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Speculative Investor Behavior and Learning[¤]

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Abstract

As traders learn about the true distribution of some asset's dividends, a speculative premium occurs as each trader anticipates the possibility of re-selling the asset to another trader before complete learning has occured. Reasonable ignorance priors lead to large bubbles during the learning process. This phenomenon explains a paradox concerning the pricing of initial public o®erings. The result casts light on the signi¯cance of the common prior assumption in economic models. JEL classi¯cation: D83, G12.
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1. Introduction

In their 1978 paper, Speculative Investor Behavior in a Stock Market with Heterogeneous Expectations, Harrison and Kreps characterize the price of a risky asset in a world where traders are risk neutral, have heterogeneous expectations about the asset's value and cannot sell the asset short. In equilibrium, the asset must - after every history - be held by the trader who values it the most after that history, and the price must equal her short term valuation of holding the asset (if the price was less that her valuation, she would demand an infinite quantity). On the other hand, the price will typically exceed the short term valuation of holding the asset to other traders (since they cannot sell it short). How will the price compare with the traders' "fundamental valuations" - that is, the value to them of holding the asset forever? Clearly the price must be at least as great as the fundamental valuation of the trader holding the asset. Harrison and Kreps observed that - given the heterogeneous expectations - the price will typically be strictly greater since the trader currently holding the asset will anticipate contingencies in the future where other traders will value the asset more, and she will be able to re-sell for strictly more than her fundamental valuation.

Harrison and Kreps interpreted this result as a formalization of the notion of speculation in Keynes (1936, chapter 12): speculation occurs when an asset is bought for its short term expected gain, at a price higher the expected discounted value of dividends. This result has apparently been largely ignored, presumably because of the assumption of (unmodelled) heterogeneity of expectations. In this paper, I consider a special case of the model of Harrison and Kreps, where traders initially have heterogeneous beliefs about the asset's fundamental value, but - because dividends are assumed to be i.i.d. - their beliefs converge over time. What becomes of the speculative premium then? In particular, how does the speculative premium depend on the initial heterogeneity of prior beliefs?

The key property of prior beliefs is the following. Say that a trader is an optimist if, after every history, she has the highest valuation of the asset. I provide a necessary and sufficient condition on priors for the existence of an optimist, and show that the price always equals the optimist's fundamental valuation. On the other hand, if no optimist exists then the price always strictly exceeds every trader's fundamental valuation but through time the price and traders' valuations all converge to an objective valuation.

This leaves open the question of how large are speculative premia before learning occurs. I show that reasonable di®erences in priors can lead to large speculative premia. I investigate the case where an asset either pays a dividend or not in any one period. Traders are uncertain of the true binomial parameter generating the process. One trader has a uniform prior on the binomial parameter and thus after observing s dividends in t periods, believes that the probability of a dividend in the next period is $\frac{s+1}{t+2}$. Another trader puts more weight on the empirical frequency and has posterior $\frac{s+\frac{1}{2}}{t+1}$. Then the price of the asset (before any returns are realized) is 8% higher than either trader's valuation of the returns from the asset. This 8% premium is generated by the option to re-sell the asset at some future date.

This phenomenon is consistent with the \hot issue" anomaly in the pricing of initial public o®erings. The opening market price of initial public o®erings appears to be too high relative to long run values. Miller (1977) has suggested that this may be because the price tends to re°ect the valuation of the most optimistic investor. My model formalizes Miller's conjecture, and makes clear that it is enough that traders have di®erent initial beliefs about the distribution of the parameters of the data generating process. It is not required that they have di®erent ex ante expected valuations of the stream of dividends from the asset.

By showing how fully rational learning is consistent with the model of Harrison and Kreps, I hope to show how heterogeneous prior beliefs can and should be used (selectively) as an assumption in understanding ¯nancial and other markets.

The paper is organized as follows. In section 2, I examine the evolution of fundamental valuations of traders with initially heterogeneous beliefs. In section 3, I present the model and main result relating prices to traders' valuations of the asset. In section 4, I present a numerical example which illustrates that the most reasonable priors imaginable lead to signi¯cant speculative premia in asset prices. In section 5, I discuss the relation to the empirical literature on initial public o®erings. In section 6, I argue that the model and examples in this paper suggest a (limited) re-appraisal of economists' suspicion of arguments which appeal to di®erences in prior beliefs. Section 7 concludes. The appendix includes proofs and a discussion of some technical issues concerning multiplicity of equilibria which were ignored in section 3.

2. Fundamental Values with Learning

A group of risk neutral traders are learning about the underlying value of a risky asset. In each period, the asset either pays a dividend of \$1, or not. The probability of a dividend in each period is μ and the dividend process is i.i.d. All this is common knowledge among the group of traders. Traders do not (initially) know the true value of μ, but have possibly heterogeneous prior beliefs about the true value of μ.

This simple dividend structure enables us to identify a trader's valuation of the asset after any given history with her point estimate $\hat{\mu}$ of the value of μ, given that history. To see why, note that $\hat{\mu}$ is also the expected dividend of the asset in the next period. Given the i.i.d. assumption, it is also the expected dividend of the asset in any future period. Thus if the interest rate from a safe asset were r, the (risk neutral) trader would value the asset at $\left(\frac{1}{1+r}\right)\hat{\mu} + \left(\frac{1}{1+r}\right)^2\hat{\mu} + ::: = \frac{1}{r}\hat{\mu}$. Thus (ignoring the constant $\frac{1}{r}$), I will refer to a trader's point estimate of μ as her fundamental valuation of the asset. Note that this is the expected value to the trader of holding the asset forever. In the next section, I will relate these fundamental valuations to market prices. In this section, I explore how these fundamental valuations evolve as traders learn the true value of μ. Key questions for the analysis of competitive markets in later sections are the following. When is it the case that one trader remains the most optimistic about the asset (i.e. has the highest fundamental valuation) after every history? Conversely, when is it the case that after every history, there is a positive probability that the trader with the highest current fundamental valuation will not have the highest valuation in the future?

To state these questions formally, I introduce notation for the set of traders. There is a ¯nite collection of risk neutral traders, $I = f1; ::; Ig$. An alternative interpretation is that I is a ¯nite set of types of traders and that there are an in¯nite number of each type. Trader i's beliefs are represented by a density $¼_i$ over possible values of μ in $[0; 1]$. Assume that each $¼_i$ is twice di®erentiable and uniformly bounded below (there exists " such that $¼_i(\mu)$ ¸ " for all i 2 I and μ 2 $[0; 1]$).

Suppose trader i observes a history of t periods in which a total of s dividends are paid. Then his posterior density over μ is given by

$$\xi_i(\theta|s,t) = \frac{\theta^s(1-\theta)^{t-s}\pi_i(\theta)}{\int_{\zeta=0}^{1} \zeta^s(1-\zeta)^{t-s}\pi_i(\zeta)d\zeta}; \tag{2.1}$$

and the probability he attaches to a dividend being paid in the next period is

$$\mu_i(s,t) = \int_{\theta=0}^{1} \theta\xi_i(\theta|s,t)d\theta = \frac{\int_{\theta=0}^{1} \theta^{s+1}(1-\theta)^{t-s}\pi_i(\theta)d\theta}{\int_{\theta=0}^{1} \theta^s(1-\theta)^{t-s}\pi_i(\theta)d\theta}. \tag{2.2}$$

For the reasons noted above, I will refer to $\mu_i(s,t)$ as trader i's fundamental valuation of the asset after history (s,t). I will be examining the properties of the following example throughout the paper.

