CARESS Working Paper #96-06

*Approximate Common Knowledge Revisited**

Stephen Morris†

September 1996

Abstract

Suppose we replace "knowledge" by "belief with probability p" in standard definitions of common knowledge. Very different notions arise depending the exact definition of common knowledge used in the substitution. This paper demonstrates those differences and identifies which notion is relevant in each of three contexts: equilibrium analysis in incomplete information games, best response dynamics in incomplete information games, and agreeing to disagree/no trade results.

1. Introduction

Suppose we replace "knowledge" in the definition of common knowledge by belief with high probability; what notion of approximate common knowledge do we get? The answer is surprisingly sensitive to the exact definition of common knowledge in that construction.

* This paper incorporates material from "Trade and Almost Common Knowledge," University of Cambridge Economic Theory Discussion Paper #194. In particular, it corrects lemma 4 of that paper, which was false. I am grateful for financial support to visit Cambridge from an ESRC research grant (R000232865) and EEC contract (SPESCT910057). This version was prepared for the 1996 SITE Summer Workshop on Epistemic and other Foundational Issues in Game Theory.

† Department of Economics, University of Pennsylvania, 3718 Locust Walk, Philadelphia PA 19104-6297. E-Mail: smorris@econ.sas.upenn.edu.

Consider (as I shall throughout this paper) the case where there are two individuals, 1 and 2. Say that one individual *p-believes* event E if he assigns it probability at least p. Event E is *common p-belief* if both p-believe E, both p-believe that both p-believe E, both p-believe that both p-believe that both p-believe E, and so on.[1] Event E is *iterated p-belief* for 1 if 1 p-believes E, 1 p-believes that 2 p-believes E, 1 p-believes that 2 p-believes that 1 p-believes E, and so on. Event E is *iterated p-belief* if it is iterated p-belief for both individuals.[2]

Common 1-belief and iterated 1-belief are equivalent to each other and to standard definitions of common knowledge.[3] When p is not equal to 1, common p-belief is not equivalent to iterated p-belief. If an event is common p-belief, it is necessarily iterated p-belief, but the converse is not true. It might nonetheless be conjectured that for any $p < 1$, there should exist some q (sufficiently close to 1) such that if an event is iterated q-belief, it must be common p-belief. This is false: in particular, I show that for any $1/2 < r \leq p < 1$ and $\varepsilon > 0$, it is possible to find events which are iterated p-belief with probability at least $1 - \varepsilon$, but which are never common r-belief.

Monderer and Samet (1989) established that common p-belief is the natural notion of approximate common knowledge when studying the robustness to equilibria to a lack of common knowledge of payoffs. I show that iterated p-belief is the relevant notion of approximate common knowledge for the study of best response dynamics in incomplete information games.

Another important application of common knowledge, starting with Aumann (1976), has been to agreeing to disagree and no trade results. The relevant notion of approximate common knowledge for both kinds of results is *weak common p-belief.* An event is said to be weak common p-belief if it is common p-belief *either* given individuals' actual information *or* if individuals ignore some of their information.[4] This notion is much weaker than common p-belief and is necessary and sufficient for both approximate agreement results and approximate no trade results.

The paper is organized as follows. Alternative notions of approximate common knowledge are

[1] Monderer and Samet (1989).

[2] This is equivalent to $(1-p, \infty)$-approximate common knowledge, in the language of Stinchcombe (1988).

[3] Verbal hierarchical descriptions of common knowledge between two individuals in the literature are typically in the form of iterated 1-belief (see Lewis (1969), Aumann (1976) and Brandenburger and Dekel (1987)).

[4] This is equivalent to weakly p-common knowledge in Geanakoplos (1994).

introduced, characterized and related in section 2. Iterated p-belief, common p-belief and weak common p-belief are introduced in sections 2.1 through 2.3 respectively; in section 2.4, it is shown that in the special case when p equals 1, all three notions are equivalent; but in section 2.5, it is shown that if $p < 1$, there is no necessary connection between common p-belief and the two weaker notions. Section 3 considers applications and shows which notion is relevant for which application. Section 4 contains discussion.

2. Approximate Common Knowledge

There are two individuals, $\{1, 2\}$; let Ω be a countable state space, with typical element ω. For each $i \in \{1, 2\}$, let $\mathcal{Q}_i$ be a partition of Ω. Write $\mathcal{F}_i$ for the σ-field generated by $\mathcal{Q}_i$. Let P be a probability on the countable state space.

Event $E \subseteq \Omega$ is *simple* if $E = E_1 \cap E_2$ and each $E_i \in \mathcal{F}_i$. Whenever event $E_1 \cap E_2$ is said to be simple, it should be understood that $E_i \in \mathcal{F}_i$, for both i. Write $Q_i(\omega)$ for the (unique) element of $\mathcal{Q}_i$ containing ω. The partition $\mathcal{Q}_i$ is interpreted as individual i's information, so that if the true state is ω, individual i knows only that the true state is an element of $Q_i(\omega)$. Write $P(\omega)$ for the probability of the singleton event $\{\omega\}$, and $P[E|F]$ for the conditional probability of event E, given event F, if $P[F] > 0$. Throughout the paper, I will assume that all information sets occur with positive probability, i.e., $P[Q_i(\omega)] > 0$ for all $\omega \in \Omega$ and $i \in \{1, 2\}$. When i represents a typical individual, j will be understood to be the other individual.

An individual p-believes an event E at state ω if the conditional probability of E, given $Q_i(\omega)$, is at least p. Writing $B_i^p E$ for the set of states where i p-believes E, we have $B_i^p E \equiv \{\omega : P[E|Q_i(\omega)] \geq p\}$. The following straightforward properties of belief operators will be used extensively:

B1: If $E \in \mathcal{F}_i$, then $B_i^p E = E$.

B2: If $E_1 \cap E_2$ is simple, then $B_i^p(E_1 \cap E_2) = E_i \cap B_i^p E_j$.

B3: If $q \geq p$, then $B_i^q E \subseteq B_i^p E$.

B4: If $E \subseteq F$, then $B_i^p E \subseteq B_i^p F$.

2.1. Iterated p-Belief

Event E is *iterated p-belief* for 1 if 1 p-believes it, 1 p-believes that 2 p-believes it, 1 p-believes that 2 p-believes that 1 p-believes it, and so on. Writing $I_i^p E$ for the set of states where E is iterated p-belief for i, we have:

$$\begin{aligned} I_1^p E &\equiv B_1^p E \cap B_1^p B_2^p E \cap B_1^p B_2^p B_1^p E \cap ... \\ I_2^p E &\equiv B_2^p E \cap B_2^p B_1^p E \cap B_2^p B_1^p B_2^p E \cap ... \end{aligned}$$

Definition 2.1. *(Hierarchical). Event E is iterated p-belief if it is iterated p-belief for both players. Thus E is iterated p-belief at state ω if $\omega \in I^p E \equiv I_1^p E \cap I_2^p E$.*

This definition corresponds to $(1-p,\infty)-$approximate common knowledge, in the language of Stinchcombe (1988). It is possible to give a rather weak "fixed point" characterization of iterated p-belief. Say that collection of events $\mathcal{E}$ is *mutually p-evident* if $B_i^p E \in \mathcal{E}$, for all events $E \in \mathcal{E}$ and both i.

Proposition 2.2. *(Fixed Point Characterization) Event E is iterated p-belief at ω if and only if there exists a mutually p-evident collection of events $\mathcal{E}$ with [1] $B_i^p E \in \mathcal{E}$ for both i; and [2] $\omega \in F$, for all $F \in \mathcal{E}$.*

Proof. (if) Suppose $\mathcal{E}$ is mutually p-evident and [1] and [2] hold. By [1], $B_1^p E \in \mathcal{E}$ and $B_2^p E \in \mathcal{E}$. Now, by $\mathcal{E}$ mutually p-evident, $B_1^p B_2^p E \in \mathcal{E}$ and $B_2^p B_1^p E \in \mathcal{E}$ and so $B_1^p \left[B_2^p B_1^p\right]^n E \in \mathcal{E}$, $B_2^p \left[B_1^p B_2^p\right]^n E \in \mathcal{E}$, $\left[B_2^p B_1^p\right]^{n+1} E \in \mathcal{E}$, $\left[B_1^p B_2^p\right]^{n+1} E \in \mathcal{E}$, for all $n \geq 0$. Since $I^p E$ is exactly the intersection of these expressions, $\omega \in I^p E$ by [2].

