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The Revelation of Information and Self-Fulfilling Beliefs[*]

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Abstract

At a rational expectations equilibrium (REE), individuals are assumed to know the map from states to prices. This hypothesis has two components, that agents agree (consensus), and that they have point expectations (degeneracy). We consider economies where agents' beliefs are described by a joint distribution on states and prices, and these beliefs are fulfilled at equilibrium. Beliefs are self-fulfilling if every price in the support of the distribution is an equilibrium price. The corresponding equilibria are Beliefs Equilibria (BE). The further restriction that agents have the same beliefs results in Common Beliefs Equilibria (CBE). We study the relationship between BE, CBE, and REE, thus isolating the role of consensus and of degeneracy in achieving rational expectations.

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1. Introduction

One of the remarkable properties of competitive equilibrium is its economy in terms of informational requirements imposed on participants in the market. Agents are only expected to know, or observe, the prices of goods, in addition to their own characteristics. While the informational requirement is minimal, it is often thought to be too demanding relative to the operation of actual markets. The prices that agents need to observe include those which arise at all dates, and all contingencies. Nevertheless, the allocations achieved by overall competitive equilibria { or Arrow-Debreu allocations { can be implemented by a simpler, and more realistic structure of markets, if agents have \rational expectations", as shown by Radner (1972, 1979). This is at some cost; in making the transition, we must give up the informational economy of the competitive paradigm.

The rational expectations hypothesis requires that agents know the map from states to prices. In sequence economies, states correspond to date-event pairs. In economies with asymmetric information, states correspond to information signals received by agents. In either case, the requirement that all markets are open simultaneously has to be replaced by the assumption that agents form expectations about prices which would realize in such markets. Expectations are ful¯lled at equilibrium, which, along with market clearing, de¯nes a Rational Expectations Equilibrium (REE).

Clearly, the hypothesis itself makes strong assumptions about what agents know, and what they agree upon. The \map from states to prices" summarizes very large amounts of information, about preferences, endowments, and technological possibilities available to all participants in each state and date. While it is natural to assume that agents know the price in each state, the assumption that they know the map is much stronger, because it implies that agents know the response of prices to perturbations of these parameters.

In addition, they are assumed to agree on the map, so that price expectations are assumed to be the same for all participants. To appreciate that this is in fact an additional assumption, imagine an economy where there are multiple competitive equilibria in each state. Knowledge of the parameters of the economy would lead agents to deduce a correspondence from states to prices. It is perfectly \rational", in such a world, for them to disagree on how equilibria are selected in each state.[1] What results is certainly not REE: the very notion of diverse expectations being self-fulfilling would appear to be internally inconsistent.[2]

Formally, the Rational Expectations Hypothesis (REH) amounts to the assumption that price expectations are single-valued, and coincident across agents. Agents agree on the realization of prices in each state and simultaneously believe that there is a single possible price in each state. In this paper, we examine the extent to which these assumptions are necessary to the characterization of outcomes.

We are concerned with REE in economies with asymmetric information. Radner (1979) showed that at a REE, prices generically reveal all information, in economies where information, or signals, take finitely many values. In addition, these fully revealing equilibria coincide with the competitive equilibria of the corresponding symmetric information economy, and inherit their usual properties, such as local uniqueness.[3]

We focus on the robustness of these characteristics of competitive equilibria in economies with asymmetric information, with respect to the two components of the rational expectations hypothesis. We also want to disentangle the impact of the two components. It is sometimes argued that consensus is a natural hypothesis in economic models (e.g. Aumann (1987)), whereas the assumption of "point expectations" is probably more questionable. It is known to be limiting in many applications.

It is necessary to start with a set of restrictions on individual rationality which are weaker than the REH. How should one describe the behaviour of agents who do not necessarily know the map from states to prices, in a world where market clearing is common knowledge? We propose to do this by describing agents as being uncertain about prices. Their beliefs are represented by a joint probability distribution on states and prices. It is then possible to associate successively stronger hypotheses about the rationality of beliefs, or expectations, in terms of the restrictions imposed on this joint distribution. This construction allows

[1] Polemarchakis and Siconolfi (1993) use the indeterminacy of equilibria in an incomplete markets economy to construct multiple maps from states to prices. Allen, Dutta and Polemarchakis (1994) show that consensus is necessary to achieve equilibrium with enough markets.

[2] Townsend (1978) provides an implicit definition, that such expectations are correct when averaged across individuals; Anderson and Sonnenschein (1982) require that they be statistically indistinguishable from outcomes, with finitely many observations.

[3] These results were extended to the infinite dimensional case by Allen (1981a).

us to ask what restrictions are imposed by individual rationality, and whether further restrictions are necessary to obtain the REH. At the same time, we need to pay attention to the characteristics of the resulting competitive equilibria. It is important to understand whether weaker hypotheses of rationality enlarge the set of equilibria, and alter their characteristics. In other words, can REE be the outcome of restrictions weaker than the REH?

Common knowledge of rationality and market clearing imposes the restriction that beliefs are concentrated on market clearing prices. We consider, ¯rst, such \Beliefs Equilibria (BE)", which result from pro¯les of beliefs which satisfy this property, and no further restrictions. As we have noted before, consensus needs to be imposed as a restriction; we call the outcomes \Common Beliefs Equilibria (CBE)", where agents agree on the joint distribution of prices and states, and these beliefs are self-ful¯lling by de¯nition. Rational Expectations Equilibria (REE) are CBE where the beliefs of every agent are degenerate, so that they assign probability one to a single price in each state.

We formally de¯ne these solution concepts in Section 2. The purpose of the paper is to use examples to explore the relation between these solution concepts. These examples are in Section 3; we are able to demonstrate most of the issues in the context of a very simple family of examples, with two agents, two goods, and state-dependent Cobb-Douglas preferences. This class of examples is familiar from related literature (e.g. Kreps (1977), Radner (1979), Allen (1981b), Radner and Jordan (1982), and Ausubel (1990)). They allow for clear and explicit solutions, and allow us to relate our solutions to previous work.

The message of the ¯rst three examples is that \rational expectations" cannot be deduced from rationality alone. Example 1 demonstrates the necessity of consensus; the set of BE is much larger than the unique REE. In Example 2, we show that consensus is not su±cient, so that the full set of restrictions of the REH are necessary for REE to be the only equilibria. This theme is explored further in Example 3, which is known to have multiple REE. We show there that randomizations over REE generate CBE.

In each of these examples, the dimension of information signals is higher than that of prices. We show, in Example 4, that in the absence of this problem, rationality alone may yield rational expectations: the unique REE is also the unique BE. Example 5 demonstrates that this logic may lead to the failure of BE to exist. This is essentially the example of Kreps (1977). The economy is such that no BE exists (and hence, no CBE or REE).

Our approach is related to a number of different strands of the existing literature. We discuss these in detail as we introduce our solution concepts and also in the concluding section 4. Closest to our work is McAllister (1990), which is also a study of the implications of common knowledge of market clearing and rationality alone. A crucial distinction is that while McAllister allows prices to reveal information about fundamentals which is not known to any agent, we require that any uncertainty about prices is uncorrelated with fundamentals not known to any agent. One implication is that if there is symmetric information, our solution concept of beliefs equilibrium reduces to competitive equilibrium of the expected economy. McAllister's solution concept expands the set of outcomes even with symmetric information. We illustrate this point in Example 6.

Our analysis of Examples 1-3 suggests that the "full revelation" property, as well as the "local uniqueness" property of REE are not preserved by BE or even CBE. It is known to be difficult to obtain examples, or characterization, of partially revealing REE (e.g. Ausubel (1990)). The first property suggests that the class of CBE may provide a natural home for the analysis of partial revelation. The second suggests similarities to the notion of "sunspot equilibria" (Cass and Shell (1983)) and the possibility of indeterminacy of competitive equilibria (Geanakoplos and Mas-Colell (1989)), both in incomplete markets. We discuss this in Section 4.

2. Framework

The framework is essentially that of Radner (1979). There are H agents and L goods. Uncertainty is represented by states of nature, $s \in S$, where $S \subseteq \Re^K$ for some finite K. Endowments as well as preferences of agents can be state-dependent. Endowments are $e^h : S \to \Re^L_+$, and utility functions are $U^h : \Re^L_+ \times S \to \Re$, for $h = 1, \cdots, H$.

