deviations don't affect the affirmative action constraint. Suppose by contradiction that in the candidate best response wages the ratio $w_i^j(\theta)/p(\theta, \pi^j)$ is not constant in θ for at least one group, without loss of generality say group a. Then we can find a positive measure set $\Theta^a \subset [\widehat{\theta}^a, 1]$ such that $w_i^a(\theta)/p(\theta, \pi^a) > w_i^a(\theta')/p(\theta', \pi^a)$ for all $\theta \in \Theta^a$, $\theta' \in [\widehat{\theta}^a, 1] \backslash \Theta^a$. It is always possible to choose Θ^a small enough so that there exists $\Theta^b \subset [\widehat{\theta}^b, 1]$ such that $\lambda^a \int_{\theta \in \Theta^a} f_{\pi^a}(\theta) d\theta = \lambda^b \int_{\theta \in \Theta^b} f_{\pi^b}(\theta) d\theta$ (i.e. the mass of workers in the two sets is the same) and $w_i^b(\theta)/p(\theta, \pi^b) \geq w_i^b(\theta')/p(\theta', \pi^b)$ for almost all $\theta \in \Theta^b$, $\theta' \in [\widehat{\theta}^b, 1] \backslash \Theta^b$. Now, suppose one firm posts zero wage to workers belonging to sets Θ^a and Θ^b. By construction, affirmative action constraint remains satisfied and qualified workers have been reduced by $R_C = \sum_{i=a,b} \lambda^i \int_{\theta \in \Theta^i} \pi^i f_q(\theta) d\theta$. The proposed deviation consists in "firing" workers belonging to sets Θ^a and Θ^b while reducing proportionally workers in the simple task to keep the factor ratio at the same level of the candidate equilibrium. Any candidate equilibrium must involve zero profits, but since wage bill per unit of production is lower, profits must be positive after the deviation so that the deviation is profitable. Formally, let C and S be the total factor inputs respectively in the complex and simple task. To keep the factor ratio constant, the deviation must reduce workers in the simple task by $R_S = S \cdot R_c/C$. Because of the affirmative action constraint, reduction of workers in the simple task must be proportionally distributed between groups. Compute then θ_{dev}^a and θ_{dev}^b to satisfy $\int_{[0, \theta_{dev}^j]} f_{\pi^j}(\theta) d\theta = \lambda^j R_s$, $j = a, b$ (we also have to make sure that $\theta_{dev}^j < \widehat{\theta}^j$, $j = a, b$ which is guaranteed by choosing Θ^a small enough). Consider the following deviation from the candidate equilibrium wage profile $w^j(\theta)$, $j = a, b$:

$$w_{dev}^j(\theta) = \begin{cases} 0 & \text{if } \theta \in \Theta^j \cup [0, \theta_{dev}^j], \\ w^j(\theta) & \text{otherwise} \end{cases} \qquad j = a, b \tag{76}$$

By construction, the factor ratio remains constant. Using constant returns to scale, production decreases by $R_c/C = R_s/S$. It is now intuitive but cumbersome to show that average wage per unit of production decrease, so that the deviation is profitable. In the simple task average wage per worker is constant by the first part of this lemma. In the complex task, define the average wage per qualified worker in the candidate equilibrium as $\overline{k}_c^j = \int_{\theta \in [\widehat{\theta}^j, 1]} w(\theta) f_{\pi^j}(\theta) d\theta / [\pi^j (1 - F_q(\widehat{\theta}^j))]$.