(only if) Suppose E is iterated p-belief at ω. Let

$$\mathcal{E} = \left\{ F \subseteq - \quad : \quad \begin{array}{c} F \in \left\{ B_1^p \left[B_2^p B_1^p\right]^n E, B_2^p \left[B_1^p B_2^p\right]^n E, \left[B_2^p B_1^p\right]^{n+1} E, \left[B_1^p B_2^p\right]^{n+1} E \right\}, \\ \text{for some } n \geq 0 \end{array} \right\}.$$

By definition of iterated p-belief, [2] holds. By construction of $\mathcal{E}$, [1] holds and $\mathcal{E}$ is mutually p-evident. □

Example 2.3. *- $= \{1, 2, 3, 4, 5, 6\}$; $\mathcal{Q}_1 = (\{1, 2\}, \{3\}, \{4\}, \{5, 6\})$; $\mathcal{Q}_2 = (\{1, 3, 4\}, \{2, 5, 6\})$; $P(\omega) = 1/6$ for all $\omega \in$ - .*

If $E^* = \{1, 2, 3\}$, then $I^{0.6}E^* = \{3\}$. Let us verify this, first using the hierarchical definition, and then using the fixed point characterization: $B_1^{0.6}E^* = \{1, 2, 3\}$; $B_2^{0.6}E^* = \{1, 3, 4\}$; $B_2^{0.6}B_1^{0.6}E^* = \{1, 3, 4\}$; and $B_1^{0.6}B_2^{0.6}E^* = \{3, 4\}$. But now since $B_1^{0.6}\{1, 3, 4\} = \{3, 4\}$ and $B_2^{0.6}\{3, 4\} = \{1, 3, 4\}$, $I_1^{0.6}E^* = \{3\}$, $I_2^{0.6}E^* = \{1, 3, 4\}$ and $I^{0.6}E^* = \{3\}$. On the other hand, consider the collection of events $\mathcal{E} = (\{1, 3, 4\}, \{3, 4\})$. Observe that [1] $B_i^{0.6}E^* \in \mathcal{E}$ for both i; [2] $3 \in E$ for all $E \in \mathcal{E}$; and [3] $B_i^{0.6}E \in \mathcal{E}$ for all $E \in \mathcal{E}$ and both i.

2.2. Common p-Belief

An event E is *common p-belief* if both p-believe it, both p-believe that both p-believe it, and so on. Formally, define a "both p-believe" operator as follows: $B_*^pE \equiv B_1^pE \cap B_2^pE$.

Definition 2.4. *(Hierarchical) Event E is common p-belief at ω if*

$$\omega \in C^pE \equiv \bigcap_{n \geq 1} [B_*^p]^n E \equiv B_*^pE \cap B_*^pB_*^pE \cap B_*^pB_*^pB_*^pE \cap ...$$

This notion can be given a tight fixed point characterization. Event F is *p-evident* if $F \subseteq B_*^pF$. Thus event F is p-evident exactly if $\mathcal{E} = \{E : F \subseteq E\}$ is mutually p-evident. By B2, a simple event $F_1 \cap F_2$ is p-evident if $F_1 \subseteq B_1^pF_2$ and $F_2 \subseteq B_2^pF_1$.

Proposition 2.5. *(Fixed Point Characterization) The following statements are equivalent; [1] event E is common p-belief at ω; [2] there exists a p-evident event F such that $\omega \in F$ and $F \subseteq B_*^pE$; [3] there exists a simple p-evident event $F_1 \cap F_2$ such that $\omega \in F_1 \cap F_2$ and $F_i \subseteq B_i^pE$ for both i.*

The equivalence of [1] and [2] is due to Monderer and Samet (1989), who *defined* common p-belief using the fixed point characterization.

Common p-belief may differ from iterated p-belief. In example 2.3, $B_1^{0.6}E^* = \{1, 2, 3\}$; $B_2^{0.6}E^* = \{1, 3, 4\}$, so $B_*^{0.6}E^* = \{1, 3\}$. Now $B_1^{0.6}B_*^{0.6}E^* = \{3\}$ and $B_2^{0.6}B_*^{0.6}E^* = \{1, 3, 4\}$, giving $[B_*^{0.6}]^2 E^* = \{3\}$. Now $B_1^{0.6}\left([B_*^{0.6}]^2 E^*\right) = \{3\}$ and $B_2^{0.6}[B_*^{0.6}]^2 E^* = \emptyset$, giving $[B_*^{0.6}]^3 E^* = \emptyset$, and thus $C^{0.6}E^* = \emptyset$.

2.3. Weak Common p-Belief

More information can reduce the degree of common p-belief of an event. Consider the following example.

Example 2.6. *- $= \{0, 1, 2, ...\}$; $\mathcal{Q}_1 = $ (-); $\mathcal{Q}_2 = $ (-); $P(\omega) = \delta (1 - \delta)^{\omega}$ for all $\omega \in$ - , where $\delta \in (0, 1)$.*

Thus individuals 1 and 2 have no information. Consider the event $E^* = \{1, 2, 3...\}$; $P[E^*] = 1 - \delta$, so for any $p \leq 1 - \delta$, $B_1^p E^* =$ - , $B_2^p E^* =$ - , so $B_*^p E^* =$ - and $C^p E^* =$ - . Thus for sufficiently small δ, E^* is always common p-belief (for *any* given $p < 1$).

Now suppose that individuals 1 and 2 received some information about the state of the world. In particular, the example becomes:

Example 2.7. *- $= \{0, 1, 2, ...\}$; $\mathcal{Q}_1 = (\{0\}, \{1, 2\}, \{3, 4\}, ..)$; $\mathcal{Q}_2 = (\{0, 1\}, \{2, 3\}, ...\})$; $P(\omega) = \delta (1 - \delta)^{\omega}$ for all $\omega \in$ - , where $\delta \in (0, 1)$.*

Now for *any* $p \geq 1/2$ and $\omega \geq 1$, $B_*^p (\{\omega, \omega + 1, ..\}) = \{\omega + 1, \omega + 2, ..\}$. Thus $B_*^p E^* = \{2, 3, ..\}$, $[B_*^p]^n E^* = \{n + 1, n + 2, ..\}$ for all $n \geq 0$ and so $C^p E^* = \emptyset$.

This suggests the following alternative notion of approximate common knowledge. Suppose that each individual i had access to information partition $\mathcal{Q}_i$, but need not acquire that information. What is the maximum attainable degree of common p-belief of a given event? Thus say that event E is *weak common p-belief* if event E is common p-belief given the individuals' information *or any worse information*. Formally, write $\mathcal{Q} \equiv (\mathcal{Q}_1, \mathcal{Q}_2)$ and index belief and common p-belief operators as follows (in this section only): $B_{\mathcal{Q}_i}^p E \equiv \{\omega : P[E | Q_i(\omega)] \geq p\}$, $B_{\mathcal{Q}}^p E \equiv B_{\mathcal{Q}_1}^p E \cap B_{\mathcal{Q}_2}^p E$ and $C_{\mathcal{Q}}^p E \equiv \bigcap_{n \geq 1} \left[B_{\mathcal{Q}}^p \right]^n E$. Say that $\mathcal{Q}'$ is a *coarsening* of $\mathcal{Q}$ if $Q_i(\omega) \subseteq Q'_i(\omega)$ for both i and all $\omega \in$ - . Write $\mathcal{C}(\mathcal{Q})$ for the coarsenings of $\mathcal{Q}$.

Definition 2.8. *(Hierarchical). Event E is weak common p-belief at ω (under $\mathcal{Q}$) if event E is common p-belief at ω under some coarsening of $\mathcal{Q}$, i.e., if $\omega \in W^p E \equiv \bigcup_{\mathcal{Q}' \in \mathcal{C}(\mathcal{Q})} C_{\mathcal{Q}'}^p E$.*

Simple event $F_1 \cap F_2$ is *weakly p-evident* if it is empty or $P[F_1 | F_2] \geq p$ and $P[F_2 | F_1] \geq p$.

Proposition 2.9. *(Fixed Point Characterization). Event E is weak common p-belief at ω if and only if there exists a weakly p-evident event $F_1 \cap F_2$ with $\omega \in F_1 \cap F_2$ and $P[E|F_i] \geq p$ for both i.*

This notion is due to Geanakoplos (1994, p. 1482) who called it weakly $p-$common knowledge.

Proof. Suppose $\omega \in W^p E$. Then $\omega \in C^p_{\mathcal{Q}'} E$ for some $\mathcal{Q}' \in \mathcal{C}(\mathcal{Q})$. By proposition 2.5, there exists simple event $F_1 \cap F_2$ with [1] $F_i \subseteq B^p_{\mathcal{Q}'_i} E$ for both i and [2] $F_i \subseteq B^p_{\mathcal{Q}'_i} F_j$ for both i. But [1] implies $P[E|F_i] \geq p$ for both i, and [2] implies $P[F_j|F_i] \geq p$ for both i. On the other hand, suppose there exists a weakly p-evident event $F_1 \cap F_2$ with $\omega \in F_1 \cap F_2$ and $F_i \subseteq B^p_i E$ for both i. Let

$$Q'_i(\omega) = \begin{cases} F_i, \text{ if } \omega \in F_i \\ - \setminus F_i, \text{ if } \omega \notin F_i \end{cases}.$$

By construction $F_i \subseteq B^p_{\mathcal{Q}'_i} E$ for both i, so $F_1 \cap F_2 \subseteq B^p_{\mathcal{Q}'} E$. But $F_1 \cap F_2 \subseteq F_i \cap B^p_{\mathcal{Q}'_i} F_j = B^p_{\mathcal{Q}'_i}(F_1 \cap F_2)$ for both i (by B2). Thus $F_1 \cap F_2 \subseteq B^p_{\mathcal{Q}'} E \subseteq \left[B^p_{\mathcal{Q}'}\right]^2 E \subseteq C^p_{\mathcal{Q}'} E$. $\square$

Corollary 2.10. *If $P[E] \geq p$, then $W^p E = -$.*

Proof. Since $- \in \mathcal{F}_i$ for both i, $-$ is weakly p-evident. $\square$

2.4. The Relation between Alternative Notions for $p = 1$

Iterated 1-belief, common 1-belief and weak common 1-belief are all equivalent

Proposition 2.11. *For all events E: $I^1 E = C^1 E = W^1 E$.*

Proof. Observe first that for each i and all collections of events $\left\{E^k\right\}_{k=1}^{\infty}$, $B^1_i\left(\underset{k\geq 1}{\cap} E^k\right) = \underset{k\geq 1}{\cap} B^1_i E^k$. Thus $I^1 E \subseteq B^1_i I^1 E$ for each i, i.e., $I^1 E$ is 1-evident. Now since $I^1 E \subseteq B^1_* E$ (by definition), $I^1 E \subseteq C^1 E$ by lemma 2.5. But lemma 2.14 below shows $C^p E \subseteq I^p E$ for all p, so $I^1 E = C^1 E$.

