CARESS Working Paper #95-17
A Rudimentary Model of Search with Divisible Money and Prices [¤]

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September, 1995

Abstract

We consider a version of Kiyotaki and Wright's monetary search model in which agents can hold arbitrary amounts of divisible money. A continuum of stationary equilibriums, indexed by the aggregate real-money stock, exist with all trading occurring at a single price. There is always a maximum level of the real money stock consistent with existence of such an equilibrium. In the limit as trading becomes faster relative to discounting, any real money stock becomes feasible in such an equilibrium. In contrast to the original Kiyotaki-Wright model, higher equilibrium real money stocks unambiguously correspond to higher welfare.

[¤] This paper was written while Ed Green was BP Visiting Professor at the London School of Economics. We thank Warren Weber, Juan Renero and Helene Rey for helpful comments on an earlier draft. The authors can be contacted at \ejg@res.mpls.frb.fed.us" and \ruilin@ysidro.sas.upenn.edu". The views expressed in this paper are those of the authors, and do not necessarily re°ect those of the Federal Reserve Bank of Minneapolis or the Federal Reserve System.

1. Introduction

The absence or rarity of mutual coincidence of wants is a difficulty that must be overcome in an economy where agents can only meet in small coalitions of short-term duration. At least some trade can occur, and the resulting allocation can Pareto dominate the endowment, if either a commodity or else a fiat money can be used as a medium of exchange. Kiyotaki and Wright (1989) formulate and study a search-equilibrium model in which money plays this role endogenously. For the sake of analytical tractability, they assume that all consumption goods, and fiat-money objects as well, come in indivisible units. Moreover, they impose the very strong constraint that an agent must only hold one object at a time. Thus an agent cannot hold a money balance of more than one unit, nor can he engage in production while holding money because the output would be impossible to store. In this paper, we formulate and begin to analyze a version of the Kiyotaki-Wright model in which money is divisible and agents face no non-market constraints on their holding of it. This generalization of the model is dual to recent generalizations by Shi (1995) and Trejos and Wright (1995), who posit divisible goods but indivisible money. Shi (1994) studies a search model in which money is assumed to be divisible from the viewpoint of the household (which consists of infinitely many individual traders), but in which each transaction involves an indivisible unit of money. Diamond and Yellin (1990) study a search model in which both money and goods are divisible, but they assume an exogenous cash-in-advance constraint rather than deriving the monetary nature of trade as an equilibrium phenomenon.

There are at least three good reasons to loosen the assumptions that Kiyotaki and Wright make about money and money holding. A first reason is to understand better the ratio of exchange between money and goods. Because Kiyotaki and Wright's assumptions fix this ratio at parity exogenously, the crucial distinction between real and nominal money balances cannot be drawn in their model. Moreover, although Kiyotaki and Wright do not follow Walras in directly imposing the "law of one price" as a part of the definition of equilibrium, these assumptions trivially imply that it must hold. But validity of the law of one price is an important question that one wants to use a search-equilibrium model to investigate. If there is no auctioneer with whom all of the agents in the economy can and must deal simultaneously, and if moreover the decentralization of exchange amplifies the heterogeneity among agents by generating dispersion in money holdings, then is it consistent to assume that all transactions occur at identical terms? If so, does the process of exchange move agents systematically toward such uniformity? These are questions that can be addressed using a search model with divisible money (and, indeed, we will answer the first question in the affirmative here), but that cannot even be posed in the context of the original Kiyotaki-Wright model. Note that this issue is also addressed by the indivisible-money models of Shi and Trejos-Wright.

Second, the combination of indivisibility and inventory constraint on money makes it difficult to formalize and study some of the properties of money about which a model is supposed to provide insight, or to analyze the equilibrium effects of some monetary phenomena in a way that does justice to intuition. The effects of paying interest on money balances would be difficult to formalize, because an agent who is holding money is unable to accept any more of it. (Admittedly the policy might be formalized in a schematic way by assuming that the interest payment is after the initial money holding has been spent.) An additional example is that, because Kiyotaki and Wright's constraints prevent the price level from rising, their model cannot be used informatively to investigate the effects of a "helicopter drop" of new money to agents in the economy. As a framework in which to formulate and study such questions about the effects of changing stocks and flows of nominal money, our model of divisible money has an evident advantage over models in which prices are introduced by making goods alone divisible.

Third, the Kiyotaki-Wright assumptions imply welfare results that it would be absurd to apply to an actual economy. The most obvious instance is that an increase in the nominal money stock can be Pareto worsening in their model because the agents who hold the newly-minted money would be unable to engage in production. Aiyagari, Wallace and Wright (1995) discuss further difficulties in this same vein, in the context of a model that builds on Trejos and Wright's (1995) model. Such paradoxical results are direct consequences of the assumptions that money is indivisible and that there is an exogenous bound on traders' money inventories. Our present model avoids both of these problematic assumptions.