Similarly, define the same average wage in the proposed deviation as $\overline{k}_{dev}^j = \int_{\theta \in [\widehat{\theta}^j, 1] \backslash \Theta^j} w_{dev}^j(\theta) f_{\pi^j}(\theta) d\theta / \int_{\theta \in [\widehat{\theta}^j, 1] \backslash \Theta^i} \pi^j f_q(\theta) d\theta$. Using $w(\theta) f_{\pi^i}(\theta) = \pi^i f_q(\theta) \cdot w(\theta)/p(\theta, \pi^j)$ and $w_{dev}^j(\cdot) = w_i^j(\cdot)$ for $\theta \in [\widehat{\theta}^j, 1] \backslash \Theta^j$, $j = a, b$ we can rewrite average wages and derive the following inequality from the fact that the ratio $w(\cdot)/p(\cdot, \pi^j)$ is higher for $\theta \in \Theta^j$, $j = a, b$

$$\frac{\int_{\theta \in [\widehat{\theta}^j, 1] \backslash \Theta^j} \pi^j f_q(\theta) \frac{w(\theta)}{p(\theta)} d\theta + \int_{\theta \in \Theta^j} \pi^j f_q(\theta) \frac{w(\theta)}{p(\theta)} d\theta}{\int_{\theta \in [\widehat{\theta}^j, 1] \backslash \Theta^j} \pi^j f_q(\theta) d\theta + \int_{\theta \in \Theta^j} \pi^j f_q(\theta) d\theta} > \frac{\int_{\theta \in [\widehat{\theta}^j, 1] \backslash \Theta^j} \pi^j f_q(\theta) \frac{w(\theta)}{p(\theta)} d\theta}{\int_{\theta \in [\widehat{\theta}^j, 1] \backslash \Theta^j} \pi^j f_q(\theta) d\theta} \tag{77}$$

i.e. $\overline{k}^j > \overline{k}^j_{dev}$, $j = a, b$. But then the total wage bill payed to the complex task workers is equal to $W_{dev} = \sum_i \overline{k}^i_{dev} \int_{\theta \in [\widehat{\theta}^i, 1] \setminus \Theta^i} \pi f_q(\theta) d\theta < (1 - R_c) \sum_i \overline{k}^i \int_{\theta \in [\widehat{\theta}^i, 1]} \pi f_q(\theta) d\theta$. Since wages decrease proportionally more than production, $w^j_{dev}(\cdot)$ implies positive profits, and the deviation is profitable. ■

Lemma III *The equilibrium wage schedules* $\left\langle w^j_1, w^j_2 \right\rangle_{j=a,b}$ *are continuous at almost all* $\theta \in [0, 1]$.

Proof. In Lemma II we established that $w^j_i(\theta) = k^j_s$ for almost all $\theta \leq \widehat{\theta}^j$ and $w^j_i(\theta) = k^j_c p(\theta, \pi^j)$ for almost all $\theta > \widehat{\theta}^j$, $j = a, b$. All we need to show is that $k^j_s = k^j_c p(\widehat{\theta}^j, \pi^j)$, $j = a, b$. The proof is by contradiction and consists of two parts: either there is a group with $k^j_s > k^j_c p(\widehat{\theta}^j, \pi^j)$, or $k_s \leq k^j_c p(\widehat{\theta}^j, \pi^j)$ for both groups, with strict inequality for at least one group.

Consider the first case. Then, there exists a positive measure interval $[\widehat{\theta}^j, \theta^*]$ such that $k_s > k^j_c p(\theta, \pi^j) = w^j_i(\theta)$ for almost all $\theta \in [\widehat{\theta}^j, \theta^*]$. In the proof of the necessity part of Proposition 1 we have already shown that there exists a profitable deviation from such a wage schedule. The deviation consists in offering a slightly higher wage to workers in the interval $[\widehat{\theta}^j, \theta^*]$, and use them to replace an equal mass of workers in the simple task who receive an higher wage k_s. Notice that his deviation does not affect the affirmative action constraint since the deviation firm is replacing expensive simple task workers with an equal mass cheaper workers stolen from the other firm.

