CARESS Working Paper #95-09
Budget-constrained Search[*]

Richard Manning Julian Manning[†]

April 25, 2000

Abstract

A consumer with diminishing marginal utility in consumption, who can search for lower prices, will balance the gains from spreading consumption evenly through time against the benefits of delaying consumption until lower prices are revealed. Optimal programs of consumption, savings and price are characterized for a general formulation of this problem. Intertemporal substitutability is measured by relative-risk aversion. That relative-risk aversion that is small is sufficient for the intuitive solution: As the best current price rises, more search and less consumption is done. The general model is adapted to special cases. Among other things, this shows that linear utility and sequential search implies ex ante calculable reservation prices and consumption only when search stops. However, this characterization is a consequence of the restriction to linear utility. Outside of this context reservation prices and consumption may not be calculable, ex ante.

[*]Jeffrey Banks, Mitch Harwitz and Seamus Hogan are thanked for their helpful comments on this problem. Special thanks are due to Peter Morgan for his comments and encouragement.

[†]Please send all correspondence to Julian Manning, Department of Economics, University of Pennsylvania, Philadelphia, PA 19104, USA, Phone: (215) 898 8486, Fax: (215) 573 2057, E-mail: Manning@econ.sas.upenn.edu

1. Introduction

Price dispersion is ubiquitous. Stigler (1961) noted this fact and introduced the theory of consumer search to explain the failure of the Law of One Price. In Stigler's account the consumer wishes to buy one unit of a commodity. Quotations are simultaneously obtained and the commodity is bought from the seller quoting the lowest price. Manning and Morgan (1982) applied this fixed-sample-size (FSS) search procedure to a consumer with a utility function and a budget constraint. Their model is an effort to integrate search theory with neo-classical utility theory. However, to the author's knowledge, there have been no further such attempts.

An alternative search procedure was proposed by McCall (1965, 1970). In the canonical formulation of sequential search the consumer wishes to buy one unit of the commodity and obtains quotations one-at-a-time until a satisfactory price is obtained. This yields a reservation-price rule. The consumer gets quotations until one is no larger than a pre-determined, constant "reservation price". Then purchase is made at that most recently quoted price. The reservation-price rule extends to consumers with more complex preferences, but Veendorp (1984) shows that it breaks down if they are subject to a budget constraint.

Most of the attention of search theorists has been applied to synthesizing pure sequential search and FSS search. Gal, Landsberger and Levykson (1981) and Morgan (1983) present models that lead to "compound strategies" for search in the labour market allowing the sample size to be chosen in each period. A more general search procedure is developed in Morgan and Manning (1985).

The general formulation, like most consumer search theory, assumes that consumption is delayed until search stops. [1] The papers of Gal, Landsberger and Levykson, Morgan, and Morgan and Manning ignore intertemporal budget allocation between consumption and search and the effects of other prices. There are exceptions. Benhabib and Bull (1983) present a model of search in the labour market where workers may search for job offers while working. The decision not to work, but to search, is endogenous. However, there is no reason, a priori, to suppose that this analysis would yield insight into the behavior of a consumer consuming while searching.

Although there are circumstances in which consumption is best delayed until search is over, the timing of consumption should be determined in the theory

[1] There are some exceptions. Kohn and Shavell (1974) allow some consumption while search proceeds, but do not consider the optimal intertemporal allocation of consumption, which is the focus here. Manning (1989) deals with optimal consumption, but uses period-by-period budget constraints, so the savings decision is avoided.

rather than imposed by assumption. This is particularly important when search can extend over many periods and consumer utility is strictly concave. In that case, the consumer has an interest in spreading consumption through time that must be balanced against the expected gains flowing from lower prices that may be obtained from further search. This paper improves on this literature by modifying the canonical model of consumer search to account for intertemporal budget allocation between consumption and search. Further, it is shown through a special case that it is only under restrictive conditions that the results of previous authors may be obtained.

Section 2 introduces a class of models where consumers are free to consume while searching. The consumer is initially endowed with wealth (money). There is at least one consumption good and the price in each period of one commodity is unknown when viewed from earlier periods. The prices of all other commodities in all periods are known at the outset. How many quotations to obtain, and how to divide his (her) remaining wealth between consumption and savings, are the decisions confronting the consumer in every period. The degree of concavity of consumer utility is measured by the degree of relative-risk aversion.

The solution when relative-risk aversion is small is characterized in Section 3: When the best price falls below a critical value, search stops and all remaining wealth is spent on consumption. At successively higher prices, one or more quotations are taken and some consumption occurs with the rest of the wealth saved for the future. This solution is reminiscent of a reservation price rule. However, the reservation prices cannot be predetermined, in general.

In a special case reservation prices can be predetermined. Section 4 shows that if the utility function is linear and the consumer searches sequentially, then consumption is delayed until search stops and a non-decreasing sequence of reservation prices can be computed in advance. This conforms with the results obtained in canonical models with no recall. However, it is shown that this result does not hold for utility functions that take other forms. For instance, if relative-risk aversion is sufficiently large and if price rises above a critical value, search stops and all remaining wealth is spent on consumption. As before, the reservation prices cannot be predetermined.

Some concluding remarks are offered in Section 5.

