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The Sequential Regularity of Competitive Equilibria and Sunspots*

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Abstract

This paper studies the robustness of a competitive equilibrium against sunspots, or endogenous uncertainty. It is shown that an equilibrium is robust if and only if it is sequentially regular.

*This paper will eventually become a joint paper with Thorsten Hens. Conversations with Piero Gottardi have been very helpful. I am solely responsible for any errors or ommisions.

1. Introduction

This paper studies the relationship between the sequential regularity of competitive equilibria in a two-period economy with one asset and the e®ect of sunspots.

First assume that there is no uncertainty about the state of economy in the second period, and thus the markets are complete. An equilibrium is said to be sequentially regular if it is regular and all the spot market equilibria are regular by themselves. In other words, in a sequentially regular equilibrium, every continuation of the equilibrium path is a regular equilibrium. An economy, identi¯ed with its endowment vectors, is sequentially regular if every equilibrium is sequentially regular. If a regular equilibrium is not sequentially regular, the spot market equilibrium would appear unstable if it is not sequentially regular.

Balasko (1983, 94) recognized the importance of the sequential regularity in a temporal equilibrium setting.[1] The de¯nition given in this paper follows Mandler (1989a,b) which studied the sequential regularity in rational expectation equilibrium with production, and the reader is referred to these papers for motivation. In our simple exchange model, the issue of sequential regularity arises since the endowments in the second period is the sum of exogenously given endowments and the yields from the asset, which is determined by asset trades in the ¯rst period.[2]

Now suppose that the agents observe sunspots at the beginning of the second period. The equilibrium considered above can be naturally identi¯ed as a degenerate case of sunspot equilibrium where the prices do not depend on sunspots { call such an equilibrium a non-sunspot equilibrium and call the original equilibrium a certainty equilibrium. Note that the regularity of a non-sunspot equilibrium is qualitatively di®erent from that of a certainty equilibrium; the former implies that the equilibrium is locally isolated from other sunspot equilibria, whereas the latter does not necessarily have such an implication. Put it di®erently, a certainty equilibrium is robust against sunspots if it is also a regular non-sunspot equilibrium.

[1] In fact, Balasko studied a stronger notion which he called the T-regularity.

[2] Geanakoplos-Polemarchakis (1986) considered a stronger notion, the strong regularity: an equilibrium is strongly regular, if the spot market economy is a regular economy. Clearly, the strong regularity will imply the sequential regularity, but not vice versa. Geanakoplos-Polemarchakis showed that an economy whose equilibria are strongly regular is generic in utility functions.

The main result of this paper is to show that a certainty equilibrium is sequentially regular if and only if it is a regular, non-sunspot equilibrium. Taking advantage of the symmetry in the system of the ¯rst order conditions that describes the equilibria (the universal system), we shall provide a very simple proof in section 3.

An interesting implication of the result is that the regularity of a non-sunspot equilibrium is independent of the structure of sunspots. More precisely, if a certainty equilibrium is sequentially regular, it is a regular sunspot equilibrium for any number over sunspots and any probability distribution of sunspots. When the asset is real, this implies that the equilibrium is locally isolated from any other sunspot equilibria. This observation is important when one wishes to think that the probability distribution of sunspots and/or the structure of sunspot states are endogenously determined. See section 4 for detail.

The implication of sequential regularity is di®erent depending on whether the asset is nominal or real. When the asset is real, the second period equilibrium can be naturally seen as an equilibrium of the exchange economy where each household's endowments are the initial endowments plus the real payo®s from the asset portfolio which is determined in the previous period. So the implication of the regularity of the equilibrium is standard. When the asset is nominal, the second period economy is an exchange economy where each household receives a transfer in unit of account, so there is one degree of indeterminacy; the normalization of prices matters. See section 4.