Now suppose $\mathcal{Q}' \in C(\mathcal{Q})$. For all events E: $B^1_{\mathcal{Q}'_i} E \subseteq B^1_{\mathcal{Q}_i} E$ for each i, so $B^1_{\mathcal{Q}'} E \subseteq B^1_{\mathcal{Q}} E$, so $C^1_{\mathcal{Q}'} E \subseteq C^1_{\mathcal{Q}} E$; thus $C^1 E \subseteq W^1 E$. But lemma 2.15 below shows that $C^p E \subseteq W^p E$ for all p, so $C^1 E = W^1 E$. $\square$

The "truth axiom" requires that 1-beliefs are always correct, i.e., $B^1_i E \subseteq E$ for all events E and each i. In our setting, the truth axiom is equivalent to requiring that P has full support, i.e.,

$P(\omega) > 0$ for all $\omega \in$ - . Under the truth axiom with $p = 1$, all three notions outlined above are equivalent to the following definition of common knowledge.

Let $\mathcal{F}^* = \mathcal{F}_1 \cap \mathcal{F}_2$. Now $\mathcal{F}^*$ is the σ-field generated by the meet of the individuals' partitions.

Definition 2.12 (Aumann (1976)). *Event E is common knowledge at ω if*

$$\omega \in \mathcal{CK}E \equiv \{\omega : \omega \in F \subseteq E, \text{ for some } F \in \mathcal{F}^*\}.$$

Lemma 2.13. *For events E: (a) $\mathcal{CK}E \subseteq I^1E = C^1E = W^1E$; (b) under the truth axiom, $\mathcal{CK}E = I^1E = C^1E = W^1E$.*

Proof. (a) If $F \in \mathcal{F}^*$, then F is 1-evident. If F is 1-evident and $F \subseteq E$, then $F \subseteq B_*^1 F \subseteq B_*^1 E$. Thus $\omega \in \mathcal{CK}E \Rightarrow \omega \in F \subseteq E$, for some $F \in \mathcal{F}^* \Rightarrow \omega \in F \subseteq B_*^1 E$, for some 1-evident $F \Rightarrow \omega \in C^1E$, by lemma 2.5.

(b) Under the truth axiom, if F is 1-evident then $F \in \mathcal{F}^*$. Under the truth axiom, if F is 1-evident and $F \subseteq B_*^1 E$, then $F \subseteq E$. Thus $\omega \in C^1E \Rightarrow \omega \in F \subseteq B_*^1 E$, for some 1-evident $F \Rightarrow \omega \in F \subseteq E$, for some $F \in \mathcal{F}^* \Rightarrow \omega \in \mathcal{CK}E$. $\square$

2.5. The Relation Between Alternative Notions for $p < 1$

The equivalence of the alternative notions of approximate common knowledge does not, in general, hold if $p < 1$. This is because the belief operator typically fails to satisfy the distributive property that if event E is believed with probability at least p, and event F is believed with probability at least p, then event $E \cap F$ is believed with probability at least p, so it is possible that $B_i^p E \cap B_i^p F \neq B_i^p (E \cap F)$.

The following two lemmas show that common p-belief is in general a stronger notion than either iterated p-belief or weak common p-belief.

Lemma 2.14. *For all events E and $p \in (0, 1]$: $C^pE \subseteq I^pE$.*

Proof. For any event E and individual i, $B_*^p E \subseteq B_i^p E$. Thus $B_*^p B_*^p E \subseteq B_2^p B_1^p E \cap B_1^p B_2^p E$; by an iterative argument, we have $[B_*^p]^{2n-1}(E) \subseteq B_1^p [B_2^p B_1^p]^{n-1} E \cap B_2^p [B_1^p B_2^p]^n E$ and $[B_*^p]^{2n}(E) \subseteq$

$[B_2^p B_1^p]^n E \cap [B_1^p B_2^p]^n E$, for all $n \geq 1$. So

$$C^p E \equiv \bigcap_{n \geq 1} [B_*^p]^n E \subseteq I_1^p E \cap I_2^p E \equiv I^p E. \quad \square$$

Lemma 2.15. *For all events E and $p \in (0,1]$: $C^p E \subseteq W^p E$.*

Proof. $C^p E \equiv C_{\mathcal{Q}}^p E \subseteq \bigcup_{\mathcal{Q}' \in \mathcal{C}(\mathcal{Q})} C_{\mathcal{Q}'}^p E \equiv W^p E.$ $\square$

2.5.1. The Unbounded State Space Case

With no restrictions on the size of the state space - , there need be no connection between common p-belief and the two weaker variants. In particular, we have:

Remark 2.16. *For all $1/2 < r \leq p < 1$ and $0 < \varepsilon < 1$, it is possible to construct an information system containing an event E with $P[I^p E] \geq 1 - \varepsilon$, $P[W^p E] \geq 1 - \varepsilon$ and $C^r E = \emptyset$.*

This is shown by the following example:

Example 2.17. *This example is parameterized by $1/2 < r \leq p < 1$ and $0 < \varepsilon < 1$. Write N for the smallest integer satisfying $N \geq Max\left\{\frac{1}{2r-1}, \frac{2}{1-p}\right\}$ and M for the smallest integer satisfying $M \geq Max\left\{\frac{N^{2(N+1)}}{\varepsilon}, \frac{N^{2(N+1)}}{1-p}\right\}$. Each individual i observes a signal $s_i \in S = \{1, ..., N+M\}$. A state consists of the pair of signals observed by the two individuals, so $\omega \equiv (s_1, s_2)$ and - $\equiv S^2$. Individuals' partitions reflect the fact that they observe only their own signals. Thus $Q_i((s_1, s_2)) = \{(s_1', s_2') \in$ - $: s_i = s_i'\}$. Let $P(\omega) = \pi(\omega) \Big/ \sum_{\omega' \in} \pi(\omega')$, for all $\omega \in$ - , where π is defined as follows:*[5]

[5]Formally, we have $\pi(s_1, s_2) = N^{2(N-n)+1}$, if $s_1 = s_2 = n$ and $n = 1, ..., N$; $\pi(s_1, s_2) = N^{2(N-n)}$, if $s_1 = n+1$, $s_2 = n$ and $n = 1, ..., N$; $\pi(s_1, s_2) = N$, if $s_1 = s_2 = n$ and $n = N, ..., N+M-1$; $\pi(s_1, s_2) = N$, if $s_1 = n+1$, $s_2 = n$ and $n = N+1, ..., N+M-2$; $\pi(s_1, s_2) = 1$, if $s_1 = N+1$ and $s_2 = 1, ..., N$; $\pi(s_1, s_2) = 1$, if $s_1 = N+M$ and $s_2 = N+1, .., N+M-1$; $\pi(1, N+M) = N^{2N}$; $\pi(N+M, N+M) = N^{2N+1}$; $\pi(s_1, s_2) = 0$, otherwise.

	1	2	3	·	N-1	N	N+1	N+2	·	N+M-2	N+M-1	N+M
1	N^{2N-1}	0	0	·	0	0	0	0	·	0	0	N^{2N}
2	N^{2N-2}	N^{2N-3}	0	·	0	0	0	0	·	0	0	0
3	0	N^{2N-4}	N^{2N-5}	·	0	0	0	0	·	0	0	0
·	·	·	·	·	·	·	·	·	·	·	·	·
N-1	0	0	0	·	N^3	0	0	0	·	0	0	0
N	0	0	0	·	N^2	N	0	0	·	0	0	0
N+1	1	1	1	·	1	1	N	0	·	0	0	0
N+2	0	0	0	·	0	0	N	N	·	0	0	0
·	·	·	·	·	·	·	·	·	·	·	·	·
N+M-2	0	0	0	·	0	0	0	0	·	N	0	0
N+M-1	0	0	0	·	0	0	0	0	·	N	N	0
N+M	0	0	0	·	0	0	1	1	·	1	1	N^{2N+1}

The following notion will be useful. Let X be some collection of possible signals, i.e., $X \subseteq \{1, 2, .., N+M\}$. Write $E_i^+(X)$ for the set of states where individual i's signal is in X, i.e., $E_i^+(X) = \{(s_1, s_2) : s_i \in X\}$; and write $E_i^-(X)$ for the set of states where individual i's signal is *not* in X, i.e., $E_i^-(X) = \{(s_1, s_2) : s_i \notin X\}$. Let $E^* = E_1^-(1)$.