Example 2.1. (Ignorance Priors). Imagine a situation where the risky asset is being traded for the first time, so that traders do not have a history of past dividends on the basis of which to form beliefs about θ. They must form some kind of "ignorance priors" about θ. I want to consider traders who are as reasonable and conservative as possible in the light of their ignorance, but I want to argue this will not pin down the exact prior they use. Suppose that trader 1 has a uniform prior over the parameter space, i.e. $\pi_1(\theta) = \theta$ for all $\theta \in [0,1]$, so that his valuation of the asset after history (s,t) is $\mu_1(s,t) = \frac{s+1}{t+2}$. This seems like a reasonable way of dealing with his ignorance. On the other hand, trader 2 has read Jeffreys' (1946) proposal for dealing with ignorance and chooses a prior which minimizes entropy. Thus $\pi_2(\theta) \sim \sqrt{\theta(1-\theta)}$, so that her valuation of the asset after history (s,t) is $\mu_2(s,t) = \frac{s+\frac{1}{2}}{t+1}$. Note that her point estimate of the dividend is thus shifted towards the observed empirical frequency $\frac{s}{t}$ (relative to trader 1's posterior). Both these posteriors seem perfectly reasonable. Despite many attempts, there is no philosophical (or other) agreement on how to assign priors in the face of ignorance.

Given the simple learning environment, fundamental valuations will converge to the "objective" value, i.e. the observed frequency of dividends.

Lemma 2.2. For all $\theta_0 \in [0,1]$, $\mu_i(\theta_0 t, t) \to \theta_0$ as $t \to \infty$.

In the ignorance priors example, as $t \to \infty$, $\mu_1(\theta_0 t, t) = \frac{\theta_0 t+1}{t+2} \to \theta_0$ and $\mu_2(\theta_0 t, t) = \frac{\theta_0 t+\frac{1}{2}}{t+1} \to \theta_0$. Thus all valuations converge to the observed frequency or common objective value. They will, however, converge at different rates. For any given history (s, t), some traders will be more optimistic, some less so. But when is there a trader who values the asset the most after every history?

Definition 2.3. Trader k is a global optimist if $\mu_k(s, t) \geq \mu_i(s, t)$ for all $i \in I$ and all histories (s, t).

In the ignorance priors example, there is no global optimist, since $\mu_1(s, t) = \frac{s+1}{t+2} > \frac{s+\frac{1}{2}}{t+1} = \mu_2(s, t)$ if $s < \frac{t}{2}$, while $\mu_1(s, t) < \mu_2(s, t)$ if $s > \frac{t}{2}$. Intuitively, trader 2 puts more weight on the data, so after "good histories" ($s > \frac{t}{2}$) she values the asset the most; trader 1, whose uniform prior puts less weight on the data, is thus more optimistic after "bad histories" ($s < \frac{t}{2}$).

Even if trader k is not a global optimist, we might be interested in a situation where at least after some history, trader k is and remains the most optimistic. Thus say that history (s', t') follows (s, t) if $t' \geq t$ and $t' - t + s \geq s' \geq s$.

Definition 2.4. Trader k is a local optimist if there exists a history (s, t) such that for all histories (s', t') following (s, t), $\mu_k(s', t') \geq \mu_i(s', t')$ for all $i \in I$.

Is it the case, in the ignorance priors example, that if the history is sufficiently bad, trader 1 not only values the asset the most, but will value it the most after every continuation history? The answer is no: after any history (s, t), suppose that $2t$ periods follow in which a dividend is always paid. Then we will be at history $(s + 2t, 3t)$. In this case, since $s + 2t > \frac{3t}{2}$, $\mu_1(s + 2t, 3t) < \mu_2(s + 2t, 3t)$. Thus 1 is not a local optimist. An analogous argument shows that 2 is not a local optimist. No matter how much they have learned about θ, they always attach positive probability to switching from a good history to a bad history, or vice versa.

The existence of global and local optimists is critical to our analysis since it determines when there is a trader who ends up holding the asset forever with no retrading. An alternative way of expressing the non-existence of a local optimist is the following.

Definition 2.5. Beliefs $\{\pi_i\}_{i \in I}$ satisfy perpetual switching if, for every $i \in I$ and history (s, t), there exists $j \neq i$ and a history (s', t') following (s, t) such that $\mu_j(s', t') > \mu_i(s', t')$.

By the definitions, beliefs satisfy perpetual switching if and only if there is no local optimist. The ignorance priors example showed that beliefs satisfy perpetual switching under the most reasonable circumstances. I will report a necessary and sufficient condition soon. But let us first record the canonical circumstance in which there is not perpetual switching.

Example 2.6. (Common Priors): $\pi_i = \pi_1$ for each $i \in I$.

Then $\mu_i(s;t) = \mu_j(s;t)$ for all $i; j \in I$ and history $(s;t)$, and each trader is a local and global optimist and thus beliefs do not satisfy perpetual switching. These examples suggest the intuition that perpetual switching will typically hold when there is sufficient heterogeneity of prior beliefs. But how different must beliefs be to allow perpetual switching? It is possible to give a precise characterization in terms of prior beliefs.

Definition 2.7. Trader k is rate dominant if $\frac{\pi_k'(\theta)}{\pi_k(\theta)} \geq \frac{\pi_i'(\theta)}{\pi_i(\theta)}$ for all $i \in I$ and $\theta \in [0;1]$.

This condition ensures that there is a single trader whose density is always increasing at the fastest rate. To see why this important, consider trader i's expected value of θ contingent on knowing that θ is in the interval $[\theta_0 - \varepsilon; \theta_0 + \varepsilon]$. For small ε, this will be approximately θ_0 plus a term of order ε including the term $\frac{\pi_i'(\theta_0)}{\pi_i(\theta_0)}$. In fact, this local property can be used to give a tight characterization of the properties that we are interested in.

Theorem 2.8. (Heterogeneous Fundamental Values). The following are equivalent claims:

- Trader k is rate dominant.
- Trader k is a global optimist.
- Trader k is a local optimist.

The proof is in the appendix (section 8.1). Since we have shown the equivalence of being a global optimist and being a local optimist, I will henceforth use the term "optimist" to mean either. So how likely is perpetual switching? Equivalently, how close to the common prior assumption is the requirement that there exists a rate dominant trader? I will argue that it is very close. For example, it might be conjectured that if some trader's prior (first order) stochastically dominated all others, then he must be an optimist. This is false, as the following example shows.

Example 2.9. (Stochastic Dominance). Suppose there are two traders 1 and 2. Trader 1 believes that $\theta = \frac{1}{5}$ with probability $\frac{1}{2}$ and $\theta = \frac{4}{5}$ with probability $\frac{1}{2}$, while trader 2 believes that $\theta = \frac{1}{5}$ with probability $\frac{1}{2}$ and $\theta = \frac{3}{5}$ with probability $\frac{1}{2}$. Trader 1's prior stochastically dominates trader 2's. However, suppose that they observe history $(s,t) = (3,6)$, i.e. they observe 3 dividends in 6 periods. Trader 1's posterior probability that $\theta = \frac{4}{5}$ is $\frac{\left(\frac{4}{5}\right)^3\left(\frac{1}{5}\right)^3}{\left(\frac{1}{5}\right)^3\left(\frac{4}{5}\right)^3+\left(\frac{4}{5}\right)^3\left(\frac{1}{5}\right)^3} = \frac{1}{2}$; thus his fundamental valuation is $\mu_1(3,6) = \frac{1}{2}\times\frac{4}{5} + \frac{1}{2}\times\frac{1}{5} = \frac{1}{2}$. Trader 2's posterior probability that $\theta = \frac{3}{5}$ is $\frac{\left(\frac{3}{5}\right)^3\left(\frac{2}{5}\right)^3}{\left(\frac{3}{5}\right)^3\left(\frac{2}{5}\right)^3+\left(\frac{1}{5}\right)^3\left(\frac{4}{5}\right)^3} = \frac{3^3 2^3}{3^3 2^3+4^3} = \frac{27}{27+8} = \frac{27}{35}$; thus his fundamental valuation is $\mu_2(3,6) = \frac{27}{35}\times\frac{3}{5} + \frac{8}{35}\times\frac{1}{5} = \frac{89}{175} > \frac{1}{2}$.

With its discrete probability distributions, this example does not satisfy the assumptions of this paper. But it would be straightforward to approximate the priors in the example with smooth densities uniformly bounded below, with the same result.

Some intuition of what lies behind the above results comes from considering a parameterized class of prior probability distributions.