2. The Model

The consumer has an initial wealth, $y \in \Re_+;\ y > 0$. This is to be spent over a finite horizon of T periods. There are n (≥ 1) consumption goods, and the price in each period of one commodity (taken to be commodity 1) is unknown when viewed from earlier periods. Commodity 1 is non-durable and may not be inventoried.[2] All prices in the first period are assumed known. The prices of all other commodities in all periods (if there are any) are known at the outset. At a cost of $c > 0$ per quotation, prices of commodity 1 in the next period can be obtained in the current period. These price quotations are independent draws from a distribution F with support $[a, b] \subset \Re$, $0 < a < b < \infty$. Price quotes in one period give no information about future prices[3] How many quotations to obtain, and how to divide his remaining wealth between consumption and savings, are the decisions confronting the consumer in every period.

Let $y_t \in \Re_+$, $y_t \geq 0$, be the wealth of the consumer at the beginning of period t, and let $e_t \in \Re_+$, $e_t \geq 0$, be the expenditure on all consumer goods in that period. Clearly,

$$y_{t+1} = y_t - e_t - cn_t, \qquad t = 1, \ldots, T - 1,$$

where $n_t \in Z_+$ is the number of quotations asked for in period t. No purchase can be made without sampling price so, if $n_t = 0$, the demand for commodity 1 in period $t + 1$ must also be zero.

The consumer's preferences are represented by the indirect utility function $v: \Re^n_{++} \times \Re_+ \mapsto \Re$. Consumption expenditure in period t is allocated optimally over all commodities, the prices of which are then known. The indirect utility from this is $v(p_t, e_t)$, where $p_t \equiv (p_{1t}, p_{?t})$, p_{1t} is the lowest price at which commodity 1 is available in period t and $p_{?t} \equiv (p_{2t}, \ldots, p_{nt}) \in \Re^{n-1}_{++}$ are the prices of commodities $2, \ldots, n$. v is assumed to be continuous in expenditure and twice differentiable. Since wealth is finite and the cost of each quotation is positive the number of quotations obtained every period is bounded. The per period discount factor is ρ, $0 < \rho \leq 1$.

The price p_{1t} in period t is a random variable ex ante. Neither consumption expenditure nor the number of quotations in any period can be decided at the beginning of the planning horizon. Therefore both must be expressed as functions

[2] Inventorying will affect the consumers best strategy. However, the point I wish to concentrate upon is the addition to the canonical model of only an intertemporal budget constraint. The addition of inventorying would complicates notation and does not alter the main conclusion of the paper; that the main predictions of search theory are critically dependant upon the restriction that consumption is delayed until search stops.

[3] If past quotations for the price of commodity 1 yield information about future prices of commodity 1 then the consumers best strategy will be affected. However, footnote 3 applies. The addition of temporal correlation between the distributions of the price of commodity 1 only complicates notation and does not alter the main conclusion of the paper.

of the lowest current price, the prices of commodities $2, \ldots, n$ and wealth. A search rule is a sequence of ordered pairs $\delta \equiv \{(e_t, n_t)\}_{t=1}^{T}$ where

$$(e_t, n_t) : X^t \mapsto \Re_+ \times Z_+ \quad \text{and} \quad X^t \equiv [a, b] \times \Re_+^n \times [a, b] \times \Re_+^n .$$

Given initial wealth, y, the expected present valued utility of the search rule δ is

$$V_1^T(y; \delta) \equiv E\left[\sum_{t=1}^{T} \rho^{t-1} v(p_t, e_t)\right].$$

Let P denote the set of all search rules. A search rule is said to be optimal if and only if there exists no other search rule which provides the searcher with a higher expected present value of utility.[4] Formally, a search rule $\delta^\star = \{(e_t^\star, n_t^\star)\}_{t=1}^{T}$ is optimal[5] if and only if, for any given y,

$$V_1^T(y; \delta^\star) = \max_{\delta \in P} V_1^T(y; \delta).$$

It is assumed that v is concave in expenditure, $\partial^2 v(p, e)/\partial e^2 \leq 0$. This ensures that any local maximum of v is the global maximum of v. A consumer is risk averse if and only if $\partial^2 v(p, e)/\partial e^2 < 0$. This ensures that any global maximum of v is unique.

Relative-risk aversion, is quantified by the Arrow-Pratt measure

$$R = -e\frac{\partial^2 v/\partial e^2}{\partial v/\partial e}.$$

x_i denotes the ordinary demand for commodity i for $i = 1, \ldots, n$. $\eta_i = (e/x_i)(\partial x_i/\partial e)$ denotes the expenditure-elasticity of commodity i for $i = 1, \ldots, n$. Relative-risk aversion is smaller than the expenditure elasticity of demand in commodity i if $R < \eta_i$ and larger than the expenditure elasticity of demand in commodity i if $R > \eta_i$. If $R < \eta_1$ ($R > \eta_1$) then relative-risk aversion is small (large).

[4] Because the value of indirect utility is always finite the value of the weighted sum of indirect utilities is always finite. This ensures comparability of any two search rules.

[5] Since initial wealth is finite expenditure in each period is chosen from a compact set. By the continuity of indirect utility in expenditure and that all prices are strictly positive, the weighted sum of indirect utilities must attain a maximum.

3. Optimal Consumption, Search and Savings

The lemmas and propositions to follow are proven using the Principle of Optimality. Applying the Principle of Optimality allows the selection of an optimal search rule by backwards induction. At any period of time t optimization consists of choosing the optimal expenditure and the optimal number of quotations given that the rules determining expenditure and the number of quotations have been chosen optimally for periods $t + 1, \ldots, T$. Lemmas 1 and 2 present results about the behavior of optimal expenditure when the number of quotations need not be optimal. Lemmas 3 and 4 characterize the relationship between the optimal number of quotations and any price and wealth.

