CARESS Working Paper #96-01

Multiplicity of Equilibria

Christian Ghiglino[*]
University of Geneva

Mich Tvede
University of Copenhagen

Abstract

In the present paper a pure exchange, general equilibrium model is considered and the equilibrium set is studied. It is shown for all total endowments and an open and dense set of preferences that if there are $l \geq 2$ commodities and $m \geq 2$ consumers then there exists a set of distributions of endowments with nonempty interior such that the associated economies have at least $l-1+\min\{l,m\}$ equilibria for $l+\min\{l,m\}$ even and at least $l-2+\min\{l,m\}$ equilibria for $l+\min\{l,m\}$ odd.

Keywords: General Equilibrium, Multiplicity of Equilibria.

JEL-classification: D51.

Correspondence: Mich Tvede, Institute of Economics, University of Copenhagen, Studiestraede 6, DK-1455 Copenhagen K, Denmark.

[*]Research support from the Center for Analytic Economics, the Center for Applied Mathematics and the Department of Economics, Cornell University, is gratefully acknowledged

1 Introduction

In general, economic agents have to coordinate their actions in order to make them mutually compatible and thereby obtain a feasible state. This coordination among agents can be obtained through a form of consistency of expectations, i.e. all agents expect the same state to prevail and therefore make mutually compatible actions. In economies with a unique equilibrium it is natural that agents expect the same state to prevail, namely the equilibrium, thus uniqueness of equilibrium leads to coordination. However for economies with multiple equilibria it is less clear how and on what agents coordinate, so multiplicity of equilibria can lead to market failures due to lack of coordination.

Market failures can take the form of sunspot equilibria, where agents coordinate their actions but the outcome of the coordination is determined by "sunspots" which are stochastic variables. Therefore it is hardly surprising that sunspot equilibria are closely related to multiplicity of equilibria. Indeed multiplicity of equilibria combined with restricted market participation lead to the existence of sunspot equilibria in general equilibrium economies as shown by Balasko, Cass and Shell in [3].

Clearly the number of equilibria depend on the form of the excess demand function for the economy in question and a crucial issue in relation to excess demand functions is whether Walras' law and homogeneity characterize the set of excess demand functions. This question was studied by Sonnenschein, Mantel and Debreu [4, 7, 9] and the most general answer was probably given by Debreu: *Any continuous function, which satisfies Walras' law and homogeneity and has compact domain in the set of positive prices, is excess demand function for some economy with at most as many consumers as commodities.* However, in order to obtain this result total endowments as well as preferences are considered to be parameters.

Excess demand functions' lack of characteristics may leave the impression that little can be said about economies beyond the theorems on existence of equilibrium and the welfare theorems. However if economies are parametrized by endowments, i.e. preferences are fixed, then they possess some structure as shown by Balasko in [2] and using this structure the issue of multiplicity of equilibria can be addressed. The fact that there exists a set of distributions of endowments with nonempty interior such that the associated economies have at least as many equilibria as commodities has been used in [3] to establish the existence of sunspot equilibria, but no formal theorem has been stated. Moreover differentiable topology can be used to show that the set of excess demand functions in some sense is a small subset of the set of functions which satisfy Walras' law and homogeneity and have the set of positive prices as domain as done by Balasko in [1].

In the present paper economies are parametrized by the distribution of endowments, i.e. total endowments and preferences are considered to be fixed, and within this framework the relation between distributions of endowments and multiplicity of equilibria is studied. Through a fibration of the equilibrium set and an application of index theory it is shown that if there are $l \geq 2$ commodities and $m \geq 2$ consumers then there exists a set of distributions of endowments with nonempty interior such that the associated economies have at least $l - 1 + \min\{l, m\}$ equilibria for $l + \min\{l, m\}$ even and at least $l - 2 + \min\{l, m\}$ equilibria for $l + \min\{l, m\}$ odd. The result is established for consumption sets without lower bounds, but subsequently it is extended to consumption sets with lower bounds through a translation of endowments and preferences. The result of the present paper deviates from the results of Debreu, Mantel and Sonnenschein in the sense that total endowments as well as preferences are considered to be fixed and in the sense that only two consumers are needed independently of the number of commodities.

The results of the present paper can be applied to static economies with restricted market participation in order to find economies with sunspot equilibria as already mentioned. For stationary dynamic economies the results can be applied to find economies with multiple steady states because stationary dynamic economies can be reduced to static economies whose equilibria are steady states as shown by Kehoe and Levine in [6]. However while the relation between local indeterminacy of steady states and the existence of sunspot equilibria has been explored by Spear, Srivastava and Woodford in [10] among others, the relation between multiplicity of steady states and sunspot equilibria is less explored. Moreover there is no relation between multiplicity of steady states and local indeterminacy of steady states.