As one makes money divisible and abolishes inventory restriction on agents' money holdings in the original Kiyotaki-Wright model, one can expect the emergence of a large class of equilibrium. In this paper, we are going to focus on a special class, stationary one-price equilibrium, to address the three issues mentioned above. Further studies on other classes of equilibrium, e.g., equilibrium with price dispersion, will deepen our understanding regarding the properties of money as well as welfare implications of alternative monetary policies.

2. The Environment

The set of agents is a nonatomic mass of measure 1. There are $k \geq 3$ types of agent. Each type $i \in \{1, \ldots, k\}$ has mass $1/k$ in the population. Time is continuous, and agents are infinite lived. There are $k+1$ goods. Of these goods, k (which we index by 1 through k) are indivisible, immediately perishable goods that are produced by the agents. The remaining good is a divisible, perfectly durable, flat-money object. The total nominal stock of this flat money is a constant M.

An agent of type i can produce one unit of good $i+1 \pmod{k}$ instantaneously and costlessly at any time. He consumes only good i, from which he derives instantaneous utility $u > 0$. Each agent maximizes the discounted expected utility of his consumption stream, with discount rate ¯.

Meetings between agents are pairwise. Each agent meets other agents randomly according to a Poisson process with parameter ¹. The distribution of partners' characteristics from which an agent' meetings are drawn matches the demographic distribution of characteristics in the entire population of the economy. A meeting partner has two characteristics, his type and the amount of money that he holds. An agent's type is observable, but not his money holding.

In this economy there is no double coincidence of wants (in the sense of trades that give strictly positive utility to both traders) between any pair of agents. Consumption goods cannot be used as commodity money because they are perishable. Thus trade must involve using ¯at money as a medium of exchange. We assume a seller-posting-price trading mechanism. When a type-i agent who possesses ¯at money meets a type-(i ¡ 1) trader who can produce his desired good, the seller (the type-(i ¡ 1) agent) posts an o®er that the buyer (the type-i agent) must either accept or reject. Trade occurs if and only if the o®er accepted, and in that case the buyer pays exactly the seller's o®er price. This speci¯c assumption about the trading protocol is crucial to the results which follow.

3. De¯nition of Stationary Equilibrium

We are going to consider stationary equilibrium in the trading environment just described. Moreover we restrict attention to equilibriums in which all agents with identical characteristics act alike, and in which all of the k types are symmetric. (Hereafter, all of our discussion will be in terms of a generic type i.) A stationary equilibrium can be characterized in terms of ¯ve theoretical constructs: agents' trading strategy, the stationary measure on traders' money holdings, the stationary distributions of o®ers and reservation prices (willingness to accept o®ers), and the value function for money holdings.

The domain of possible money holdings is $\mathbb{R}_+$, the set of nonnegative real numbers. A type-i agent's trading strategy is a pair of real-valued functions on $\mathbb{R}_+$, !(´) that speci¯es the o®er that he will make as a seller when his current money holding is ´ and he meets a type-(i + 1) agent, and ½(´) that speci¯es the maximum o®er price that he will accept as a buyer when his current money holding is ´ and he meets a type-(i ¡ 1) agent. A buyer cannot accept an o®er that exceeds his money holding so the feasibility constraint will be imposed that

½(´) · ´: (3.1)

The stationary distribution of money holdings is given by a measure H de¯ned on $\mathbb{R}_+$.

A strategy (or, more precisely, a symmetric strategy profile) and a stationary distribution of money holdings imply stationary distributions of offers and reservation prices. Note that a buyer's willingness to pay depends on his current money holding, so a trader's reservation price as a function of his money holding is the solution of an optimization problem. Thus we will often refer to a trader's optimal reservation price.

Define the stationary distribution of offers by

$$\Omega(x) = H(\omega^{-1}([0;x])) \tag{3.2}$$

and the stationary distribution of reservation prices by

$$R(x) = H(\rho^{-1}([0;x))): \tag{3.3}$$

Note that, for convenience, we are defining R to be continuous from the left, rather than from the right as would be conventional.

The value function $V^*: \mathbb{R}_+ \to \mathbb{R}_+$ of money holdings specifies the expected discounted utility that an agent will receive, given his current money holding, if he adopts an optimal trading strategy.

The value function is studied in terms of its Bellman equation. Intuitively the Bellman equation states that $V^*(\eta)$ is the discounted expected value of $W + V^*(\eta')$, where η' is the agent's money holding immediately after his next meeting with a potential trading partner, and $W = 0$ if that transaction will not result in a purchase but $W = u$ if it will result in a purchase (and hence will be accompanied by consumption). By a potential trading partner of an agent of type i, we mean either an agent of type $i - 1$ from whom the agent might make a purchase, or else an agent of type $i + 1$ to whom he might make a sale. Since the mass of those two types together in the population is $2/k$, the Poisson parameter for the frequency of meetings with them is $2\mu/k$. Therefore the value function with appropriate discount factor is given by

$$V^*(\eta) = \int_0^\infty e^{-\beta t} E[W + V^*(\eta')|\eta] \frac{2\mu}{k} e^{-(2\mu/k)t}\, dt \tag{3.4}$$