Lemma 1: For any period $t \in \{1, \ldots, T - 1\}$ and given that the consumer is risk averse and exactly r quotations are taken in period t, the optimal expenditure in period t on consumption is increasing in wealth, and is decreasing (constant, increasing) in the best current price of commodity 1 if relative-risk aversion is smaller (equals, larger) the expenditure elasticity of demand in commodity 1.

The intuition for Lemma 1 can be found in the following remark.

Remark: $\partial^2 v / \partial p_- \partial e \lesseqgtr 0$ as $R \lesseqgtr \eta_-$.

The remark says that if relative-risk aversion is small the marginal utility of expenditure is declining in the best current price of commodity 1. Therefore optimal expenditure on consumption is decreasing in the best current price of commodity 1 if relative-risk aversion is small. This is exactly what Lemma 1 says. Similar reasoning explains why current expenditure on consumption increases with the best current price of commodity 1.

In Lemmas 2, 3 and 4 and in Proposition 1 it is assumed that relative-risk aversion is small to permit a particular general characterization of the optimal search rule. A special characterization of the optimal search rule when relative-risk aversion is not small is made in Section 4.

Lemma 2: If the consumer is risk averse and relative-risk aversion is small, then in any period $t \in \{1, \ldots, T\}$ the optimal expenditure on consumption is decreasing in the number of quotations purchased in period t.

The intuition for Lemma 2 can also be found in the remark. If relative-risk aversion is small then an increase in the best current price of commodity 1 will lead to a reduction in the marginal utility of current expenditure and so a reduction in current expenditure. If the number of quotations increases then the expectation for the best price of commodity 1 for next period decreases. This would lead to an increase in the marginal utility of expenditure next period and so to a decrease in current expenditure as saving increases.

Lemma 3: Suppose that the consumer is risk averse, and that relative-risk aversion is small. For any period $t \in \{1, \dots, T-1\}$ the price support [a,b] can be partitioned into intervals

$$[p^{r^0}, p^{r^1}), [p^{r^1}, p^{r^2}), \dots, [p^{r^{m-1}}, p^{r^m}]; p^{r^0} = a, p^{r^m} = b,$$

so that $0 \le r^1 < r^2 < \dots < r^m \le N$; $n_t^{\star}(p_t, y_t) = r^k$ if $p_t \in [p^{r^{k-1}}, p^{r^k})$ $k = 1, \dots, m-1$ and $n_t^{\star}(p_t, y_t) = r^m$ if $p_t \in [p^{r^{m-1}}, p^{r^m}]$.

The partition of the price support into a sequence of intervals in which search is successively increasing depends on the periods remaining, on wealth y_t and on the discount factor ρ. These arguments are omitted in Lemma 3 to simplify the notation.

Lemma 4: Suppose that the consumer is risk averse, and that relative-risk aversion is small. Each partitioning price $p_1^{r^k}$ is a decreasing function of wealth. Suppose further that relative risk aversion is smaller than the expenditure elasticity of demand in commodity i, $i \neq 1$. Each partitioning price $p_1^{r^k}$ is a decreasing function of the price of commodity i, $i \neq 1$.

Lemma's 1 through 4 permit a characterization of optimal search, consumption and savings for the case of a consumer with a relative-risk aversion small. Lemma 3 shows that the number of quotations is non-decreasing in the best current price of commodity 1. Lemma 4, when combined with Lemma 3, shows that, as wealth increases the number of quotations is non-decreasing. Consumption expenditure is decreasing in the best current price of commodity 1, according to Lemma 1 (for a fixed number of quotations), and Lemma 1, 2 and 3 (as the number of quotations changes). These conclusions are summarized as

Proposition 1: If the consumer is risk averse, and has relative-risk aversion small, then in each period $t \in \{1, \dots, T-1\}$ the number of quotations is a non-decreasing function of, and consumption expenditure declines in, the best current price of commodity 1.

This Proposition is intuitively appealing. As the best current price of commodity 1 rises, more search and less consumption is done. This is because when relative-risk aversion is small the marginal unit of commodity 1 in the current period is a close substitute for future marginal units of commodity 1. Note, however, that this conclusion relies on the assumption of small relative-risk aversion. For example, Lemma 1 reveals that consumption expenditure will decrease with price in commodity 1 (for a given number of quotations) if and only if relative-risk aversion is small. More significantly, the conclusion in Lemma 2, that consumption expenditure will fall when more quotations are taken, may be reversed unless relative-risk aversion is assumed to be small. If it does not hold, then Lemma

3's partition of the price support into intervals in which various numbers of quotations are optimal becomes more problematic. In particular, higher prices may be associated with fewer quotations and more consumption expenditure than are optimal at a low price. Indeed, it may happen that search stops and all wealth is consumed at a high price even though search would continue at lower prices, if relative-risk aversion is not small. In the special case of purely sequential search an even stronger conclusion emerges.

Inspection of the proof of Lemma 2 reveals that when relative risk aversion is large the direction of change in consumption given a change in the number of quotations is ambiguous. This is the source of the difficulty in establishing a general theory of search when relative risk aversion is large. For instance, as price increases the optimal number of quotations may drop. As a consequence, where relative risk aversion is large, the price support may not be partitioned into intervals ordered by the number of quotations. Further, no definite conclusion can be drawn about the dependence of partitioning prices on wealth.

4. Special Cases

Search is purely sequential if and only if the consumer is constrained to obtain no more than one quotation every period. However, when search need not be purely sequential and if relative risk aversion is not small then a partition of price into intervals is not generally possible. Further, it it becomes difficult, in general, to determine the response of any price partition to changes in wealth. When search is purely sequential, it is straightforward to describe the optimal search and consumption expenditure of a consumer with a large relative-risk aversion. If search is purely sequential then Lemma 1 characterizes the response of optimal expenditure to changes in wealth and best current price. However, more definite results can be obtained.