We first characterize $I^p E^*$. Some calculations for this example are summarized in the appendix on page 24; in particular, the following properties of the operator B_i^p are verified:

$$B_2^p\left(E_1^-(n)\right) = E_2^-(n), \text{ for all } n = 1, .., N, \tag{2.1}$$

$$B_1^p\left(E_2^-(n)\right) = E_1^-(n+1), \text{ for all } n = 1, .., N-1, \tag{2.2}$$

$$B_1^p\left(E_2^-(N)\right) = \text{-} \; . \tag{2.3}$$

Since $E^* \in \mathcal{F}_1$, $B_1^p E^* = E^*$ (by B1); by (2.1), $B_2^p B_1^p E^* = B_2^p E^* = E_2^-(1)$; by (2.2), $B_1^p B_2^p B_1^p E^* = B_1^p B_2^p E^* = E_1^-(2)$; by (2.1), $[B_2^p B_1^p]^2 E^* = B_2^p B_1^p B_2^p E^* = E_2^-(2)$. Iteratively applying (2.1) and (2.2) gives

$$[B_1^p B_2^p]^{n-1} B_1^p E^* = [B_1^p B_2^p]^{n-1} E^* = E_1^-(n) \text{ and } [B_2^p B_1^p]^n E^* = [B_2^p B_1^p]^{n-1} B_2^p E^* = E_2^-(n). \tag{2.4}$$

for all $n = 1, .., N$. By (2.4) and (2.3),

$$[B_1^p B_2^p]^N E^* = B_1^p \left[[B_2^p B_1^p]^{N-1} B_2^p E^* \right] = B_1^p \left[E_2^- (N) \right] = - \ .$$

Thus $I_1^p E^* = E_1^- (1, ..., N)$, $I_2^p E^* = E_2^- (1, ..., N)$ and $I^p E^* = E_1^- (1, ..., N) \cap E_2^- (1, ..., N)$. So $P[I^p E] = P[E_1^- (1, ..., N) \cap E_2^- (1, ..., N)] \geq 1 - \varepsilon$ (see appendix).

Now we characterize $C^r E^*$. The following properties of the operator B_i^r are verified in the appendix:

$$B_2^r \left(E_1^- (1, .., n) \right) \subseteq E_2^- (n) \text{, for all } n = 1, .., N + M, \tag{2.5}$$

$$B_1^r \left(E_2^- (1, .., n) \right) \subseteq E_1^- (n+1) \text{, for all } n = 1, .., N + M - 1. \tag{2.6}$$

Now by B1 and (2.5),

$$B_*^r E^* = B_1^r E^* \cap B_2^r E^* \subseteq E_1^- (1) \cap E_2^- (1) \ . \tag{2.7}$$

$$\begin{aligned} \text{So } [B_*^r]^2 E &= B_1^r B_*^r E \cap B_2^r B_*^r E \\ &\subseteq B_1^r \left[E_1^- (1) \cap E_2^- (1) \right] \cap B_2^r \left[E_1^- (1) \cap E_2^- (1) \right] \text{, by (2.7) and B4} \\ &= E_1^- (1) \cap B_1^r E_2^- (1) \cap B_2^r E_1^- (1) \cap E_2^- (1) \text{, by B2} \\ &\subseteq E_1^- (1,2) \cap E_2^- (1) \text{, by (2.6).} \end{aligned}$$

Iteratively applying (2.5) and (2.6), we have

$$\begin{aligned} [B_*^r]^{2n-2} E^* &\subseteq E_1^- (1, .., n) \cap E_2^- (1, .., n-1) \\ \text{and } [B_*^r]^{2n-1} E^* &\subseteq E_1^- (1, .., n) \cap E_2^- (1, .., n) \ . \end{aligned}$$

for all $n = 2, ..., N + M$. Thus $C^r E^* = [B_*^r]^{2N+2M-1} E^* = \emptyset$.

Finally observe that $P[E^*] \geq p$ (see appendix), so, by corollary 2.10, $W^p E^* =$ - and $P[W^p E^*] = 1$.

The assumption in remark 2.16 that $r > 1/2$ is important: if $r < 1/2$, then event E is common r-belief with high probability whenever it is iterated p-belief with high probability.

Remark 2.18. *If $r < 1/2$ and $r \leq p < 1$, then for all events E: $P[C^r E] \geq 1 - (1 - P[I^p E]) \left(\frac{1-r}{1-2r} \right)$.*

Proof. Kajii and Morris (1995) have shown that if $r < 1/2$, then for every simple event F:

$$P\left[C^r F\right] \geq 1 - (1 - P[F]) \left(\frac{1-r}{1-2r}\right). \tag{2.8}$$

But $B_*^p E$ is simple and $I^p E \subseteq B_*^p E$. So

$$\begin{aligned} P\left[C^r E\right] &= P[C^r B_*^r E] \\ &\geq P[C^r B_*^p E] \\ &\geq 1 - (1 - P[B_*^p E]) \left(\frac{1-r}{1-2r}\right), \text{ by (2.8)} \\ &\geq 1 - (1 - P[I^p E]) \left(\frac{1-r}{1-2r}\right). \quad \square \end{aligned}$$

On the other hand, for any $0 < r \leq p < 1$, it is possible to construct an information system with $\omega \in I^p E$ but $\omega \notin C^r E$, for some state ω and event E.

Remark 2.19. *For all $1/2 < r \leq p < 1$ and $\varepsilon > 0$, it is possible to construct an information system containing an event E with $P\left[W^p E\right] = 1$ and $I^r E = C^r E = \emptyset$.*

Consider example 2.7, with $\delta < \min\{\varepsilon, 1-p\}$. For any $r \geq 1/2$, $B_1^r E^* = E^*$, $B_2^r B_1^r E^* = B_2^r E^* = \{2, 3, ..\}$, $B_1^r B_2^r B_1^r E^* = B_1^r B_2^r E^* = \{3, 4, ..\}$, etc.. Thus $I^r E^* = C^r E^* = \emptyset$. But $P\left[E^*\right] = 1 - \delta > p$, so E^* is weakly p-evident, $W^p E^* = E^*$ and $P\left[W^p E^*\right] = P[E^*] = 1 - \delta \geq 1 - \varepsilon$.

2.5.2. The Bounded State Space Case

If the state space is bounded, proposition 2.2 can be used to give a bound on the difference between iterated p-belief and common p-belief.

Proposition 2.20. *Suppose - has n elements. Then for all events E: $I^p E \subseteq C^{1-2^n(1-p)} E$ and $I^{1-2^{-n}(1-p)} E \subseteq C^p E$.*

Propostion 2.20 implies in particular that for any $p < 1$, there exists some $q < 1$ (which depends on p and n) such that whenever an event is iterated q-belief, it is also common p-belief.

Proof. Suppose $E^1, ..., E^K$ is an arbitrary collection of events and $\omega \in B_i^p E^1 \cap B_i^p E^2 \cap .. \cap B_i^p E^K$. Then $P\left[E^k \middle| Q_i(\omega)\right] \geq p$ for each $k \Rightarrow P\left[\left(- \backslash E^k\right) \middle| Q_i(\omega)\right] \leq 1 - p$ for each $k \Rightarrow$

$P\left[\bigcup_{k=1}^{K}\left(- \setminus E^k\right)\middle| Q_i(\omega)\right] \leq K(1-p) \Rightarrow P\left[\bigcap_{k=1}^{K} E^k \middle| Q_i(\omega)\right] = \left[- \setminus\left(\bigcup_{k=1}^{K}\left(- \setminus E^k\right)\right)\middle| Q_i(\omega)\right] \geq$ $1-K(1-p)$. Thus $\omega \in B_i^{1-K(1-p)}\left(E^1 \cap E^2 \cap .. \cap E^K\right)$; so

$$B_i^p E^1 \cap B_i^p E^2 \cap .. \cap B_i^p E^K \subseteq B_i^{1-K(1-p)}\left(E^1 \cap E^2 \cap .. \cap E^K\right). \tag{2.9}$$

By proposition 2.2, $\omega \in I^p E$ implies there exists a mutually p-evident collection of events $\mathcal{E}$ with $\omega \in A = \bigcap_{F \in \mathcal{E}} F$, $B_1^p E \in \mathcal{E}$ and $B_2^p E \in \mathcal{E}$; thus $A \subseteq B_i^p E \subseteq B_i^{1-K(1-p)} E$ for each i; $\mathcal{E}$ has at most 2^n elements, so, by (2.9), $A = \bigcap_{F \in \mathcal{E}} F \subseteq \bigcap_{F \in \mathcal{E}} B_i^p F \subseteq B_i^{1-2^n(1-p)}\left(\bigcap_{F \in \mathcal{E}} F\right) = B_i^{1-2^n(1-p)} A$ for each i: thus A is $(1-2^n(1-p))$-evident and $\omega \in C^{1-2^n(1-p)} E$ by proposition 2.5. □

This result gives a (very loose) lower bound on the number of states required to allow a given divergence between iterated and common p-belief. Thus if there are n states and there exists a state ω with $\omega \in I^p(E)$ and $\omega \notin C^r(E)$, then corollary 2.20 implies that $r \geq 1 - 2^n(1-p)$, so that $n \geq \log_2(1-r) - \log_2(1-p)$. For example, if $p = 0.999$ and $r = 0.501$, then we must have $n \geq 9$. On the other hand, example 2.17 gives a (very loose) upper bound on the number of states required to allow a given divergence. If $p = 0.999$ and $r = 0.501$ (and $\varepsilon \geq 0.001$), then the construction of example 2.17 has approximately 5×10^{13213} states!