Agents maximize expected utility, expectations being conditional on information available to them. Let $\Delta(S)$ be the class of probability distributions on S. For each $\beta \in \Delta(S)$, and $x : S \to \Re^L_+$, define

$$V^h(\beta; x) = \int_S U^h(x(s); s) d\beta$$

as the expected utility of agent h from the state-dependent consumption bundle x if she holds belief β, defining a probability distribution on S.

We assume that agents share a common prior distribution on the fundamental state space; call this $\pi \in \Delta(S)$. In addition, they receive private signals $\sigma^h : S \to \mathbb{R}$. Endowments are assumed measurable with respect to private information, so that $\sigma^h(s) = \sigma^h(s') \Rightarrow e^h(s) = e^h(s')$. We refer to $\pi^h(\cdot|\sigma^h) \in \Delta(S)$ as the (signal-dependent) prior of agent h. The state of information is $\sigma = (\cdots;\sigma^h;\cdots) \in \Sigma \subseteq \mathbb{R}^H$. For any realization $\sigma \in \Sigma$, the posterior distribution $\pi_*(\cdot|\sigma) \in \Delta(S)$ is the full-information posterior distribution.[4]

Agent h observes his private signal, σ^h, and the vector of prices p. Prices can vary across payoff-relevant information states: specifically, agent h does not observe $\sigma^{-h} = (\sigma^1;\cdots;\sigma^{h-1};\sigma^{h+1};\cdots;\sigma^H)$, but can infer something about them from her observation of p. Suppose that conditional beliefs for agent h were given by $\beta^h : \Sigma^h \times \mathbb{R}^L_{++} \to \Delta(S)$, so that $\beta^h\left(\cdot|\sigma^h;p\right)$ is h's probability distribution over states if he observed σ^h and p. Later, we will want to derive these beliefs endogenously, but let us first consider when markets clear if we fix a vector of such conditional beliefs, $B = \left(\beta^1;\cdots;\beta^h;\cdots;\beta^H\right)$.

Agent h's net trade is thus a σ^h measurable function such that $z^h\left(s;p|\beta^h\right)$ maximizes

$$V^h\left(\beta^h\left(\cdot|\sigma^h(s);p\right);z^h+e^h\right)$$

subject to the budget constraints $p.z^h \leq 0$. Prices, p, clear markets, so that

$$\sum_{h=1}^{H} z^h(s;p|\beta^h) = 0;$$

for each $s \in S$. Note that the true state s is typically unobserved at the time of choosing z^h. Let $P_C(B;s)$ be the set of prices which clear markets in state s given beliefs B, i.e.

$$P_C(B;s) = \left\{ p \in \mathbb{R}^{L-1}_{++} \,\middle|\, \sum_{h=1}^{H} z^h\left(s;p|\beta^h\right) = 0 \right\}:$$

The Rational Expectations Hypothesis assumes, at this point, that every agent knows a deterministic map from states to prices. This map can then be used to derive B. We want to allow for uncertainty about the price process. We replace the rational expectations hypothesis with one where each agent's beliefs are described by a conditional probability distribution on prices given states, written as $\delta^h(p|s)$.

[4] In many of our examples, we assume that $S = \Sigma$, so that π_* is degenerate.

A belief for agent h is a mapping $\delta^h : S \to \Delta(\Re^{L-1}_{++})$. For each $s \in S$, $\delta^h(\cdot|s)$ is a conditional probability distribution on the $L-1$ vector of relative prices p. We assume that beliefs are measurable with respect to the joint information of all agents, i.e. $\sigma(s) = \sigma(s') \Rightarrow \delta^h(\cdot|s) = \delta^h(\cdot|s')$. This is analogous to the usual assumption in rational expectations equilibrium. Let $\delta = (\delta^1; \cdots; \delta^h; \cdots; \delta^H)$ be a profile of beliefs. We assume that they have common support and write

$$P_*(\delta; s) = \left\{ p \in \Re^L_{++} \,\middle|\, \delta^h(p|s) > 0 \right\} \text{ for all } h = 1; \ldots; H:$$

Given h's beliefs δ^h, we can deduce her posterior beliefs β^h by Bayes rule. Define β^h_δ by

$$\beta^h_\delta(s|p; \sigma^h) = \frac{\pi(s|\sigma^h)\delta^h(p|s)}{\int_S \delta(p|s)d\pi(s|\sigma^h)}: \tag{2.1}$$

whenever $\int_S \delta(p|s)\pi(s|\sigma^h) > 0$; we will not be evaluating β^h_δ when this condition is not satisfied. Write $\delta = (\delta^1; \cdots; \delta^h; \cdots; \delta^H)$ and $B_\delta \equiv \left(\beta^1_{\delta^1}; \ldots; \beta^H_{\delta^H}\right)$. We are now in a position to define our solution concepts.

Definition 2.1. Beliefs profile δ is self-fulfilling if, for each $s \in S$, markets clear at every p in the common support of $\delta^h(s)$, i.e.

$$P_*(\delta; s) \subseteq P_C(B_\delta; s) \text{ for all } s \in S:$$

Notice that, because of our common support assumption on δ, conditional beliefs B are always uniquely defined by (2.1) for all s in $P_*(\delta; s)$. On the other hand, if we dropped the common support assumption, the definition would be sensitive to beliefs following unexpected realizations of prices.

Definition 2.2. Beliefs profile δ is degenerate if there exists $\phi : \Sigma \to \Re^L_{++}$ such that $P_*(\delta; s) = \{\phi(\sigma(s))\}$ and thus, for all h, $\delta^h(p|s) = \begin{cases} 1; & \text{if } p = \phi(\sigma(s)) \\ 0, & \text{otherwise} \end{cases}$.

Definition 2.3. A Beliefs Equilibrium is a self-fulfilling beliefs profile. A Common Beliefs Equilibrium is a Beliefs Equilibrium with $\delta^h = \delta$ for $h = 1; 2; \cdots; H$. A Rational Expectations Equilibrium is a common beliefs equilibrium with degenerate beliefs.

Remarks

- Our first definition explores the rationality of beliefs held by agents. These are, after all, theories about prices, which are an outcome of the economic system. If they live in a world where prices clear markets, this should be incorporated into their model of the world. Self-fulfilling beliefs support only equilibrium prices in each state.

- Self-fulfilling beliefs are necessary, but may not be strong enough to yield rational expectations equilibria. Definition 2.3 indicates the successive strengthening which is required to achieve that. At a rational expectations equilibrium, agents' beliefs agree, and, in addition, these beliefs are degenerate.

- Beliefs are theories about prices in each information state, s. Agreement on beliefs δ is not sufficient for agreement on posteriors B. This last is typically true only at fully revealing equilibria (formally defined below).

- The property that beliefs are self-fulfilling is true of the entire profile; one person's theory is fulfilled by the simultaneous actions of all agents, which depend on their beliefs.

- We have specified that self-fulfilling beliefs have support entirely on market-clearing prices. This definition could be weakened. Anderson and Sonnenschein (1982) require beliefs to contain equilibrium prices, i.e. $P_* \cap P_C \neq \emptyset$, so that rationality imposes the relatively weak restriction that agents never observe prices which they had thought impossible. They may nevertheless continue to assign positive probability to impossible events. Clearly, our equilibria continue to be admissible in this definition, and the class of equilibria so obtained should be much larger. We explore the implications of the relatively tighter notion.

- We do not impose the stronger restriction that $P_* = P_C$. If P_C is not a singleton, this would rule out the REH, as agents would be restricted to putting positive probability on all equilibria.

- Note that in the definition of self-fulfilling beliefs, we require merely that every price in the support be an equilibrium price. Given this, the auctioneer may choose to randomize in any way over the equilibria, because market clearing must be achieved state by state, and not in expected or in probabilistic terms. In the case of CBE, this is really a demonstration that common beliefs can be fulfilled, and not necessarily that they will be.

- The requirement that each $\delta^h(\cdot|s)$ be measurable with respect to σ is analogous to the usual rational expectations equilibrium assumption that prices are measurable with respect to the join of agents' information. McAllister's (1990) definition of admissible beliefs is given in a rather different setting from this one. But the essential difference from our beliefs equilibrium is that he does not make this assumption.

Definition 2.4. Belief profile δ is fully revealing if an outside observer could deduce all signals from prices, i.e. $P_*(\delta; s) \cap P_*(\delta; s') = \emptyset$ if $\sigma(s) \neq \sigma(s')$.