The paper is organized as follows: In section 2 the model is introduced; in section 3 the equilibrium set is endowed with a fibre bundle structure and some useful results are established; in section 4 index theory is introduced and the main result is established; in section 5 the main result is extended to consumption sets with lower bounds; and finally in section 6 some final remarks are offered.

2 The Model

A general equilibrium model with $l \geq 2$ commodities and $m \geq 2$ consumers is considered. Consumer i is described by her consumption set X_i, her endowment ω_i and her utility function u_i. The consumers are supposed to verify the following assumptions

–

(A.1) $X_i = \mathbf{R}^l$.

(A.2) $\omega_i \in \mathbf{R}^l$.

(A.3) $u_i \in C^\infty(X_i, \mathbf{R})$ and $u_i^{-1}(a)$ is bounded from below for all $a \in \mathbf{R}$.

(A.4) u_i has positive derivatives, $Du_i \in C^\infty(X_i, \mathbf{R}^l_{++})$.

(A.5) u_i has negative definite Hessian, $y^T D^2 u_i(x) y \in \mathbf{R}_{--}$, for all $x \in X_i$ and $y \in \mathbf{R}^l \setminus \{0\}$.

All assumptions are standard for the differentiable general equilibrium model except (A.5) which is a little stronger than the usual assumption of strict quasi-concavity. The set of utility functions is endowed with the Whitney topology and the set of economies is endowed with the product topology. In section 5 it is discussed how the results can be extended to models in which consumption sets are bounded from below.

Consumer i maximizes her utility function subject to her budget constraint

$$\begin{array}{ll} \max & u_i(x) \\ \text{s.t.} & p \cdot x_i \;=\; p \cdot \omega_i \;=\; w_i, \end{array}$$

where $x_i \in \mathbf{R}^l$ is her consumption and $p \in \mathbf{R}^l_{++}$ is the price. The solution to this problem is her demand and if the price is varied then her demand function $f_i : \mathbf{R}^l_{++} \times \mathbf{R} \to \mathbf{R}^l$ is obtained. The demand function satisfies the budget constraint and homogeneity and it is smooth. If

$$\sum_{i=1}^{m} f_i(p, p \cdot \omega_i) - \omega_i \;=\; 0$$

then $(p, \omega_1, \ldots, \omega_m)$ is an equilibrium.

3 Fibration of the Equilibrium Set

It seems to be hard to study how the number of equilibria is related to the distribution of endowments due to the fact that demand functions depend on endowments in a quite complicated way. Therefore, the equilibrium set is endowed with a fibre bundle structure, see [2].

Definition 1 *For $(p, w) \in \mathbf{R}^l_{++} \times \mathbf{R}^m$ the fibre $F(p, w) \subset \mathbf{R}^l_{++} \times \mathbf{R}^{lm}$ consists of all pairs (p, ω) for which*

(B.1) $p \cdot \omega_i = w_i$ *for all* $i \in \{1, \ldots, m\}$.

(B.2) $\sum_{i=1}^m f_i(p, w_i) = \sum_{i=1}^m \omega_i$.

It is easy to verify that fibres are linear manifolds of dimension $(l-1)(m-1)$ embeded in the equilibrium set. Call the projection of fibres, $F(p, w)$, on their second coordinates ω-fibres, $G(p, w)$, i.e. $F(p, w) = \{p\} \times G(p, w)$.

Every fibre contains one and only one no trade equilibrium characterized by $\omega_i = f_i(p, w_i)$ for all $i \in \{1, \ldots, m\}$, while there is some exchange for all other distributions of endowments.

Lemma 1 *Suppose that*

(C.1) $\sum_{i=1}^m f_i(p(h), w_i(h)) = r$ *for all* $h \in \{1, \ldots, n\}$.

(C.2) *rank* $[\ p(1)\ \cdots\ p(n)\ ] = n$.

then the intersection of the n ω-fibres $(G(p(h), w(h)))_{h=1}^n$ is a non-empty linear manifold of dimension $(l-n)(m-1)$.

Proof The intersection is characterized by the following system of equations

$$\begin{aligned} p(h) \cdot \omega_i &= w_i(h) \text{ for all } h \text{ and } i \\ \sum_{i=1}^m \omega_i &= r \end{aligned}$$

Clearly, if the n prices are linearly independent then $l - n$ coordinates of endowment for the consumers are determined by the other n coordinates and the endowment of one consumer is determined by endowments of the other $m - 1$ consumers. Q.E.D.

According to lemma 1 no ω-fibres intersect if and only if all Pareto optimal allocations are supported by the same prices. Thus the question of whether or not economies have multiplicity of equilibria is equivalent to the question of whether or not economies have Pareto optimal allocations which are supported by different prices.