Suppose instead $k^j_s \leq k^j_c p(\widehat{\theta}^j, \pi^j)$ for $j = a, b$ with strict inequality for at least one group, say group a. We propose a deviation on wages of both groups that keeps production constant and maintains the affirmative action constraint satisfied. Take and define $\theta^{a\prime\prime} \in (\theta^{a\prime}, \widehat{\theta}^a)$ as the value that divides the mass of workers with $\theta \in [\theta^{a\prime}, \widehat{\theta}^a]$ in two equal parts, i. e. $\theta^{a\prime\prime}$ is the solution of the following equation: $[F_{\pi^a}(\widehat{\theta}^a) - F_{\pi^a}(\theta^{a\prime})]/2 = [F_{\pi^a}(\theta^{a\prime\prime}) - F_{\pi^a}(\theta^{a\prime})]$. In the proposed deviation, workers with $\theta \in [\theta^{a\prime}, \theta^{a\prime\prime})$ will be assigned to the simple task, and workers with $\theta \in [\theta^{a\prime\prime}, \widehat{\theta}^a]$ to the complex task. We want to keep mass of productive workers in the complex task constant. For this purpose, compute $\theta^{a*} > \widehat{\theta}^a$ so that $\int_{\theta^{a\prime\prime}}^{\widehat{\theta}^a} f_q(\theta) d\theta = \left(\int_{\widehat{\theta}^a}^{\theta^{a*}} f_q(\theta) d\theta \right) / 2$ and consider the following deviation:

$$
\begin{aligned}
w^a_{dev}(\theta) &= \begin{cases} k^a_s + \epsilon & \text{for } \theta \in [\theta^{a\prime}, \widehat{\theta}^a] \\ 0 & \text{for } \theta \in [\widehat{\theta}^a, \theta^{a*}] \\ w^a_i & \text{otherwise} \end{cases} \\
t^a_{dev}(\theta) &= \begin{cases} 0 & \text{for } \theta \in [0, \theta^{a\prime\prime}] \\ 1 & \text{for } \theta \in (\theta^{a\prime\prime}, 1] \end{cases}
\end{aligned}
\tag{78}
$$

By construction, the mass of workers employed in the simple task and the mass of qualified workers employed in the complex task are unchanged. On the other hand, if we consider this deviation alone, the affirmative action constraint will not be satisfied because the mass of workers employed in the complex task is increased by the following amount (a formal argument is omitted, but it will be symmetric to the argument used in Proposition I, Lemma II showing that employing the same mass of workers with higher average θ increases the mass of qualified workers):

$$
\Psi' - \Psi = \lambda^a \left(\int_{\theta^{a\prime\prime}}^{\widehat{\theta}^a} f_{\pi^a}(\theta) d\theta - \frac{\int_{\widehat{\theta}^a}^{\theta^{a*}} f_{\pi^a}(\theta) d\theta}{2} \right) \tag{79}
$$

To keep the affirmative action constraint unchanged we have to deviate also on wages offered to group b. The idea is to compute $\theta^{b'}$ and θ^{b*} with $\widehat{\theta}^b < \theta^{b'} < \theta^{b*} < 1$ to attract from the other firm workers with $\theta \in [\widehat{\theta}^b, \theta^{b'})$, get rid of workers with $\theta \in [\theta^{b*}, 1]$ so as to keep the mass of productive workers employed in the complex task constant (equation (80)) and to satisfy the affirmative action constraint (equation (81)). Formally, compute $\theta^{b'}$ and θ^{b*} in order to satisfy the following set of equations:

$$
\int_{\widehat{\theta}^b}^{\theta^{b'}} f_q(\theta) d\theta = \int_{\theta^{b*}}^{1} f_q(\theta) d\theta \tag{80}
$$

$$\int_{\widehat{\theta}^b}^{\theta^{b\prime}} f_{\pi^b}(\theta)d\theta - \int_{\theta^{b*}}^{1} f_{\pi^b}(\theta)d\theta = \frac{\Psi' - \Psi}{\lambda^a} \tag{81}$$

