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Financial Innovation and Expectations[*]: Endogenous Incompleteness and Real Indeterminacy.

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Abstract

This paper analyzes an incomplete …nancial markets model with price-taking utility-maximizing …nancial innovators and no-short sales restrictions. It is shown that, given the indeterminacy of the no arbitrage price conjecture of innovators, …nancial markets can remain incomplete in equilibrium. As a consequence, real indeterminacy of degree at least equal to int (S/2)(S-(S/2)) results, where S is the number of spots in the future. The dimension of innovators' beliefs giving rise to I newly introduced …nancial assets is I(S-I), with an equal degree of real indeterminacy.

1. Introduction

In the standard model of general equilibrium with incomplete …nancial markets, the set of (insu¢ciently many) available assets is exogenously given. This note is an attempt to make the asset characteristics a result of the maximizing behavior of individuals. In particular, if one could show that markets are endogenously incomplete, this would add robustness to the well-known results on equilibria with incomplete markets. The lack of existence, generic local uniqueness or Pareto

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optimality of equilibrium allocations, and the presence of sunspot equilibria crucially depend on the absence of spot contingent …nancial trades. Moreover, once some markets are missing, the equilibrium set strongly depends on the particular pro…le of asset payo¤s (asset payo¤ matrix), and not just on the number of assets with uncorrelated payo¤s (its rank).

There are many studies now available on endogenous asset formation, making various assumptions on the innovation process. The following is not meant to be a complete list of papers or hypotheses, but it captures the main features of a group of models including, among others, works by Allen and Gale ([1] and [2]), Du¢e and Jackson [7], Pesendorfer [11] and Bisin [5].[1] Innovators choose a …nancial structure before investors make their trading plans, and have complete knowledge of the resulting competitive equilibria. In this two-stage game, they maximize pro…ts at time zero, then disappear. There is a cost of innovation for the innovator (cost of producing and marketing) and possibly for the buyer (commissions and "copyrights"). As a warranty to honor the payment related to the newly issued asset, each innovator has to provide a collateral in terms of preexisting assets and/or goods owned or produced. Bisin allows a bid-ask spread to be charged by the innovator.

Some of these assumptions are departures from the competitive paradigm on which models of incomplete markets are usually built. In this note we try to analyze the problem of innovation in a general equilibrium framework. Individuals maximize, markets clear and, in particular, innovators are price takers. We do believe that in certain cases innovators are aware of the strategic e¤ects of their choices. More generally, a strategic approach to some aspects of the problem may be useful to capture some "realistic" features of …nancial innovation. On the other hand, a pure competitive approach may help focus on reasons for market incompleteness other than monopolistic or oligopolistic behavior. Transaction costs or asymmetric information are also already known to cause lack of …nancial markets. Nevertheless, many models of incomplete markets display symmetric information and zero transaction costs. We will show that, in this case, market incompleteness is essentially a problem of expectations and of coordination of beliefs among innovators. When innovators are price takers, and there are no frictions in the market, …nancial markets can turn out to be incomplete. This result would extend, in our opinion, previous work on endogenous asset creation by Allen and Gale [1], and on real indeterminacy of equilibria by Balasko and Cass [4].

[1] A symposium issue of JET is now dedicated to …nancial innovation. For a survey of the topics emphasized in that issue, see Du¢e and Rahi [8]. A more extended treatment of the subject is found in Allen and Gale [3].

Therefore we choose to model innovation in a simple, impersonal and frictionless way.

An asset can be “produced”, then sold and purchased at no cost. When an asset is introduced in the economy, everyone can freely trade it. No short sales restrictions are imposed, and no collateral is required. Innovation is simply the addition of a yield vector to a preexisting yield matrix. An asset is introduced if it is marketed (purchased or sold) by the innovator.[2] The innovator’s incentive to innovate is embedded in the desire to enlarge the space of attainable wealth transfers in the future. There is no explicit pro…t maximizing motive, contrary to the maintained assumption in the …nance literature. This choice derives from the problem of de…ning pro…t maximization when …nancial markets are incomplete, a problem bypassed in the existing literature by assuming that innovators disappear after introducing the securities. Our innovators are utility maximizers, and can be thought of as individuals or single-ownership …rms.