Example 2.10. (Beta Distributions). Each trader has a prior in the set of beta distributions, i.e. for each $i \in I$,

$$\pi_i(\theta) = \frac{\theta^{\alpha_i-1}(1-\theta)^{\beta_i-1}}{\int_{\zeta=0}^{1}\zeta^{\alpha_i-1}(1-\zeta)^{\beta_i-1}d\zeta}, \text{ for some } \alpha_i > 0 \text{ and } \beta_i > 0$$

In this example, trader k is an optimist if and only if

$$\alpha_k \geq \alpha_i \text{ and } \beta_k \leq \beta_i \text{ for all } i \in I. \tag{2.3}$$

There are two ways to show this. First, it can be proved directly from the following implied fundamental valuations (see Hartigan (1983) pages 76-78):

$$\mu_i(s,t) = \frac{s+\alpha_i}{t+\alpha_i+\beta_i}. \tag{2.4}$$

Second, theorem 2.8 can be used to check this from the rate dominance condition. Note that

$$\frac{\pi_i'(\theta)}{\pi_i(\theta)} = \frac{(\alpha_i - 1)\theta^{\alpha_i - 2}(1-\theta)^{\beta_i - 1} - (\beta_i - 1)\theta^{\alpha_i - 1}(1-\theta)^{\beta_i - 2}}{\theta^{\alpha_i - 1}(1-\theta)^{\beta_i - 1}} = \frac{\alpha_i - 1}{\theta} - \frac{\beta_i - 1}{1-\theta};$$

so

$$\frac{\pi_k'(\theta)}{\pi_k(\theta)} - \frac{\pi_i'(\theta)}{\pi_i(\theta)} = \frac{\alpha_k - \alpha_i}{\theta} + \frac{\beta_i - \beta_k}{1-\theta}.$$

Thus (2.3) implies that trader k is rate dominant. For the converse, note that as $\theta \to 0$, $\theta\frac{\pi_i'(\theta)}{\pi_i(\theta)} \to \alpha_i - 1$ while as $\theta \to 1$, $(1-\theta)\frac{\pi_i'(\theta)}{\pi_i(\theta)} \to \beta_i - 1$. Thus if $\alpha_i > \alpha_k$, $\frac{\pi_i'(\theta)}{\pi_i(\theta)} > \frac{\pi_k'(\theta)}{\pi_k(\theta)}$ for θ sufficiently close to 0, while if $\beta_i < \beta_k$, $\frac{\pi_i'(\theta)}{\pi_i(\theta)} > \frac{\pi_k'(\theta)}{\pi_k(\theta)}$ for θ sufficiently close to 1. Thus if trader k is rate dominant, (2.3) holds.

So within this two dimensional family of prior distributions, a two dimensional restriction (i.e. equation 2.3) must hold to ensure the existence of an optimist. Note that the ignorance priors case (example 2.1) falls in the class of beta distributions: trader 1's prior has $\alpha_1 = \beta_1 = 1$, while trader 2's prior has $\alpha_2 = \beta_2 = \frac{1}{2}$. Thus we can check directly from condition (2.3) that there is no optimist.

3. Market Prices with Learning

In the previous section, traders' "fundamental valuations" of the risky asset were characterized. Fundamental valuations reflect the (expected) value to each trader of holding the asset forever. In a market setting, however, where the asset can be re-sold, traders will want to take into account the possibility that they can sell the asset at a price higher than their fundamental valuations in some future contingency.

Suppose there are two infinitely lived assets: the risky asset studied in the previous section and a riskless asset with interest rate $r > 0$. Traders buy and sell the risky asset in a competitive market in each time period $t = 0, 1, \ldots$ after any dividend is paid. I assume that they cannot sell the risky asset short. I assume that the total quantity of the risky asset is sufficiently small and traders' endowments of the riskless asset are sufficiently large so that no trader ever needs to go short in the riskless asset. Thus if there were short sales constraints in the riskless asset (i.e. borrowing constraints), they would never bind.

In this section, I informally describe a set of equilibrium prices, and study its properties. These prices are the unique set of prices in the infinite economy which are the limit of prices in finite truncations of the economy. In the appendix (section 8.3), I show the existence of other equilibrium prices which entail "Ponzi schemes".

Write $P(s;t;r)$ for the price of the risky asset (in terms of current dollars) after history $(s;t)$. Write $\mu^*(s;t)$ for the most optimistic fundamental valuation of the asset of any trader after history $(s;t)$, i.e.

$$\mu^*(s;t) = \max_{i \in I} \mu_i(s;t)$$

Then equilibrium prices must satisfy

$$P(s;t;r) = \frac{1}{1+r} \begin{bmatrix} \mu^*(s;t)\{1 + P(s+1;t+1;r)\} \\ + (1 - \mu^*(s;t))\, P(s;t+1;r) \end{bmatrix} \tag{3.1}$$

This condition states that the price of the asset after history $(s;t)$ is equal to the highest expected discounted return (among all traders) of holding it to the next period. If the price of the risky asset was strictly higher than any trader's expected return from holding it to the next period, then no one will hold the asset and prices cannot be equilibrium prices. On the other hand, if the price was strictly lower than the highest expected return, then the trader with that highest expected return would want to hold infinite quantities, so markets would not clear.

Notice that the current dollar price of the riskless asset must be $\frac{1}{r}$, so if we write $p(s;t;r)$ for the price of the risky asset in terms of the riskless asset, we have $p(s;t;r) = \frac{P(s;t;r)}{1/r} = rP(s;t;r)$. Substituting in equation (3.1) gives:

$$\frac{p(s;t;r)}{r} = \frac{1}{1+r} \begin{bmatrix} \mu^*(s;t)\left\{1 + \frac{p(s+1;t+1;r)}{r}\right\} \\ + (1 - \mu^*(s;t))\frac{p(s;t+1;r)}{r} \end{bmatrix} \tag{3.2}$$

or

$$p(s;t;r) = \frac{1}{1+r} \begin{bmatrix} \mu^*(s;t)\{r + p(s+1;t+1;r)\} \\ + (1 - \mu^*(s;t))p(s;t+1;r) \end{bmatrix} \tag{3.3}$$

Following Harrison and Kreps (1978), a price scheme satisfying equation (3.3) can be explicitly calculated as follows. Set $p^0(s;t;r) = 0$ for all $s \leq t; r \in \Re_{++}$; define $p^n(s;t;r)$ recursively by

$$p^{n+1}(s;t;r) = \frac{1}{1+r}\left[\begin{array}{c} \pi_*(s;t)\{r + p^n(s+1;t+1;r)\} \\ +(1-\pi_*(s;t))p^n(s;t+1;r) \end{array}\right] \qquad (3.4)$$

Now let $p^*(s;t;r) = \lim_{n \to \infty} p^n(s;t;r)$. To show the limit exists, I first show by induction that $p^n(s;t;r) \leq 1$, for all $n; s; t; r$: this is clearly true for $n = 0$; if it is true for n, then $p^{n+1}(s;t;r) \leq \frac{1}{1+r}\{r+1\} = 1$. Since $p^n(s;t;r)$ is non-decreasing in n, the limit exists. Since p^* is a fixed point of (3.4), it certainly satisfies equation (3.3).

Harrison and Kreps showed the existence of such a "minimal pricing scheme" in a more general setting and showed that the price is no less than any trader's valuation i.e. - in this model - $p^*(s;t;r) \geq \pi_i(s;t)$ for all histories $(s;t)$, all traders $i \in I$ and all interest rates r. I can use the extra structure of the learning model to prove some stronger results.

Theorem 3.1. (Speculative Premia). (i) If trader k is an optimist, then $p^*(s;t;r) = \pi_k(s;t)$ for all histories $(s;t)$ and interest rates r; (ii) if there is no optimist, then $p^*(s;t;r) > \pi_i(s;t)$ for all histories $(s;t)$, interest rates r and traders $i \in I$; (iii) as $t \to \infty$, $p^*(\theta_0 t;t;r) \to \pi_i(\theta_0 t;t) \to \theta_0$, for all $\theta_0 \in [0;1]$, interest rates r and traders $i \in I$; (iv) as $r \to \infty$, $p^*(s;t;r) \to \pi_*(s;t)$ for all histories $(s;t)$.