(choosing $\theta^{a\prime}$ close enough to $\widehat{\theta}^a$ guarantees existence of $\theta^{b'}$ and θ^{b*} solving the system of equations). Consider deviation $\langle w^a_{dev}, t^a_{dev}\rangle$ together with the following deviation:

$$w^b_{dev}(\theta) = \begin{cases} k^b_s + \epsilon & \text{for } \theta \in [\widehat{\theta}^b, \theta^{b\prime}] \\ 0 & \text{for } \theta \in [\theta^{b*}, 1] \\ w^b_i & \text{otherwise} \end{cases} \tag{82}$$

Since the proposed deviations make sure that production remain constant, change in profits will depend only on change in wages. Letting ϵ terms go to zero, we have:

$$\lim_{\epsilon \downarrow 0} \Delta w_{dev} = \frac{1}{2}\int_{\theta^{a\prime}}^{\widehat{\theta}^a} w^a_i(\theta) f_{\pi^a}(\theta)d\theta - \frac{1}{2}\int_{\widehat{\theta}^a}^{\theta^{a*}} w^a_i(\theta) f_{\pi^a}(\theta)d\theta + \frac{1}{2}\int_{\widehat{\theta}^b}^{\theta^{b\prime}} w^b_i(\theta) f_{\pi^b}(\theta)d\theta - \frac{1}{2}\int_{\theta^{b*}}^{1} w^b_i(\theta) f_{\pi^b}(\theta)d\theta$$

Observe now that if $\theta^{a\prime}$ is close enough to $\widehat{\theta}^a$ there is $h < k^a_c$ satisfying $k^a_s + \epsilon < hp(\theta, \pi^a)$ for every $\theta \in [\theta^{a\prime}, \widehat{\theta}^a]$. We can then conclude using the usual relation $f_{\pi^j}(\cdot) = \pi^j f_q(\cdot)$:

$$\lim_{\epsilon \downarrow 0} \Delta w_{dev}(\epsilon) < \frac{1}{2}\int_{\theta^{a\prime}}^{\widehat{\theta}^a} h\pi^a f_q(\theta)d\theta - \frac{1}{2}\int_{\widehat{\theta}^a}^{\theta^{a*}} k^a_c \pi^a f_q(\theta)d\theta + \frac{1}{2}k^b_c\pi^b \left(\int_{\widehat{\theta}^b}^{\theta^{b\prime}} f_q(\theta)d\theta - \int_{\theta^{b*}}^{1} f_q(\theta)d\theta \right) \tag{83}$$

The last term on the right hand side is equal to zero by construction, so the expression reduces to $2\lim_{\epsilon\downarrow 0}\Delta w_{dev}(\epsilon) < \pi^a(h - k^a_c)\int_{\theta^{a\prime}}^{\widehat{\theta}^a} f_q(\theta)d\theta < 0$

We can then choose ϵ small enough so that the change in wages is negative and the deviation is profitable. ■

Proof of Proposition 5 *(necessity)* Using the result shown in Lemma III, the total wage bill paid to workers of group j is equal to $k^j_c = [p(\widehat{\theta}^j, \pi^j)F_{\pi^j}(\widehat{\theta}^j) + 1 - F_q(\widehat{\theta}^j)]$ and it is strict increasing in k^j_c. As shown in the proof of sufficiency part of this proposition, $k^a_c = k^b_c = y_1(C,S)$ implies zero profits. To show that a wage schedule with $k^j_c \neq y_1(C,S)$ cannot be an equilibrium, suppose for example that $k^a_c > y_1(C,S)$. Then zero profits condition implies $k^b_c < y_1(C,S)$. Then we can construct a profitable deviation that deals only with workers assigned to the complex task. The idea of the deviation comes from the observation that the cost of labor per productive worker is higher in group a than in group b. We substitute high test result workers in group a with low test result workers of the same group, taking care of keeping the number of workers employed in the skilled task constant this will reduce the number of qualified workers of group a; if we construct a symmetric deviation for group b in order to restore the original mass of qualified workers, then the total wage bill will be lower than in the candidate equilibrium without changing total production and keeping the affirmative action constraint satisfied. Formally, define $\theta^{a\prime}$ and θ^{a*} with $\widehat{\theta}^a < \theta^{a\prime} < \theta^{a*} < 1$ as the solution of the following equation:

$$F_{\pi^a}(\theta^{a\prime}) - F_{\pi^a}(\widehat{\theta}^a) = 1 - F_{\pi^a}(\theta^{a*}) \tag{84}$$