Notice that there may exist belief equilibria and common belief equilibria which are fully revealing but are not rational expectations equilibria. However, such fully revealing belief equilibria will simply be randomizations over the competitive equilibria of the complete information economy.

3. Beliefs Equilibria: Some Examples

We have defined the successively stronger solution concepts of BE, CBE, and REE which are justified by strengthening the hypothesis about the nature of beliefs and about consensus in such beliefs. We have argued that common knowledge of rationality and market clearing implies only that beliefs are self-fulfilling, so that rationality of beliefs corresponds to the notion of Beliefs Equilibria. We want to unpack the Rational Expectations Hypothesis into two components: that of consensus – the restriction that beliefs are the same across market participants, and that of degeneracy – that these beliefs are restricted to be point expectations of prices in each state. Imposing consensus yields CBE, and the further restriction of degeneracy yields REE.[5]

It remains to be seen whether a competitive economy yields equilibria – CBE or even BE – other than the Rational Expectations Equilibria. Otherwise, our arguments could be seen as strong justification for the notion of an REE, which obtain as outcomes of restrictions much weaker than the REH.

[5] It is theoretically possible to impose degeneracy without consensus, at the cost of obtaining equilibria which cannot fulfill all expectations at once, by construction.

Do self-fulfilling beliefs necessarily yield rational expectations equilibria? We explore this issue in some examples. The examples are simple enough to allow for complete characterization of competitive equilibria with alternative restrictions. To do this, we have specialized to $H = 2$ and $L = 2$. The two goods are labelled x and y. Endowments are state-independent; preferences are assumed to be state dependent. In each example, preferences are of the form

$$V(\pi^h, x^h, y^h) = \int_S (a^h(s) \ln x^h(s) + b^h(s) \ln y^h(s))\, d\pi^h.$$

Endowments are assumed to be state-independent

$$e^1(s) = e^1, \quad e^2(s) = e^2.$$

This family is familiar from several papers on rational expectations, including Kreps (1977), Radner (1978), and Ausubel (1990). Allen (1981b) sets out a class of economies where the dimension of the state space is greater than that of prices; she further restricted these economies to satisfy monotonicity conditions with respect to the signals, as well as the property that rational expectations equilibria be unique. Our examples satisfy the monotonicity conditions. As it happens, this family is rich enough to display the properties that we need to establish. We believe that their explicit characterization of equilibria allows insights into the analysis of BE and CBE in more abstract settings.

A brief overview of our examples is as follows.

1. Example 1 is an economy with a unique REE. We show there that the set of BE is much larger, while the unique CBE coincides with the REE. This is a clear demonstration that consensus is necessary to support REE. It also suggests that there are economies where consensus implies degeneracy; their characterization is an open question.

2. The property that the REE is unique is true of the class of economies in Example 2 as well. However, for an open set of parameter values within this class, the economy has a continuum of CBE, which are partially revealing. Both properties are of interest: the fact that the CBE are indeterminate, and that they "hide" information which would have been revealed by prices at an REE. The CBE are non-degenerate, equilibrium price distributions, and can be understood as "sunspot equilibria".

3. The economy of Example 3 has multiple REE; it is really a reformulation of an example due to Radner (1979). This example allows a better characterization of the class of CBE. We show that the class of equilibrium price distributions is convex; as a consequence, randomizations over REE generate CBE, but do not exhaust them. The property of multiple equilibria is known to be non-generic in Cobb-Douglas economies, but certainly not so in more general ones. The example demonstrates that non-degenerate CBE may be typical of such situations, so that the power of the consensus hypothesis in forcing REE is likely to be limited.

4. Example 4 is contrary to the spirit of the previous ones. The only BE are REE, so that the weakest restriction of rationality suffices to yield rational expectations. Consensus and degeneracy are consequences of rationality. The example has one-sided uncertainty, which suggests that dimensions of signals are likely to be of critical importance in justifying rational expectations.

5. We demonstrate, in Example 5, that the argument of Example 4 may lead to the failure of existence of BE: the example of Kreps (1977), which is known not to possess REE, also has no BE. Beliefs simply cannot be self-fulfilling. It is also a demonstration that our solution concept is tight: weakening the solution concept in the manner suggested by Anderson and Sonnenschein (1982) would restore existence.

6. We demonstrate, in Example 6, that the measurability restriction on the join of agents' information is - in general - binding. We conjecture that this explains why McAllister (1990) is able to get a general existence result for a related solution concept.

We will assume, throughout, that the objective probability distributions π on $s \in S$, or equivalently, on $(a^1(s); b^1(s); a^2(s); b^2(s))$ are common knowledge, so that agents agree on their priors. In order to recover information from prices, they need to form posterior probability distributions $\beta^h(s|p;\sigma^h)$. Rationality of beliefs implies restrictions on δ^h, and therefore on β^h. In the examples below, we state the direct restrictions on the posteriors, which are compatible with common priors π on S. If, in addition, agents are assumed to have common beliefs, this implies that the posterior probabilities are derived from a common joint distribution on states and prices.

3.1. Example 1: Consensus is Necessary

For our first family of examples, we assume that agents have the same preferences in each state s. Let

$$a^h(s) = a > 0; \quad b^h(s) = b > 0;$$

and $s = (a; b) \in S \equiv \Re^2_{++}$. Preferences are

$$U^h(x^h(s); y^h(s); s) = a \ln x^h(s) + b \ln y^h(s); \quad h = 1; 2:$$

Further,

$$\sigma^1(s) = a; \quad \sigma^2(s) = b;$$

so that the two agents each observe different co-ordinates of the state s. By construction, $\sigma(s) = s$.

Let $\pi \in \Delta(\Re^2_{++})$ be the objective probability distribution on the state space known to both. We will assume that a and b are not perfectly correlated, i.e. that π is such that $var(a|b) > 0$ and $var(b|a) > 0$, for every pair $(a; b)$. Let $(1; p)$ be the prices of the two goods. Finally, endowments are

$$e^1 = [1; 0]; \quad e^2 = [0; 1]:$$

We refer to this as Economy 1.

Proposition 3.1. The following characteristics are true of Economy 1.

1. There is a unique Rational Expectations Equilibrium, where $p = \frac{b}{a}$.
2. There is no Common Beliefs Equilibrium other than the REE.
3. A profile of beliefs $\delta = (\delta^1; \delta^2)$ is self-fulfilling whenever

$$E_{\delta^1} b|a; p = ap; \quad E_{\delta^2} a|b; p = \frac{b}{p}:$$

There exist a continuum of Beliefs Equilibria satisfying this condition.

The proof is constructive. Let $\delta^1; \delta^2$ be the beliefs profile. Agent 1 observes a and p. Let $b_1(a; p) = E_1 b|a; p$ be the conditional expectation of a given b and p, derived from π,δ^1 by Bayes updating; similarly, $a_2(b,p) = E_2 b|a,p$ is the conditional expectation of b given a and p, derived from π,δ^2. Agent 1 demands

$$x^1 = \frac{a}{a + b_1(a,p)}, \quad py^1 = \frac{b_1(a,p)}{a + b_1(a,p)},$$

agent 2 demands

$$x^2 = p\frac{a_2(b,p)}{a_2(b,p) + b}, \quad y^2 = \frac{b}{a_2(b,p) + b}.$$

Given a realization (a,b) and expectations formulae $b_1(.,.), a_2(.,.)$, market clearing prices must satisfy

$$p = \frac{1 + \frac{b}{a_2(b,p)}}{1 + \frac{a}{b_1(a,p)}}. \tag{3.1}$$

Lemma 3.2. At any BE, the following conditions must be satisfied:

$$b_1(a,p) = a\phi_1(p), \quad a_2(b,p) = b\phi_2(p),$$

for some pair of strictly positive-valued functions ϕ_1 and ϕ_2; and

$$P_\delta \subseteq \{p \in \Re_{++} : p\phi_2(p)(1 + \phi_1(p)) - \phi_1(p)(1 + \phi_2(p)) = 0\}.$$

Proof:

The market clearing condition, (3.1), implies

$$\frac{a}{b_1(a,p)} = \Psi_1\left(p, \frac{b}{a_2(b,p)}\right) = \frac{b}{p.a_2(b,p)} + \frac{1-p}{p}$$

$$\frac{b}{a_2(b,p)} = \Psi_2\left(p, \frac{b}{b_1(a,p)}\right) = \frac{p.a}{b_1(a,p)} + p - 1$$

The quantity a_2 is an expectation conditional on b,p, so that $\frac{b}{a_2}$ should be invariant to a; similarly, $\frac{a}{b_1}$ should be invariant to b. The restrictions $\frac{\partial b_1}{\partial b} = \frac{\partial a_2}{\partial a} = 0$ are true at each $p \in P_\delta$ and all (a,b) if, and only if, $b_1(a,p) = a\phi_1(p)$ and $a_2(b,p) = b\phi_2(p)$ for some pair of functions ϕ_1, ϕ_2. These must be positive valued to ensure $p > 0$. Finally, these restrictions imply that equation (3.1) rewrites as

$$p = \frac{1 + \frac{1}{\phi_2(p)}}{1 + \frac{1}{\phi_1(p)}}.$$

Self-fulfilling beliefs put full support on market clearing prices, which yields the last restriction. □

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At a BE, agents can have different beliefs about prices. If agents hold the same beliefs, so that $\delta^1 = \delta^2$, their subjective probability distributions must be coherent with a (common) joint distribution on $(p; a; b)$. The additional restrictions imposed by common beliefs are implications of this. In characterizing CBE, we drop the subscripts $1; 2$ from the expectations operators to indicate that they are derived from the same joint distribution.