where the first exponential expression inside the integrand is the discount factor, and the second is the exponential waiting time implied by the Poisson process. Note that the expectation does not have to be conditioned on t because, since we are assuming stationarity, the expectation does not depend on the time at which it is taken. Evaluation of the integral yields

$$V^*(\eta) = \frac{2\mu}{k\beta + 2\mu} E[W + V^*(\eta')|\eta]: \tag{3.5}$$

Since agents of types $i - 1$ and $i + 1$ are equally numerous in the population, the probability that the first potential trading partner will be a seller is $1/2$. If the type-i trader's reservation price is r, then the conditional expectation of $W + V^*(\eta')$ in that event is

$$\int_0^r [u + V^*(\eta - x)]\, d\Phi(x) + (1 - \Phi(r))V^*(\eta). \tag{3.6}$$

In the complementary event that the first potential trading partner will be a buyer, if the type i trader makes offer o, then the conditional expectation of $W + V^*(\eta')$ is

$$R(o)V^*(\eta) + (1 - R(o))V^*(\eta + o). \tag{3.7}$$

Substitution of the equally weighted average of the optimized values of (3.6 and (3.7) for the expectation in (3.5) yields

$$\begin{aligned} V^*(\eta) = \frac{1}{k\beta + 2\mu} \Bigg[& \max_{r \in [0,\eta]} \left[\int_0^r [u + V^*(\eta - x)]\, d\Phi(x) + (1 - \Phi(r))V^*(\eta) \right] \\ & + \max_{o \in \mathbb{R}_+} \left[R(o)V^*(\eta) + (1 - R(o))V^*(\eta + o) \right] \Bigg]. \end{aligned} \tag{3.8}$$

Equation (3.8) is the Bellman equation for V^*. Standard arguments establish that it has a unique solution in the space of bounded measurable functions, and that this solution does indeed specify the optimal expected discounted value of each possible level of money holding.

An equilibrium consists of a measure on money holdings and offer and reservation-price strategies that jointly satisfy conditions of feasibility, optimization and stationarity. The feasibility condition (3.1) has already been stated. The optimization condition is that

$$\begin{aligned} V^*(\eta) = \frac{1}{k\beta + 2\mu} \Bigg[& \left[\int_0^{\rho(\eta)} [u + V^*(\eta - x)]\, d\Phi(x) + (1 - \Phi(\rho(\eta)))V^*(\eta) \right] \\ & + \left[R(\omega(\eta))V^*(\eta) + (1 - R(\omega(\eta)))V^*(\eta + \omega(\eta)) \right] \Bigg]. \end{aligned} \tag{3.9}$$

where Φ, R, and V^* that are defined by (3.2), (3.3), and (3.8) respectively.

Implicitly we are describing an economy in which the money holding of each agent is a continuous-time, pure-jump Markov process on the state space $\mathbb{R}_+$. The transition probabilities are the probabilities of transactions occurring, induced by the optimal strategies (ω, ρ). The stationarity condition of equilibrium is that the measure H is a stationary initial distribution of this process.

In the equilibriums that we are going to study in this paper, the support of H will be the discrete set $\{0, p, 2p, 3p, \ldots\}$. Giving an exact statement of the stationarity condition is much easier in this special case than in general, so we will state the formal condition of stationarity in the context of this class of equilibriums.

4. One-Price Equilibrium

In this section we are going to characterize a sufficient condition for a one-price equilibrium to exist. We are going to begin by supposing that all trades occur at a single price p, and that all agents' money holdings are integer multiples of p. Also we assume that agents always offer to sell at p, and that every agent who holds money is willing to purchase at p. We characterize the stationary measure on traders' money holdings under these assumptions. Then we find the corresponding solution for the value function and use it to calculate the optimal reservation-price function. Finally we find a sufficient condition such that the optimal offer function is constant at price p. Thus, under this condition, the offer function, reservation-price function, and stationary measure that we have found are an equilibrium.

4.1. Stationary Measure on Traders' Money Holdings

Consider the formulation of equilibrium just given, in the special case that all trades occur at a single price p, and that the support of the population measure H of money holdings is on the discrete set of points $p\mathbb{N} = \{0; p; 2p; 3p; \ldots\}$.[1] In this case, define

$$h(n) = H(\{np\}). \tag{4.1}$$

That is, $h(n)$ is the measure of the set of agents who hold precisely np units of fiat money. Now, instead of working with the measure H on $\mathbb{R}_+$, we can work with the equivalent measure h on $\mathbb{N}$. We will say that a trader is in state n when his money holding is np. The proportion of agents who hold positive money holdings is defined to be

$$m \equiv \sum_{n=1}^{\infty} h(n). \tag{4.2}$$

Note that

$$h(0) = 1 - m. \tag{4.3}$$

In a one-price equilibrium, an agent only moves into state n (the state of having money holding np) by either making a sale from state $n - 1$ or making a purchase from state $n + 1$. He moves out of state n by either making a purchase or a sale. Clearly an agent of type i will make a sale whenever he meets an agent of type $i + 1$ whose money holding is positive, and he will make a purchase whenever he meets an agent of type $i - 1$ if his own money holding is positive. Stationarity requires that the sum of time rates of inflow to state n from all other