Given the genericity result o®ered by Cass (1992) and Lisboa (1995),[3] it immediately follows that the set of sequentially regular economies in the non-sunspot model regular is open and full measure, when the asset is nominal. The result is of course consistent with the generic sequential regularity shown in Balasko (1994), Mandler (1989 a,b), and Mas-Colell.[4]

In a model where there is no consumption in the ¯rst period, Hens (1990) shows that the regularity of non-sunspot equilibrium does not depend on probability and asset structure for a ¯xed number of sunspot states. Our result is not immediately comparable to his, since the restriction of no ¯rst period consumption involves a little di®erent structure; most importantly, asset payo®s must depend on sunspots in order to have non-degenerate sunspot equilibria in such a model if markets are to be complete except for sunspots. As a technical remark, Hens' proof is very complicated since he described the regularity in terms

[3] See also Pietra (1992) and Suda-Tallon-Villanacci (1992).

[4] In his presidential address to the econometric society, July 1993.

of excess demand function.[5]

2. The Model

We consider a competitive two-period exchange economy with one asset. In period 0, commodities and the asset are exchanged. In period 1, the asset pays off and commodities are exchanged. The model is similar to that in Cass (1992).

2.1. The Sunspot Economy

We assume that there are S; $S \geq 1$, sunspot states in the second period. Sunspot s occurs with a publicly known probability $\pi^s > 0$. Spot commodity markets open in the first and second period, and there are C commodities in each spot, labelled by $c = 1, 2, \ldots, C$. We label each spot by $s = 0, \ldots, S$, spot zero corresponding to the first period. There are H households, labelled by $h = 1, 2, \ldots H$. Household h receives endowments e^0_h in the first period and e^1_h in the second period.

There is one asset which pays $a(p) \in \Re$ in unit of account in the second period in any state. So the payoffs do not directly depend on sunspots. The asset is in zero net supply. It is assumed that a is C^1 function of $p \in \Re^C_{++}$. The two special cases of interest are

- $a(p) = 1$ for all p (nominal, inside money)
- there exists a vector $a \in \Re^C$ such that $a(p) = p \cdot a$ for all p (real asset)

The following summarizes the notation:

x^s_h is the consumption plan in state s by household h. $x_h \equiv (x^s_h)^S_{s=0}$, $x \equiv (x_h)^H_{h=1}$

p^s are the price vector of commodities in spot s, $p \equiv (p^s)^S_{s=0}$,

the price of the asset is denoted by q

b_h is the demand for the asset by household h.

Preferences of household h over consumption plans are represented by the expected utility function $U_h(x_h) = \sum^S_{s=1} \pi^s u_h(x^0_h, x^s_h)$.

We assume:

[5] He has recently notified us that his argument can be dramatically simplified by directly working with the universal system as in our proof.

Assumption 2.1. u_h is smooth, differentiably strictly increasing (i.e. $Du_h \gg 0$), differentiably strictly concave (i.e. $D^2 u_h(x_h)$ is negative definite) and with the closure of the indifference surfaces contained in $\mathbb{R}^{2C}_{++}$.

Assumption 2.2. $e_h \gg 0$.

For each $s = 0, 1, \dots, S$, let $\Xi^s = \mathbb{R}^{HC}_{++} \times \mathbb{R}^{H}_{++} \times \mathbb{R}^{C}_{++}$, with its generic element $\xi^s = (x^s, \lambda^s, p^s)$. Let $\Xi^m = \mathbb{R}^H \times \mathbb{R}$ with generic element $\xi^m = (b, q)$. Finally set $\Xi = \Xi^m \times \Xi^0 \times \Xi^1 \times \cdots \times \Xi^S$ and denote its generic element by $\xi = \left(\xi^m, \xi^0, \dots, \xi^S\right) = \left((b, q), \left(x^s, \lambda^s, p^s\right)_{s=0}^{S}\right)$.