Proposition 2: Suppose that search is purely sequential. If relative-risk aversion is large, then there are functions $p_{1t}(p_{*t}; y_t)$ and $e_t(p_t; y_t)$ such that the optimal number of quotations and expenditure on consumption are given by the rule:

$$\text{If} \quad p_{1t} \geq p_{1t}(p_{*t}; y_t); \quad \text{then} \quad n_t^* = 0 \quad \text{and} \quad e_t^* = y_t; \quad t = 1, \ldots, T.$$

$$\text{If} \quad p_{1t} < p_{1t}(p_{*t}; y_t); \quad \text{then} \quad n_t^* = 1 \quad \text{and} \quad e_t^* = e_t(p_t; y_t); \quad t = 1, \ldots, T.$$

Consumption expenditure is constant or increases as the best current price goes up. The functions p_{1t} are increasing in wealth and decreasing in the price of each commodity $i = 2, \ldots, n$ and $t = 1, \ldots, T.$

For the case of purely sequential search, Propositions 1 and 2 show that the nature of the solution changes abruptly as the value of relative-risk aversion exceeds expenditure elasticity of demand of commodity 1. As the current price for commodity 1 drops the quantity of commodity 1 that would have to be forgone to search for future prices of commodity 1 drops. Because relative-risk aversion is large the marginal unit of commodity 1 in the current period is less substitutable for future marginal units of commodity 1. As a consequence, the loss of utility associated with not consuming commodity 1 in any future period is large. Further, as the current quantity of commodity 1 that would have to be forgone to obtain quotes for future prices of commodity 1 and the marginal utility associated with the consumption of commodity 1 drop, the utility loss associated with the consumption of commodity 1 and search alone drops. Therefore, when relative-risk aversion is large, if the current price for commodity 1 drops enough search will start.

There is, of course, no reason why relative-risk aversion should be always small, or always large. The solution in the general case is much less structured, even with purely sequential search, than these two propositions might be mistaken to imply. Their proper interpretation is that the intuitively appealing solution to the search and consumption problem is correct only in particular circumstances .

If there is only one commodity, then the indirect utility function has the special form

$$v(p; e) = u\left(\frac{e}{p}\right)$$

where $u(x)$ is the utility from consuming x units of the commodity. u is concave. A consequence of there being only one commodity is that the expenditure-elasticity of demand, ζ, is identically unity.

Risk aversion has been assumed in the two propositions established so far. In each situation the consumer simultaneously searches and consumes. Evidently, the common assumption in search theory, that consumption is delayed until search stops, seriously misrepresents the behavior of a risk-averse consumer. The final result shows that consumption is delayed by a risk-neutral consumer. Furthermore, even with a budget constraint, a sequence of reservation prices can be calculated at the outset. This is the closest that the general model presented here can come to the canonical model of search theory. Proposition 3 illustrates how dependent the reservation price rules obtained in past literature are upon the restriction to search procedures where consumption is exogenously delayed until search stops.

Proposition 3: Suppose that there is one commodity, that search is purely sequential, and the consumer's utility function is linear. There is a non-decreasing sequence $\{\bar{p}_{11}, \bar{p}_{12}, \ldots, \bar{p}_{1T}\}$, calculable ex ante, such that for $t = 1, \ldots, T$ the optimal number of quotations and expenditure on consumption are given by the rule:

$$\text{If} \quad p_{1t} \le \bar{p}_{1t}, \quad \text{then} \quad n_t^\star = 0 \quad \text{and} \quad e_t^\star = y_t;$$

$$\text{If} \quad p_{1t} > \bar{p}_{1t}, \quad \text{then} \quad n_t^\star = 1 \quad \text{and} \quad e_t^\star = 0.$$

For a consumer with a linear utility function who must search sequentially, the optimal search consumption (and savings) behavior is that which is derived in purely sequential search theory: That is, consumption is delayed until search stops and there is a reservation-price rule which specifies when search should stop.

5. Concluding Remarks

The magnitude of relative-risk aversion has been shown to be of major importance in determining the nature of the solution to the general problem of consumer search and savings. (Manning and Morgan, for FSS search, and Veendorp, for sequential search, previously remarked on the role of relative-risk aversion in special cases). Only when strong assumptions are made about relative-risk aversion is it possible to obtain straightforward characterizations of the optimal search and consumption policies. In general, the optimal search and consumption policies will enjoy little structure. There exist models of markets that support price-dispersed equilibria. These models confront price-setting oligopolists with consumers who follow some search strategy: See, for example, Burdett and Judd (1983) and Carlson and McAfee (1983). Since they seek to show that price-dispersed equilibria are possible, the demand side has been kept simple in these models. Now that the logical possibility of price-dispersed equilibria is not in question, it is appropriate to complicate the demand side of such market models by admitting more general consumption behavior such as that described here. There would result a model of generality similar to the familiar supply and demand paradigm, and Stigler's insight would be validated.