3. Applications

3.1. Game Theory

To illustrate the significance of approximate common knowledge in game theory, I will focus on simple examples. In particular, I will be interested in symmetric two player, two action, games with two strict Nash equilibria:

$\mathcal{G}$	0	1
0	x_{00}, x_{00}	x_{01}, x_{10}
1	x_{10}, x_{01}	x_{11}, x_{11}

where $x_{00} > x_{10}$, $x_{11} > x_{01}$. The best response dynamics are completely characterized by the probability q such that each player is indifferent between his two actions if the other plays action

0 with probability q, i.e.,

$$q = \frac{(x_{11} - x_{01})}{(x_{11} - x_{01}) + (x_{00} - x_{10})}$$

Our analysis will depend only on the parameter q. Thus our analysis of the general game $\mathcal{G}$ would be the same if we restricted attention to:

$\mathcal{G}'$	0	1
0	$1-q, 1-q$	$0, 0$
1	$0, 0$	q, q

3.1.1. Best Response Dynamics and Iterated p-Belief

Two individuals are endowed with the information structure discussed earlier in the paper. They are playing the (degenerate) incomplete information game where each has the two actions 0 and 1 available and payoffs are always given by the matrix $\mathcal{G}$. A pure strategy for individual i would usually be written as a $\mathcal{Q}_i$-measurable function $\sigma_i : - \to \{a, b\}$. I will find it useful, however, to identify an individual's strategy with the set of states where he plays action 0, i.e., $E_i = \{\omega : \sigma_i(\omega) = 0\}$. Player i's pure strategy set is thus $\mathcal{F}_i$.

I want to study the incomplete information game best response dynamics. Assume that q is generic so that there is a unique best response. Suppose individual 1 is choosing strategy E_1, i.e., playing 0 at all states in E_1 and 1 at all states not in E_1. We can characterize best response functions in terms of belief operators. If 2 assigns probability more than q to the event E_1, his best response is to play 0; if he assigns probability less than q, his best response would be to play 1. Thus $B_2^q E_1$ is 2's best response to E_1 and $B_1^q E_2$ is 1's best response to E_2. Thus we have a best response function, $\rho : \mathcal{F}_1 \times \mathcal{F}_2 \to \mathcal{F}_1 \times \mathcal{F}_2$, with $\rho(E_1, E_2) = (B_1^q E_2, B_2^q E_1)$. Now we have:

Proposition 3.1. *If players initially chose strategies (E_1, E_2) and revise their strategies by best response dynamics, then action profile $(0, 0)$ is always played if and only if the events E_1 and E_2 are iterated q-belief, i.e.,*

$$\bigcap_{n \geq 0} [[\rho^n]_1 (E_1, E_2) \cap [\rho^n]_2 (E_1, E_2)] = I^q E_1 \cap I^q E_2.$$

Proof. First observe that (by B2) $I_1^q E_1 = E_1 \cap B_1^q B_2^q E_1 \cap ..$; while $I_1^q E_2 = B_1^q E_2 \cap B_1^q B_2^q B_1^q E_2 \cap ..$, so

$$\begin{aligned} \underset{n\geq 0}{\cap} [\rho^n]_1 (E_1, E_2) &= E_1 \cap B_1^q E_2 \cap B_1^q B_2^q E_1 \cap B_1^q B_2^q B_1^q E_2 \cap .. = I_1^q E_1 \cap I_1^q E_2 \\ \text{and } \underset{n\geq 0}{\cap} [\rho^n]_2 (E_1, E_2) &= E_2 \cap B_2^q E_1 \cap B_2^q B_1^q E_2 \cap B_2^q B_1^q B_2^q E_1 \cap .. = I_2^q E_1 \cap I_2^q E_2. \end{aligned}$$

Thus $\underset{n\geq 0}{\cap} [[\rho^n]_1 (E_1, E_2) \cap [\rho^n]_2 (E_1, E_2)] = I_1^q E_1 \cap I_1^q E_2 \cap I_2^q E_1 \cap I_2^q E_2 = I^q E_1 \cap I^q E_2$. □

The following example is in the spirit of Rubinstein (1989).

Example 3.2. *Let the information structure be that of example 2.7. Suppose that initially player* 2 *played action* 0 *everywhere except at states* 0 *and* 1, *and player* 1 *played action* 0 *everywhere except at state* 0. *Thus* $E_1 = \{1, 2, ..\}$ *and* $E_2 = \{2, 3, ..\}$. *Now suppose that* $q > 1/2$, *so that* $(1, 1)$ *is the risk dominant equilibrium of the game. Then best response dynamics gives us:*

$$\rho^n (E_1, E_2) = \begin{cases} (\{n+1, n+2, ...\}, \{n+2, n+3....\}), \text{ if } n \text{ is even} \\ (\{n+2, n+3, ...\}, \{n+1, n+2....\}), \text{ if } n \text{ is odd} \end{cases}.$$

Thus $\underset{n\geq 0}{\cap} ([\rho^n]_1 (E_1, E_2) \cap [\rho^n]_2 (E_1, E_2)) = I^q E_1 \cap I^q E_2 = \emptyset$.

The incomplete information game best response dynamics do not have a natural interpretation. However, Morris (1996) shows that incomplete information game best response dynamics are formally equivalent to best response dynamics in a certain class of local interaction games. The latter has been a topic of recent research.

3.1.2. Equilibrium, Iterated Deletion of Dominated Strategies and Common p-Belief

Consider the following related problem. Suppose individuals are endowed again with the information system discussed earlier. Now they are playing an incomplete information game where payoffs are given by the matrix $\mathcal{G}$ at all states, *except* that each individual i has a dominant strategy to play 1 at all states not in event $E_i \in \mathcal{F}_i$. As before, identify individual i's strategy with the set of states where he plays 0.

Proposition 3.3. *$\left(B_1^q C^q\left(E_1 \cap E_2\right), B_2^q C^q\left(E_1 \cap E_2\right)\right)$ is a pure strategy equilibrium of this game. On the other hand, if pure strategy F_i survives iterated deletion of strictly dominated strategies, then $F_i \subseteq B_i^q C^q\left(E_1 \cap E_2\right)$.*

This is a version of results in Morris, Rob and Shin (1995). Monderer and Samet (1989) first proved general results relating common p-belief to equilibria of incomplete information games.

Proof. [1] I will show that strategy $B_1^q C^q\left(E_1 \cap E_2\right)$ is a best response to $B_2^q C^q\left(E_1 \cap E_2\right)$. If $\omega \in B_1^q C^q\left(E_1 \cap E_2\right) \subseteq E_1$, player i attaches probability at least q to player 2 choosing action 0. Since payoffs are given by matrix $\mathcal{G}$, action 0 is a best response. If $\omega \in E_1 \backslash B_1^q C^q\left(E_1 \cap E_2\right)$, player i attaches probability at most q to player 2 choosing action 0. Since payoffs are given by matrix $\mathcal{G}$, action 1 is a best response. Finally, if $\omega \notin E_1$, action 1 is a dominant action.

[2] Let $\mathcal{U}_i^n \subseteq \mathcal{F}_i$ be the set of player i strategies which survive n rounds iterated deletion of strictly interim dominated strategies. Clearly, $F_i \in \mathcal{U}_i^1 \Rightarrow F_i \subseteq E_i$. I will show by induction on $n \geq 2$ that $F_i \in \mathcal{U}_i^n \Rightarrow F_i \subseteq B_i^q\left[B_*^q\right]^{n-2}\left(E_1 \cap E_2\right)$. Suppose $F_i \in \mathcal{U}_i^2$: since $\mathcal{U}_i^2 \subseteq \mathcal{U}_i^1$, $F_i \subseteq E_i$; since player i attaches positive probability only to strategies $F_j \subseteq E_j$, we must have $F_i \subseteq B_i^q E_j$. So $F_i \subseteq E_i \cap B_i^q E_j = B_i^p(E_1 \cap E_2)$ (by B2). Now suppose that the inductive hypothesis is true for n. Suppose $F_i \in \mathcal{U}_i^{n+1}$: since $\mathcal{U}_i^{n+1} \subseteq \mathcal{U}_i^n$, $F_i \subseteq B_i^q\left[B_*^q\right]^{n-2}\left(E_1 \cap E_2\right)$; since player i attaches positive probability only to strategies $F_j \subseteq B_j^q\left[B_*^q\right]^{n-2}\left(E_1 \cap E_2\right)$, we must have $F_i \subseteq B_i^q B_j^q\left[B_*^q\right]^{n-2}\left(E_1 \cap E_2\right)$. So $F_i \subseteq B_i^q\left[B_*^q\right]^{n-2}\left(E_1 \cap E_2\right) \cap B_i^q B_j^q\left[B_*^q\right]^{n-2}\left(E_1 \cap E_2\right) = B_i^q\left(\left[B_*^q\right]^{n-1}\left(E_1 \cap E_2\right)\right)$ (by B2). □