$x \in \left(\mathbb{R}^{C(S+1)}\right)^H$ is called an equilibrium allocation if there exists $p \in \mathbb{R}^{C(S+1)}_{++}$, $q \in \mathbb{R}$, $b \in \mathbb{R}^H$ such that:

(H) for each h, (x_h, b_h) solves the following given p and q:

$$\begin{aligned}
&\max_{x_h, b_h} U_h(x_h) \\
&\text{subject to} \\
&p^0(x_h^0 - e_h^0) + q b_h = 0 \\
&p^s(x_h^s - e_h^1) - a(p^s) b_h = 0 \text{ for } s = 1, \dots, S
\end{aligned}$$

(M) markets clear, i.e.

$$\begin{aligned}
&\textstyle\sum_{h=1}^{H} (x_h^0 - e_h^0) = 0 \\
&\textstyle\sum_{h=1}^{H} (x_h^s - e_h^1) = 0 \text{ for } s = 1, \dots, S \\
&\textstyle\sum_{h=1}^{H} b_h = 0
\end{aligned}$$

An equilibrium is characterized by the following system of equations: let Φ^s; $s = 1, \dots, S$, be given by:

$$\Phi^s(\xi) = \begin{bmatrix} \begin{pmatrix} \vdots \\ \frac{\partial}{\partial x_h^s} u_h(x_h^0, x_h^s) - \lambda_h^s p^s \\ \vdots \end{pmatrix} \\ \begin{pmatrix} \vdots \\ -p^s(x_h^s - e_h^s) + a(p) b_h \\ \vdots \end{pmatrix} \\ \sum_h x_h^{sn} - e_h^{sn} \end{bmatrix} \tag{2.1}$$

and

$$\phi^0(\xi) = \begin{bmatrix} \begin{pmatrix} \vdots \\ \sum_s \pi^s \frac{\partial}{\partial x_h^0} u_h(x_h^0; x_h^s) - \lambda_h^0 p^0 \\ \vdots \end{pmatrix} \\ \begin{pmatrix} \vdots \\ -p^0(x_h^0 - e_h^0) - q b_h \\ \vdots \end{pmatrix} \\ \sum_h x_h^{0\backslash} - e_h^{0\backslash} \end{bmatrix}$$

and let

$$\phi^m(\xi) = \begin{pmatrix} -\lambda_h^0 q^0 + \sum_{s=1}^S \pi^s \lambda_h^s a(p^s) \\ \sum_h b_h \end{pmatrix} \tag{2.2}$$

and set

$$\Phi(\xi) = \begin{pmatrix} \phi^m \\ \phi^0 \\ \vdots \\ \phi^S \end{pmatrix} \tag{2.3}$$

where $x_h^{s\backslash}$ is the $C - 1$ dimensional vector that is obtained by dropping the element corresponding to commodity C in spot $s = 0, 1, \ldots, S$. Notice that (2.1) describe the spot market equilibrium in state s. It is well known and can be readily proved that $\bar{x}$ is an equilibrium allocation if and only if there exists $\xi = \left((b, q), (x^s, \lambda^s, p^s)_{s=0}^S\right)$ with $x = \bar{x}$ such that $\Phi(\xi) = 0$. So we shall say ξ is an equilibrium if $\Phi(\xi) = 0$.

Definition 2.3. An equilibrium ξ of economy e is regular if $D_\xi \Phi(\xi)$ has full (row) rank.[6] An economy e is regular if every equilibrium is regular.

Definition 2.4. A non-sunspot equilibrium is an equilibrium ξ with the property $p^s = p^1$ for every $s = 1, \ldots, S$.

2.2. The Certainty Economy

The certainty economy is the degenerate sunspot economy where $S = 1$. So it is a complete market economy. The corresponding equilibrium system of equations

[6] The matrix $D_\xi \Phi(\xi; e)$ has $S + 1$ more rows than columns.

is as follows: Let

$$\hat{\Phi}^1(\hat{\xi}) = \begin{bmatrix} \begin{pmatrix} \vdots \\ \frac{\partial}{\partial x_h^1} u_h(\hat{x}_h^0; \hat{x}_h^1) - \hat{\lambda}_h^1 \hat{p}^1 \\ \vdots \end{pmatrix} \\ \begin{pmatrix} \vdots \\ -\hat{p}^1(\hat{x}_h^1 - e_h^1) + a(\hat{p})\hat{b}_h \\ \vdots \end{pmatrix} \\ \sum_h \hat{x}_h^{1n} - e_h^{1n} \end{bmatrix} \tag{2.4}$$