Consider the following deviation of the wage function for group a:

$$w^a_{dev}(\theta) = \begin{cases} w^a_i(\theta) + \epsilon & \text{for } \theta \in [\widehat{\theta}^a, \theta^{a'}] \\ 0 & \text{for } \theta \in [\theta^{a*}, 1] \\ w^a_i(\theta) & \text{otherwise} \end{cases}$$

The mass of *qualified* workers of workers that belong to group a decrease since we substitute workers with high test result with an equal mass of workers with low test result (the formal proof is similar to the one used in Proposition 1, Lemma II. We can quantify the loss in productive workers employed in the complex task as $C' - C = \frac{\lambda^a}{2}\left(\int_{\theta^{a*}}^{1}\pi^a f_q(\theta) - \int_{\widehat{\theta^a}}^{\theta^{a\prime}}\pi^a f_q(\theta)\right)$: Next, consider the following deviation for group b:

$$w_{dev}^b(\theta) = \begin{cases} 0 & \text{for } \theta \in [\widehat{\theta^b}, \theta^{b\prime}] \\ w_i^b(\theta) + \epsilon & \text{for } \theta \in [\theta^{b*}, 1] \\ w_i^b(\theta) & \text{otherwise} \end{cases}$$

With $\theta^{b\prime}$ and θ^{b*} satisfying the following two equations:

$$F_{\pi^b}(\theta^{b\prime}) - F_{\pi^b}(\widehat{\theta^b}) = 1 - F_{\pi^b}(\theta^{b*}) \tag{85}$$

$$\int_{\theta^{b*}}^{1} \pi^b f_q(\theta) - \int_{\widehat{\theta^b}}^{\theta^{a\prime}} \pi^b f_q(\theta) = \frac{2}{\lambda^b}(C' - C) \tag{86}$$

Notice that if $\theta^{a\prime}$ is chosen to be close enough to $\widehat{\theta}^a$ (which implies $C' - C$ close enough to zero) then a solution to the system of equations specified above exists). Equation (85) guarantees that the number of employed workers remains constant, and (86) that the gain in productive workers employed in the complex task obtained with deviation $w_i^{b\prime}(\cdot)$ is equal to the loss due to $w_i^{a\prime}(\cdot)$. Under the proposed deviations, productions remains constant, so difference in profits depends uniquely on difference in wage bill:

$$\begin{aligned} w(\epsilon) - w &= \frac{\lambda^a}{2}\int_{\widehat{\theta^a}}^{\theta^{a\prime}} \left(w_i^a(\theta) + 2\epsilon\right) f_{\pi^a}(\theta)d\theta - \frac{\lambda^a}{2}\int_{\theta^{a*}}^{1} w_i^a(\theta) f_{\pi^a}(\theta)d\theta \\ &\quad - \frac{\lambda^b}{2}\int_{\widehat{\theta^b}}^{\theta^{b\prime}} w_i^b(\theta) f_{\pi^b}(\theta)d\theta + \frac{\lambda^b}{2}\int_{\theta^{b*}}^{1}\left(w_i^b(\theta) + 2\epsilon\right) f_{\pi^b}(\theta)d\theta \\ &= k_c^a(C' - C) - k_c^b(C' - C) + \epsilon\left[\lambda^a \int_{\widehat{\theta^a}}^{\theta^{a\prime}} \pi^a f_q(\theta)d\theta + \lambda^b \int_{\theta^{b*}}^{1} \pi^b f_q(\theta)d\theta\right] \end{aligned}$$

Since $k_c^b < y_1(C, S) < k_c^a$ and by construction $C' - C < 0$, then there is an ϵ small enough such that the difference in wage bill is negative and the deviation is profitable. ■

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