[1] This is evidently the simplest stationary equilibrium. We do not know whether there are other one-price equilibriums in which the support of H is not $p\mathbb{N}$.

states must equal the time rate of outflow from state n. The time rate of a population flow is the instantaneous transition probability for an individual multiplied by the population of the state from which the transition occurs. The time derivative of $h(n)$, for all $n > 0$ is thus

$$\dot{h}(n) = \mu h(n+1) + \mu m h(n-1) - \mu(m+1)h(n): \tag{4.4}$$

The time derivative of $h(0)$ is

$$\dot{h}(0) = \mu h(1) - \mu m h(0): \tag{4.5}$$

Setting these two derivatives equal to zero and arguing recursively, it is seen that the only candidates for stationary measures are of the form

$$\forall n \in \mathbb{N} \quad h(n) = m^n(1-m) \tag{4.6}$$

for some $m \in (0;1)$. Given this geometric functional form specified by (4.6), the quantities p, m, and M are related by the equation

$$M = p\sum_{n=1}^{\infty} n h(n) = \frac{m}{1-m}p: \tag{4.7}$$

This characterization of stationarity by equations (4.6) and (4.7) is the remaining equilibrium condition that was postponed from the end of the preceding section.

4.2. Equilibrium Value Function

Now we solve equation (3.9) for the equilibrium value function. To begin, recall that the presumed optimal strategy in one-price equilibrium is that agents are always willing to sell at p, and that every agent who holds money is willing to purchase at p. Formally this assumption means that

$$\sigma(p) = 1 \quad \text{and} \quad \forall x < p \;\; \sigma(x) = 0 \tag{4.8}$$

and

$$R(p) = 1 - m: \tag{4.9}$$

(Note that the assumption that B is left-continuous, defined by (3.3), is applied here.) It will be convenient to define $V(n) = V^*(np)$. Then, using (4.8) and (4.9), (3.9) simplifies for $n = 0$ to

$$V(0) = \frac{\mu}{k\beta + 2\mu}[V(0) + (1-m)V(0) + mV(1)]; \tag{4.10}$$

which yields

$$V(0) = \frac{\mu m}{k\beta + \mu m}V(1): \tag{4.11}$$

For all positive n, (3.9) simplifies to

$$V(n) = \frac{\mu}{k\beta + 2\mu}\Big[[u + V(n-1)] + [(1-m)V(n) + mV(n+1)]\Big]. \qquad (4.12)$$

The system of equations (4.11) and (4.12) defines the value function implicitly in terms of five parameters: μ, k, β, m, and u. Note that $\mu/(k\beta)$ actually functions as a single parameter. To simplify further computations, define

$$\phi = \frac{\mu}{k\beta}. \qquad (4.13)$$

Equation (4.12) can be rewritten in matrix form as

$$\begin{pmatrix} V(n+1) \\ V(n) \\ u \end{pmatrix} = \begin{pmatrix} \left[\frac{1}{\phi m} + \frac{1}{m} + 1\right] & \frac{-1}{m} & \frac{-1}{m} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V(n) \\ V(n-1) \\ u \end{pmatrix} \qquad (4.14)$$

Equation (4.14) is an inhomogeneous second-order linear difference equation. Its family of solutions is given in terms of eigenvectors of the matrix

$$D = \begin{pmatrix} \left[\frac{1}{\phi m} + \frac{1}{m} + 1\right] & \frac{-1}{m} & \frac{-1}{m} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (4.15)$$

in (4.14), and the correct solution is determined by means of two endpoint conditions.[2] One of these endpoint conditions is equation (4.11). The other condition is that V is bounded. It is bounded below because it is nonnegative and above because, even if a trader's rate of consumption were not constrained by his need to pay for the goods that he acquires in trade, he would still have only discrete consumption opportunities that would occur at times separated by μ on average, and the utility of which would thus be discounted.

The matrix D has three distinct eigenvectors, all of which have real eigenvalues. The solution of equation (4.14) is therefore determined by a linear combination of the eigenvectors for which (because V is bounded) the eigenvalue has absolute value at most 1. These two eigenvectors can be expressed as

$$D\begin{pmatrix} \phi \\ \phi \\ 1 \end{pmatrix} = \begin{pmatrix} \phi \\ \phi \\ 1 \end{pmatrix} \quad \text{and} \quad D\begin{pmatrix} \lambda \\ 1 \\ 0 \end{pmatrix} = \lambda\begin{pmatrix} \lambda \\ 1 \\ 0 \end{pmatrix}, \qquad (4.16)$$

where

$$\lambda = \frac{1}{2}\left(\frac{1}{\phi m} + \frac{1}{m} + 1 - \sqrt{\left(\frac{1}{\phi m} + \frac{1}{m} + 1\right)^2 - \frac{4}{m}}\right) \in (0,1). \qquad (4.17)$$

[2] See Lefschetz (1977, Chapter III) for a discussion of the continuous-time theory, which is completely analogous.