Lemma 3.3. At a CBE, the following restrictions hold:

$$E b|a; p = b(a; p) = ap;$$

$$E a|b; p = a(b; p) = \frac{b}{p}:$$

Proof: Common beliefs implies that $E a|p = E a(b; p)|p$ and $E b|p = E b(a; p)|p$ with respect to some common conditional distribution $F(a; b|p)$. From Lemma (3.2), this implies

$$E b|p = \phi_1(p) E a|p; \quad E a|p = \phi_2(p) E b|p:$$

This is true if, and only if,

$$\phi_1(p)\phi_2(p) = 1:$$

Market clearing implies

$$p = \frac{1 + \frac{1}{\phi_2(p)}}{1 + \frac{1}{\phi_1(p)}} = \phi_1(p)$$

whenever $\phi_1(p)\phi_2(p) = 1$. The stated conditions follow. □

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These restrictions have a strong implication, stated below.

Lemma 3.4. (Common Beliefs are Degenerate) There is a unique CBE, at which

$$p = \frac{b}{a}$$

with probability 1 for each pair $a; b$.

Proof: From the Lemma (3.3), a CBE implies

$$E\,b|a; p = ap; \; E\,a|b; p = \frac{b}{p}.$$

Suppose $\delta(p|a; b)$ is non-degenerate for some $a; b$. Let $z = p.\frac{a}{b}$. The first restriction implies $E\frac{1}{z}|a; p = 1$ for each such a and the second restriction implies $Ez|b; p = 1$ for each such b. If $\delta(.|(a; b)$ is degenerate for some $(a; b)$, Lemma 1 implies that $p = \frac{b}{a}$ so that $z = 1$ with probability 1. It follows that at a CBE, we must necessarily have $Ez = 1$ and $E\frac{1}{z} = 1$. By Jensen's inequality, this is possible only if the random variable z equals 1 with probability 1. The result follows. $\square$

$\cdots$

In this class of examples, common beliefs suffice to yield rational expectations. The REE is unique, with the price function $p = \frac{b}{a}$. At each state $(a; b)$, allocations are

$$x^1 = \frac{a}{a+b}; \quad y^1 = \frac{b}{a+b};$$

$$x^2 = \frac{b}{a+b}; \quad y^2 = \frac{a}{a+b}.$$

Given the information structure of the economy, every agent knows the state, even though prices do not reveal all information.

It remains to characterize the class of BE.

Lemma 3.5. A profile of beliefs $\delta = (\delta^1; \delta^2)$ is self-fulfilling whenever the following hold:

$$E_1 b|a; p = ap$$

$$E_2 a|b; p = \frac{b}{p}$$

for each $(a; b)$ and every $p \in \Re_{++} = P_\delta$.

Proof: The restrictions satisfy the condition of Lemma (3.2). If they hold, any $p > 0$ satisfies the market clearing condition, (3.1).$\Box$

$\cdots$

To demonstrate that non-trivial BE exist, it suffices to note that $P_\delta = \Re_{++}$, i.e. all positive prices are admissible, and that δ_i should be such that $E_{\delta_1} p | a = \frac{E_\pi b}{E_\pi a}$ and $E_{\delta_2} \frac{1}{p} | b = \frac{E_\pi b}{E_\pi a}$. Clearly, there exist a continuum of beliefs profiles satisfying these moment conditions.[6]

3.2. Example 2: Partially Revealing CBE

In this section, we are concerned with common beliefs equilibria which allow outcomes which could not occur in any rational expectations equilibrium. We will construct a robust class of economies where there is a unique rational expectations equilibrium, but there is a continuum of common belief equilibria.

This example has two sided uncertainty - i.e. there are two agents each of whom knows something which the other does not. There are finitely many states of nature. Specifically, $S = \{1; 2; 3; 4\}$. We assume that $a^h(s) + b^h(s) = 1$, so that preferences are

$$U^h(x^h(s); y^h(s); s) = a^h(s) \ln x^h(s) + (1 - a^h(s)) \ln y^h(s); \quad h = 1; 2:$$

Let us parameterize this economy by the pair $(\pi; a) \in \Delta(S) \times (0; 1)^8$ where $a = \{a^h(s) | h = 1; 2; s \in S\}$. Agents receive signals which lead to different partitions of S. Define $\sigma^h(s)$ as follows :

$$\sigma^1(s) = 1 \quad \text{if} \quad s \in \{1; 2\};$$

$$\sigma^1(s) = 0 \quad \text{if} \quad s \in \{3; 4\};$$

and

$$\sigma^2(s) = 1 \quad \text{if} \quad s \in \{1; 3\};$$

$$\sigma^2(s) = 0 \quad \text{if} \quad s \in \{2; 4\}:$$

[6] In an earlier version, Dutta and Morris (1994), we demonstrated the characteristics of BE in fully parametrized families for the distribution π.

Note that $\sigma(s) = (\sigma^1(s), \sigma^2(s))$ is a one-to-one map. State-independent endowments are

$$e^1 = e^2 = [1, 1].$$

It is convenient to normalize prices to lie in the unit simplex, so that the price vector is $(p, 1 - p)$. This completes the specification of Economy 2.

In constructing CBE, we need to impose further restrictions on the utility parameters a as follows:

$$a^1(s) + a^2(s) \neq a^1(s') + a^2(s') \quad \text{if} \quad s \neq s', s, s' \in S. \tag{3.2}$$

$$\mathrm{Max}[a^1(1), a^1(2)] > \mathrm{Min}[a^1(3), a^1(4)], \tag{3.3}$$

$$\mathrm{Min}[a^1(1), a^1(2)] < \mathrm{Max}[a^1(3), a^1(4)], \tag{3.4}$$

$$\mathrm{Max}[a^2(1), a^2(3)] > \mathrm{Min}[a^2(2), a^2(4)], \tag{3.5}$$

$$\mathrm{Min}[a^2(1), a^2(3)] < \mathrm{Max}[a^2(2), a^2(4)]. \tag{3.6}$$

Note that there exists an open set of parameter values $a \in (0, 1)^8$ for which these conditions hold. For example, the inequalities $a^1(1) > a^1(3) > a^1(2) > a^1(4)$ and $a^2(1) > a^2(2) > a^2(3) > a^2(4)$ are sufficient.

Proposition 3.6. The following characteristics are true of Economy 2.

1. There exists a unique fully revealing REE if the restriction (3.2) holds. This is true for generic a (for all π).

2. There is an open set of (π, a) for which there exist a continuum of partially revealing CBE. Specifically, there is such a continuum if a satisfies restrictions (3.2)-(3.6), and $\pi(s) > 0$ for each $s \in S$.

This will be proved via a series of lemmas. Let $a^h(\sigma^h, p) = E a^h(s) | \sigma^h, p$. Agents demand $x^h = \frac{a^h(\sigma^h, p)}{p}$ and $y^h = \frac{1 - a^h(\sigma^h, p)}{(1 - p)}$. Market clearing implies

$$p = \frac{a^1(\sigma^1, p) + a^2(\sigma^2, p)}{2}.$$

Lemma 3.7. A fully revealing REE exists if the restriction (3.2) holds.