In words:

- If there is an optimist (part i), he will always end up holding the asset after every history. Therefore there is no possibility of re-selling the asset and the asset price will always reflect his fundamental valuation of the asset. Note that in this case the price of the asset is independent of the interest rate by normalization.

- If there is no optimist (part ii), then there is eternal switching. Thus at every date, every holder of the asset attaches positive probability to being able to re-sell the asset at a price higher than his own valuation in some future contingency. Thus the price is always strictly greater than any trader's fundamental valuation.

- Now suppose that we look at histories where a dividend is realized in proportion θ_0 of time periods (part iii). As $t \to \infty$, all traders' posteriors converge to θ_0; thus in particular, the differences in fundamental valuations converge to 0. So the option value of being allowed to re-sell the asset goes to zero, and the price converges to $\mu_*(\theta_0 t; t)$ which converges to θ_0. In particular, $p^* \to \theta_0$ with probability one if the true value of θ is θ_0.

- As the interest rate increases (part iv), the speculative premium goes to zero as the discounted value of the expected dividend tomorrow swamps the discounted value of any option to re-sell.

This result obviously depends on some extreme assumptions which have been made: in particular, traders were assumed to be short sales constrained in the risky asset but not in the riskless asset. No doubt some traders in asset markets are liquidity constrained and cannot buy as much of a risky asset as they would like. Thus the conclusion that the risky asset is systematically driven up above fundamental valuations is not especially robust. But what is most interesting about theorem 3.1 is its characterization of how the existence and long run behavior of speculative premia depends on the heterogeneity of prior beliefs in the population of traders. This type of result is robust to the various assumptions.

The theorem is proved in the appendix (section 8.2). The proof also demonstrates why the price scheme p^* is the limit as $n \to \infty$ of the unique equilibrium prices of an n period truncation of the economy.

When there is no optimist, I do not have an analytic solution for p^*. However, it is possible to numerically calculate p^* using equation (3.4): this is done in the next section.

4. Reasonable Priors lead to Significant Speculative Premia

Say that the speculative premium at history $(s; t)$ is $p^*(s; t) - \mu_*(s; t)$. This is non-negative by theorem 3.1. If there is a strictly positive speculative premium, then $p^*(s; t) > \mu_*(s; t)$ and thus $p^*(s; t) > \mu_i(s; t)$ for all $i \in I$. In this case, the price exceeds every trader's fundamental valuation of the asset. Theorem 3.1 establishes that - in the absence of an optimist - a speculative premium exists after every history. On the other hand, theorem 3.1 also establishes that at $t \to \infty$, that speculative premium tends to zero. The purpose of this section is to establish numerically that apparently innocuous differences in prior beliefs lead to significant speculative premia. In particular, I return to the ignorance prior case (example 2.1) which was intended to capture the most reasonable possible priors in the face of ignorance.

Recall the two reasonable priors implied $\mu_1(s;t) = \frac{s+1}{t+2}$, $\mu_2(s;t) = \frac{s+\frac{1}{2}}{t+1}$ and thus $\mu_*(s;t) = \max\left\{\frac{s+1}{t+2}, \frac{s+\frac{1}{2}}{t+1}\right\}$. However we noted in section 2 that neither trader is an optimist, so by theorem 3.1, we must have $p^*(s;t;r) > \max\left\{\frac{s+1}{t+2}, \frac{s+\frac{1}{2}}{t+1}\right\}$ for all histories $(s;t)$ and interest rates r. In particular, we must have $p^*(0;0;r) > \mu_*(0;0) = \frac{1}{2}$ for all interest rates r. On the hand, theorem 3.1 also shows that as $r \to \infty$ (so that $\frac{1}{1+r} \to 0$), the speculative premium becomes insignificant in pricing the asset: thus $p(0;0;r) \to \frac{1}{2}$ as $r \to \infty$. Figure 1 plots $p(0;0;1)$ as r varies from 0.05 to 40.

Note that, because of the numerical procedure, it is not possible to calculate $p(0;0;r)$ accurately for arbitrarily small r. However, $p(0;0;0.05) = 0.54$, so that at an interest rate of 5%, a speculative premium of 8% is generated.

Another prediction of theorem 3.1 is that $p(\mu_0 t;t;r) \to \mu_0$ as $t \to \infty$. Figure 2 plots $p\left(\frac{t}{2};t;0.05\right)$ for t in the interval $[0;50]$.

5. Initial Public Offerings

There are two empirical puzzles associated with initial public offerings (Ritter (1991)). Offer prices tend to be significantly lower than initial market prices (the "under pricing" anomaly). Initial market prices tend to be high relative to long run prices (the "hot issue" anomaly). The model presented in section 3 provides one explanation for the latter phenomenon. When is it the case that an asset is traded in a situation where traders must form beliefs in the absence of historical data on the performance of returns? An initial public offering is an obvious example. While there may exist plentiful information on which to form an assessment, the lack of historical data creates scope for different traders to have different beliefs on the basis of the same information.

This explanation for the hot issue anomaly was presented some years ago by Miller (1977, page 1156). It is worth quoting in detail:

> The prices of new issues, as of all securities, are set not by the appraisal of the typical investor, but by the small number who think

> highly enough of the investment merits of the new issue to include it in their portfolio. The divergence of opinion about a new issue are greatest when the stock is issued. Frequently the company has not started operations, or there is uncertainty about the success of new products or the pro¯tability of a major business expansion. Over time, this uncertainty is reduced.... With the passage of time, and the reduction of uncertainty, the appraisal of the top x percent of the investors is likely to decline even if the average assessment is not changed. This would explain the poor performance of a group of new issues when compared to a group of stocks about which the uncertainty does not decrease over time.[1]

The model of section 3 can be seen as a formalization of Miller's argument. However, the ignorance prior example makes clear that the result does not require that traders have di®erent prior valuations of the asset. It is enough that their \ignorance" priors over unknown parameters do not have the same shape.

Ritter provides further evidence for Miller's hypothesis which has been formalized here. He ¯nds that initial public o®erings under perform the market by an average of 17% in their ¯rst three years. But this average varies with the age of the company - i.e. the number of years between its founding and the initial public o®ering. Controlling for industry, the initial overpricing of initial public o®erings decreases monotonically from 34% for ¯rms that are less than one year old to 4% for ¯rms which are more than twenty years old. It is more plausible that traders have di®erent beliefs on the basis of the same public information when the ¯rm going public also has a shorter record under private ownership.

6. The Common Prior Assumption in Economic Theory

When Miller proposed his explanation of the \hot issue" anomaly, economists were less explicit about the origin of di®erent (posterior) beliefs than they are today.

[1] Miller also suggests that this argument o®ers a partial but non-strategic explanation for the under-pricing anomaly: \Incidently, if underwriters ignore the above arguments and price new issues on the basis of their own best estimates of the prices of comparable seasoned securities, they will typically underprice new issues. The mean of their appraisals will resemble the mean appraisal of the typical investor, and this will be below the appraisals of the most optimistic investors who actually constitute the market for the security. This may be a partial explanation for the underpricing of new issues by underwriters."

Now we make a clear distinction between di®erences in posterior beliefs which are explained by di®erential information, and those which are unexplained by private information and thus represent a violation of the common prior assumption. The di®erences in beliefs in sections 2 through 4 cannot be interpreted as di®erences in information. If they were, then a \no trade" theorem along the lines of Milgrom and Stokey (1982) would guarantee no trade and no speculative premia.

I will argue that the model of this paper helps explain why and how economists might - selectively - allow di®erences in prior beliefs to be used to understand economic phenomena[2]. I will do so by discussing in turn each of three strands of argument which are made in support of the common prior assumption, and arguing why they are not compelling at least in this and certain other contexts.

6.1. \Anything can happen when priors di®er"

This puzzling opinion is highly prevalent in the economics folklore. Of course allowing di®erences in prior beliefs introduces another degree of freedom into modelling. Allowing di®erences in utility functions and information also introduces extra degrees of freedom. \Anything can happen" in many economic models under some appropriate assumptions about the heterogeneity of utility functions and information. Nonetheless, we remain interested in explaining phenomena in terms of heterogeneity of utility functions and information, even though we do not necessarily explicitly model the source of that heterogeneity. Heterogeneity of prior beliefs seems to be a victim of some kind of double standard.