Now, by Lefschetz (1977), there are two coefficients θ_1 and θ_2 such that

$$\begin{pmatrix} V(1) \\ V(0) \\ u \end{pmatrix} = \theta_1 \begin{pmatrix} \phi \\ \phi \\ 1 \end{pmatrix} + \theta_2 \begin{pmatrix} \lambda \\ 1 \\ 0 \end{pmatrix}. \tag{4.18}$$

It follows from (4.11) and (4.18) that

$$\theta_1 = u; \qquad \theta_2 = V(0) - \phi u; \qquad V(0) = \frac{(1-\lambda)m}{(1/\phi) + (1-\lambda)m}\,\phi u. \tag{4.19}$$

Moreover, by induction, for all $n \geq 0$

$$\begin{aligned} \begin{pmatrix} V(n+1) \\ V(n) \\ u \end{pmatrix} &= D^n \begin{pmatrix} V(1) \\ V(0) \\ u \end{pmatrix} \\ &= u \begin{pmatrix} \phi \\ \phi \\ 1 \end{pmatrix} + \lambda^n (V(0) - \phi u) \begin{pmatrix} \lambda \\ 1 \\ 0 \end{pmatrix}. \end{aligned} \tag{4.20}$$

In particular, the second row states that for all n

$$V(n) = \lambda^n V(0) + (1 - \lambda^n)\phi u. \tag{4.21}$$

Equations (4.19) and (4.21) imply the following lemma.

Lemma 4.1. V is increasing and satisfies the concavity condition that, for all $j > 0$, $V(n+j) - V(n)$ is a decreasing function of n.

Although we are characterizing a class of equilibriums in which each trader's money holding is always an integer multiple of p, the value function is defined for non-integer multiples as well. Given the presumed optimal trading strategy, the value function is evidently a step function. Specifically, if $[x]$ denotes the integer part of x (that is, $x = [x] + \epsilon$ for some $\epsilon \in [0, 1)$) then

$$V^*(\eta) = V([\eta/p]). \tag{4.22}$$

This completes the derivation of the value function.

4.3. Equilibrium Strategy

Suppose that an agent a of type i with money holding η meets a trading partner of type $(i-1)$ with money holding η'. Each observes the other's type but neither observes the other's money holding. Independently of one another, a chooses a reservation price r and the partner posts an offer o. If $r \geq o$, then the partner supplies a unit of good i to a in exchange for amount o of money, and otherwise no transaction takes place. Agent a should choose an optimal reservation price that solves the maximization problem with respect to r that occurs on the right hand side of the Bellman equation (3.8). The solution to this maximization problem may not be unique. Therefore we assume that a buyer always accepts an offer when he is indifferent. That is,

$$\rho(\eta) = \max\{r \in [0, \eta] \mid u + V^*(\eta - r) \geq V^*(\eta)\}. \tag{4.23}$$

This reservation price is a's full value for a unit of good $(i+1)$. (Analogously, in Vickrey's (1960) analysis of a second-price auction, to bid this full value is the buyer's weakly dominant action.) An alternative assumption, that a buyer only accepts an offer when he gains strictly from it, would not change our general conclusions. The following lemma provides further information about the reservation-price function ρ that will be used below.

Lemma 4.2. The reservation-price function ρ specified by equation (4.23) satisfies the conditions that $\rho(p) = p$ and that, for every positive integer multiple np of p, $\rho(np)$ is an integer multiple of p. It also satisfies the condition that $[\rho(\eta)/p]$ is a nondecreasing function of η.

Proof. Equations (4.19) and (4.23) imply that $\rho(p) = p$. By (4.22) and (4.23), $V^*(\eta) = V([\eta/p])$ and $\eta - \rho(\eta) = jp$ for some $j \in \mathbb{N}$. To prove that $[\rho/p]$ is nondecreasing, suppose that $\eta < \eta'$. By (4.23), $V^*(\eta) - V^*(\eta - \rho(\eta)) \leq u$. By (4.22), then, $V([\eta/p]) - V((\eta - \rho(\eta))/p) = V([\eta/p]) - V([\eta/p] - [\rho(\eta)/p]) \leq u$. By the concavity of V established in lemma 4.1, $V([\eta'/p]) - V([\eta'/p] - [\rho(\eta)/p]) \leq u$. That is, this concavity condition implies that $V^*(\eta') - V^*(\eta' - p[\rho(\eta)/p]) \leq u$. Thus by (4.23) and the increasingness assertion in lemma 4.1, $\rho(\eta') \geq p[\rho(\eta)/p]$. Therefore $[\rho(\eta)/p] \leq [\rho(\eta')/p]$. Q.E.D.

Lemma 4.2 has the following, intuitively obvious, consequence.

Lemma 4.3. For an agent whose money holding is at least p, it is optimal to accept an offer of p if all sellers' offers are almost surely at price p.

Lemma 4.3 establishes one of the two presumptions about the equilibrium strategy, stated at the beginning of subsection B, that we have used to derive the value function. The other presumption is that all offers are made at price p. The following lemma establishes a sufficient condition for this latter presumption to characterize agents' optimizing behavior in an equilibrium.