Proof: At a fully revealing equilibrium,

$$p(s) = \frac{a^1(s) + a^2(s)}{2}:$$

The result follows. 2

¢¢¢

De¯ne now the quantities

$$®_{1;1} = Ea^1(s)j_{(¾^1=1)} = \frac{¼(1)a^1(1) + ¼(2)a^1(2)}{¼(1) + ¼(2)};$$

$$®_{1;0} = Ea^1(s)j_{(¾^1=0)} = \frac{¼(3)a^1(3) + ¼(4)a^1(4)}{¼(3) + ¼(4)};$$

$$®_{2;1} = Ea^2(s)j_{(¾^2=1)} = \frac{¼(1)a^2(1) + ¼(3)a^1(3)}{¼(1) + ¼(3)};$$

$$®_{2;0} = Ea^2(s)j_{(¾^2=0)} = \frac{¼(2)a^2(2) + ¼(4)a^2(4)}{¼(2) + ¼(4)}:$$

These are the conditional expectations which are relevant if agents do not recover information from prices.

Lemma 3.8. A fully non-revealing REE exists if and only if

$$®_{1;1} = ®_{1;0} \quad \text{and} \quad ®_{2;1} = ®_{2;0}:$$

Proof: A fully non-revealing REE has

$$p(¾) = \frac{Ea^1j¾^1 + Ea^2j¾^2}{2}:$$

Agent 1's demand is non-revealing if $®_{1;1} = ®_{1;0}$ and agent 2's demand is non-revealing if $®_{2;1} = ®_{2;0}$. Both conditions are necessary for a non-revealing equilibrium. 2

¢¢¢

The condition that $\alpha_{h,1} = \alpha_{h,0}$ for $h = 1, 2$ is non-generic. If these fail, the REE is unique and fully revealing as long as restriction (3.2) holds.

To construct CBE, we need to construct posterior probability distributions $\beta \in \Delta(S)$ such that $E_\beta a^h|\sigma^h$ are invariant to σ^h. For any $\beta \in \Delta(S)$, and for $h = 1, 2$, $\sigma^h \in \{0, 1\}$, define

$$\alpha_{h,\sigma^h}(\beta) = E_\beta a^h|\sigma^h.$$

Lemma 3.9. Suppose conditions (3.3)-(3.6) hold. There exist a continuum of probability distributions $\beta \in \Delta(S)$ such that

$$\alpha_{1,1}(\beta) = \alpha_{1,0}(\beta)$$

and

$$\alpha_{2,1}(\beta) = \alpha_{2,0}(\beta).$$

Proof: Note that

$$\alpha_{1,1}(\beta) = \frac{\beta(1)a^1(1) + \beta(2)a^1(2)}{\beta(1) + \beta(2)}$$

and

$$\alpha_{1,0}(\beta) = \frac{\beta(3)a^1(3) + \beta_4 a^1(4)}{\beta(3) + \beta(4)}.$$

The equality $\alpha_{1,1}(\beta) = \alpha_{1,0}(\beta)$ is satisfied for some $\beta \in \Delta(S)$ as long as equations (3.3)-(3.4) hold. By analogy, equations (3.5) and (3.6) guarantee that $\alpha_{2,1}(\beta) = \alpha_{2,0}(\beta)$. Finally, there are a continuum of β achieving this because this imposes two restrictions on three unknowns $\beta(1), \beta(2), \beta(3)$, with $\sum_{s=1}^{4} \beta(s) = 1$ by definition. $\Box$

$\cdots$

Let $B^* \subset \Delta(S)$ be the set of probabilities which satisfy the requirement of Lemma 3.9: call this the set of confounding distributions.

Lemma 3.10. Suppose $\pi(s) > 0$ for $s = 1, 2, 3, 4$, and that a satisfies restrictions (3.3)-(3.6). There exist a continuum of partially revealing CBE.

Proof: Let $\beta \in B^*$ be any confounding distribution. Choose $\epsilon(s) \in (0,1)$ for $s = 1,2,3,4$ such that $\beta(s) = \frac{\pi(s)\epsilon(s)}{\sum_s \pi(s)\epsilon(s)}$. Let $q(\beta) = \frac{\alpha_{1,1}(\beta)+\alpha_{2,1}(\beta)}{2}$, which is the state-independent price if the posterior distribution is β, and let $p(s) = \frac{a^1(s)+a^2(s)}{2}$ be the fully revealing REE. Now consider the common beliefs $\delta(p|s)$ described by $\mathrm{Prob}(p = p(s)|s) = 1 - \epsilon(s)$ and $\mathrm{Prob}(p = q(\beta)|s) = \epsilon(s)$. This is a CBE as long as $q(\beta) \neq p(s)$. By construction, each such $\beta \in B^*$ corresponds to a CBE. □

$\cdots$

This example is robust to perturbations of the utility functions and endowments. Thus we have generated a robust class of economies where there is a unique rational expectations equilibrium which is fully revealing, but there are a continuum of partially revealing common belief equilibria.

3.3. Example 3: Multiple Equilibria and Price Distributions

Suppose the economy admits multiple REE. Can we construct non-degenerate CBE from randomizations over equilibria? This example shows that this is indeed the case; however, the class of CBE is larger than lotteries over REE. Typically, randomizations over deterministic equilibria are not ex-ante equilibrium distributions in competitive economies. This example suggests that the recovery of information from prices can allow for such randomizations to be self-fulfilling.

We assume that $a^1(s) + b^1(s) = 1, a^2(s) + b^2(s) = 1$. It is useful to write

$$U^1(x^1, y^1, s) = (1 - b)\ln x^1(s) + b \ln y^1(s);$$

$$U^2(x^2, y^2, s) = a \ln x^2(s) + (1 - a)\ln y^2(s).$$

States of nature are $s = (a, b) \in S \equiv (0,1)^2$. Further, $\sigma_1(s) = s = (a, b)$; $\sigma_2(s) = b$. Agent 1 is fully informed; agent two observes only one co-ordinate of the state, and needs to know the other. Endowments are

$$e_1 = [1, 0]; \quad e_2 = [0, 1].$$

The probability distribution on S is π. We assume that a and b are not perfectly correlated, i.e. that π is such that $\mathrm{var}(a|b) > 0$. Prices are $[1, p]$. We refer to this as Economy 3.

This example has a family resemblance to those analyzed by Radner (1979), and in Ausubel (1990). In particular, the restriction $b = \frac{1}{2}$ yields one of the examples from Radner (1979). The restriction that the prior puts support only on states (a, b) with either $b = a$ or $b = a^n$ for some $0 < n \neq 1$ generates a class of examples studied in Ausubel (1990).[7]

Proposition 3.11. The following characteristics are true of Economy 3.

1. There are multiple Rational Expectations Equilibria.
2. The class of price distributions at Common Beliefs Equilibria is convex, and contains probability mixtures over the Rational Expectations Equilibria.
3. There are Common Beliefs Equilibria which are not probability mixtures of Rational Expectations Equilibria.

Agent 1 observes the state s. Her demands are

$$x^1 = (1 - b); \quad py^1 = b;$$

agent 2 observes b and p. Let $a_2(b, p) = E_2 a|b, p$. He demands

$$x^2 = a_2(b, p)p; \quad y^2 = (1 - a_2(b, p)).$$

Given (a, b) and the expectations formula $a_2(\cdot, \cdot)$, market clearing implies

$$p = \frac{b}{a_2(b, p)}. \tag{3.7}$$

We turn to the characterization of CBE for this class of economies. Agent 1 is fully informed, so that any self-fulfilling belief δ_2 will support a CBE.

Lemma 3.12. Let $\delta(p|s)$ be a conditional probability distribution on $\Re_{++}$ for each $s \in (0, 1)^2$. δ supports a CBE if, and only if, the following restriction holds:

$$E_\delta a|b, p = \frac{b}{p}$$

for (almost) every (b, p).

[7] The examples in Ausubel (1990) differ slightly in the information structure, in that the second individual is assumed to recieve a null signal. The change in assumption is far from innocent, as we have multiple REE , which proves important in the construction of non-trivial CBE.

Proof: Let $a(b,p) = E_{\delta} a|b,p$. From equation (3.7), market clearing implies

$$a(b,p) = \frac{b}{p}.$$

For any $p > 0$, markets clear only if this condition holds. Any joint distribution of (p,a,b) satisfying this restriction is a CBE. $\square$

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This Lemma indicates a natural way to construct price distributions supporting a CBE. If the random variable a is non-degenerate, we can always construct another random variable, z, with the property that

$$E a|b,z = z.$$

This requires that z and a satisfy second order stochastic dominance, conditional on each b. Now define

$$p(z,b) = \frac{b}{z}.$$

By construction, $E a|b,p = E a|b,z = \frac{b}{p}$.