An interesting feature of the process of …nancial innovation is the mechanism of price determination of the newly introduced assets. Innovators must have beliefs about objects which may not be introduced in equilibrium (like their own, possibly new, asset). In our model each innovator has beliefs about the prices of all goods and about prices and payo¤s of assets introduced by other innovators. Then, consistently with these beliefs, the innovator forms further beliefs on the relationship between yields and prices of the assets she plans to issue, essentially following a no-arbitrage rule. Since these beliefs are based on all the information available to the consumer/innovator at the time she makes a decision, they are rational. In equilibrium, individuals maximize, markets clear and beliefs are consistent (see De…nition 2.1, and preceding comments).

A wide-spread interpretation, especially in the macroeconomic literature, of how equilibria arise in competitive models of …nancial markets with uncertainty is that individuals know the model and can therefore compute the equilibrium price correspondence as the modeler does. This is one justi…cation of rational expectations equilibria. The sunspots literature has cast doubt on it. It has shown how ”crazy” expectations can be self-ful…lling, i.e., consistent with market clearing, and how the huge multiplicity of equilibria so people couldn’t possibly be solving the model to come out with equilibrium prices. That literature also stresses the vast role of beliefs in explaining economic phenomena. The equilibrium beliefs can be rational, in the sense of being based on the best use of the information available to individuals, and consistent. But beliefs do not have to be based on the knowledge of the whole economy. We think that price-taking behavior should

[2] We will assume away the case of innovation that is not bene…cial to the asset designer, by postulating a sunk cost related to this activity.

convey the idea that individuals have very little information about the economy, besides prices and other common variables. This captures the idea of decentralization of decisions, which is a fundamental reason for being interested in the study of competitive markets.

Therefore this different, but plausible, interpretation can be given. Each household knows its preferences and endowments. To solve the maximization problem, it forms beliefs about prices. If these beliefs happen to be equal to the equilibrium prices, then households' behavior leads to market clearing.[3]

The idea of limited information is even more appealing in the case of financial innovation. Indeed, it seems plausible to assume that financial intermediaries have limited information about the economy. But the existing literature modelling innovation as a strategic game has not yet dealt with the problem of incomplete information.[4] In this sense the informational requirement of our innovators is weaker. Moreover, price-taking innovators cannot compute the equilibrium price correspondence starting from the knowledge of the individual demand schedules, as functions of prices and asset payoffs. This is because the innovator's problem is not well defined unless the innovator has a conjecture about the relationship between y, the vector of the possibly-introduced asset payoffs, and q, its price, for given q. One can choose a conjecture based on the individual knowledge of the, say $j-th$, innovator. In this case, one cannot escape the arbitrariness of the conjecture, and the only reasonable restriction (not arbitrary) is that this conjecture $f_j : \Re^S \to \Re$, where S is the number of contingencies in the future, be consistent with the no arbitrage condition, i.e.,

$$f_j(y^j) = \delta_j y^j \text{ for } \delta_j \in \left\{ \delta \in \Re^S \mid q^{\backslash j} = \delta Y^{\backslash j} \right\}$$

given beliefs about $Y^{\backslash j}$ (the choice of the other innovators), and the other asset prices. δ_j is the vector of previsions over the implicit prices of wealth in the future.

This conjecture will be the same for all the innovators only in equilibria where markets are complete, otherwise not, at least typically. We will assume that δ_j be constant, although more generally it could depend on all the variables known by j. At the current stage, we believe that the results of the analysis wouldn't be altered by allowing more general functions for δ_j, as long as they are restricted

[3] Of course, this story does not explain why we observe equilibrium, and in this respect leaves the competitive equilibrium concept on the same level as the Nash equilibrium. Contrary to the Nash solution, it does not say what happens if beliefs do not cause markets to clear.

[4] In Allen and Gale [2], for instance, innovators select out of a given price system the equilibrium price corresponding to a certain financial structure. The price system is perfectly known, meaning that the innovators know the economy.

to depend on variables observed by the innovator. The generality of the function only translates into a slightly more elaborated formal argument, which does not affect the logic of the model.

Given our framework, we show that no innovation, partial innovation and innovation which makes markets complete are all possible equilibrium situations, at least typically in the space of endowments, which parametrize these economies. More precisely, for any possible number I of introduced independent assets, the set of equilibrium allocations exhibits a degree of real indeterminacy equal to the number of missing markets, $S - I$, times these newly introduced assets, I. This is the dimension of all the possible combinations of new assets to be introduced, provided only I turn out to be independent. Moreover, the set of equilibrium beliefs exhibits the same degree of indeterminacy. So, in particular, we show that almost all possible return matrices can be supported by appropriate beliefs held by the innovators, and market incompleteness is relatively likely to arise in equilibrium.