Appendix

Let $w_t^\star(p_{1t}, y_t)$ be the maximum expected utility from period t onwards, given the price p_{1t} and wealth y_t. By the Principle of Optimality

$$w_t^\star(p_t; y_t) \equiv \max_{0 \le e_t \le y_t - cn_t; n_t \ge 0} \left(v(p_t; e_t) + \rho E_{p_{1t+1}} \left[w_{t+1}^\star(p_{t+1}; y_{t+1}) | n_t \right] \right) : \qquad (1)$$

The expectation of utility from period $t+1$ onwards is taken over the minimum of the n prices quoted at the beginning of that period. The minimum price has as its density

$$g(p_{1t+1}|n_t) = n_t f(p_{1t+1})(1 - F(p_{1t+1}))^{n_t - 1}:$$

Equation (1) can be decomposed into a two stage optimization. Consider

$$w_t(p_t; y_t; r) \equiv \max_{0 \le e_t \le y_t - cr} \left(v(p_t; e_t) + \rho E_{p_{1t+1}} \left[w_{t+1}^\star(p_{t+1}; y_t - e_t - cr) | r \right] \right) \qquad (2)$$

which is the maximum expected utility from period t onwards, given current price and wealth and that r quotations are sought in that period, but optimal policies are followed in the future. Often $w_t(p_t; y_t; r)$ will be written $w_t^r(p_t; y_t)$ or w_t^r.

The first order condition to (2) makes it possible to characterize the response of consumption to changes in wealth and best current price for any risk averse consumer. Only interior solutions to (2) are of interest.

Let $e_t(p_t; y_t; r)$ solve (2). Uniqueness of $e_t(p_t; y_t; r)$ is guaranteed by $\partial^2 v / \partial e^2 < 0$. Often $e_t(p_t; y_t; r)$ will be written $e_t^r(p_t; y_t)$ or e_t^r: This is the optimal expenditure on all consumption in period t if r quotations are taken and optimal policies followed subsequently. Substituting $e_t(p_t; y_t; r)$ into (2) results in

$$\begin{aligned} & w_t(p_t; y_t; r) \\ & \quad \equiv v(p_t; e_t(p_t; y_t; r)) + \rho E_{p_{1t+1}} \left[w_{t+1}^\star(p_{t+1}; y_t - e_t(p_t; y_t; r) - cr) \mid r \right] : \end{aligned} \qquad (3)$$

Using equation (3) we can write

$$w_t^\star(p_t; y_t) \equiv \max_{r \ge 0} \left\{ w_t(p_t; y_t; r) \right\} : \qquad (4)$$

Let $n_t^\star(p_t; y_t)$ solve (4). Sometimes it will be convenient to write $n_t^\star(p_t)$, suppressing wealth.

Proof of Lemma 1:

The first-order condition for the maximum of (2) implies that

$$\frac{\partial v(p_t,e_t)}{\partial e_t} \begin{array}{c} < \\ = \\ > \end{array} \rho E_{p_{1t+1}} \left[\frac{\partial w^{*}_{t+1}(p_{1t+1}, p_{*t+1}, y_t - e_t - cr)}{\partial y_{t+1}} \,\middle|\, r \right] \quad \text{as } e_t \in \begin{cases} \{0\} \\ [0, y_t - cr] \\ \{y_t - cr\} \end{cases} . \tag{5}$$

The assumption of risk aversion ensures that the maximand in (2) is strictly concave, so the first-order condition describes the global maximum. Only interior solutions are of interest. In addition, the Implicit Function Theorem applied to (5) yields a differentiable function $e^r_t(p_t, y_t)$, for which

$$0 \le \frac{\partial e^r_t}{\partial y_t} = \frac{\rho E_{p_{1t+1}} \left[\frac{\partial^2 w^{*}_{t+1}(p_{1t+1}, p_{*t+1}, y_t - e^r_t - cr)}{\partial y^2_{t+1}} \,\middle|\, r \right]}{\frac{\partial v^2(p_t, e^r_t)}{\partial e^2_t} + \rho E_{p_{1t+1}} \left[\frac{\partial^2 w^{*}_{t+1}(p_{1t+1}, p_{*t+1}, y_t - e^r_t - cr)}{\partial y^2_{t+1}} \,\middle|\, r \right]} \le 1 \tag{6}$$

and

$$\frac{\partial e^r_t}{\partial p_{it}} = - \frac{\frac{\partial^2 v(p_t, e_t)}{\partial e_t \partial p_{it}}}{\frac{\partial v^2(p_t, e^r_t)}{\partial e^2_t} + \rho E_{p_{1t+1}} \left[\frac{\partial^2 w^{*}_{t+1}(p_{1t+1}, p_{*t+1}, y_t - e^r_t - cr)}{\partial y^2_{t+1}} \,\middle|\, r \right]} \begin{array}{c} < \\ = \\ > \end{array} 0 \quad \text{as } R \begin{array}{c} < \\ = \\ > \end{array} \varrho_i, \; i = 1, \ldots, n, \tag{7}$$

in view of the remark. ∎

Proof of Remark:

Roy's identity is

$$\frac{\partial v}{\partial p_i} \equiv -x_i \frac{\partial v}{\partial e}$$

where x_i is ordinary demand for commodity i. Therefore

$$\frac{\partial^2 v}{\partial p_i \partial e} = -x_i \frac{\partial^2 v}{\partial e^2} - \frac{\partial x_i}{\partial e}\frac{\partial v}{\partial e}$$

from which the result is obvious.∥

Proof of Lemma 2:

The dependence of e_t^r on the number of quotations r follows from (5). By treating r as a continuous variable (for this purpose only) the Implicit Function Theorem can be applied. This gives

$$\frac{\partial e_t^r}{\partial r} = \frac{\rho \partial E_{p_{1t+1}}\left[\frac{\partial w^*_{t+1}(p_{1t+1}, p_{*t+1}, y_t - e_t^r - cr)}{\partial y_{t+1}} \Big| r\right] / \partial r}{\frac{\partial v^2(p_t, e_t^r)}{\partial e_t^2} + \rho E_{p_{1t+1}}\left[\frac{\partial^2 w^*_{t+1}(p_{1t+1}, p_{*t+1}, y_t - e_t^r - cr)}{\partial y_{t+1}^2} \Big| r\right]}. \tag{8}$$