In addition, mixtures of CBE are also CBE, as the next Lemma shows.

Lemma 3.13. Let δ_i and δ_j be conditional probability distributions on $\Re_{++}$ which support CBE. Let $\epsilon \in (0,1)$ and define

$$\delta_\epsilon = \epsilon\delta_i + (1-\epsilon)\delta_j.$$

δ_ϵ also supports a CBE.

Proof: Let $\pi(a,b)$ be the common, objective probability distribution on (a,b), and define

$$F_i(a|b,p) = \frac{\delta_i(p|a,b)\pi(a,b)}{\int_a \delta_i(p|a,b)\pi(da,b)},$$

$$F_j(a|b,p) = \frac{\delta_j(p|a,b)\pi(a,b)}{\int_a \delta_j(p|a,b)\pi(da,b)},$$

as the conditional probability distributions associated with δ_i and δ_j; similarly,

$$F_\epsilon(a|b,p) = \frac{\delta_\epsilon(p|a,b)\pi(a,b)}{\int_a \delta_\epsilon(p|a,b)\pi(da,b)}.$$

>From the definitions,

$$F_\varepsilon = \lambda(b,p)F_i + (1-\lambda(b,p))F_j$$

where $\lambda(b,p) = \frac{\varepsilon\int_a F_i\,\pi(da,b)}{\int_a(\varepsilon F_i + (1-\varepsilon)F_j)\,\pi(da,b)}$. Clearly, $0 \le \lambda(b,p) \le 1$ for each (b,p), and

$$E_\varepsilon[a|b,p] = \int_a a\,dF_\varepsilon = \lambda(b,p)E_i(a|b,p) + (1-\lambda(b,p))E_j(a|b,p).$$

If δ_i, δ_j support CBE, $E_i[a|b,p] = E_j[a|b,p] = \frac{b}{p}$, implying $E_\varepsilon[a|b,p] = \frac{b}{p}$. By Lemma (3.12), δ_ε supports a CBE. $\Box$

$\cdots$

Lemma (3.13) has an important implication about the possibility of non-degenerate CBE. Convex combinations of CBE are also CBE. If there are at least two REE, this guarantees a family of non-degenerate CBE. This is formally stated next.

Lemma 3.14. Let $p_1(a,b), p_2(a,b)$ be price functions which support rational expectations equilibria. For each $\varepsilon \in (0,1)$, let $\delta_\varepsilon(p|a,b)$ be the distribution of a random variable p defined by

$$p = p_1(a,b) \quad \text{with} \quad \text{probability} \quad \varepsilon;$$

$$p = p_2(a,b) \quad \text{with} \quad \text{probability} \quad 1-\varepsilon.$$

The belief δ_ε supports a CBE for each $\varepsilon \in (0,1)$.

$\cdots$

It remains to show that there are multiple REE. Lemma (3.12) characterizes all CBE; and the every REE must correspond to a price function $p(a,b)$ satisfying the equilibrium condition. We can use this as a constructive method to find equilibria.

Lemma 3.15. There is a fully revealing REE, at which

$$p(a,b) = \frac{b}{a}$$

for each $(a,b) \in (0,1)^2$.

::

There are many more REE, in such economies, as we show next. In order to construct the equilibria, we need to define a class of price functions $p_k(a;b)$. Let $I_k = \{I_{k1}; \cdots; I_{kj}; \cdots I_{kn_k}\}$ be a partition of the interval $(0;1)$, such that the cells I_{kj} are mutually exclusive and exhaustive. Let

$$\alpha_k(a;b) = E a|b; a \in I_{kj} \quad \text{if} \quad a \in I_{kj}; \quad j = 1; \cdots n_k:$$

Clearly, for each partition I_k, α_k is a function. The candidate price functions are

$$p_k(a;b) = \frac{b}{\alpha_k(a;b)}:$$

Lemma 3.16. For each partition I_k of the interval $(0;1)$, the price function

$$p_k(a;b) = \frac{b}{\alpha_k(a;b)}$$

supports a REE.

Proof: It suffices to note that $E a|b; a \in I_{kj} = E a|b; p_k$ whenever $a \in I_{kj}$; that p_k is a non-deterministic function, and that $E a|b; p_k(a;b) = \frac{b}{p_k(a;b)}$. $\square$

::

For each partition of the unit interval, we can construct an REE which is measurable with respect to that partition. There are at least countably infinite such partitions, so that the number of REE is very large in this class of economies.

The coarsest partition has a single cell, $I_0 = \{I_{01} = (0;1)\}$, and the corresponding price function is

$$p_0(b) = \frac{b}{E a|b}:$$

This particular price function is referred to as the non- revealing equilibrium price in Radner (1979). The price function corresponding to the finest partition is $p = \frac{b}{a}$, which yields the fully revealing REE. For each intermediate partition, we can find an REE which is partially revealing; the equilibria differ in the information content of prices, as well the expected utility of equilibrium allocations.

To demonstrate that the class of CBE contain equilibria which are not randomizations over REE, we specialize to set $b = \frac{1}{2}$ with probability 1. The argument can be extended, but this simpler economy will suffice for the demonstration. Let

$$Z_1(a) = (1 - \epsilon) + (2\epsilon - 1)a; \quad Z_2(a) = (1 - \epsilon) + (2\epsilon - 1)(1 - a):$$

Define a random variable z, as follows. For each a, $z = Z_1(a)$ with probability ϵ and $z = Z_2(a)$ with probability $(1 - \epsilon)$. Let $p = \frac{b}{z} = \frac{1}{2z}$ be the candidate price function. By construction,

$$E[a|z] = \epsilon \frac{z - (1 - \epsilon)}{2\epsilon - 1} + (1 - \epsilon)\left(1 - \frac{z - (1 - \epsilon)}{2\epsilon - 1}\right) = z;$$

and

$$E[a|p] = \frac{1}{2p} = \frac{b}{p}:$$

For any $\epsilon \in (0; 1)$, this yields a price distribution which supports a CBE. For each a, this distribution has support points $\frac{1}{2Z_1(a)}; \frac{1}{2Z_2(a)}$. Now

$$E[a|Z_1] = \frac{Z_1 - (1 - \epsilon)}{2\epsilon - 1} \neq Z_1;$$

$$E[a|Z_2] = 1 - \frac{Z_2 - (1 - \epsilon)}{2\epsilon - 1} \neq Z_2;$$

the price functions $p_i(a) = \frac{1}{2Z_i(a)}$ are not REE.

3.4. Example 4: When Do Self-Fulfilling Beliefs Imply Rational Expectations?

If there is no uncertainty and there is a unique competitive equilibrium price vector then in the unique beliefs equilibrium, that price vector is chosen with probability one, and thus there is both full revelation and a REE. But it is straightforward to identify more general classes of economies where all BE are fully revealing and REE.

Suppose there is one sided uncertainty, so that a single agent is informed. Suppose the informed agent's endowment is state independent and suppose that his demand for some good strictly increasing in the state (at any give price).

Formally, let 1 be the informed agent. So $\sigma^1(s) = s$ and $\sigma^h(s)$ is a constant function for all $h \neq 1$. Thus we may identify $\sigma(s)$ with $\sigma^1(s)$. Assume that 1's endowment is state independent and that her excess demand $z^1(p; \sigma^1)$ is strictly monotonic in σ^1 for each p. Let Economy 4 be any economy satisfying these restrictions : they include Cobb- Douglas economies of the previous examples with $E a^1(s)|\sigma^1(s)$ being monotone in σ^1.

Proposition 3.17. In Economy 4, all Beliefs Equilibria satisfy full revelation. If each complete information economy has a unique equilibrium, then every Beliefs Equilibrium is a Rational Expectations Equilibrium.

Proof: Suppose δ is a self-fulfilling belief, and $p \in P_*(\delta; s) \cap P_*(\delta; s')$ for some $s \neq s'$ (assume w.l.o.g. $s > s'$). All agents other than 1 have excess demand for good 1 at price p that is independent of the state s; but 1's demand for good 1 is strictly greater at price p and state s than at price p and state s'. So the market for good 1 cannot clear at both s and s'. At BE with full revelation, beliefs are just randomizations over competitive prices. If these are unique, we must have a REE. $\square$

3.5. Example 5: Non-Existence of Beliefs Equilibrium

Rational expectations equilibria do not always exist. Here we confirm that Beliefs Equilibria may not exist also. The following example is essentially that of Kreps (1977). This presentation follows Allen (1986).