Lemma 2 *Suppose that the economy $(r, (u_i)_{i=1}^m)$ has n Pareto optimal allocations $(p(h), x(h))_{h=1}^n \in \mathbf{R}_{++}^l \times \mathbf{R}^{lm}$ with*

$$\textit{rank}\ [\ p(1)\ \cdots\ p(n)\ ]\ =\ n,$$

where the $p(h)$'s are supporting prices, then there exists a distribution of endowments on all ω-fibres with $\sum_{i=}^m f_i(p, w_i) = r$ for which the associated economy has at least n equilibria.

Proof There exists a $h \in \{1, \ldots, n\}$ such that

$$\text{rank}\ [\ p(1)\ \cdots\ p(h-1)\ p\ p(h+1)\ \cdots\ p(n)\ ]\ =\ n,$$

According to lemma 1 the intersection of the n fibres is non-empty and economies with distributions of endowments in the intersection have n equilibria. Q.E.D.

Remark Lemma 2 can be deduced from [3].

As shown in the next lemma most economies have Pareto optimal allocations which are supported by different prices, thus most economies have multiple equilibria.

Lemma 3 *For all $r \in \mathbf{R}^l$ and all $n \in \{1, \ldots, l\}$ there exists an open and dense set of utility functions such that the economy $(r, (u_i)_{i=1}^m)$ has n Pareto optimal allocations $(p(h), x(h))_{h=1}^n \subset \mathbf{R}_{++}^l \times \mathbf{R}^{lm}$ with*

$$\textit{rank}\ [\ p(1)\ \cdots\ p(n)\ ]\ =\ n,$$

where the $p(h)$'s are supporting prices.

Proof There is an isomorphism between Pareto optimal allocations and $\prod_{i=2}^{m}]-\infty,\bar{u}_i[$, where $\bar{u}_i=\sup_x u_i(x)$ for $i\in\{2,\ldots,m\}$, due to the fact that Pareto optimal allocations are solutions to the following problem

$$\begin{array}{rl} \max & u_1(x_1) \\ \text{s.t.} & u_2(x_2) \ \geq \ u_2,\ldots,u_m(x_m) \ \geq \ u_m \\ & \sum_{i=1}^{m} x_i \ = \ r \end{array}$$

where $(u_2,\ldots,u_m)\in\prod_{i=2}^{m}]-\infty,\bar{u}_i[$.

"Openness" This part of the proof is omitted due to its triviality.

"Density" All economies $(r,(u_i)_{i=1}^{m})$ have a continuum of Pareto allocations. Consider n pairs of prices and Pareto optimal allocations $(p(h),(x_i(h))_{i=1}^{m})_{h=1}^{n}$ with $x_i(h)\neq x_i(h')$ for all i and $h\neq h'$.

Let $j_y\in C^\infty(\mathbf{R}^l,[0,1])$ be a partition of unity with center y, outer radius r_o and inner radius r_i, i.e.

$$j_y(x) \ = \ \begin{cases} 0 & \text{for } \|y-x\|\geq r_o \\ 1 & \text{for } \|y-x\|\leq r_i \end{cases}$$

where $r_o=\frac{1}{3}\min\|x_i(h)-x_i(h')\|$ and $r_i=\frac{1}{2}r_o$. For all $(y)_{i=1}^{m}, z\in\mathbf{R}^l$ there exists $\epsilon\in\mathbf{R}_{++}$ such that $u_i(x)+\delta\|Du_i(y_i)\|j_{y_i}(x)z\cdot x$ satisfies (A.3), (A.4) and (A.5) for all $i\in\{1,\ldots,m\}$ and all $\delta<\epsilon$.

Suppose that

$$\begin{array}{rcl} \text{rank}\ [\ p(1)\ \cdots\ p(h)\ ] & = & h \\ p(h+1) & \in & \text{span}\ [\ p(1)\ \cdots\ p(h)\ ] \end{array}$$

and perturb $u_i(x)$ into $u_i(x)+\delta\|Du_i(x_i(h+1))\|j_{x_i(h+1)}(x)z\cdot x$ then $(x_i(h+1))_{i=1}^{m}$ is also a Pareto optimal allocation for the "perturbed" economy because the derivatives of the perturbed utility functions are proportional. If $z\notin$ span $[\ p(1)\ \cdots\ p(h)\ ]$ then $Du_i(x_i(h+1))+\delta\|Du_i(x_i(h+1))\|z\notin[\ p(1)\ \cdots\ p(h)\ ]$ and $(x_i(h+1))_{i=1}^{m}$ is supported with a linearly independent price. Q.E.D.

Lemma 1, lemma 2 and lemma 3 can be combined in order to obtain the following corollary.

Corollary 1 *For all total endowments and an open and dense set of utility functions there exist distributions of endowments on all ω-fibres such that the associated economies have at least as many equilibria as the number of commodities.*

In order to obtain stronger results than corollary 1 demand functions as well as properties of the equilibrium set have to be exploited.