In any case, heterogeneity of prior beliefs is just a particular form of heterogeneity of (ex ante) utility functions. Morris (1992) showed how the model of Harrison and Kreps which was the basis for this paper can be extended to allow for risk aversion and many goods in a full general equilibrium model (it is essentially the same model with di®erences in utility functions and endowments playing the role of di®erences of prior beliefs). This model is then similar to many models (including models of money) where short sales constraints drive a wedge in ¯rst order conditions to create phenomena which can be interpreted as bubbles. The increased generality of allowing arbitrary utility functions is admirable. But if we think (as is surely the case for initial public o®erings) that di®erent priors

[2] Following Lintner (1969), there have been a number of attempts to allow for heterogeneous prior beliefs in ¯nance: see Biais and Boessarts (1993) and Harris and Raviv (1993) for recent contributions. Morris (1993) contains a more detailed discussion of the role of the common prior assumption in economic theory.

are the primary source of the ex ante gains from trade, why not exploit the linear structure that focussing on beliefs allows?

Finally, consider theorem 3.1. Heterogeneity of prior beliefs, far from leading to \anything happening", leads to prices with remarkable qualitative properties that are do not depend on the particular priors.

6.2. \Rationality entails the common prior assumption"

There is a widespread intuition that di®erences in beliefs between rational people must be a consequence of private information. This intuition con°icts with the usual economists' notion of rationality as consistency (Savage (1954)), and other attempts to formalize the intuition mathematically or philosophically have met with little success (see Morris (1993, section 3)). On a more practical level, I presented in section 2 the thought experiment of imagining traders forming priors about the dividends of an as yet unobserved asset. It was extremely hard to conceive of any criteria - rational or otherwise - that might require traders to have the same prior. A number of di®erent priors seem entirely reasonable.

An alternative way of presenting the \rationality implies common priors" argument can be told in the context of initial public o®erings. The explanation of the over-pricing anomaly has something of a \winner's curse" °avor. It is tempting to argue that any trader holding the asset should want to revise downwards his valuation of the asset in the light of others' willingness to sell to him at that price. It is tempting, in other words, to interpret the di®erent priors as di®erent information.

No doubt many apparent di®erences in prior beliefs are explainable by di®erent information at some level. But consider how implausible this argument is in the particular context of the ignorance priors example. Two traders with di®erent priors over the value of µ must each conclude that the other knows something (about some true underlying distribution generating distributions over µ) that he does not. This would make sense only if we believed that there was some stationary distribution from which the value of initial public o®erings was being drawn. But if we believe that each initial public o®ering contains something genuinely new (not predictable from past data), then genuine di®erences in priors are surely reasonable.

6.3. "Learning implies the common prior assumption"

We are justified in assuming common priors, the argument goes, because past experience will have removed differences in beliefs unexplained by differences of information. The common prior assumption is then justified when learning has finished, so that everyone has learned the true underlying data generating process. But presumably we live in a world where rational learning is still taking place. One reason why this is true is that there are new types of events whose distribution cannot be predicted from past experience. In the economy, there may be some data generating processes which have been learned. Initial public offerings are presumably a situation where learning has not been completed.

Indeed, the argument that learning justifies the common prior assumption can be turned around. Suppose we want to test the idea that learning has (typically) led to a world in which all differences in posteriors are explained by information. Then consider those (rare) situations where there has not been a chance for complete learning to occur. Presumably we should expect to find distinctive behavior in those situations reflecting the heterogeneity of priors. In that sense, the over-pricing of initial public offerings is consistent with the learning rationale for the common prior assumption. But since it may take some time for full learning to occur, there remains a role for investigating what happens before learning is complete.

7. Conclusion

In this paper, I considered a simple environment where traders' fundamental valuations of an asset could be identified with their point estimates of the single parameter of the dividend process. I showed that even a small amount of heterogeneity in traders' prior beliefs implied "eternal switching," so that every trader after every history attaches positive probability to someone else valuing the asset strictly more after some continuation history. This in turn implies that even if traders' posterior beliefs are converging to the true value, the speculative premium never disappears. I also showed that the most reasonable imaginable differences in prior beliefs lead to significant speculative premia. In evaluating the existence and size of speculative premia, what matters is not just who initially has the highest expected value of the dividend in the next period. The speculative premium depends on differences in beliefs at all possible future contingencies. This explains why intuitively small differences in prior beliefs matter a lot.

Using unexplained differences in prior beliefs in economics has been out of fashion for some time. Initial public offerings represent the canonical situation where past experience will not have removed differences in beliefs. They thus represent an ideal test of whether it is possible to make interesting predictions from economic models with unexplained - but reasonable - differences in prior beliefs.

8. Appendix

8.1. Proof of Heterogenous Fundamental Values Theorem

To prove theorem 2.8, I must show the equivalence of the following.

1. Trader k is rate dominant.
2. Trader k is a global optimist.
3. Trader k is a local optimist.

By the definitions, (2) implies (3). So it suffices to show that (3) implies (1) and (1) implies (2).

(3) implies (1): Write $\nu_i(s;t)$ for the approximation of $\mu_i(s;t)$ obtained by the second order Taylor series expansion of $\pi_i(\theta)$, i.e. setting $\pi_i(\theta) = \pi_i(\theta_0) + (\theta - \theta_0)\pi_i'(\theta_0) + \frac{1}{2}(\theta-\theta_0)^2\pi_i''(\theta_0)$. Thus:

$$\nu_i(s;t) = \frac{\int_{\theta=0}^{1}\binom{t}{s}\theta^{s+1}(1-\theta)^{t-s}\left(\pi_i(\theta_0) + (\theta-\theta_0)\pi_i'(\theta_0) + \frac{1}{2}(\theta-\theta_0)^2\pi_i''(\theta_0)\right)d\theta}{\int_{\theta=0}^{1}\binom{t}{s}\theta^{s}(1-\theta)^{t-s}\left(\pi_i(\theta_0) + (\theta-\theta_0)\pi_i'(\theta_0) + \frac{1}{2}(\theta-\theta_0)^2\pi_i''(\theta_0)\right)d\theta}. \tag{8.1}$$

Writing $I(a;b) = \int_{\theta=0}^{1}\theta^a(1-\theta)^b d\theta$, this equals:

$$\nu_i(s;t) = \frac{\left\{\begin{array}{c}\pi_i(\theta_0)I(s+1;t-s) + \pi_i'(\theta_0)I(s+2;t-s) - \theta_0\pi_i'(\theta_0)I(s+1;t-s)\\ +\frac{1}{2}\pi_i''(\theta_0)I(s+3;t-s) - \pi_i''(\theta_0)\theta_0 I(s+2;t-s) + \frac{1}{2}\pi_i''(\theta_0)\theta_0^2 I(s+1;t-s)\end{array}\right\}}{\left\{\begin{array}{c}\pi_i(\theta_0)I(s;t-s) + \pi_i'(\theta_0)I(s+1;t-s) - \theta_0\pi_i'(\theta_0)I(s;t-s)\\ +\frac{1}{2}\pi_i''(\theta_0)I(s+2;t-s) - \pi_i''(\theta_0)\theta_0 I(s+1;t-s) + \frac{1}{2}\pi_i''(\theta_0)\theta_0^2 I(s;t-s)\end{array}\right\}}:$$