Suppose that there are two states of nature, so that $s \in S = \{s_1; s_2\}$; $\pi(s_1) = \pi(s_2) = \frac{1}{2}$. Preferences are

$$U^1(x^1(s_1); y^1(s_1); s_1) = \frac{1}{3} \ln x^1(s_1) + \frac{2}{3} \ln y^1(s_1) \tag{3.8}$$

$$U^1(x^1(s_2); y^1(s_2); s_2) = \frac{2}{3} \ln x^1(s_2) + \frac{1}{3} \ln y^1(s_2) \tag{3.9}$$

$$U^2(x^2(s_1); y^2(s_1); s_1) = \frac{2}{3} \ln x^2(s_1) + \frac{1}{3} \ln y^2(s_1) \tag{3.10}$$

$$U^2(x^2(s_2); y^2(s_2); s_2) = \frac{1}{3} \ln x^2(s_2) + \frac{2}{3} \ln y^2(s_2): \tag{3.11}$$

Agent 1 observes the state $\sigma^1(s) = s$ and agent 2 observes an uninformative signal $\sigma^2(s) = 0$ for $s \in S$. State-independent endowments are

$$e_1 = [1; 0] \quad e_2 = [0; 1]$$

and prices are $[1; p]$. This specifies our Economy 5.

Proposition 3.18. There is no beliefs equilibrium in Economy 5.

Proof: Suppose δ is a self-fulfilling belief, and $p \in P_*(\delta; s_1) \cap P_*(\delta; s_2)$. Agent 2's demand at price p is independent of the state s; but 1's demand for good 1 is strictly greater at price p and state s_2 than in state s_1. So the same price p cannot clear the market in both states. Thus we must have full revelation. But if there is full revelation, the price must be $p = 1$ in both states, which contradicts full revelation. $\square$

$\cdots$

It is useful to note, at this point, that our solution concept is distinct from that of Anderson and Sonnenschein (1982). In particular, they allow admissible beliefs to assign positive probability to prices which do not occur, and cannot clear markets in the relevant states. The following illustration is adapted from Anderson and Sonnenschein (1982).

Suppose we were to find prices $p_1; p_2$ such that agent 2 believes that both prices can occur in both states, whereas we require, at equilibrium, only that they occur in some state. In particular, let β_i summarise agent 2's posterior beliefs as follows:

$$\beta_1 = \mathrm{Prob}(s = s_1 | p = p_1); \quad \beta_2 = \mathrm{Prob}(s = s_1 | p = p_2):$$

It follows that we can find

$$p_1 = \frac{2}{1 + \beta_1}; \quad p_2 = \frac{1}{1 - \beta_2}:$$

These prices are distinct as long as $\beta_1 + 2\beta_2 \neq 1$. This restores the existence of equilibrium, in situations where beliefs assign positive probability to impossible events (i.e. either $\beta_1 < 1$ or $\beta_2 > 0$ or both.)

3.6. Example 6: The Importance of Join Measurability

Consider another two state economy, but now suppose that neither agent has any information. Thus $S = \{s_1; s_2\}$; $\pi(s_1) = \pi(s_2) = \frac{1}{2}$. Preferences are given by

$$U^1(x^1(s_1); y^1(s_1); s_1) = \frac{1}{2}\ln x^1(s_1) + \frac{1}{2}\ln y^1(s_1) \tag{3.12}$$

$$U^1(x^1(s_2); y^1(s_2); s_2) = \frac{1}{2}\ln x^1(s_2) + \frac{1}{2}\ln y^1(s_2) \tag{3.13}$$

$$U^2(x^2(s_1); y^2(s_1); s_1) = \frac{2}{3}\ln x^2(s_1) + \frac{1}{3}\ln y^2(s_1) \tag{3.14}$$

$$U^2(x^2(s_2); y^2(s_2); s_2) = \frac{1}{3}\ln x^2(s_2) + \frac{2}{3}\ln y^2(s_2): \tag{3.15}$$

Both agents observe uninformative signals $\sigma^h(s) = 0$ for all h and $s \in S$. Again, state-independent endowments are $e_1 = [1; 0]$ and $e_2 = [0; 1]$; and prices are $[1; p]$. This specifies our Economy 6.

The unique beliefs equilibrium of this economy has $\delta^h(1|s) = 1$ and $x^h(s) = \frac{1}{2}$ for all h and s. This is a consequence of the measurability condition on the price process which requires $\delta^h(\cdot|s_1) = \delta^h(\cdot|s_2)$ for each h, since no one can distinguish states s_1 and s_2.

However, if the measurability assumption was relaxed, we would have a continuum of common belief equilibria (parameterized by $\alpha \in [0; 1]$) where $\delta\left(\frac{3}{4-2\alpha}\middle|s_1\right) = 1-\alpha$, $\delta\left(\frac{3}{2+2\alpha}\middle|s_1\right) = \alpha$, $\delta\left(\frac{3}{4-2\alpha}\middle|s_2\right) = \alpha$, and $\delta\left(\frac{3}{2+2\alpha}\middle|s_2\right) = 1-\alpha$. To check, observe that 1's demand for good 1 is always $\frac{1}{2}$. If 2 observes price $\frac{3}{4-2\alpha}$, he attaches probability $1-\alpha$ to state s_1, and thus demands $\left(\frac{2}{3}(1-\alpha)+\frac{1}{3}\alpha\right)\frac{3}{4-2\alpha} = \frac{1}{2}$. If 2 observes price $\frac{3}{2+2\alpha}$, he attaches probability α to state s_1, and thus demands $\left(\frac{2}{3}\alpha+\frac{1}{3}(1-\alpha)\right)\frac{3}{2+2\alpha} = \frac{1}{2}$.

Such indeterminacy of equilibria comes from a different source from that identified in Examples 2 and 3. In those examples, uncertainty about others' signals allowed information to be hidden. Here, it is simply correlation between completely unobserved fundamentals and the price process.

McAllister (1990), in a somewhat different setting from ours, shows existence of "admissible beliefs", which are essentially the same as our beliefs equilibria. However, he allows prices to depend on information which no one knows and requires a regularity condition (on page 351) which ensures that sufficient payoff relevant information is not known by anybody.

4. Remarks

We have demonstrated, by means of examples, that, in the presence of asymmetric information, rational expectations must be assumed rather than deduced. The examples point to a number of issues discussed below.

4.1. Justifying Rational Expectations

We believe that this research agenda is important in evaluating the significance of the Rational Expectations Hypothesis. In our definitions, we unpacked this assumption into three parts.

First, following McAllister (1990), there is the requirement agents only put positive probability on prices that clear markets (this defines Beliefs Equilibria). Morris (1995) gives a formal argument why this is exactly what is entailed by assuming common knowledge of market clearing and price taking behaviour. Second, there is the requirement that there is consensus (common probability distributions) about the price process (adding this gave Common Beliefs Equilibria). Finally, degeneracy (the restriction that there is a unique possible price for each state) is required to generate Rational Expectations Equilibria. Our Examples 2 and 3 demonstrated that, in the presence of asymmetric information, rational expectations must be assumed rather than deduced. But we used examples to show that in some circumstances market clearing and consensus (Example 1) and market clearing alone (Example 4) are sufficient for rational expectations.

4.2. The Broader Hypothesis of Rational Expectations

The assumption of rational expectations is used in two quite different contexts. In sequence economies, where markets open at different points of time, the REH entails that participants know the market clearing prices which will occur at future dates, possibly infinitely far into the future. In addition, it is used to define equilibria in economies with asymmetric information, where participants are assumed to know what prices would have been in every information state. Guesnerie (1989) analyses situations where the REH can be deduced from weaker rationality restrictions (in his language, when rationalizable expectations imply the REH). That work clearly suggests that there are circumstances where it can be so deduced, and others where it cannot. We are concerned with the same kind of question in the context of REE with asymmetric information. It is of some interest to study whether solution concepts corresponding to CBE can be rationalized in situations where REE cannot. Kurz (1994) analyses the properties of rational beliefs in sequence economies; rational beliefs are those which cannot be contradicted by past data. This is similar to the notion used by Anderson and Sonnenschein (1982) in the context of static economies with asymmetric information. As we have pointed out before, our rationality restriction is more demanding, as Example 5 demonstrates.