Due to Walras' law, only equilibrium conditions for the first $l-1$ commodities have to be considered and due to homogeneity only prices for the first $l-1$ commodities have to be considered, i.e. the price for the last commodity can be normalized to one. Let g_i denote the demand function for consumer i without the last coordinate and let q denote the price without the last coordinate.

On fibres the Jacobian matrices of the excess demand functions are affine functions, i.e.

$$D_q \sum_i g_i(p, p\cdot\omega_i) - \omega_i \;=\; M + KY$$

where

$$M = \begin{bmatrix} \sum_{i=1}^{m} \frac{\partial f_i^1}{\partial p^1} + \frac{\partial f_i^1}{\partial w_i} f_i^1 & \cdots & \sum_{i=1}^{m} \frac{\partial f_i^1}{\partial p^l} + \frac{\partial f_i^1}{\partial w_i} f_i^{l-1} \\ \vdots & & \vdots \\ \sum_{i=1}^{m} \frac{\partial f_i^{l-1}}{\partial p^1} + \frac{\partial f_i^{l-1}}{\partial w_i} f_i^1 & \cdots & \sum_{i=1}^{m} \frac{\partial f_i^{l-1}}{\partial p^{l-1}} + \frac{\partial f_i^{l-1}}{\partial w_i} f_i^{l-1} \end{bmatrix}$$

$$K = \begin{bmatrix} \frac{\partial f_1^1}{\partial w_1} - \frac{\partial f_m^1}{\partial w_m} & \cdots & \frac{\partial f_{m-1}^1}{\partial w_{m-1}} - \frac{\partial f_m^1}{\partial w_m} \\ \vdots & & \vdots \\ \frac{\partial f_1^{l-1}}{\partial w_1} - \frac{\partial f_m^{l-1}}{\partial w_m} & \cdots & \frac{\partial f_{m-1}^{l-1}}{\partial w_{m-1}} - \frac{\partial f_m^{l-1}}{\partial w_m} \end{bmatrix}$$

$$Y = \begin{bmatrix} \omega_1^1 - f_1^1 & \cdots & \omega_1^{l-1} - f_1^{l-1} \\ \vdots & & \vdots \\ \omega_{m-1}^1 - f_{m-1}^1 & \cdots & \omega_{m-1}^{l-1} - f_{m-1}^{l-1} \end{bmatrix}$$

thus M and K are constant on fibres.

Consider $l-1$ fibres $(F(p(h), w(h)))_{i=1}^{l-1}$, if

$$\text{rank}\ [\ q(1)\ \cdots\ q(l-1)\ ]\ =\ l-1,$$

then the endowments for consumer i have to satisfy

$$(\omega_i^1, \ldots, \omega_i^{l-1})\ =\ (w_i(1) - \omega_i^l, \ldots, w_i(l-1) - \omega_i^l)Q^{-1}$$

in order to be in the intersection of the ω-fibres, where

$$Q\ =\ [\ q(1)\ \cdots\ q(l-1)\ ].$$

Hence on the intersection of the $l-1$ fibres the determinants of the Jacobian matrices are affine functions, i.e.

$$|\ D_q \sum_{i=1}^{m} g_i(p(h), p(h) \cdot \omega_i) - \omega_i\ |\ =\ s(h) + \sum_{i=1}^{m-1} s_i(h)\omega_i^l$$

for $h \in \{1, \ldots, l-1\}$. In the second part of the appendix the coefficients $s(h), s_1(h), \ldots, s_{m-1}(h)$ are calculated.

4 The Main Theorem

At an equilibrium (p, ω), if

$$\text{rank}\ D_q \sum_{i=1}^{m} g_i(p, p \cdot \omega_i)\ =\ l-1$$

then there exist an open neighborhood, - , of ω and a smooth function, $p : -\ \to \mathbf{R}_{++}^{l-1} \times \{1\}$, such that

$$\sum_{i=1}^{m} g_i(p(\omega), p(\omega) \cdot \omega_i) - \omega_i\ =\ 0$$

for all $\omega \in -$ and the equilibrium is called regular, otherwise it is called singular. The index of a regular equilibrium is

$$\text{i}(p, \omega)\ =\ (-1)^{l-1}\text{sign}\ |\ D_q \sum_{i=1}^{m} g_i(p, p \cdot \omega_i)\ |.$$

and if all equilibria of an economy are regular then the sum of their indices is $+1$ according to the Poincaré-Hopf Theorem, see [8].

Theorem 1 *For all prices and incomes, (p, w), and an open and dense set of utility functions, $(u_i)_{i=1}^m$, there exists a set of distributions of endowments with nonempty interior in the ω-fibre associated with the prices and incomes, $G(p, w)$, such the associated economies have at least $l - 1 + \min\{l, m\}$ equilibria for $l + \min\{l, m\}$ even and at least $l - 2 + \min\{l, m\}$ equilibria for $l + \min\{l, m\}$ odd.*

Proof Let $r = \sum_i f_i(p, w_i)$ and consider $l - 1$ Pareto optimal allocations, $(p(h), (x_i(h))_{i=1}^m)_{h=1}^{l-1}$ with $p(1) = p$, $w_i(1) = w_i$ for all i and $x_i(h) \neq x_i(h')$ for all i and $h \neq h'$ for the economy $(r, (u_i)_{i=1}^m)$.