Now substituting:

$$I(a,b) = \frac{a!\,b!}{(a+b+1)!},$$

we get

$$\rho_i(s,t) = \frac{\left\{ \begin{array}{c} \pi_i(\theta_0)\frac{(s+1)!(t-s)!}{(t+2)!} + \pi_i'(\theta_0)\frac{(s+2)!(t-s)!}{(t+3)!} - \theta_0\pi_i'(\theta_0)\frac{(s+1)!(t-s)!}{(t+2)!} \\ +\frac{1}{2}\pi_i''(\theta_0)\frac{(s+3)!(t-s)!}{(t+4)!} - \pi_i''(\theta_0)\theta_0\frac{(s+2)!(t-s)!}{(t+3)!} + \frac{1}{2}\pi_i''(\theta_0)\theta_0^2\frac{(s+1)!(t-s)!}{(t+2)!} \end{array} \right\}}{\left\{ \begin{array}{c} \pi_i(\theta_0)\frac{s!(t-s)!}{(t+1)!} + \pi_i'(\theta_0)\frac{(s+1)!(t-s)!}{(t+2)!} - \theta_0\pi_i'(\theta_0)\frac{s!(t-s)!}{(t+1)!} \\ +\frac{1}{2}\pi_i''(\theta_0)\frac{(s+2)!(t-s)!}{(t+3)!} - \pi_i''(\theta_0)\theta_0\frac{(s+1)!(t-s)!}{(t+2)!} + \frac{1}{2}\pi_i''(\theta_0)\theta_0^2\frac{s!(t-s)!}{(t+1)!} \end{array} \right\}}.$$

Cancelling out $\frac{s!(t-s)!}{(t+1)!}$ gives:

$$\rho_i(s,t) = \frac{\left\{ \begin{array}{c} \pi_i(\theta_0)\frac{s+1}{t+2} + \pi_i'(\theta_0)\frac{(s+1)(s+2)}{(t+2)(t+3)} - \theta_0\pi_i'(\theta_0)\frac{s+1}{t+2} \\ +\frac{1}{2}\pi_i''(\theta_0)\frac{(s+1)(s+2)(s+3)}{(t+2)(t+3)(t+4)} - \pi_i''(\theta_0)\theta_0\frac{(s+1)(s+2)}{(t+2)(t+3)} + \frac{1}{2}\pi_i''(\theta_0)\theta_0^2\frac{s+1}{t+2} \end{array} \right\}}{\left(\begin{array}{c} \pi_i(\theta_0) + \pi_i'(\theta_0)\frac{s+1}{t+2} - \theta_0\pi_i'(\theta_0) \\ +\frac{1}{2}\pi_i''(\theta_0)\frac{(s+1)(s+2)}{(t+2)(t+3)} - \pi_i''(\theta_0)\theta_0\frac{s+1}{t+2} + \frac{1}{2}\pi_i''(\theta_0)\theta_0^2 \end{array} \right)}.$$

Thus $\rho_i(s,t) - \frac{s+1}{t+2} =$

$$\frac{\pi_i'(\theta_0)\frac{s+1}{t+2}\left(\frac{s+2}{t+3} - \frac{s+1}{t+2}\right) + \frac{1}{2}\pi_i''(\theta_0)\frac{(s+1)(s+2)}{(t+2)(t+3)}\left(\frac{s+3}{t+4} - \frac{s+1}{t+2}\right) - \pi_i''(\theta_0)\theta_0\frac{s+1}{t+2}\left(\frac{s+2}{t+3} - \frac{s+1}{t+2}\right)}{\pi_i(\theta_0) + \pi_i'(\theta_0)\frac{s+1}{t+2} - \theta_0\pi_i'(\theta_0) + \frac{1}{2}\pi_i''(\theta_0)\frac{(s+1)(s+2)}{(t+2)(t+3)} - \pi_i''(\theta_0)\theta_0\frac{s+1}{t+2} + \frac{1}{2}\pi_i''(\theta_0)\theta_0^2}.$$

Observe that as $t \to \infty$,

- $\frac{\theta_0 t+1}{t+2} \to \theta_0$; $\frac{\theta_0 t+2}{t+3} \to \theta_0$; $\frac{\theta_0 t+2}{t+3} - \frac{\theta_0 t+1}{t+2} = \frac{(1-\theta_0)t+1}{(t+2)(t+3)} \to \frac{1-\theta_0}{t}$.
- $\left(\frac{\theta_0 t+3}{t+4} - \frac{\theta_0 t+1}{t+2}\right) = \frac{\theta_0 t^2+3t+2\theta_0 t+6-\theta_0 t^2-t-4\theta_0 t-4}{(t+2)(t+4)} = \frac{2(1-\theta_0)t-2}{(t+2)(t+4)} \to \frac{2(1-\theta_0)}{t}$.
- $\frac{\theta_0 t+2}{t+3} - \frac{\theta_0 t+1}{t+2} = \frac{\theta_0 t^2+2t+2\theta_0 t+4-\theta_0 t^2-t-3\theta_0 t-3}{(t+2)(t+3)} = \frac{(1-\theta_0)t-1}{(t+2)(t+3)} \to \frac{1-\theta_0}{t}$.

Thus as $t \to \infty$,

$$\begin{aligned} \rho_i(\theta_0 t,t) &\to \theta_0 + \frac{\pi_i'(\theta_0)\theta_0\frac{1-\theta_0}{t} + \frac{1}{2}\pi_i''(\theta_0)\theta_0^2\frac{2(1-\theta_0)}{t} - \pi_i''(\theta_0)\theta_0^2\frac{1-\theta_0}{t}}{\pi_i(\theta_0) + \pi_i'(\theta_0)\theta_0 - \theta_0\pi_i'(\theta_0) + \frac{1}{2}\pi_i''(\theta_0)\theta_0^2 - \pi_i''(\theta_0)\theta_0^2 + \frac{1}{2}\pi_i''(\theta_0)\theta_0^2} \\ &= \theta_0 + \frac{\theta_0(1-\theta_0)}{t}\frac{\pi_i'(\theta_0)}{\pi_i(\theta_0)}. \end{aligned}$$

Now as $t \to \infty$, trader i's posterior $\xi_i(\theta|\theta_0 t; t)$ concentrates mass around θ_0 (recall that each π_i was uniformly bounded below), so

$$\mu_i(\theta_0 t; t) \to \nu_i(\theta_0 t; t) \to \theta_0 + \frac{\theta_0(1-\theta_0)}{t}\frac{\pi_i'(\theta_0)}{\pi_i(\theta_0)}. \qquad (8.2)$$

Suppose π_k is not rate dominant. Then (by continuity) there exists $i \in I$ and a rational number $\theta_0 \in (0;1)$ such that $\frac{\pi_i'(\theta_0)}{\pi_i(\theta_0)} > \frac{\pi_k'(\theta_0)}{\pi_k(\theta_0)}$. Let $\theta_0 = \frac{a}{b}$, for two strictly positive integers $a; b$. But now for any $(s; t)$, there exists T sufficiently large such that history $(aT; bT)$ follows $(s; t)$ and (by 8.2) $\mu_k(aT; bT) < \mu_i(aT; bT)$. Thus k is not a local optimist at $(s; t)$.

(1) implies (2): Suppose trader k is rate dominant. Let

$$\beta_i(\theta) = \frac{\pi_i(\theta)}{\int_{\theta'=\theta}^{1} \pi_i(\theta')\, d\theta'}.$$

By trader k rate dominant, we have (for any $i \in I$) $\frac{\pi_k(\theta)}{\pi_i(\theta)}$ non-decreasing in θ, so that

$$\frac{\pi_k(\theta')}{\pi_k(\theta)} \geq \frac{\pi_i(\theta')}{\pi_i(\theta)}; \text{ for all } \theta' \geq \theta.$$

Thus

$$\frac{\int_{\theta'=\theta}^{1} \pi_k(\theta')\, d\theta'}{\pi_k(\theta)} \geq \frac{\int_{\theta'=\theta}^{1} \pi_i(\theta')\, d\theta'}{\pi_i(\theta)}; \text{ for all } \theta \in [0;1];$$

so

$$\beta_k(\theta) = \frac{\pi_k(\theta)}{\int_{\theta'=\theta}^{1} \pi_k(\theta')\, d\theta'} \leq \frac{\pi_i(\theta)}{\int_{\theta'=\theta}^{1} \pi_i(\theta')\, d\theta'} = \beta_i(\theta); \text{ for all } \theta \in [0;1]. \qquad (8.3)$$