Example 3.4. *Suppose the game $\mathcal{G}$ is as follows:*

$\mathcal{G}$	*0*	*1*
0	$9,9$	$-10,0$
1	$0,-10$	$0,0$

Note that $q = 10/19$, so equilibrium $(1,1)$ is (just) risk dominant; but equilibrium $(0,0)$ is Pareto-dominant. Now suppose the information structure is given by example 2.17, where $1/2 < r < 10/19$. Let player 2's payoffs always be given by matrix $\mathcal{G}$; player 1's payoffs are given by matrix $\mathcal{G}$, except that player has a dominant strategy to play action 1 if he observes signal 1. Thus $E_1 = E^* = E_1^-(1)$ and $E_2 =$ - . Since $C^q\left(E_1 \cap E_2\right) \subseteq C^r\left(E_1 \cap E_2\right) = C^r\left(E_1^-(1)\right) = \emptyset$ (by B4), the unique strategy *surviving iterated deletion of dominated strategies for each player is $\emptyset$. Thus action 0 is never played despite the fact that it is iterated p-belief that payoffs are given by $\mathcal{G}$, with high probability and for any p.*

3.2. Agreeing to Disagree, No Trade and Weak Common p-Belief

Write $\mathcal{X}$ for the set of functions, $x : - \to [0,1]$. Let $\mathbf{E}(x|F)$ be the expected value of $x \in \mathcal{X}$ given event F with $P[F] > 0$:

$$\mathbf{E}(x|F) = \frac{\left(\sum_{\omega \in F} x(\omega) P(\omega)\right)}{\left(\sum_{\omega \in F} P(\omega)\right)}.$$

Let $\mathbf{E}_i$ be the expectation operator for individual i, so that $\mathbf{E}_i(x|\omega) = \mathbf{E}(x|Q_i(\omega))$. Let $\Pi_i^+(x,q)$ be the set of states where individual i's expected value of x is at least q, let $\Pi_i^-(x,q)$ be the set of states where individual i's expected value of x is at most q, and let $\Pi_i(x,q)$ be the set of states where individual i's expected value of x is exactly q:

$$\begin{aligned} \Pi_i^+(x,q) &= \{\omega : \mathbf{E}_i(x|\omega) \geq q\} \\ \Pi_i^-(x,q) &= \{\omega : \mathbf{E}_i(x|\omega) \leq q\} \\ \Pi_i(x,q) &= \{\omega : \mathbf{E}_i(x|\omega) = q\} = \Pi_i^+(x,q) \cap \Pi_i^-(x,q) \end{aligned}$$

Let $\mathcal{T}(x,q_1,q_2)$ be the set of states where individual 1's expected value of x is at least q_1, while individual 2's expected value is no more than q_2:

$$\mathcal{T}(x,q_1,q_2) = \Pi_1^+(x,q_1) \cap \Pi_2^-(x,q_2) = \{\omega : \mathbf{E}_1(x|\omega) \geq q_1 \text{ and } \mathbf{E}_2(x|\omega) \leq q_2\}.$$

If $\mathcal{T}(x,q_1,q_2)$ is empty for all $x \in \mathcal{X}$ and all q_1 and q_2 with q_1 significantly bigger than q_2, then we say there is *approximate no trade.*

Let $\mathcal{D}(x,q_1,q_2)$ be the set of states where individual 1's expected value of x is exactly q_1, while individual 2's expected value is exactly q_2.

$$\mathcal{D}(x,q_1,q_2) = \Pi_1(x,q_1) \cap \Pi_2(x,q_2) = \{\omega : \mathbf{E}_1(x|\omega) = q_1 \text{ and } \mathbf{E}_2(x|\omega) = q_2\}$$

Thus 1 and 2 disagree by $|q_1 - q_2|$ about the expected value of x. If $\mathcal{D}(x,q_1,q_2)$ is empty for all $x \in \mathcal{X}$ and all q_1 and q_2 with $|q_1 - q_2|$ large, then we say there is *approximate agreement.*

Proposition 3.5. *If there is weak common p-belief that individuals are prepared to trade, then the gains from trade must be small for large p. Specifically, if* $W^p(\mathcal{T}(x, q_1, q_2)) \neq \emptyset$*, then* $q_1 - q_2 \leq 2(1-p)$.

Since $\mathcal{D}(x, q_1, q_2) \subseteq \mathcal{T}(x, q_1, q_2)$, the trade result extends to agreeing to disagree.

Corollary 3.6. *If there is weak common p-belief that individuals disagree, then the disagreement must be small for large p. Specifically, if* $W^p(\mathcal{D}(x, q_1, q_2)) \neq \emptyset$*, then* $q_1 - q_2 \leq 2(1-p)$.

Monderer and Samet (1989) first proved a version of corollary 3.6, for common p-belief. Neeman (1996) improved the bound to $1-p$. Geanakoplos (1994) observed that essentially the same proof works for weak common p-belief. Sonsino (1995) showed a version of proposition 3.5 for common p-belief (but see Neeman (1995) for an argument why results like proposition 3.5 implicitly assume that individuals are irrational).

Proof. (of proposition 3.5). First observe that $\mathcal{T}(x, q_1, q_2)$ is a simple event, by construction. Thus $W^p(\mathcal{T}(x, q_1, q_2))$ is non-empty if and only if there exists $F_1 \in \mathcal{F}_1 \backslash \emptyset$ and $F_2 \in \mathcal{F}_2 \backslash \emptyset$ with $F_1 \subseteq \Pi_1^+(x, q_1)$, $F_2 \subseteq \Pi_2^-(x, q_2)$, $P[F_1|F_2] \geq p$ and $P[F_2|F_1] > p$. Observe that

$$\begin{aligned} q_1 &\leq \mathbf{E}(x|F_1) \\ &= \mathbf{E}(x|F_1 \cap F_2).P[F_2|F_1] + \mathbf{E}(x|(F_1 \backslash F_2)).(1 - P[F_2|F_1]) \\ &\leq \mathbf{E}(x|F_1 \cap F_2).P[F_2|F_1] + 1 - P[F_2|F_1] \\ &\leq \mathbf{E}(x|F_1 \cap F_2) + (1-p), \end{aligned}$$

$$\begin{aligned} \text{while } q_2 &\geq \mathbf{E}(x|F_2) \\ &= \mathbf{E}(x|F_1 \cap F_2).P[F_1|F_2] + \mathbf{E}(x|(F_2 \backslash F_1)).(1 - P[F_1|F_2]) \\ &\geq \mathbf{E}(x|F_1 \cap F_2).P[F_1|F_2] \\ &\geq \mathbf{E}(x|F_1 \cap F_2) - (1-p). \end{aligned}$$

Thus $q_1 - q_2 \leq 2(1-p)$. □

The following is a partial converse to proposition 3.5.

Proposition 3.7. *Suppose E is a finite simple event, $P[E] > 0$ and $W^p E = \emptyset$. Then there exists $x \in \mathcal{X}$ such that $E \subseteq \mathcal{D}(x, 1/2 + (1/4)(1-p), 1/2 - (1/4)(1-p))$.*

Since $\mathcal{D}(x, q_1, q_2) \subseteq \mathcal{T}(x, q_1, q_2)$, the agreeing to disagree result extends to trade.

Corollary 3.8. *Suppose E is a finite simple event, $P[E] > 0$ and $W^p E = \emptyset$. Then there exists $x \in \mathcal{X}$ such that $E \subseteq \mathcal{T}(x, 1/2 + (1/4)(1-p), 1/2 - (1/4)(1-p))$.*

Proof. Write $E = E_1 \cap E_2$, each $E_i \in \mathcal{F}_i$. Write $\mathcal{Q}_i^* = \{F \in \mathcal{Q}_i : F \subseteq E_i\}$ and $T_i = \mathcal{Q}_i^* \cup \{- \setminus E_i\}$. Note that T_i is a finite partition of - which coarsens $\mathcal{Q}_i$. For any $(F_1, F_2) \in T_1 \times T_2$, let $\pi(F_1, F_2) = \sum\limits_{\omega \in F_1 \cap F_2} P(\omega)$. Consider the following linear programming problem. Choose $y : T_1 \times T_2 \to [0,1]$ and $\delta \in [0, 1/2]$ to maximize δ subject to

$$\begin{array}{l} \text{[i]} \sum\limits_{F_2 \in T_2} y(F_1, F_2)\, \pi(F_1, F_2) = \left(\frac{1}{2} + \delta\right) \sum\limits_{F_2 \in T_2} \pi(F_1, F_2), \text{ for all } F_1 \in \mathcal{Q}_1^* \\ \text{[ii]} \sum\limits_{F_1 \in T_1} y(F_1, F_2)\, \pi(F_1, F_2) = \left(\frac{1}{2} - \delta\right) \sum\limits_{F_1 \in T_1} \pi(F_1, F_2), \text{ for all } F_2 \in \mathcal{Q}_2^* \\ \text{[iii]} \; y(F_1, F_2) \geq 0, \text{ for all } (F_1, F_2) \in T_1 \times T_2 \\ \text{[iv]} \; y(F_1, F_2) \leq 1, \text{ for all } (F_1, F_2) \in T_1 \times T_2 \end{array} \tag{3.1}$$

Observe first that the maximand δ is less than $1/2$. If $\delta = 1/2$, then we would have $\mathbf{E}(x|E_1) = 1$ and $\mathbf{E}(x|E_2) = 0$, which implies $P[E_1 \cap E_2] = 0$, a contradiction.