First, suppose that

$$\text{rank } Q \; = \; l - 1,$$

and let endowments be in the intersection of the $l-1$ ω-fibre associated with the $l - 1$ Pareto optimal allocations, then the determinants of the Jacobian matrices for the $l - 1$ equilibria are affine functions, i.e.

$$\mid D_q \sum_{i=1}^m g_i(p(h), p(h) \cdot \omega_i) \mid \; = \; s(h) + \sum_{i=1}^{m-1} s_i(h) \omega_i^l$$

for all $h \in \{1, \ldots, l - 1\}$.

Second, if $m > l$ then fix endowments for some of the consumers such that that $m = l$ and suppose that rank $S = m - 1$, where

$$S \; = \; \begin{bmatrix} s_1(1) & \cdots & s_{m-1}(1) \\ \vdots & & \vdots \\ s_1(m-1) & \cdots & s_{m-1}(m-1) \end{bmatrix},$$

then there exist ρ, μ and ν in the intersection of the $l - 1$ ω-fibres such that

$$s(h) + \sum_{i=1}^{m-1} s_i(h) \rho_i^l \;\; = \;\; 0$$

$$s(h) + \sum_{i=1}^{m-1} s_i(h) \mu_i^l \;\; < \;\; 0$$

$$s(h) + \sum_{i=1}^{m-1} s_i(h) \nu_i^l \;\; > \;\; 0$$

for all $h \in \{1, \ldots, m-1\}$. Moreover for all $\epsilon \in \mathbf{R}_{++}$ there exist μ and ν with the desired properties and $\|\rho - \mu\| \leq \epsilon$ and $\|\rho - \nu\| \leq \epsilon$.

Thirdly, suppose for ρ that either: 1) There are at least l regular equilibria for $l+m$ even or at least $l-1$ regular equilibria for $l+m$ odd or; 2) all equilibria but the first $m-1$ equilibria are regular and thus can be written $(p_h(\omega), \omega)$ where $h \in \{m, \ldots, n\}$, $p_h \in C^\infty(- \ , \mathbf{R}^l_{++})$ and - is a neighborhood of ρ. Let μ and ν in - be such that

$$i(p_h(\rho), \rho) \;=\; i(p_h(\mu), \mu) \;=\; i(p_h(\nu), \nu)$$

for all $h \in \{m, \ldots, n\}$ and suppose for either μ or ν that either: 3) there are at least $l - 1 + \min\{l, m\}$ regular equilibria for $l + m$ even or at least $l - 2 + \min\{l, m\}$ regular equilibria for $l + m$ odd or; 4) all equilibria are regular.

In case either 1) or 3) is satisfied the claim of the theorem is established. In case neither 1) nor 3) is satisfied the argument is slightly more complicated. Due to the fact that $(\rho_i, u_i)_{i=1}^m$ has at least $l-1$ equilibria and to the Poincaré-Hopf Theorem: On the one hand if the sum of the indices of all equilibria except the $m-1$ equilibria associated with the first $m-1$ Pareto allocations is even then either $(\mu_i, u_i)_{i=1}^m$ or $(\nu, u_i)_{i=1}^m$ has at least m more equilibria than $(\rho_i, u_i)_{i=1}^m$ and; on the other hand if the sum of the indices of all equilibria except the $m-1$ equilibria associated with the first $m-1$ Pareto optimal allocations is odd then either $(\mu_i, u_i)_{i=1}^m$ or $(\nu, u_i)_{i=1}^m$ has at least $m-1$ more equilibria than $(\rho_i, u_i)_{i=1}^m$.

In the appendix it is shown that Q has rank $l-1$, S has rank $m-1$ and the equilibria at ρ, μ and ν have the desired properties with respect to regularity. Q.E.D.

Since the perturbations involved in the proof of theorem 1 do not change demand at (p, w), prices and incomes can be replaced with total endowments in the theorem.

Corollary 2 *For all total endowments, r, and an open and dense set of utility functions, $(u_i)_{i=1}^m$, there exists a set of distributions of endowments with nonempty interior such the associated economies have at least $l - 1 + \min\{l, m\}$ equilibria for $l + \min\{l, m\}$ even and at least $l - 2 + \min\{l, m\}$ equilibria for $l + \min\{l, m\}$ odd.*

5 Consumption Sets with Lower Bounds

The analysis has been restricted to economies with consumption sets without lower bounds, however by a translation of preferences and endowments it is extended to consumption sets with lower bounds.