Let

$$\alpha_i(\theta) = \frac{\int_{\theta'=\theta}^{1} \theta' \pi_i(\theta')\, d\theta'}{\int_{\theta'=\theta}^{1} \pi_i(\theta')\, d\theta'}.$$

Thus

$$\frac{d\alpha_i}{d\theta} = -\frac{\theta\pi_i(\theta)}{\int_{\theta'=\theta}^{1}\pi_i(\theta')\,d\theta'} + \frac{\pi_i(\theta)\int_{\theta'=\theta}^{1}\theta'\pi_i(\theta')\,d\theta'}{\left(\int_{\theta'=\theta}^{1}\pi_i(\theta')\,d\theta'\right)^2} = \beta_i(\theta)\,(\alpha_i(\theta) - \theta). \tag{8.4}$$

Now (8.4) and (8.3) imply that if $\alpha_k(\theta) = \alpha_i(\theta)$, then $\frac{d\alpha_k}{d\theta} \leq \frac{d\alpha_i}{d\theta}$. Thus since $\alpha_j(1) = 1$ for all $j \in I$, α_k can never fall below α_i as θ goes from 1 to 0. Thus $\mu_k(0;0) = \alpha_k(0) \geq \alpha_i(0) = \mu_i(0;0)$. Thus a rate dominant trader is the most optimistic after the null history $(0;0)$. But traders' posteriors after history $(s;t)$ are

$$\xi_i(\theta|s;t) = \frac{\theta^s(1-\theta)^{t-s}\pi_i(\theta)}{\int_{\zeta=0}^{1}\zeta^s(1-\zeta)^{t-s}\pi_i(\zeta)d\zeta}; \tag{8.5}$$

so that

$$\xi_i'(\theta|s;t) = \frac{d\xi_i(\theta|s;t)}{d\theta} = \frac{s\theta^{s-1}(1-\theta)^{t-s}\pi_i(\theta) - (t-s)\,\theta^s(1-\theta)^{t-s-1}\pi_i(\theta) + \theta^s(1-\theta)^{t-s}\pi_i'(\theta)}{\int_{\zeta=0}^{1}\zeta^s(1-\zeta)^{t-s}\pi_i(\zeta)d\zeta}.$$

Thus

$$\frac{\xi_i'(\theta|s;t)}{\xi_i(\theta|s;t)} = \frac{s}{\theta} - \frac{t-s}{1-\theta} + \frac{\pi_i'(\theta)}{\pi_i(\theta)};$$

so

$$\frac{\xi_k'(\theta|s;t)}{\xi_k(\theta|s;t)} - \frac{\xi_i'(\theta|s;t)}{\xi_i(\theta|s;t)} = \frac{\pi_k'(\theta)}{\pi_k(\theta)} - \frac{\pi_i'(\theta)}{\pi_i(\theta)}.$$

Thus ξ_k inherits the rate dominance property. So, by the above argument, $\mu_k(s;t) \geq \mu_i(s;t)$, for every history $(s;t)$, so trader k is a global optimist.

8.2. Proof of the Speculative Premium Theorem

I will give a simple algebra argument to simultaneously prove theorem 3.1 and provide some further intuition. In section 3, the sequence of prices $\{p^n(s;t;r)\}_{n=1}^{\infty}$ were used as a computation device. They can also be given an economic interpretation. Suppose that, instead of being infinitely lived, the asset was only going to survive for a further n periods. Then, by backward induction and equation (3.4), $p^n(s,t,r)$ would be the unique equilibrium price for a risky asset with n periods to live after history (s,t). Now compare these prices with trader i's valuation of the risky asset if he was going to hold it for n periods without the opportunity to re-sell. Write $h^n(r)$ for the discounted value of a dollar in each of the next n periods (with $h^0(r) = 0$ by convention), i.e.

$$h^n(r) = \frac{1}{1+r} + \left(\frac{1}{1+r}\right)^2 + \cdots + \left(\frac{1}{1+r}\right)^n = \frac{1}{r}\left(1 - \left(\frac{1}{1+r}\right)^n\right). \tag{8.6}$$

The fundamental valuation of trader i of the n period risky asset (in terms of the infinitely lived riskless asset) after history (s,t) is then $rh^n(r)\mu_i(s,t)$. Now write $\alpha_i^n(s,t,r)$ for the “speculative premium” by which the market price exceeds i's valuation, i.e.

$$\alpha_i^n(s,t,r) \equiv p^n(s,t,r) - rh^n(r)\mu_i(s,t), \tag{8.7}$$

and write $\alpha_i^*(s,t,r)$ for the (well-defined) $\lim_{n\to\infty} \alpha_i^n(s,t,r)$. Observe that $\alpha_i^n(s,t,r) = 0$ and

$$\alpha_i^*(s,t,r) = p^*(s,t,r) - \mu_i(s,t) \tag{8.8}$$

Substituting (8.7) into the inductive definition of p^{n+1} (i.e. equation 3.4) gives:

$$\begin{bmatrix} \alpha_i^{n+1}(s,t,r) \\ +rh^{n+1}(r)\mu_i(s,t) \end{bmatrix} = \frac{1}{1+r}\begin{bmatrix} \mu_*(s,t)\left\{\begin{array}{c} r+\alpha_i^n(s+1,t+1,r) \\ +rh^n(r)\mu_i(s+1,t+1) \end{array}\right\} \\ +(1-\mu_*(s,t))\left\{\alpha_i^n(s,t+1,r)+rh^n(r)\mu_i(s,t+1)\right\} \end{bmatrix}. \tag{8.9}$$

By definition of h^n (equation 8.6),

$$h^{n+1}(r) = \frac{1}{1+r}(1+h^n(r)) \tag{8.10}$$

Substituting (8.10) in (8.9) and re-arranging gives:

$$\alpha_i^{n+1}(s,t,r) = \frac{1}{1+r}\left[\begin{array}{c} \mu_*(s,t)\,\alpha_i^n(s+1,t+1,r) + (1-\mu_*(s,t))\alpha_i^n(s,t+1,r) \\ +r\,h^n(r)\left(\begin{array}{c} +r\,(\mu_*(s,t)-\mu_i(s,t)) \\ \mu_*(s,t)\,\mu_i(s+1,t+1) - \\ (1-\mu_*(s,t))\,\mu_i(s,t+1) - \mu_i(s,t) \end{array}\right) \end{array}\right]. \tag{8.11}$$

But by the assumption that dividends are i.i.d., trader i at time t has identical expectations of the return of the asset at times $t+1$ and $t+2$. Thus

$$\mu_i(s,t) = \mu_i(s,t)\,\mu_i(s+1,t+1) + (1-\mu_i(s,t))\,\mu_i(s,t+1); \tag{8.12}$$

Substituting (8.12) in (8.11) and re-arranging gives:

$$\alpha_i^{n+1}(s,t,r) = \frac{1}{1+r}\left[\begin{array}{c} \mu_*(s,t)\,\alpha_i^n(s+1,t+1,r) + (1-\mu_*(s,t))\alpha_i^n(s,t+1,r) \\ +r\,(\mu_*(s,t)-\mu_i(s,t)) \\ +r\,h^n(r)\,(\mu_*(s,t)-\mu_i(s,t))\,(\mu_i(s+1,t+1)-\mu_i(s,t+1)) \end{array}\right]. \tag{8.13}$$

Equation (8.13) can now be used to prove each of the first three components of the theorem.

(i) Now if trader k is an optimist, then $\mu_*(s,t) = \mu_k(s,t)$ for all histories (s,t). Thus we have:

$$\alpha_k^{n+1}(s,t,r) = \frac{1}{1+r}\left[\mu_*(s,t)\,\alpha_k^n(s+1,t+1,r) + (1-\mu_*(s,t))\,\alpha_k^n(s,t+1,r)\right]. \tag{8.14}$$

But since $\alpha_k^0(s,t,r) = 0$ for all $s \le t, r$, we have by induction $\alpha_k^n(s,t,r) = 0$ for all $s \le t, r, n$ and thus $\alpha_k^*(s,t,r) = 0$ for all $s \le t, r$. So by (8.8), $p^*(s,t,r) = \mu_k(s,t)$ for all $s \le t, r$.