By standard linear programming arguments, we have that if (y, δ) is a solution to this problem, we must have $\lambda_1 : T_1 \to \Re$, $\lambda_2 : T_2 \to \Re$, $\zeta : T_1 \times T_2 \to \Re_+$ and $\xi : T_1 \times T_2 \to \Re_+$, such that:

$$\begin{array}{l} \text{[i]} \; \lambda_1(F_1)\pi(F_1, F_2) - \lambda_2(F_2)\pi(F_1, F_2) + \zeta(F_1, F_2) - \xi(F_1, F_2) = 0, \text{ for all } (F_1, F_2) \in \mathcal{Q}_1^* \times \mathcal{Q}_2^*. \\ \text{[ii]} \; \lambda_1(F_1)\pi(F_1, - \setminus E_2) + \zeta(F_1, - \setminus E_2) - \xi(F_1, - \setminus E_2) = 0, \text{ for all } F_1 \in \mathcal{Q}_1^*. \\ \text{[iii]} \; -\lambda_2(F_2)\pi(- \setminus E_1, F_2) + \zeta(- \setminus E_1, F_2) - \xi(- \setminus E_1, F_2) = 0, \text{ for all } F_2 \in \mathcal{Q}_2^*. \\ \text{[iv]} \; \zeta(F_1, F_2) > 0 \Rightarrow y(F_1, F_2) = 0. \\ \text{[v]} \; \xi(F_1, F_2) > 0 \Rightarrow y(F_1, F_2) = 1. \end{array} \tag{3.2}$$

First suppose that $\lambda_i(F_i) \leq 0$ for all $F_i \in \mathcal{Q}_i^*$ and both i. Then δ would remain a solution if we replace [i] and [ii] in (3.1) with:

$$\begin{array}{l} \text{[i']} \sum_{F_2 \in T_2} y(F_1,F_2)\,\pi(F_1,F_2) \le \left(\frac{1}{2}+\delta\right) \sum_{F_2 \in T_2} \pi(F_1,F_2)\,, \text{ for all } F_1 \in \mathcal{Q}_1^*. \\ \text{[ii']} \sum_{F_1 \in T_1} y(F_1,F_2)\,\pi(F_1,F_2) \ge \left(\frac{1}{2}-\delta\right) \sum_{F_1 \in T_1} \pi(F_1,F_2)\,, \text{ for all } F_2 \in \mathcal{Q}_2^*. \end{array} \quad (3.3)$$

But this revised problem has solution 1/2 (e.g., set $y(F_1, F_2) = 1/2$, for all $(F_1, F_2) \in T_1 \times T_2$). This contradicts our earlier result that $\delta < 1/2$.

Now suppose that there exists F_i^* with $\lambda_i(F_i^*) > 0$ and $\lambda_i(F_i^*) > \lambda_j(F_j)$ for all $F_j \in \mathcal{Q}_j^*$. Without loss of generality, take $i = 1$. But now parts [i] and [ii] of (3.2) imply that $\xi(F_1^*, F_2) > 0$ for all $F_2 \in T_2$; so by part [v] of (3.2), $y(F_1^*, F_2) = 1$ for all $F_2 \in T_2$; so by part [i] of (3.1), $\delta = 1/2$, again a contradiction.

So if we let λ^* be the largest value in the range of λ_1 and λ_2, and let $\mathcal{Q}_i^{**} = \{F_i \in \mathcal{Q}_i^* : \lambda_i(F_i) = \lambda^*\}$, we know that each $\mathcal{Q}_i^{**}$ is non-empty. By (3.2), we must have $y(F_1, F_2) = 1$ if $F_1 \in \mathcal{Q}_1^{**}$ and $F_2 \notin \mathcal{Q}_2^{**}$; and $y(F_1, F_2) = 0$ if $F_2 \in \mathcal{Q}_2^{**}$ and $F_1 \notin \mathcal{Q}_1^{**}$. So parts [i] and [ii] of (3.1) become:

$$\begin{array}{l} \text{[i]} \sum_{F_2 \in \mathcal{Q}_2^{**}} y^*(F_1,F_2)\,\pi(F_1,F_2) + \sum_{F_2 \in T_2 \setminus \mathcal{Q}_2^{**}} \pi(F_1,F_2) = \left(\frac{1}{2}+\delta\right) \sum_{F_2 \in T_2} \pi(F_1,F_2)\,, \text{ for all } F_1 \in \mathcal{Q}_1^{**}. \\ \text{[ii]} \sum_{F_1 \in \mathcal{Q}_1^{**}} y^*(F_1,F_2)\,\pi(F_1,F_2) = \left(\frac{1}{2}-\delta\right) \sum_{F_1 \in T_1} \pi(F_1,F_2)\,, \text{ for all } F_2 \in \mathcal{Q}_2^{**}. \end{array}$$

Now let $F_i^* = \bigcup_{F_i \in \mathcal{Q}_i^{**}} F_i$, $\alpha = \mathbf{E}(x|F_1^* \cap F_2^*)$, $p_i = P\left[F_j^*|F_i^*\right]$ for each $j \neq i$ and $x(\omega) = y^*(T_1(\omega), T_2(\omega))$, where $T_i(\omega)$ is the element of T_i containing state ω.

We have $1/2 - \delta = \mathbf{E}_2(x|F_2^*) \le \alpha$, so

$$\begin{aligned} 1/2 + \delta &= \mathbf{E}_1(x|F_1^*) \\ &= p_1\alpha + (1 - p_1) \\ &\ge p_1(1/2 - \delta) + (1 - p_1). \end{aligned}$$

Re-arranging gives $\delta \ge (1/2)(1 - p_1)/(1 + p_1) \ge (1/4)(1 - p_1)$. Analogously, we have $1/2 + \delta = \mathbf{E}_1(x|F_1^*) \ge \alpha$, so

$$\begin{aligned} 1/2 - \delta &= \mathbf{E}_2(x|F_2^*) \\ &= p_2\alpha \\ &\le p_2(1/2 + \delta)\,. \end{aligned}$$

Re-arranging gives $\delta \geq (1/2)(1-p_2)/(1+p_2) \geq (1/4)(1-p_2)$. But since $E_1 \cap E_2$ is not weakly p-evident, we must have either p_1 or p_2 less than p. Thus $\delta \geq (1/4)(1-p)$. □

4. Notes

1. **The Common Prior Assumption**. Throughout this paper, I assumed the common prior assumption. For the characterizations of common p-belief, iterated p-belief and the game theoretic results, it made no difference. On the other hand, the characterization of weak common p-belief and the no trade / agreement results depend on the common prior. Assuming a common prior made it harder to show the large divergence between common p-belief and iterated p-belief.

2. **Many Individuals**. I focussed on the case of two individuals for simplicity. Many of the results generalize to many individuals. For example, iterated p-belief is naturally defined (hierarchically) as follows. Let $\mathcal{I}$ be a collection of individuals, each with a partition $\mathcal{Q}_i$ giving belief operator B_i^p. Let $F(n)$ be the collection of functions $f : \{1, ..., n\} \to \mathcal{I}$. Define

$$I^p E \equiv \bigcap_{n \geq 1,\ f \in F(n)} B_{f(1)}^p B_{f(2)}^p .... B_{f(n)}^p E.$$

Say that collection of events $\mathcal{E} \subseteq 2$ is mutually p-evident if $B_i^p F \in \mathcal{E}$ for all $F \in \mathcal{E}$. Then proposition 2.2 remains true essentially as stated: Event E is iterated p-belief at ω if and only if there exists a mutually p-evident collection of events $\mathcal{E}$ with [1] $B^p E \in \mathcal{E}$ for all $i \in \mathcal{I}$; and [2] $\omega \in F$, for all $F \in \mathcal{E}$.

3. **Yet Another Notion of Approximate Common Knowledge**. Börgers (1994) and Monderer and Samet (1990) use the following notion of *repeated common p-belief*:

$$R^p E \equiv B_*^p E \cap B_*^p \left(E \cap B_*^p E\right) \cap B_*^p \left(E \cap B_*^p \left(E \cap B_*^p E\right)\right) \cap ...$$

More formally, define an operator $B_*^p(.;E) : 2 \to 2$ by $B_*^p(F;E) = B_1^p(F \cap E) \cap B_2^p(F \cap E)$, and let $R^p E = \bigcap_{n \geq 1} \left[B_*^p(.;E)\right]^n E$. By definition, for all events E, $R^p E \subseteq C^p E$. As noted by Monderer and Samet (1990), we have $C^p E \subseteq R^{2p-1} E$ for all events E.[6] Thus common p-belief and repeated common p-belief will deliver the same results for p close to 1.

4. **The Relation Between Iterated p-Belief and Weak Common p-Belief**. Common p-belief implies iterated p-belief and weak common p-belief (see lemmas 2.14 and 2.15). Iterated p-belief may be much weaker than common p-belief (see remark 2.16). Weak common p-belief may be much weaker than both iterated p-belief and common p-belief (see remark 2.19). However, we did not show whether iterated p-belief is necessarily a stronger requirement than weak common p-belief. Specifically, for arbitrary $1/2 < r \leq p < 1$, is it possible to have $\omega \in I^p E$ but $\omega \notin W^r E$? This remains an open question.

5. **Attaining Approximate Common Knowledge**. Can approximate common knowledge be achieved in practice? Consider first common p-belief. Presumably it is not possible to infinite levels of belief by decentralized communication. In practice, it is the observation of almost public events (i.e., p-evident events) that will generate common p-belief. Iterated p-belief has no such natural fixed point characterization. Thus it seems unlikely that iterated p-belief (with p close to one) will often be attained in settings where common p-belief is not (our examples notwithstanding). On the other hand, if there is a common prior, we have given one natural sufficient condition for an event to be weak common p-belief with high probability: it is enough that the event has high ex ante probability (see corollary 2.10). No almost public event is required to attain weak common p-belief.