$$\hat{\Phi}^0(\hat{\xi}) = \begin{bmatrix} \begin{pmatrix} \vdots \\ \frac{\partial}{\partial x_h^0} u_h(\hat{x}_h^0, \hat{x}_h^1) - \hat{\lambda}_h^0 \hat{p}^0 \\ \vdots \end{pmatrix} \\ \begin{pmatrix} \vdots \\ -\hat{p}^0(\hat{x}_h^0 - e_h^0) - \hat{q}\hat{b}_h \\ \vdots \end{pmatrix} \\ \sum_h x_h^{0n} - e_h^{0n} \end{bmatrix}$$

and let

$$\hat{\Phi}^m(\hat{\xi}) = \begin{pmatrix} -\hat{\lambda}_h^0 \hat{q} + \hat{\lambda}_h^1 a(\hat{p}) \\ \sum_h b_h \end{pmatrix}$$

and set

$$\hat{\Psi}(\hat{\xi}) = \begin{pmatrix} \hat{\Phi}^m \\ \hat{\Phi}^0 \\ \hat{\Phi}^1 \end{pmatrix}$$

As before, $\hat{\xi}$ is said to be an equilibrium if $\hat{\Psi}\left(\hat{\xi}\right) = 0$.

Definition 2.5. An equilibrium $\hat{\xi}$ is sequentially regular if it is regular and $D_{\hat{\xi}^1}\hat{\Phi}^1$ has full (row) rank. The certainty economy is said to be sequentially regular if every equilibrium is sequentially regular.

The definition above is equivalent to the condition that the derivative of the Arrow-Debreu excess demand function of the economy has full rank.[7]

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[7] See Mas-Colell (1995) for instance.

2.3. Certainty Equilibrium as a Non-sunspot Equilibrium

The following is straightforward.

Lemma 2.6. If ξ is a non-sunspot equilibrium, then for every h and $s = 1; \ldots; S$; $x_h^1 = x_h^s$ and $\frac{\lambda_h^s}{\lambda_h^0} = \frac{\lambda_1^s}{\lambda_1^0}$.

In view of Lemma 2.6, there is a natural one-to-one relation between a certainty equilibrium $\hat\xi$ and a non-sunspot equilibrium ξ given by

$$\left((\hat b; \hat q); \left(\hat x^s; \hat\lambda^s; \hat p^s\right)_{s=0}^{1}\right) \longleftrightarrow \left((\hat b; \hat q); (x^s; \lambda^s; p^s)_{s=0}^{S}\right) \tag{2.5}$$

where $x^0 = \hat x^0$, $x^s = \hat x^1$ for $s = 1; \ldots; S$, and $\lambda_h^0 = \hat\lambda_h^0$ and $\lambda_h^s = \hat\lambda_h^1$ for every h. A non-sunspot equilibrium ξ can therefore be naturally seen as an equilibrium $\hat\xi$ of the (associated) certainty economy by the rule (2.5); conversely, a certainty equilibrium can be naturally seen as a non-sunspot equilibrium.

3. Main Result

Our main result is:

Proposition 3.1. Let ξ be a non-sunspot equilibrium and $\hat\xi$ be the corresponding certainty equilibrium given by (2.5). Then $\hat\xi$ is sequentially regular if and only if ξ is a regular (sunspot) equilibrium.

Proof is directly by computation. Let ξ be a non-sunspot equilibrium, and $\hat\xi$ be the corresponding certainty equilibrium obtained by the identification rule (2.5).