Suppose the economy $(X_i, \omega_i, u_i)_{i=1}^m$ has at least $l-1+\min\{l,m\}$ equilibria for $l+\min\{l,m\}$ even and at least $l-2+\max\{l,m\}$ equilibria for $l+\min\{l,m\}$ odd and $\omega_i \in \mathbf{R}^l$ for all $i \in \{1,\ldots,m\}$. Let y_i be a lower bound for the indifference surface through ω_i and let $h_i : \mathbf{R}^l \to \mathbf{R}_{++}$ be the ratio between the norm of x and the norm of the point in the intersection between the straight line through origin and x and the indifference surface through ω_i translated by $-y_i$, i.e.

$$h_i(x) \;=\; \frac{\|x\|}{\|\{y|\exists t : tx = y \text{ and } u_i(y+y_i) = u_i(\omega_i)\}\|}.$$

Consider the economy $(\hat{X}_i, \hat{\omega}_i, \hat{u}_i)_{i=1}^m$, where $\hat{X}_i = \mathbf{R}^l_{++}$, $\hat{\omega}_i = \omega_i - y_i$ and

$$\hat{u}_i(x) \;=\; \begin{cases} h_i(x)^{\frac{1}{2}} u_i(\omega_i) & \text{for} \quad u_i(x+y_i) < u_i(\omega_i) \\ \\ u_i(x+y_i) & \text{for} \quad u_i(x+y_i) \geq u_i(\omega_i) \end{cases}$$

Clearly consumption sets are bounded from below and the utility functions are strictly concave and continuous. Moreover utility functions are smooth except at the indifference surface through $\hat{\omega}_i$. Let $\hat{x}_i(p,\hat{\omega}_i)$ be the consumption on the indifference surface through $\hat{\omega}_i$ with support p, i.e. $\hat{x}_i(p,\hat{\omega}_i)$ is the solution to

$$\begin{aligned} D\hat{u}_i(x) - \lambda p &= 0 \\ \\ \hat{u}_i(x) - \hat{u}_i(\hat{\omega}_i) &= 0, \end{aligned}$$

then the demand function is

$$\hat{f}_i(p,w_i) \;=\; \begin{cases} \dfrac{w_i}{p \cdot \hat{x}_i(p,\hat{\omega}_i)} \hat{x}_i(p,\hat{\omega}_i) & \text{for} \quad w_i < p \cdot \hat{x}_i(p,\hat{\omega}_i) \\ \\ f_i(p, w_i + p \cdot y_i) - y_i & \text{for} \quad w_i \geq p \cdot \hat{x}_i(p,\hat{\omega}_i) \end{cases}$$

Thus the excess demand functions for the two economies $(X_i, \omega_i, u_i)_{i=1}^m$ and $(\hat{X}_i, \hat{\omega}_i, \hat{u}_i)_{i=1}^m$ are identical, so they have the same equilibria. Corollary 2 and the construction in the present section can be combined to the following corollary for economies with consumption sets with lower bounds.

Corollary 3 *For an open set of utility functions, $(u_i)_{i=1}^m$, there exists a set of distributions of endowments with nonempty interior such the associated economies have at least $l-1+\min\{l,m\}$ equilibria for $l+\min\{l,m\}$ even and at least $l-2+\min\{l,m\}$ equilibria for $l+\min\{l,m\}$ odd.*

6 Final Remarks

In the present paper multiplicity of equilibria has been studied, but no characterization of the set of the distributions of endowments for which the associated economies have multiplicity of equilibria has been offered. However simple examples, e.g. $l=2$ and $m=2$ and $l=2$ and $m=3$, suggest that the set is quite large. In case $l=2$ and $m=2$ fibres are straight lines and on most fibres there is a unique point for which the equilibrium is singular and on the one side the index is +1 and on the other side it is -1. If the index is -1 then there are at least two more equilibria provided all equilibria are regular.

7 Appendix

Q has rank $l-1$

Proof See the proof of lemma 3. Q.E.D.

S has rank $m-1$

Through an application of the implicit function theorem on the first-order conditions of the consumers, the Slutsky matrix, $M_i(x)$, and the derivatives with respect to income, $D_w f_i$, are seen to be determined by

$$\begin{bmatrix} M_i(x) & D_w f_i(p,w) \\ \Lambda_i(p,w) & D_w\lambda_i(p,w) \end{bmatrix} = \begin{bmatrix} D^2u_i(x) & -p \\ p^T & 0 \end{bmatrix}^{-1} \begin{bmatrix} \lambda_i(p,w)I & 0 \\ 0 & 1 \end{bmatrix},$$

where $\Lambda_i(p,w) = D_p\lambda_i(p,w) + f_i(p,w)D_w\lambda(p,w)$. Some tedious calculations show that the Slutsky matrix and the derivative with respect to income are

$$\begin{aligned} M_i(x) &= \lambda_i(p,w) D^2 u_i(x)^{-1} [\ I - a[\ p \quad p\ ] D^2 u_i(x)^{-1}\ ] \\ D_w f_i(p,w) &= a_i D^2 u_i(x)^{-1} p, \end{aligned}$$

where $a_i = (p^T D^2 u_i(x)^{-1} p)^{-1}$ and

$$[p \quad p] = \begin{bmatrix} p^1 p^1 & \cdots & p^1 p^l \\ \vdots & & \vdots \\ p^l p^1 & \cdots & p^l p^l \end{bmatrix}.$$