4.3. Consensus and Common Priors

The notion of CBE is stricter than that of BE, because it imposes consensus as a further restriction. The common prior assumption is certainly an important part of the REH; it is often appealed to as a "natural" assumption in economic modelling with uncertainty (see Aumann (1987) and Morris (1993)). Some justification is made by the logic that participants who observe the same data will eventually agree on their models of the world. One of the messages of our analysis is that (in this context) this argument is circular. In order to learn the same things from the data, it is necessary that their models are the same in the first place. In addition, probability distributions are unlikely to be learned fully with finitely many data points, except in very special circumstances. We have restricted participants to have the same priors for exogenous states, so that $\pi \in \Delta(S)$ is common to all, even in BE; for CBE, they also agree on the model of prices, conditional on states. While the common priors restriction is useful in imposing intellectual discipline in the analysis of some economic phenomena, it may well cloud our understanding of others, such as equilibrium price variability, as Example 1 suggests. Lack of consensus presents rather different issues with degenerate beliefs (e.g. Townsend (1978)), which we have not analysed here.

4.4. Sunspot Equilibria

Cass and Shell (1983) introduced the idea of "sunspot equilibria" where prices may depend on payoff-irrelevant or extrinsic states. It is well-known that all equilibria of a model of equilibrium price uncertainty with common priors, such as CBE, are formally equivalent to the some equilibrium of a model with sunspots.

It su±ces to index the variation of prices within states by a random variable, i.e. augment the state space, and call this random variable the realization of a such a sunspot. Let us consider the connection in the context of our model. The equilibrium of example 2 identi¯ed in the proof of Lemma 3.10 would also be a rational expectations equilibrium of the economy with an expanded state space. For example, if we added high or low sunspot activity as part of the description of the state, and chose the appropriate probabilities of the sunspot states, we could construct an equilibrium where the price was fully revealing if there was high sunspot activity and non-revealing if there was low sunspot activity. However, this re-interpretation itself raises a couple of questions.

Importantly, the sunspot an equilibrium phenomenon. Arbitrary sunspots will not work in a ¯nite state setting. Radner's (1979) generic revelation theorem will presumably continue to hold for a generic choice of sunspots and sunspot probabilities. Translated into the language of sunspots, our result would say that for a robust class of economies, there exists a set of sunspots and a (non-generic) probability distribution over sunspots for which there is a non-revealing equilibrium. Put another way, not every distribution is an equilibrium distribution. While every CBE can be reinterpreted as a sunspot equilibrium (and every BE as a sunspot equilibrium with imperfectly correlated signals) we would prefer to understand them as describing competitive equilibrium price distributions.

On the other hand, if a continuum of sunspots is permitted, it is not restrictive to ¯x the sunspot states ex ante.

Another problem with the sunspot re-interpretation is that the usual de¯nition of rational expectations equilibrium requires that prices cannot reveal any information which is not known to any agent. Notice that the re-interpretation requires that it does: in the example, each agent learns from the price whether sunspot activity is high or low, something no one knew in advance. Notice that our de¯nitions implicitly assumed that prices cannot reveal anything about fundamentals that is not known by all agents collectively, but allows prices to reveal extrinsic information.

Finally, note that our result ¯ts well with the intuition that imperfections are required for sunspots. Cass and Shell (1983) showed that sunspots are irrelevant to competitive equilibria when all agents agree on the probability distribution and there are complete markets. A folk theorem in that literature holds that sunspot equilibria exist if there are \imperfections" in the market. Disagreement on probabilities is one such imperfection. In our common belief equilibria, this disagreement is caused by di®erent information, not di®erent priors. Thus non-revelation of information is necessary to sustain di®erences in beliefs and thus sunspots. But the sunspots themselves are necessary for the non-revelation.

4.5. Noisy Rational Expectations

Our results have a clear connection with models of \noisy rational expectations". The early generic revelation results (Radner (1979) and Allen (1981a)) ran counter to many economists' intuitions, and there were a number of attempts to alter the model in ways that got around full revelation but still allowed existence (Allen (1985) and Anderson and Sonnenschein (1982)). There was some dissatisfaction with the fact that this noise, which prevents the revelation of information by prices, is introduced in arbitrary ways, often at odds with full rationality. One interpretation of our results is that we show how it is possible to introduce \noise" endogenously with no appeal to either incomplete rationality or unexplained exogenous factors.

4.6. Robust Partially Revealing Equilibria

Prices appear to convey information but not to fully reveal all information. This, at least, seems to be intuition of many who have studied rational expectations equilibria. Yet it is extremely hard to come up with robust examples of economies with partially revealing REE: see Ausubel (1990). We have shown that allowing endogenous noise makes it easy to justify partial revelation. Indeed, enlarging the class of equilibria to include CBE has the added advantage that non-trivial CBE which hide information occur in economies where fully revealing REE exist, which allows for our standard intuition about price-stabilization being welfare improving. This is obviously not a universal property, but may allow for relatively simple characterization.

4.7. Multiple Equilibria

It is sometimes argued that multiple competitive equilibria present a problem for the Rational Expectations Hypothesis (e.g. Hahn (1991)). Agents who \know" the structure of the economy can deduce the correspondence from states to prices. Their happening to agree on a degenerate selection must be an act of faith. That particular criticism can be incorporated in many ways; for example, Allen, Dutta and Polemarchakis (1994) show that with enough markets, prices will reveal the selection of equilibrium, so that the REH can be a consequence of rationality and complete markets at least in sequence economies. Consensus is necessary, but degeneracy is a consequence.

In economies of asymmetric information, the argument has force, as Example 3 suggests. In particular, randomizations over multiple REE generate equilibrium price distributions. Interestingly, that economy has other CBE as well: this has some similarity to Aumann's (1987) demonstration that the payo®s of correlated equilibria can lie outside the convex hull of the Nash equilibrium payo®s.

4.8. Indeterminacy of Equilibria

Examples 2 and 3 establish that there are a continuum of CBE in such economies. Recently, the possibility of competitive equilibria being indeterminate in economies with incomplete markets and nominal assets (e.g. Cass (1985) and Geanakoplos and Mas-Colell (1989)) has raised perplexing questions. Our results suggest that in economies where CBE exist, they are likely to be indeterminate, because the market-clearing conditions impose a relatively small number of moment restrictions on the equilibrium probability distribution, which is a much higher-dimensional object. The source of this indeterminacy appears to be rather di®erent from that which arises with incomplete markets, though this clearly needs a more precise technical apparatus than we use.

It is useful to note, in this context as well as that of sunspot equilibria, that the usual argument with incomplete markets is that at di®erent equilibria, agents choose di®erent transfers of revenue across states. In consequence, the distributions of endowments across states are distinct, which support distinct spot market equilibria. In our analysis, the corresponding work is done by preferences, rather than endowments. In economies with asymmetric information, agents recover information from prices. At distinct equilibria, their posterior probabilities are distinct, so that their expected utilities, and actions are distinct as well.

Polemarchakis and Siconol¯ (1993) explore the possibility of the simultaneous existence of fully revealing, partially revealing, and fully non-revealing equilibria in an incomplete markets economy where competitive equilibria are indeterminate. The initial part of their argument is similar to Geanakoplos and Mas-Colell (1989), where nominal assets are the source of this indeterminacy. Selections from this set of equilibria are associated with ¯rst period asset prices which can have arbitrary information content. In our analysis, the source of the indeterminacy is precisely that agents recover information from prices, and that different price distributions, which differ in information content, are legitimate equilibria.

4.9. Further Work

In this paper, we used examples to examine the relationship between alternative solution concepts for competitive economies with asymmetric information. This work suggested (at least) three areas of further research.

- The clearest setting where generic revelation of information occurs in rational expectations equilibria is the finite state case. We hope to characterize the degree of indeterminacy of common belief equilibria in this setting.

- We saw in section 3.5 that generalizing from REE to beliefs equilibria does not solve the existence problem. However, this failure was for a non-generic economy. When the dimension of states equals the dimension of prices, not only does generic revelation break down, but so does generic existence (Jordan and Radner (1982)). We plan to investigate when CBE exist in this setting.

- We showed in this paper that in a simple, robust class of economies rationality and market clearing do not imply rational expectations equilibria. However, we also identified classes of economies where all common belief equilibria were REE and where all rationalizable belief equilibria are REE. In doing so, we identified settings where consensus and unique prices are consequences (not assumptions) of the model. It should be possible to give general conditions for these results.

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