Let us denote

$$\frac{\partial}{\partial\hat\xi}\hat\Phi(\hat\xi; e) = \begin{pmatrix} \frac{\partial}{\partial\hat\xi^m}\hat\chi^m & \frac{\partial}{\partial\hat\xi^0}\hat\chi^m & \frac{\partial}{\partial\hat\xi^1}\hat\chi^m \\ \frac{\partial}{\partial\hat\xi^m}\hat\chi^0 & \frac{\partial}{\partial\hat\xi^0}\hat\chi^0 & \frac{\partial}{\partial\hat\xi^1}\hat\chi^0 \\ \frac{\partial}{\partial\hat\xi^m}\hat\chi^1 & \frac{\partial}{\partial\hat\xi^0}\hat\chi^1 & \frac{\partial}{\partial\hat\xi^1}\hat\chi^1 \end{pmatrix} \equiv \begin{pmatrix} A_{mm} & B_0 & B_1 \\ C_0 & A_{00} & A_{01} \\ C_1 & A_{10} & A_{11} \end{pmatrix}$$

Note that for every $s = 1; \ldots; S$:

$$\frac{\partial}{\partial\xi^s}\phi^s(\xi; e) = \frac{\partial}{\partial\hat\xi^1}\hat\chi^1(\hat\xi; e)$$

$$\frac{\partial}{\partial \xi^0}\phi^s(\xi;e) = \frac{\partial}{\partial \hat{\xi}^0}\hat{\phi}^1(\hat{\xi};e)$$

$$\frac{\partial}{\partial \xi^m}\phi^s(\xi;e) = \frac{\partial}{\partial \hat{\xi}^m}\hat{\phi}^1(\hat{\xi};e)$$

and if $s, s' \geq 1$, $s \neq s'$:

$$\frac{\partial}{\partial \xi^{s'}}\phi^s(\xi;e) = 0$$

Also, for any $s = 1, \ldots, S$:

$$\frac{\partial}{\partial \xi^s}\phi^0(\xi;e) = \pi^s \frac{\partial}{\partial \hat{\xi}^1}\hat{\phi}^0(\hat{\xi};e)$$

$$\frac{\partial}{\partial \xi^s}\phi^m(\xi;e) = \pi^s \frac{\partial}{\partial \hat{\xi}^1}\hat{\phi}^m(\hat{\xi};e)$$

So, it is readily verified that $\frac{\partial}{\partial \xi}\Phi(\xi;e)$ has the form:

$$\frac{\partial}{\partial \xi}\Phi = \begin{pmatrix} \frac{\partial}{\partial \xi^m}\phi^m & \frac{\partial}{\partial \xi^0}\phi^m & \frac{\partial}{\partial \xi^1}\phi^m & \cdots & \frac{\partial}{\partial \xi^S}\phi^m \\ \frac{\partial}{\partial \xi^m}\phi^0 & \frac{\partial}{\partial \xi^0}\phi^0 & \frac{\partial}{\partial \xi^1}\phi^0 & & \frac{\partial}{\partial \xi^S}\phi^0 \\ \frac{\partial}{\partial \xi^m}\phi^1 & \frac{\partial}{\partial \xi^0}\phi^1 & \frac{\partial}{\partial \xi^1}\phi^1 & & \frac{\partial}{\partial \xi^S}\phi^1 \\ \vdots & & & \ddots & \\ \frac{\partial}{\partial \xi^m}\phi^S & \frac{\partial}{\partial \xi^0}\phi^S & \frac{\partial}{\partial \xi^1}\phi^S & & \frac{\partial}{\partial \xi^S}\phi^S \end{pmatrix}$$

$$= \begin{pmatrix} A_{mm} & B_0 & \pi^1 B_1 & \pi^2 B_1 & & \pi^S B_1 \\ C_0 & A_{00} & \pi^1 A_{01} & \pi^2 A_{01} & \cdots & \pi^S A_{01} \\ C_1 & A_{10} & A_{11} & 0 & & 0 \\ C_1 & A_{10} & 0 & A_{11} & & 0 \\ \vdots & & \vdots & & \ddots & \vdots \\ C_1 & A_{10} & 0 & 0 & & A_{11} \end{pmatrix} \tag{3.1}$$