Let

$$\begin{bmatrix} v_{11} & \cdots & v_{1l-1} \\ \vdots & & \vdots \\ v_{l-11} & \cdots & v_{l-1l-1} \end{bmatrix} = Q^{-1} \text{ and } v_k = \sum_{j=1}^{l-1} v_{jk}$$

then $v_k \neq 0$ for some k, because Q has rank $l-1$, and some tedious calculations show that the Jacobian matrix and the determinant of the Jacobian matrix take the following forms

$$\begin{aligned} M(h) + K(h)Y(h) &= A(h) + \\ &+ [\ v_1 \sum_{i=1}^{m-1} \omega_i^l K_i(h) \ \cdots\ v_{l-1} \sum_{i=1}^{m-1} \omega_i^l K_i(h)\ ] \\ \mid M(h) + K(h)Y(h) \mid &= \mid A(h) \mid + \\ &+ \sum_{i=1}^{m-1} \omega_i^l \sum_{j=1}^{l-1} v_j \sum_{k=1}^{l-1} (-1)^{j+k} \mid A^{(j,k)}(h) \mid K_i^k(h), \end{aligned}$$

where

$$\begin{aligned} A(h) &= M(h) - K(h)[\ g_1 \ \cdots\ g_{m-1}\ ] + \\ &+ [\ \sum_{i=1}^{m-1} w_i(1) K_i(h) \ \cdots\ \sum_{i=1}^{m-1} w_i(l-1) K_i(h)\ ] Q^{-1} \end{aligned}$$

and $A^{(j,k)}(h)$ is $A(h)$ without the j'th row and the k'th column. Thus

$$\begin{aligned} s(h) &= \mid A(h) \mid \\ s_i(h) &= \sum_{j=1}^{l-1} v_j \sum_{k=1}^{l-1} (-1)^{j+k} \mid A^{(j,k)}(h) \mid K_i^k(h) \end{aligned}$$

for all $i \in \{1, \ldots, m-1\}$.

Let

$$B = \begin{bmatrix} \sum_{j=1}^{l-1} v_j(1) K_1^j(1) & \cdots & \sum_{j=1}^{l-1} v_j(1) K_{m-1}^j(1) \\ \vdots & & \vdots \\ \sum_{j=1}^{l-1} v_j(m-1) K_1^j(m-1) & \cdots & \sum_{j=1}^{l-1} v_j(m-1) K_{m-1}^j(m-1) \end{bmatrix}$$

According to [5, Lemma A.2], the first $l-1$ elements of $D_w f_i(p,w)$ can be perturbed in all directions without altering $f_i(p,w)$, therefore it is assumed that the matrix B has rank $m-1$.

There exists an $\epsilon \in \mathbf{R}_{++}$ such that if $u_m(x)$ is perturbed into $u_m(x) + \sum_{k=1}^{m-1} j_{x_m(h)}(x)(x - x_m(h))^T C(h,\delta)(x - x_m(h))$, where $j_{x_m(h)}$ is defined as in the proof of lemma 3, $r_o < \min_{h \neq h'} \|x_m(h) - x_m(h')\|$ and $C(h,\cdot) \in C^0(\mathbf{R}, \mathbf{R}^{(l-1)(l-1)})$ with $C(h,0) = 0$, then the perturbed utility function satisfies (A.3), (A.4) and (A.5) for all $\delta < \epsilon$ and the equilibria $(p(h),\rho)_{h=1}^{l-1}$ are not changed by the perturbation.

Let $C(h,\delta)$ be defined implicitly by

$$[\ D^2 u_m(x_m(h) + C(h,\delta)\ ]^{-1} = D^2 u_m(x_m(h))^{-1} + \delta D(h)$$

$$D = \begin{bmatrix} I & -q(h) \\ -q(h)^T & \sum_{j=1}^{l-1} p^j(h) \end{bmatrix},$$

then the matrix $S(\delta)$ and the determinant of $S(\delta)$ take the forms

$$\begin{aligned} S(\delta) &= S + \ldots + \delta^{l-2} B \\ \mid S(\delta) \mid &= \mid S \mid + \ldots + \delta^{(l-2)(m-1)} \mid B \mid. \end{aligned}$$

Thus $S(\delta)$ has rank $m-1$ for all values of δ, except at most $(l-2)(m-1)$ values. Q.E.D