Lemma 4.4. If all agents' reservation prices are integer multiple of p, then the optimal offer $\omega(\eta)$ is an integer multiple of p for every η. If the proportion of agents with positive money holdings in a stationary measure of form (4.6) is $m \le 1/2$, and if all agents with positive money holdings have reservation price at least p, then it is optimal for an agent always to offer to sell at p.

Proof. The first assertion is obvious. To prove the second assertion, define the set of money holdings η at which an offer at price o would be accepted by $A(o) = \rho^{-1}([o, \infty)) \subset \mathbb{R}$, and define $a(o) = \min\{n | np \in A(o)\}$. Note that $a(o) \ge [o/p]$ because an agent's reservation price cannot exceed his money holding. By lemma 4.1 and (4.23), $\{n | np \in A(o)\} = \{a(o), a(o) + 1, \ldots\}$. Thus, by (4.1) and (4.6), $H(A(o)) = m^{a(o)} \le m^{[o/p]}$. If the seller's money holding is η, then his expected value of offering o is

$$\begin{aligned} W(\eta; o) &= H(A(o))V^*(\eta + o) + (1 - H(A(o)))V^*(\eta) \\ &\le m^{[o/p]}V^*(\eta + o) + (1 - m^{[o/p]})V^*(\eta). \end{aligned} \tag{4.24}$$

By the first assertion of this lemma, there must be an optimal offer of the form $o = jp$, so we can restrict attention to this case and also assume that $\eta = np$, and simplify the upper bound on the expected value of offering o to $W(\eta; o) \le m^j V(n + j) + (1 - m^j)V(n)$. By the concavity of V established in lemma 4.1, $V(n + j) < V(n) + j(V(n + 1) - V(n))$. Therefore $W(\eta; o) \le V(n) + jm^j(V(n + 1) - V(n))$. If $m \le 1/2$, then $jm^j \le m$ for all $j > 1$, so $W(\eta; o) \le V(n) + m(V(n + 1) - V(n)) = H(A(p))V^*(\eta + p) + (1 - H(A(p)))V^*(\eta) = W(\eta; p)$. Q.E.D.

4.4. Existence and Indeterminacy of Equilibrium

We began section 4 by making two assumptions about the form of a possible stationary equilibrium. We assumed that all offers are made at a single price p and that reservation prices of all agents with positive money holdings are at least p. Under these assumptions, we have shown in subsection 4.1 that every geometrically-distributed measure on money holdings that are nonnegative integer multiples of p is stationary. In subsection 4.2, we have characterized the equilibrium value function. Finally, in subsection 4.3, we have shown that the characterizations of stationarity and optimality imply that our assumptions regarding reservation prices and offers are implied by agents' optimizing decisions if $m \le 1/2$. Equation (4.7) establishes that $m \le 1/2$ if $p \ge M$. Thus we have proved the following theorem.

Theorem 4.5. In every trading environment described in section 2, for every price $p \ge M$, there is a stationary equilibrium in which all transactions occur at price p and all traders' money holdings are integer multiples of p. The proportion of agents with positive money holdings is an increasing function of M/p, so there is a continuum of distinct stationary-equilibrium allocations.

Theorem 4.5 states that if the price level p is not below M (that is, essentially, if the per capita real money stock is not greater than 1), then a stationary one-price equilibrium exists. This is a sufficient condition, not a necessary condition for existence. The question remains, then, whether there is any maximum real money stock that is consistent with equilibrium. In fact, for any ϕ, such a maximum real money stock does exist. We prove this via two lemmas.

Lemma 4.6. Suppose that the following condition holds.

$$m^{a(2)} > \frac{1}{1+\lambda}. \tag{4.25}$$

Then there exists an n such that $\omega(np) \geq 2p$.

Proof. Using notation developed in the proof of lemma 4.4, a seller with money holding np will offer at least $2p$ if $W(np; 2p) > W(np; p)$. Substituting (4.24) into this inequality yields $m^{a(2)}[V(n+2) - V(n)] > m^{a(2)+1}[V(n+2) - V(n)] > m[V(n+1) - V(n)]$. Thus the inequality between the second and third terms is a sufficient condition for an offer of at least $2p$. Applying (4.21) yields the equivalent condition (4.25). Q.E.D.

Lemma 4.7. For any given ϕ, there exists some $J \in \mathbb{N}$ such that, for all m, $a(2) \leq J$.

Proof. By equation (4.23), an agent with money holding np will have reservation price at least $2p$ if $u + V(n-2) \geq V(n)$. That is,

$$u \geq \frac{1}{1+\phi(1-\lambda)m}\phi u(\lambda^{n-2} - \lambda^{n}). \tag{4.26}$$

For this inequality to hold, it is sufficient that $u \geq \phi u \lambda^{n-2}$, or

$$n \geq 2 - \frac{\ln(\phi)}{\ln(\lambda)}. \tag{4.27}$$

Noting that λ is a function of m and that the right hand of (4.27) reaches a finite maximum at $m = 1$, J can be taken to be the first natural number greater than that maximum. Since $a(2)$ is the smallest number satisfying (4.26) and J satisfies (4.26), $a(2) \leq J$. Q.E.D.