(ii) If there is no optimist, then for any trader $i \in I$ and any history (s,t), there exists a history (s',t'), following (s,t), such that $\mu_i(s',t') < \mu_*(s',t')$; thus $\alpha_i^n(s',t',r) > 0$, for all $n \ge 1$; this implies $\alpha_i^n(s,t,r) > 0$ for all $n \ge t' - t$; thus $\alpha_i^*(s,t,r) > 0$, so by (8.8), $p^*(s,t,r) > \mu_k(s,t)$ for all histories (s,t) and interest rates r.

(iii) For any $\varepsilon > 0$, we can choose T such that $|\mu_*(s;t) - \mu_i(s;t)| < \varepsilon$ for all $i \in I$, $s \le t$, and $t \ge T$. Let $\alpha^n_*(T;r) = \sup_{s \le t,\, t \ge T,\, i \in I} \alpha^n_i(s;t;r)$, and $\alpha^*_*(T;r) = \limsup_{n \to \infty} \alpha^n_*(T;r)$. Thus for all $s \le t$, $t \ge T$ and $i \in I$,

$$p^*(s;t;r) - \mu_i(s;t) = \alpha^*_i(s;t;r) \le \alpha^*_*(T;r). \qquad (8.15)$$

By equation (8.13),

$$\alpha^{n+1}_*(T;r) \le \frac{1}{1+r}\alpha^n_*(T;r) + \frac{r}{1+r}\left[1 + h^k(r)\right]\varepsilon \le \frac{1}{1+r}\alpha^n_*(T;r) + \varepsilon. \qquad (8.16)$$

Thus

$$\alpha^n_*(T;r) \le \varepsilon + \frac{1}{1+r}\varepsilon + \dots + \left(\frac{1}{1+r}\right)^{n-1}\varepsilon;$$

so that

$$\alpha^*_*(T;r) = \lim_{n \to \infty} \alpha^n_*(T;r) \le \varepsilon + \frac{1}{1+r}\varepsilon + \left(\frac{1}{1+r}\right)^2 \varepsilon + \dots = \varepsilon \frac{1}{1 - \frac{1}{1+r}} = \varepsilon\left(1 + \frac{1}{r}\right).$$

Thus for any given r and $\delta > 0$, we can choose T such that $p^*(s;t) - \mu_i(s;t) < \delta$ for all $s \le t$, $t \ge T$ and $i \in I$.

(iv) Since $p^n(s;t;r) \in [0;1]$ for all $s;t;r$, equation (3.4) ensures that $p^{n+1}(s;t;r) \in \left[\frac{r}{1+r}\mu_*(s;t); \frac{r}{1+r}\mu_*(s;t) + \frac{1}{1+r}\right]$; thus as $r \to \infty$, $p^{n+1}(s;t;r) \to \mu_*(s;t)$ for all $n \ge 0$, so $p^*(s;t;r) \to \mu_*(s;t)$.

8.3. Other Equilibrium Prices

The purpose of this section is to identify possible equilibrium prices, other than those of section 3, but to argue that they are in some sense unreasonable. The argument of this section is a special case of results in Harrison and Kreps (1978). In section 3, I assumed that prices do not depend on the historical pattern of dividends, only on their number. Here I want to allow for "sunspot" behavior, where the apparently irrelevant pattern of dividends effects prices.

A history of risky asset realizations is a vector $h^t = \{d_1; d_2; \dots d_t\} \in H^t = \{0;1\}^t$ (write h^0 for the empty history). Write $s(h^t)$ for the number of dividends paid during history h^t, i.e. $s(h^t)$ is the number of elements of $\{\tau \in \{1; \dots; t\} | h^t_\tau =$

$1\}$. Write $Q(h^t,r)$ for the price of the risky asset after history h^t (in current dollars). As before, prices Q are equilibrium prices if they are non-negative, no type of trader would strictly prefer to hold more of the asset and some trader is indifferent between holding the asset and not holding it. Thus prices Q occur in some competitive equilibrium if and only if $Q \geq 0$ and

$$Q(h^t,r) = \frac{1}{1+r} \max_{i \in I} \left[\mu_i\left(s\left(h^t\right),t\right)\left\{1 + Q\left(\left(h^t,1\right),r\right)\right\} + \left(1 - \mu_i\left(s\left(h^t\right),t\right)\right) Q\left(\left(h^t,0\right),r\right)\right]. \tag{8.17}$$

Again, normalizing by the price of the riskless asset, let $q(h^t,r) = rQ(h^t,r)$. Now we have:

$$q(h^t,r) = \frac{1}{1+r} \max_{i \in I} \left[\mu_i\left(s\left(h^t\right),t\right)\left\{r + q\left(\left(h^t,1\right),r\right)\right\} + \left(1 - \mu_i\left(s\left(h^t\right),t\right)\right) q\left(\left(h^t,0\right),r\right)\right]. \tag{8.18}$$

The equilibrium prices identified in section 3 certainly satisfied this property. But there exist other equilibrium prices. More precisely, letting $q^*(h^t,r) = p^*(s(h^t),t,r)$, q^* satisfies (8.18). But this difference equation does not rule out many kinds of apparently odd behavior. For example, it is possible that prices q satisfying equation (8.18) have:

- prices depending on the order of dividends i.e. $q(h^t,r) \neq q(g^t,r)$, for some $g^t, h^t \in H^t$ and $s(h^t) = s(g^t)$;
- prices decreasing in the number of dividends i.e. $q(h^t,r) < q(g^t,r)$, for some $g^t, h^t \in H^t$ and $s(h^t) > s(g^t)$;
- prices higher than the value of the maximum conceivable payment of dividends, i.e. $q(h^t,r) > 1$, for all $h^t \in H^t$.

But all these properties depend on the infinite horizon and thus entail "Ponzi schemes". In particular, any q satisfying (8.18) can be re-written in the form $q(h^t,r) = q^*(h^t,r) + \rho(h^t,r)$, where $\rho \geq 0$ is a Ponzi scheme satisfying

$$\rho(h^t,r) = \frac{1}{1+r}\left[\mu_{i(h^t)}\left(s\left(h^t\right),t\right)\rho\left(\left(h^t,1\right),r\right) + \left(1 - \mu_{i(h^t)}\left(s\left(h^t\right),t\right)\right)\rho\left(\left(h^t,0\right),r\right)\right]; \tag{8.19}$$

where

$$i\left(h^t\right) \in \arg\max_{i\in I} \left[\mu_i\left(s\left(h^t\right);t\right)\left\{r + q\left((h^t;1);r\right)\right\} + \left(1-\mu_i\left(s\left(h^t\right);t\right)\right) q\left((h^t;0);r\right)\right]: \tag{8.20}$$

Thus if q satisfies (8.18), $q(h^t;t) \geq q^*(h^t;t)$ for all $h^t \in H^t$, $t \geq 0$; this is why Harrison and Kreps labelled q^* the "minimal consistent price scheme".

Let us conclude by checking the claim that ρ must be non-negative. For any price function q, define a new price function $f(q)$ by:

$$[f(q)](h^t;r) = \frac{1}{1+r} \max_{i\in I} \left[\mu_i\left(s\left(h^t\right);t\right)\left\{r + q\left((h^t;1);r\right)\right\} + \left(1-\mu_i\left(s\left(h^t\right);t\right)\right) q\left((h^t;0);r\right)\right] \tag{8.21}$$

Thus q satisfies (8.18) if and only if $q = f(q)$. Now define sequence of prices q^k inductively by setting $q^0(h^t;r) = 0$, for all $h^t \in H^t$ and $r \in \Re_{++}$, and $q^{k+1} = f(q^k)$. Observe that $q^k(h^t;r)$ is increasing in k and $q^* = \lim_{k\to\infty} q^k$ by construction. Now consider some arbitrary non-negative prices q' satisfying (8.18). Since $q' \geq q^0$ and $f(q') = q'$, we have by induction and monotonicity of f, $q'(h^t;r) \geq q^k(h^t;r)$ for all k and so $q'(h^t;r) \geq q^*(h^t;r)$:

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