References

[1] Aumann, R.: Agreeing to Disagree. *The Annals of Statistics* 4, 1236-1239 (1976)

[2] Börgers, T.: Weak Dominance and Approximate Common Knowledge. *Journal of Economic Theory* 64, 265-276 (1994)

[6] $\omega \in R^p E$ if and only if there exists F such that $\omega \in F \subseteq B^p_* (E \cap F)$; but if $\omega \in C^p E$, then there exists F with $\omega \in F$ and, for both i, $F \subseteq B^p_i F$ and $F \subseteq B^p_i E$; the latter implies $F \subseteq B^{2p-1}_i (E \cap F)$ for both i (see equation 2.9), so $\omega \in R^{2p-1} E$.

[3] Brandenburger, A. and Dekel E.: Common Knowledge with Probability One. *Journal of Mathematical Economics* 16, 237-245 (1987)

[4] Geanakoplos, J.: Common Knowledge. chapter 40 of *Handbook of Game Theory, Volume 2*, edited by R. Aumann and S. Hart. Amsterdam: Elsevier Science (1994)

[5] Kajii, A. and Morris S.: The Robustness of Equilibria to Incomplete Information. *CARESS Working Paper #95-19* (1995)

[6] Lewis, D.: *Conventions*. Cambridge, MA: Harvard University Press (1969)

[7] Monderer, D. and Samet D.: Approximating Common Knowledge with Common Beliefs. *Games and Economic Behavior* 1, 170-190 (1990)

[8] Monderer, D. and Samet D.: Proximity of Information in Games with Incomplete Information. forthcoming in *Mathematics of Operations Research* (1990)

[9] Morris, S.: Strategic Local Interaction. (1996)

[10] Morris, S., Postlewaite, A. and Shin, H.: Depth of Knowledge and the Effect of Higher Order Uncertainty. *Economic Theory* 6, 453-467 (1995)

[11] Morris, S., Rob, R. and Shin, H.: p-Dominance and Belief Potential. *Econometrica* 63, 145-147 (1995)

[12] Neeman, Z.: Common Beliefs and the Existence of Speculative Trade. forthcoming in *Games and Economic Behavior* (1995)

[13] Neeman, Z.: Approximating Agreeing to Disagree Results with Common p-Beliefs. *Games and Economic Behavior* 12, 162-164 (1996)

[14] Rubinstein, A.: The Electronic Mail Game: Strategic Behavior under Almost Common Knowledge. *American Economic Review* 79, 385-391 (1989)

[15] Sonsino, D.: 'Impossibility of Speculation' Theorems with Noisy Information. *Games and Economic Behavior* 8, 406-423 (1995)

[16] Stinchcombe, M.: Approximate Common Knowledge. (1988)

5. Appendix

We present some properties used in example 2.17.

First note that the definitions of N and M imply:

[1] $\dfrac{1}{N+1} \leq \dfrac{2}{N+2} \leq \dfrac{1}{2}$; [2] $\dfrac{N+1}{2N+1} \leq r$; [3] $\dfrac{N^{2N+1}}{N^{2N+1}+M} \leq \dfrac{1}{2}$; [4] $\dfrac{N}{N+1} \geq p$; and [5] $\dfrac{2N-1}{2N} \geq p$.

These inequalities will be used extensively in the following calculations.

Ex Ante Probabilities

- $P\left[E_1^-(1,..,N) \cap E_2^-(1,..,N)\right] \geq \max\left(1-\varepsilon, p\right)$.

Write $F = E_1^-(1,..,N) \cap E_2^-(1,..,N)$.

$$\begin{aligned} \pi\left[F\right] &= M - 1 + (2M-3)N + N^{2N+1} \\ &\geq M \\ &\geq \tfrac{N^{2(N+1)}}{\varepsilon}. \end{aligned}$$

$$\begin{aligned} \pi\left[\text{-} \setminus F\right] &= N - 1 + \left(1 + N + .. + N^{2N}\right) \\ &= N - 1 + \tfrac{N^{2N+1}-1}{N-1} \\ &\leq N + N^{2N+1} \\ &\leq N^{2(N+1)}. \end{aligned}$$

Thus $P\left[F\right] = \frac{\pi[F]}{\pi[F]+\pi[\ \setminus F]} \geq \frac{N^{2(N+1)}/\varepsilon}{N^{2(N+1)}/\varepsilon + N^{2(N+1)}} = \frac{1}{1+\varepsilon} \geq 1-\varepsilon$. A symmetric argument shows $P[F] \geq p$.

- $P\left[E^*\right] \geq p$.

$E_1^-(1,..,N) \cap E_2^-(1,..,N) \subseteq E^*$, so $P\left[E^*\right] \geq P\left[E_1^-(1,..,N) \cap E_2^-(1,..,N)\right] \geq p$.

Properties of B_i^p

- $B_2^p\left(E_1^-(n)\right) = E_2^-(n)$, for all $n = 1, .., N$.

For $n = 1$,

$$P\left[E_1^-(n)\middle|E_2^+(m)\right] = \begin{cases} \frac{N^{2N-2}+1}{N^{2N-1}+N^{2N-2}+1} \leq \frac{2}{N+2} \leq \frac{1}{2} < p, \text{ if } m = 1 \\ \frac{N^{2N+1}}{N^{2N}+N^{2N+1}} = \frac{N}{N+1} \geq p, \text{ if } m = M+N \\ 1, \text{ for all other } m \end{cases}.$$

For $n = 2, .., N-1$,

$$P\left[E_1^-(n)\middle|E_2^+(m)\right] = \begin{cases} \frac{N^{2(N-n)}+1}{N^{2(N-n)+1}+N^{2(N-n)}+1} \leq \frac{2}{N+2} \leq \frac{1}{2} < p, \text{ if } m = n \\ \frac{N^{2(N-n)+1}+1}{N^{2(N-n)+1}+N^{2(N-n)}+1} \geq \frac{N}{N+1} \geq p, \text{ if } m = n-1 \\ 1, \text{ for all other } m \end{cases}.$$

For $n = N$,

$$P\left[E_1^-(n)\middle|E_2^+(m)\right] = \begin{cases} \frac{1}{N+1} \leq \frac{1}{2} < p, \text{ if } m = N \\ \frac{N^3+1}{N^3+N^2+1} \geq \frac{N}{N+1} \geq p, \text{ if } m = N-1 \\ 1, \text{ for all other } m \end{cases}.$$

- $B_1^p\left(E_2^-(n)\right) = E_1^-(n+1)$, for all $n = 1, .., N-1$.

For $n = 1, .., N-1$,

$$P\left[E_2^-(n)\middle|E_1^+(m)\right] = \begin{cases} \frac{1}{N+1} \leq \frac{1}{2} < p, \text{ if } m = n+1 \\ \frac{N}{N+1} \geq p, \text{ if } m = n \\ \frac{2N-1}{2N} \geq p, \text{ if } m = N+1 \\ 1, \text{ for all other } m \end{cases}.$$

- $B_1^p\left(E_2^-(N)\right) =$ - .

$$P\left[E_2^-(N)\middle|E_1^+(m)\right] = \begin{cases} \frac{N}{N+1} \geq p, \text{ if } m = N \\ \frac{2N-1}{2N} \geq p, \text{ if } m = N+1 \\ 1, \text{ for all other } m \end{cases}.$$

Properties of B_i^r

- $B_2^r\left(E_1^-(1,..,n)\right) \subseteq E_2^-(n)$, for all $n = 1,..,N+M$.

For $n = 1,..,N-1$,

$$P\left[E_1^-(1,..,n)\middle|E_2^+(n)\right] = \frac{N^{2(N-n)}+1}{N^{2(N-n)+1}+N^{2(N-n)}+1} \leq \frac{2}{N+2} \leq \frac{1}{2} < r.$$

For $n = N$,

$$P\left[E_1^-(1,..,n)\middle|E_2^+(n)\right] = \frac{1}{N+1} \leq \frac{1}{2} < r.$$

For $n = N+1,...,N+M-2$,

$$P\left[E_1^-(1,..,n)\middle|E_2^+(n)\right] = \frac{N+1}{2N+1} < r.$$

For $n = N+M-1$,

$$P\left[E_1^-(1,..,n)\middle|E_2^+(n)\right] = \frac{1}{N+1} \leq \frac{1}{2} < r.$$

For $n = N+M$,

$$P\left[E_1^-(1,..,n)\middle|E_2^+(n)\right] = 0 < r.$$

- $B_1^r\left(E_2^-(1,..,n)\right) \subseteq E_1^-(n+1)$, for all $n = 1,..,N+M-1$.

For $n = 1,..,N-1$,

$$P\left[E_2^-(1,..,n)\middle|E_1^+(n+1)\right] = \frac{1}{N+1} \leq \frac{1}{2} < r.$$

For $n = N,...,N+M-2$,

$$P\left[E_2^-(1,..,n)\middle|E_1^+(n+1)\right] = \frac{1}{2} < r.$$

For $n = N+M-1$,

$$P\left[E_2^-(1,..,n)\middle|E_1^+(n+1)\right] = \frac{N^{2N+1}}{N^{2N+1}+M} \leq \frac{1}{2} < r.$$