The denominator is negative. The numerator equals

$$-c\rho E_{p_{1t+1}}\left[\frac{\partial^2 w^*_{t+1}(p_{1t+1}, p_{*t+1}, y_t - e_t^r - cr)}{\partial y_{t+1}^2} \Big| r\right] + \rho \int_a^b \frac{\partial w^*_{t+1}(p_{1t+1}, p_{*t+1}, y_t - e_t^r - cr)}{\partial y_{t+1}} \frac{\partial g(p_{1t+1}|r)}{\partial r} dp_{1t+1}$$

The first term is positive by concavity. Differentiating (3), and using the first order condition (5) of (2), yields

$$\frac{\partial w_t^r(p_t, y_t)}{\partial p_{it}} = \frac{\partial v(p_t, e_t^r(p_t, y_t))}{\partial p_{it}} \le 0, \quad i = 1, \ldots, n. \tag{9}$$

Then, recalling (4), almost everywhere the following derivative is defined:

$$\frac{\partial w_t^*\left(p_t, y_t\right)}{\partial p_{1t}} = \frac{\partial w_t^{n^*(p_t)}\left(p_t, y_t\right)}{\partial p_{1t}} = \frac{\partial v\left(p_t, e_t^{n^*(p_t)}\left(p_t, y_t\right)\right)}{\partial p_{1t}} \le 0. \tag{10}$$

using (9).

The second term is non-negative, by an application of Appendix Theorem 1 of Manning and Morgan (1982),[6] since (10) implies that

[6] $\int h(p) \frac{\partial g(p|n)}{\partial n} dp > 0$ if h is monotonic decreasing.

$$\frac{\partial^2 w^*_{t+1}\left(p_{t+1}, y_t - e^r_t - cr\right)}{\partial y_{t+1}\partial p_{1t}} = \frac{\partial^2 v\left(p_t, e_t^{n^*(p_t)}\left(p_t, y_t\right)\right)}{\partial p_{1t}\partial e_{t+1}} \frac{\partial e_{t+1}^{n^*(p_t)}}{\partial y_{t+1}} \leq 0.$$

The sign is a consequence of the assumption of a small relative-risk aversion and Lemma 1. It follows that consumption expenditure falls as the number of quotations increases. That is,

$$e^r_t(p_t, y_t) > e^{r+1}_t(p_t, y_t); \quad r = 0, 1, \ldots, N - 1. \tag{11}$$

∥

Proof of Lemma 3:

From (9), (11) and the assumption of small relative-risk aversion

$$\frac{\partial w^r_t(p_t, y_t)}{\partial p_{it}} < \frac{\partial w^{r+j}_t(p_t, y_t)}{\partial p_{it}}; \quad i = 1, \ldots, n; \; j = 1, \ldots, N - r. \tag{12}$$

Recall, from (4), that w^*_t is the upper envelope of the family w^r_t. Perhaps one member of family itself is the envelope:

$$w^*_t(p_t, y_t) \equiv w^r_t(p_t, y_t); \quad \text{for all} \quad p_t \in [a, b] \times \mathbb{R}^{n-1}_{++}. \tag{13}$$

But then

$$n^*_t(p_t) = r; \quad \text{for all} \quad p_t \in [a, b] \times \mathbb{R}^{n-1}_{++}. \tag{14}$$

The envelope consists of no more than $N + 1$ segments, in general. (Otherwise (12) is contradicted.) These segments partition the price support $[a, b]$ as claimed since, if two segments have the same number of quotations optimal, then (12) would be contradicted.∥

Proof of Lemma 4:

The dividing price $p_1^{r^k}$ satisfies

$$w_t^{r^k}\left(p_t, y_t\right) - w_t^{r^k+1}\left(p_t, y_t\right) = 0. \tag{15}$$

This implicitly defines p^{r^k} as a function of $p_{2t}, \ldots, p_{nt}$ and y_t, for which

$$\frac{\partial p^{r^k}}{\partial y_t} = -\frac{\frac{\partial w_t^{r^k}(p_t;y_t)}{\partial y_t} - \frac{\partial w_t^{r^{k+1}}(p_t;y_t)}{\partial y_t}}{\frac{\partial w_t^{r^k}(p_t;y_t)}{\partial p_t} - \frac{\partial w_t^{r^{k+1}}(p_t;y_t)}{\partial p_t}} < 0; \tag{16}$$

$$\frac{\partial p^{r^k}}{\partial p_{it}} = -\frac{\frac{\partial w_t^{r^k}(p_t;y_t)}{\partial p_{it}} - \frac{\partial w_t^{r^{k+1}}(p_t;y_t)}{\partial p_{it}}}{\frac{\partial w_t^{r^k}(p_t;y_t)}{\partial p_t} - \frac{\partial w_t^{r^{k+1}}(p_t;y_t)}{\partial p_t}} < 0; \quad i = 2; \ldots; n: \tag{17}$$

By (12), the denominator and the numerator of (17) are negative. The numerator of (16) is also negative. To see this, differentiate (3) and note that the first-order condition yields

$$\frac{\partial w_t^r\left(p_t; y_t\right)}{\partial y_t} = \frac{\partial v\left(p_t; e_t^r\left(p_t; y_t\right)\right)}{\partial e_t^r}:$$

Then risk-aversion and (11) imply that the numerator of (16) is negative.∥

Proof of Proposition 2:

Consider the solution to

$$H_t \equiv w_t^0\left(p_t; y_t\right) - w_t^1\left(p_t; y_t\right) \equiv 0 \tag{18}$$

where $w_t^0\left(p_t; y_t\right)$ and $w_t^1\left(p_t; y_t\right)$ are defined in (2). Using (9)

$$\frac{\partial H_t}{\partial p_{it}} = \frac{\partial v(p_t; y_t)}{\partial p_{it}} - \frac{\partial v(p_t; e_t^1(p_t; y_t))}{\partial p_{it}} > 0; \quad i = 1; \ldots; n; \tag{19}$$

where the sign is a consequence of the assumption of large relative-risk aversion and that $e_t^1 \leq y_t - c < y_t$: Thus the solution is unique and defines a function $p_{1t}(p_{*t}; y_t)$ for which

$$n_t^* = \begin{cases} 0 \\ 1 \end{cases} \quad \text{if and only if} \quad p_{1t} \gtrless p_{1t}(p_{*t}; y_t): \tag{20}$$

This establishes the optimality of the rule.

From the Implicit Function Theorem

$$\frac{\partial p_{1t}(p_{*t}; y_t)}{\partial y_t} = -\frac{\frac{\partial v(p_t;y_t)}{\partial e_t} - \frac{\partial v(p_t;e_t^1(p_t;y_t))}{\partial e_t}}{\partial H_t / \partial p_{1t}} > 0: \tag{21}$$

The numerator is derived using (5). The sign is a consequence of (18) and risk aversion.

$$\frac{\partial p_{1t}(p_{?t}; y_t)}{\partial p_{it}} = -\frac{\frac{\partial v(p_t;y_t)}{\partial p_{it}} - \frac{\partial v(p_t;e_t^1(p_t;y_t))}{\partial p_{it}}}{\partial H_t/\partial p_{1t}} < 0;\ i = 2;\ldots;n. \tag{22}$$

The sign is a consequence of (19). Lemma 1 gives the conclusion on consumption expenditure.∥

Proof of Proposition 3:

Consider the decision in the second-last period. Suppress the subscript of commodity 1. If no quotation is taken, then

$$w^0_{T-1}(p_{T-1}; y_{T-1}) = \frac{y_{T-1}}{p_{T-1}}. \tag{23}$$

If one quotation is taken, then

$$w^1_{T-1}(p_{T-1}; y_{T-1}) = \max_{0 \le e^1_{T-1} \le y_{T-1} - c} \frac{e^1_{T-1}}{p_{T-1}} + \rho E_{p_T}\left[\frac{y_{T-1} - e^1_{T-1} - c}{p_T}\right]. \tag{24}$$

Clearly

$$e^1_{T-1} \in \begin{cases} \{0\}; & \text{if } 1/p_{T-1} < \rho E_{pt}[1/p_T] \\ [0; y_{T-1} - c]; & \text{if } 1/p_{T-1} = \rho E_{p_T}[1/p_T] \\ \{y_{T-1} - c\}; & \text{if } 1/p_{T-1} > \rho E_{p_T}[1/p_T] \end{cases} \tag{25}$$

Therefore:

$$w^1_{T-1}(p_{T-1}; y_{T-1}) = \begin{cases} \rho(y_{T-1} - c)E_{p_T}[1/p]; & \text{if } 1/p_T \le \rho E_{p_T}[1/p_T] \\ (y_{T-1} - c)/p_{T-1}; & \text{if } 1/p_{T-1} > \rho E_{p_T}[1/p_T] \end{cases} \tag{26}$$

Define p^1_{T-1} to satisfy

$$\frac{y_{T-1}}{p_{T-1}} = \rho(y_{T-1} - c)E_{p_T}[1/p_T] \quad \Rightarrow \quad p^1_{T-1} = \frac{y_{T-1}}{\rho(y_{T-1} - c)E_p[1/p]}. \tag{27}$$

No quotations can be taken if $y_{T-1} < c$. Therefore the decision rule has the form:

$$\text{If}\quad p_{T-1} \le p^1_{T-1} \quad \text{or} \quad y_{T-1} \le c; \quad \text{then} \quad n^\star_{T-1} = 0 \quad \text{and} \quad e^\star_{T-1} = y_{T-1};$$

$$\text{if}\quad p_{T-1} > p^1_{T-1} \quad \text{and} \quad y_{T-1} > c; \quad \text{then} \quad n^\star_{T-1} = 1 \quad \text{and} \quad e^\star_{T-1} = 0.$$

Note that p^1_{T-1} is a function of y_{T-1} (amongst other things). The induction hypothesis is that the decision rule has this form in every period. Repeated applications of the hypothesis yield the maximum expected utility from period t onwards:

$$w^\star_t(p_t; y_t) = \begin{cases} \frac{y_t}{p_t}; & \text{if } p_t \le p^1_t(y_t) \text{ or } y_t \le c \\ \sum_{i=1}^{j} \prod_{k=1}^{i-1} \rho\left(1 - F(p^1_{t+k}(y_{t+k}))\right)(y_t - ic) \int_a^{p^1_{t+i}(y_{t+i})} \frac{1}{p} dF(p); \\ \qquad \text{if } p_t > p^1_t(y_t) \text{ and } jc < y_t \le kc \end{cases} \tag{28}$$

where $j = 1, 2, \ldots, J$ and $J = \min\{T - t; \text{biggest integer less than } y_t/c\}$; and where $k = j + 1$ if $J \neq T - t$; and $k = 1$ if $J = T - t$.