Theorem 4.8. For any given ϕ, there is some $m^* < 1$ such that, for any $m > m^*$, a stationary one-price equilibrium does not exist.

Proof. Since $a(2) \leq J$ (where J is described in lemma 4.7), $m^{a(2)} \geq m^J$. Combining this inequality with (4.25),

$$m^J > \frac{1}{1+\lambda} \tag{4.28}$$

is a sufficient condition for there to exist an n such that $\omega(np) \geq 2p$. Both sides of (4.28) are continuous functions of m, and condition (4.28) is not satisfied at $m = 1/2$ but it is satisfied at $m = 1$. These two facts, combined with the fact that the set of values m at which (4.28) is not satisfied is closed, imply that the set has a maximum m^* which is strictly less than 1. Q.E.D.

5. The Limiting Case

Theorem 4.8 shows that, for any given ϕ and for sufficiently high m (or equivalently, for sufficiently high M/p), a stationary one-price equilibrium cannot exist. However, in an economy where the parameter ϕ is large, reflecting high frequency of meetings or insignificance of discounting, the minimum price level for such an equilibrium to exist is actually arbitrarily low.

The intuition for this result is that, if a buyer is confident that he will almost immediately meet another seller whose offer is very close to the minimum offer in the market, then he should be unwilling presently to accept a high offer unless his money holding is huge. The key to the formal derivation is a closer examination of the optimal reservation price, characterized in (4.23) by

$$\rho(\eta) = \max\{r \in [0, \eta] \mid u + V^*(\eta - r) \geq V^*(\eta)\}.$$

It is clear from this formula and the formula (4.22) for V^* that, if η is an integer multiple of p, then $\rho(\eta)$ must also be an integer multiple of p. Define, for the optimal reservation-price function ρ in an economy with parameters μ, k, and β satisfying (4.13) and with proportion m of agents having positive money holdings,

$$\rho(np) = r(n; \phi)p \tag{5.1}$$

We want to study what happens in the limit as ϕ approaches infinity. As in the preceding section, we assume that all offers are at price p and then verify that (asymptotically, in this section) such offers are indeed optimal given agents' optimal reservation-price functions.

Lemma 5.1. If all offers are at price p, then for every natural number $n \geq 3$, there exists a $\phi_n \in \mathbb{R}$ such that

$$\forall \phi \geq \phi_n \quad \forall j \in \{1, \ldots, n-1\} \quad r(j; \phi) \leq 1. \tag{5.2}$$

Proof. By lemma 4.2, it is sufficient to show that $\forall \phi \geq \phi_n \; r(n-1; \phi) \leq 1$. Equation (4.23) implies that, if $r(n-1; \phi) > 1$ (that is, the optimal reservation price is at least 2), then $V(n -$

$1) - V(n-3) \leq u$. Note that, by equations (4.19) and (4.21),

$$
\begin{aligned}
V(n-1) - V(n-3) &= [\phi u - V(0)](\lambda^{n-3} - \lambda^{n-1}) \\
&= \frac{u(1-\lambda^2)\lambda^{n-3}}{(1/\phi) + (1-\lambda^2)m}.
\end{aligned} \tag{5.3}
$$

Substitution of the value of λ defined in (4.17) into 5.3), and application of l'Hôpital's rule, yield $\lim_{\phi\to\infty}(V(n-1) - V(n-3)) = 2u$. Thus the lemma follows from lemma 4.2 and (4.23), since $\lim_{\phi\to\infty}(V(n-1) - V(n-3)) > u$ implies that the trader's unique optimal reservation price is 1 for sufficiently large ϕ. Q.E.D.

Now we use lemma 5.1 to show that, for sufficiently large values of ϕ, the optimum offer for all sellers is p. Recall that $W(\eta; o)$ was defined in (4.24) to be the expected value, to a seller of type i holding quantity η of money, of making an offer of o to a buyer of type $(i+1)$.

Theorem 5.2. For every $m \in (0,1)$ there exists a $\phi^*_m \in \mathbb{R}$ such that

$$
\forall \phi \geq \phi^*_m \quad \forall \eta \in \mathbb{R}_+ \quad W(\eta; p) = \max_{o \in \mathbb{R}_+} W(\eta; o). \tag{5.4}
$$

Proof. Note first that $\lim_{x\to\infty} xm^x = 0$. Thus we can choose $n \in \mathbb{N}$ such that $\max_{n \leq x} xm^x < m$. Let ϕ^*_m be the value of ϕ_n satisfying (5.2). As in the proof of theorem 4.5, we can restrict attention to the case where η and o are both integer multiples of p. Specifically let $\eta = ip$ and $o = jp$. Then

$$
W(\eta; o) = \begin{cases} V(i) & : \ j = 0; \\ V(i) + m(V(i+1) - V(i)) & : \ j = 1; \\ V(i) + m^{a(o)}(V(i+j) - V(i)) & : \ 1 < j; \end{cases} \tag{5.5}
$$

where $a(o)$, which has been defined formally in the proof of lemma 4.4, corresponds to the least level of money holding at which an offer of o will be less than the optimal reservation price. The last alternative can be bounded by using the inequalities (derived from concavity, as in lemma 4.4)

$$
\begin{aligned}
m^{a(o)}(V(i+j) - V(i)) &\leq nm^n(V(i+1) - V(i)) \ : \ 1 < j \leq n; \\
m^{a(o)}(V(i+j) - V(i)) &\leq jm^j(V(i+1) - V(i)) \ : \ n < j.
\end{aligned} \tag{5.6}
$$

With these bounds, p is seen to be the optimal reservation price by choice of n. Q.E.D.