Now we have, by adding the column blocks corresponding to states $2, \ldots, S$ to the third column block:

$$\text{rank}\begin{pmatrix} A_{mm} & B_0 & \pi^1 B_1 & \pi^2 B_1 & & \pi^S B_1 \\ C_0 & A_{00} & \pi^1 A_{01} & \pi^2 A_{01} & \cdots & \pi^S A_{01} \\ C_1 & A_{10} & A_{11} & 0 & & 0 \\ C_1 & A_{10} & 0 & A_{11} & & 0 \\ \vdots & & \vdots & & \ddots & \vdots \\ C_1 & A_{10} & 0 & 0 & & A_{11} \end{pmatrix}$$

$$
= \operatorname{rank}\begin{pmatrix}
A_{mm} & B_0 & B_1 & \pi^2 B_1 & & \pi^S B_1 \\
C_0 & A_{00} & A_{01} & \pi^2 A_{01} & \cdots & \pi^S A_{01} \\
C_1 & A_{10} & A_{11} & 0 & & 0 \\
C_1 & A_{10} & A_{11} & A_{11} & & 0 \\
\vdots & & \vdots & & \ddots & \\
C_1 & A_{10} & A_{11} & 0 & & A_{11}
\end{pmatrix}
$$

and by subtracting the third row block from each row block below the third row block,

$$
= \operatorname{rank}\begin{pmatrix}
A_{mm} & B_0 & B_1 & \pi^2 B_1 & & \pi^S B_1 \\
C_0 & A_{00} & A_{01} & \pi^2 A_{01} & \cdots & \pi^S A_{01} \\
C_1 & A_{10} & A_{11} & 0 & & 0 \\
0 & 0 & 0 & A_{11} & & 0 \\
\vdots & \vdots & \vdots & & \ddots & \\
0 & 0 & 0 & 0 & & A_{11}
\end{pmatrix}
$$

Therefore, the matrix (3.1) has full rank if and only if both $\begin{pmatrix} A_{mm} & B_0 & B_1 \\ C_0 & A_{00} & A_{01} \\ C_1 & A_{10} & A_{11} \end{pmatrix}$ and $\begin{pmatrix} A_{11} & & 0 \\ & \ddots & \\ 0 & & A_{11} \end{pmatrix}$ have full rank, and the latter matrix has full rank if and only if A_{11} has full rank. This completes the proof.

4. Applications and Remarks

4.1. Nominal v.s. Real Assets

What does the sequential regularity mean? It obviously depends upon the implication of the condition that the matrix $\frac{\partial}{\partial \xi^1}\chi^1(\hat\xi, e)$ has full row rank.

Let us start with the case of real asset; say $a(p) = p \cdot a$ for some vector $a \gg 0$. The it is easy to see that $\frac{\partial}{\partial \xi^1}\chi^1(\hat\xi, e)$ has full rank if and only if $(\bar p^1, \bar x^1)$ constitutes a regular competitive equilibrium of the second period (spot) economy, where each household h is endowed with $e_h + a b_h$; that is, $(\bar p^1, \bar x^1)$ is a regular equilibrium of the economy $(e_h + a b_h)_{h=1}^H$. This of course does not imply that the second period "continuation" economy $(e_h + a b_h)_{h=1}^H$ is a regular economy; that is, the sequential regularity is weaker than the strong regularity studied in Geanakoplos-Polemarchakis (1986).

The case of nominal asset is trickier. Let us assume $a(p) = 1$ for any p. Then it is easy to see that $\hat{\chi}^1(\hat{\xi}; e) = 0$ holds if and only if $\left(\hat{p}^1; \hat{x}^1\right)$ constitutes a competitive equilibrium of the (spot) economy, where each household h is endowed with e_h and a transfer b_h in the unit of account. Choose any vector a with $p \cdot a = 1$. Then $\left(\hat{p}^1; \hat{x}^1\right)$ constitutes a competitive equilibrium of the economy, where each household h is endowed with $e_h + ab_h$.