Regularity of equilibria at ρ, μ and ν

Proof Suppose that $(\hat{p}, \omega)$ is a singular equilibrium, where $\omega \in \{\rho, \mu, \nu\}$, then $\hat{x}_i \neq \omega_i$ for at least one consumer. As stated in the previous section of the appendix

$$\begin{aligned} D_p f_i(p, p \cdot \omega_i) &= \lambda D^2 u_i(x_i)^{-1}[\ I - a_i[\ p \quad p\ ] D_2 u_i(\hat{x}_i)^{-1}\ ] + \\ & \quad a_i D^2 u_i(x_i)^{-1}[\ p \quad (x_i - \omega_i)\ ], \end{aligned}$$

where $a_i = (p^T D^2 u_i(\hat{x}_i)^{-1} p)^{-1}$. If $\hat{x}_i = \omega_i$ then $D_p g_i(\hat{p}, \hat{p} \cdot \omega_i)$ has rank $l-1$ because it is the Slutsky matrix. Hence $D_p g_i(\hat{p}, \hat{p} \cdot \omega_i)$ is of the form $A + B$ where A has rank $l-1$, so if $D^2 u_i(\hat{x}_i)$ is perturbed into $(1+\delta) D^2 u_i(\hat{x}_i)$ then demand for the price $\hat{p}$ is unchanged and $D_p g_i(\hat{p}, \hat{p} \cdot \omega_i)$ is perturbed into $(1+\delta)^{-1} A + B$ and this matrix has rank $l-1$ for all values of δ except at most $l-1$ values.

There exists an $\epsilon \in \mathbf{R}_{++}$ such that if $u_i(x)$ is perturbed into $u_i(x) + \frac{\delta}{2} j_{\hat{x}_i}(x)(x - \hat{x}_i)^T D^2 u_i(\hat{x}_i)(x - \hat{x}_i)$, where $j_{\hat{x}_i}$ is defined as in the proof of lemma 3 and $r_o < \min_h \|x_i(h) - \hat{x}_i\|$, then the perturbed utility function satisfies (A.3), (A.4) and (A.5) for all $\delta < \epsilon$ and the equilibria $(p(h), \omega)_{h=1}^{m-1}$ are not changed by the perturbation. Moreover the first-order derivative is $D u_i(\hat{x}_i)$ and the second-order derivative is $(1+\delta) D^2 u_i(\hat{x}_i)$ at $\hat{x}_i$ for the perturbed utility function.

Let

$$C = D_p \sum_{i' \neq i} g_{i'}(\hat{p}, \hat{p} \cdot \omega_{i'})$$

then

$$D_p \sum_{i=1}^{m} g_i(\hat{p}, \hat{p} \cdot \omega_i) = (1+\delta)^{-1} A + B + C$$

for the "perturbed" economy. Hence for all values of δ, except at most $l-1$ values, $D_p \sum_{i=1}^{m} g_i(\hat{p}, \hat{p} \cdot \omega_i) - \omega_i$ has rank $l-1$ and $(\hat{p}, \omega)$ is a regular equilibrium.

This procedure is continued until either there are the desired number of equilibria or all equilibria but the first $m-1$ equilibria for ρ are regular. Q.E.D.

8 References

1. Balasko Y.: *The Class of Aggregate Excess Demand Functions*, in A. Mas-Colell (ed.), Contributions to Mathematical Economics: In Honor of Gerard Debreu, North Holland, Amsterdam, Netherlands, (1986).
2. Balasko Y.: *Foundations of the Theory of General Equilibrium*, Academic Press, Boston, MA, (1988).
3. Balasko Y., Cass D. and Shell K.: Market Participation and Sunspot Equilibria. *Review of Economic Studies* 62: 491-512 (1995).
4. Debreu G.: Excess Demand Functions, *Journal of Mathematical Economics* 1: 15-21 (1974).
5. Ghiglino C. and Tvede M.: Endowments, Stability and Fluctuations in OG Models. *Journal of Economic Dynamics and Control* 19: 621-653 (1995).
6. Kehoe T. and Levine D.: Comparative Statics and Perfect Foresight in Infinite Horizon Economies. *Econometrica* 53: 433-453 (1985).
7. Mantel R.: On the Characterization of Aggregate Excess Demand. *Journal of Economic Theory* 7: 348-353 (1974).
8. Milnor J.: *Topology from the Differentiable Point of View*. University of Virginia Press, Charlottesville, (1965).
9. Sonnenschein H.: Do Walras' Identity and Continuity Characterize the Class of Community Excess Demand Functions? *Journal of Economic Theory* 6: 345-354 (1973).
10. Spear S., Srivastava S. and Woodford M.: Indeterminacy of Stationary Equilibriuum in Stochastic Overlapping Generations Models. *Journal of Economic Theory* 50: 265-284 (1990).