6. Welfare

This version of the Kiyotaki-Wright model with divisible money, and without inventory constraints on the holding of it, provides a more natural environment to study welfare questions. In contrast to the original version, here stationary equilibriums with higher real money stocks always provide higher levels of welfare. Intuitively, the fewer agents there are without money, the fewer trading opportunities will be foregone, and therefore the higher welfare will be.

To show this formally, we consider the standard welfare measure of summing agents' utility levels. That is, our welfare measure is

$$U(m, \phi, u) = \sum_{n=0}^{\infty} h(n)V(n) = (1 - m) \sum_{n=0}^{\infty} m^n V(n). \tag{6.1}$$

Substituting the values of $V(0)$ and $V(n)$ given in (4.19) and (4.21) into equation (6.1) yields

$$U(m, \phi, u) = m\phi u. \tag{6.2}$$

Given this equation and the fact that ϕ and u are parameters of the model, welfare would be maximized by selecting the highest level of m consistent with existence of equilibrium. By equation (4.7), welfare is maximized by minimizing the price level given a fixed nominal money stock M (that is, by maximizing the real money stock in economy).

Our strong conclusion that a higher real money stock unambiguously corresponds to higher steady-state welfare arguably results from our having abstracted from production costs. If such a cost were to be modeled successfully, it might possibly turn out that traders would not produce when they were holding high real money balances. An intuition for such a result would be that the cost of production must be borne immediately, while the benefit from acquiring additional money would be discounted to the time of its expenditure. Since an increase in the economy's real money stock implies an increase in the number of traders holding high real money balances, such an increase might cause aggregate production to decrease. In contrast to the technological incompatibility between money holding and production in models incorporating the Kiyotaki-Wright inventory constraint, such an equilibrium incentive effect of the real money stock on production would presumably disappear in the limit as ϕ approaches infinity.

7. Discussion

We have studied a very preliminary, schematic version of a search-equilibrium model with divisible money. We show that there is always a continuum of stationary monetary equilibriums where all transactions occur at a single price. Agents in this model economy set their prices strategically rather than taking prices to be parametric as in the Walrasian model. The prevalence of a single price results from self-fulfilling beliefs of the agents.

One aspect of this finding is that it is a positive result about the law of one price in a search economy. Viewed in this way, it provides support for the Kiyotaki-Wright analysis in which the parity of exchange between money and goods is exogenous.

Besides the one-price equilibriums of the model, we conjecture that there may be other stationary equilibriums in which prices are dispersed. Nonstationary equilibrium may also exist. A theory of these equilibriums will be needed to address fully some of the issues that we have raised in the introduction. Moreover, given the indeterminacy of equilibrium that we have encountered, we may have little to say without a theory of equilibrium selection.

As in all search-equilibrium models without production costs, there is a non-monetary equilibrium in this model. Each agent simply gives his good away for free to anyone he meets who wants it. The existence of this equilibrium is directly traceable to our schematic assumption that production of goods is completely costless. Introduction of any production cost, however small, removes it from the equilibrium set. Also this equilibrium does not seem to be robust to "trembling-hand" types of perturbation. However, it is noteworthy that this equilibrium has a greater amount of trade (and hence provides a higher level of welfare) than does any monetary equilibrium. This is, in a sense, a paradoxical equilibrium.

In order to remove the paradoxical non-monetary equilibrium, one can introduce a "utility cost" of production into the basic model. We can show that, with this modification alone, a stationary, one-price equilibrium does not exist. However, we conjecture that an equilibrium with a single, stationary price may be a limit as ϕ and the cost parameter are taken to infinity and zero respectively.

As we have pointed out earlier, the existence of a one-price equilibrium depends on the seller-posting-price mechanism that we have assumed. With other forms of mechanism (for example, a sequential bargaining mechanism that gives some power to both the buyer and the seller) , it seems clear that price dispersion would exist in equilibrium. We conjecture that such dispersion would vanish in the limit as ϕ is taken to infinity. If a buyer is confident that he will almost immediately meet a seller whose offer is very close to the minimum offer in the market, then he should be unwilling to accept any outcome of bargaining with his current trading partner that would force him to make a high payment, unless his money holding is huge. This is the same insight on which our present analysis of frequent meetings is based.

Besides the immediate reasons for studying a search-equilibrium model of divisible money that can be held without non-market constraint, our present work is an intermediate step toward a model in which both consumption goods and money are divisible. By providing a tractable search-equilibrium model of agents' unconstrained money holdings, we hope to have contributed to the eventual study of this issue.

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