Does the condition that $\frac{\partial}{\partial \hat{\xi}^1}\hat{\chi}^1(\hat{\xi}; e)$ has full rank imply that $\left(\hat{p}^1; \hat{x}^1\right)$ is a regular equilibrium of $e_h + ab_h$? Not necessarily; imagine an Edgeworth box. Recall that the markets are complete, thus the set of equilibrium allocations is invariant of the choice of the vector a. So a is arbitrary, and hence the vector $e_h + ab_h$ can be at any point on the line $\{z : p \cdot z = p \cdot x_h\}$. So unless the economy $(e_h + ab_h)_{h=1}^H$ has a unique regular equilibrium for any a, which is certainly not a generic situation, there will be some a for which x^s is a singular equilibrium allocation. See Figure 1 below: with asset return $\hat{a}$, x_1 is a (unique) regular equilibrium of the spot market, but at a, x_1 is no longer a regular equilibrium allocation since the number of equilibria is even.

Figure 1

:

On the other hand, the following is an straightforward consequence of the well known fact that the regularity of equilibrium does not depend on the normalization of prices, $p \cdot a = 1$, in the Arrow-Debreu setting:

Proposition 4.1. Assume $a(p) = 1$ for all p. x^s is a regular equilibrium allocation for $(e_h + ab_h)_{h=1}^H$ with $p \cdot a = 1$, if and only if the square matrix $\begin{bmatrix} \frac{\partial}{\partial \xi^1} \chi^1(\hat{\xi}; e) \\ \frac{\partial}{\partial \xi^1} \left(p^1 \cdot a \right) \end{bmatrix}$ is non-singular. So in particular, if $\frac{\partial}{\partial \xi^1} \chi^1(\hat{\xi}; e)$ does not have full rank, then x^s is a singular equilibrium allocation for any a.

A straightforward proof is omitted; see for instance, Mas-Colell (1985), chapter 5. An immediate corollary is that if $\hat{x}$ is a sequentially regular equilibrium allocation of the economy with the nominal asset, then for a generic choice of a, x is a sequentially regular equilibrium allocation of the economy with the real asset that yields a in the second period.

4.2. Generic Sequential Regularity

Fix utility functions, and parameterize economies by endowment vectors; the set of economies E is therefore given by $E = \left\{ e = \left(e_h^0, e_h^1 \right)_{h=1}^H \in \mathbb{R}_{++}^{2CH} \right\}$. Cass (1992) shows that the set of economies whose non-sunspot equilibria are regular is open and dense in E. Extending Cass' result using a similar technique as in Balasko (1992), Lisboa (1994) shows that the set is in fact a full measure set. So Proposition 3.1 implies:

Corollary 4.2. The set of sequentially regular economies is open and full measure.

4.3. Endogenous Uncertainty

If one interprets sunspots as some randomization device which represents "price", or "endogenous" uncertainty, then it is awkward to fix the number of sunspots a priori. Notice that Proposition 3.1 implies that if $\hat{\xi}$ is a sequentially regular equilibrium of the certainty economy, then it will be a regular non-sunspot equilibrium for any sunspot structure.

Let us consider the case of real asset. Let $\hat{e}$ be a sequentially regular economy. Then by the standard argument, there are finitely many sequentially regular equilibria $\hat{\xi}_1(e), \ldots, \hat{\xi}_K(e)$, where each $\hat{\xi}_\cdot(e)$ is a smooth function of e in a neighborhood V of $\hat{e}$. Fix a sunspot structure arbitrarily; that is, fix a finite sunspot states. For any given probabilities $\pi = (\pi^s)_{s=1}^S$, denote by $\xi_k(e; \pi)$ the non-sunspot equilibrium version of $\hat{\xi}(e)$. By Proposition 3.1, every $\xi_k(e;\pi)$ is a regular sunspot equilibrium for any $e \in V$. Notice that for any $e \in V$, the set of non-sunspot equilibria of economy e is exactly $\{\xi_1(e), \ldots, \xi_K(e)\}$ and they are locally isolated from other sunspot equilibria, no matter what π may be. In this sense, non-sunspot equilibria of a sequentially regular economy is locally determinate under any sunspot structure.

In particular, consider any sequence of probabilities π^n where $\pi^n \to (1; 0; \ldots 0)$. This can be seen as a small perturbation of the certainty model. The observation above then implies that there is no sunspot equilibrium that converges to a non-sunspot equilibrium as $n \to \infty$. In this sense, a sequentially regular equilibrium is robust against sunspots.

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