Consider now the decision in $t - 1$. If no quotation is taken and all wealth is spent on consumption, then utility is

$$v^0_{t-1}(p_{t-1}; y_{t-1}) = \frac{y_{t-1}}{p_{t-1}}. \tag{29}$$

This is the only outcome if $y_{t-1} \le c$. If $y_{t-1} > c$ another quotation can be taken. The maximum expected utility is then

$$w^1_{t-1}(p_{t-1}; y_{t-1}) = \max_{0 \le e^1_{t-1} \le y_{t-1} - c} \left(\frac{e^1_{t-1}}{p_{t-1}} + \rho E_{p_t}\left[w^\star_t(p_t; y_{t-1} - e^1_{t-1} - c) \right] \right) \tag{30}$$

where (28) gives $w^\star_t(p_t; y_t)$. Therefore

$$e^1_{t-1} \in \begin{cases} \{0\} \\ [0; y_{t-1} - c] \\ \{y_{t-1} - c\} \end{cases} \text{as}$$

$$\begin{matrix} < \\ = \\ > \end{matrix} \int_a^{p^1_t(y_{t-1}-c)} \frac{1}{p} dF(p) + \sum_{i=1} \prod_{k=0}^{i-1} \left(1 - F\left(p^1_{t+k}\left(y_{t+k}\right)\right)\right) \int_a^{p^1_{t+i}(y_{t+i})} \frac{1}{p} dF(p). \tag{31}$$

Let the right hand side of (31) be γ, say. Substitution of (31) into (30) gives

$$w^1_{t-1}(p_{t-1}, y_{t-1}) = \begin{cases} (y_{t-1} - c)/p_{t-1}, & \text{if } \frac{1}{p_{t-1}} > \gamma \\ \rho(y_{t-1} - c) \int_a^{p^1_t(y_t)} \frac{1}{p} dF(p) \\ \quad + \sum_{i=1}^{j} \prod_{k=0}^{i-1} \rho A (y_{t-1} - (i+1)c) B, & \text{otherwise} \end{cases} \tag{32}$$

where

$$A \equiv \left(1 - F\left(p^1_{t+k}\left(y_{t+k}\right)\right)\right) \quad \text{and} \quad B \equiv \int_a^{p^1_{t+i}(y_{t+i})} \frac{1}{p} dF(p).$$

Define p^1_{t-1} as that price which equates (29) with the second branch of (32). Thus

$$\frac{1}{p^1_{t-1}} = \rho \left(1 - \frac{c}{y_{t-1}}\right) \int_a^{p^1_t(y_t)} \frac{1}{p} dF(p) + \rho \sum_{i=1} \prod_{k=0}^{i-1} A \left(1 - \frac{(i+1)c}{y_{t-1}}\right) B. \tag{33}$$

Clearly p^1_{t-1} is unique and is decreasing in y_{t-1}, and the decision rule has the form:

If $p_{t-1} \le p^1_{t-1}$ or $y_{t-1} \le c$, then $n^*_{t-1} = 0$ and $e^*_{t-1} = y_{t-1}$;

if $p_{t-1} > p^1_{t-1}$ or $y_{t-1} > c$, then $n^*_{t-1} = 1$ and $e^*_{t-1} = 0$.

This rule holds for all periods.

Consider now the sequence $\{p^1_1(y), p^1_2(y_2), \ldots, p^1_T(y_T)\}$. This can be calculated in advance. Specifically, if $p_1 > p^1_1(y)$ then $y_2 = y - c$, so that $p^1_2(y_2)$ can be calculated. Proceeding in this way builds up a deterministic sequence in which $\overline{p}_t = p^1_t(y_t)$, and which gives the stated decision rule.

Finally, the sequence is monotonic non-decreasing. For every t, p^1_t is decreasing in wealth. In addition, it is obvious that p^1_t is non-decreasing in t: The fewer periods that remain the lower is the marginal utility of postponed consumption. Since wealth decreases as search continues, together these properties of p^1_t imply monotonicity of the sequence of reservation prices. $\|$

References

[1] Benhabib, J., Bull, C.: Job search: The choice of intensity. J Pol. Econ 91, 747-64 (1983).

[2] Burdett, K., Judd, K.: Equilibrium price dispersion. Econometrica 51, 955-69 (1983).

[3] Carlson, J., McAfee, P.: Discrete equilibrium price dispersion. J. Pol. Econ. 91, 480-93 (1983).

[4] Gal, S., Landsberger, M., Levykson, B.: A compound strategy for search in the labour market. Inter. Econ. Rev. 22, 597-608 (1981).

[5] Kohn, M., Shavell, S.: The theory of consumer search. J Econ Theory 9, 93-123 (1974).

[6] Manning, R.: Search while consuming. Econ. Studies Quarterly 40, 97-108 (1989).

[7] Manning, R., Morgan, P.B.: Search and consumer theory. Rev. Econ. Studies 49, 203-216 (1982).

[8] McCall, J.J.: The economics of information and optimal stopping rules. J. Business 38, 300-17(1965).

[9] McCall, J.J.: The economics of information and job search. Quarterly J. Econ. 84, 113-26 (1970).

[10] Morgan, P.B.: Search and optimal sample sizes. Rev. Econ. Studies 50, 659-75 (1983).

[11] Morgan, P.B., Manning, R.: Optimal search. Econometrica 53, 923-44 (1985).

[12] Stigler, G.J.: The economics of information. J. Pol. Econ. 69, 213-25 (1961).

[13] Veendorp, E.C.H.: Sequential search without reservation price. Econ. Letters 6, 53-8 (1984).