As a corollary to this approach and to the results in Cass and Citanna [6], typically private financial innovation may imply either Pareto improvements or Pareto impairments.

Section 2 introduces the model and transforms it into a more manageable form. Section 3 contains the existence result. Section 4 characterizes the effects of innovation and the likelihood of beliefs that give rise to incomplete markets.

2. Set-up of the Model.

We consider a competitive two-period exchange economy with numéraire assets[5] and uncertainty. We assume that there are S, $S > 1$, possible states of the world in the second period. Spot commodity markets open in the first and second period, and there are C, $C > 1$, commodities in each spot, labelled by $c = 1, 2, \ldots, C$. We label each spot by $s = 0, \ldots, S$, spot zero corresponding to the first period. There are K agents, labelled by $k = 1, 2, \ldots K$.

Partition the set of all households in two subsets H and J: H is the set of pure households, who only consume and invest; J is the set of households who can introduce new assets, i.e., the set of potential innovators. More precisely, innovator $j \in J$ can introduce a new asset whose yield matrix is denoted by y^j and whose price vector q^j has to be determined in equilibrium - as well as all

[5] We could as well treat the case of real assets. The nature of the results would not be affected by the assets characteristics, while the proofs would require a more complex mathematical language, essentially following Duffie and Shafer [9]. We believe this would confuse the reader instead of helping keep the focus on innovation and the role of beliefs.

other prices.

We assume that there are:

A1 no preexisting assets,

A2 S potential innovators, i.e., $J = S$;

A3 one potentially introduced asset per innovator.

We conjecture that none of the above assumptions is really restrictive. Everything we prove below would go through even if there were preexisting assets. We need to have at least S potential innovators in order to leave open the possibility of completing the markets. As for the number of potentially introduced asset per innovator, it will be clear from the analysis below that in this model it is in the innovator's best interest to introduce an asset that matches the value of her excess demand in each spot, and using more than one instrument to accomplish that is not going to make any difference. Anyway, we should hasten to say that the assumptions are to be considered as a starting point for the analysis, and a more careful study of the implications of dropping each one of them is needed.

We assume that there is no cost of innovation for either innovator or pure household. The innovation consists in "a costless addition of new columns to the yield matrix".

With abuse of notation, we define $\mathcal{K} = \{1, ..., K\}$, $\mathcal{J} = \{1, ..., J\}$, and $\mathcal{H} = \{J+1, ..., J+H = K\}$. Each innovator can introduce an asset with vector yield $y^j = \left(y^{s,j}\right)_{s=1}^S$ and price q^j.

Define

$$Y^{\setminus j} = \left(\left(y^{j'}\right)_{j' \neq j}\right)_{S \times (J-1)}; \quad q^{\setminus j} = \left(q^{j'}\right)_{j' \neq j}; \quad q \equiv (q^i)_{i=1}^J$$

$b_j^{\setminus j}$ as the demand of asset other than j' by innovator j; $b_j \equiv \left(b_j^{\setminus j}; b_j^j\right)$;

b_h^j as the demand of asset j by household h; $b_h \equiv \left(b_h^j\right)_{j \in \mathcal{J}}$; $b = (b_j; b_h)_{j \in \mathcal{J}, h \in \mathcal{H}}$.

In the first period, innovators decide the offer of new assets, and plans are made for consumption and investment in period two. First-period commodities and assets are then exchanged and first period consumption takes place. Then uncertainty is resolved, assets pay their yields, goods are exchanged and finally agents consume second-period commodities. $x_k^{s,c}$ is the consumption of commodity c in state s by agent k; similar notation is used for the endowments, $e_k^{s,c}$. Both consumption and endowments are elements of $\Re_{++}^G$ for each agent, where $G = (S+1)C$. Let $E = \Re_{++}^{GK}$. We assume that assets pay in each state in units of a numéraire commodity, which is assumed to be the first commodity in each spot.

The following notation is also used:

$x_k^s \equiv (x_k^{s,c})_{c=1}^C$, $x_k \equiv (x_k^s)_{s=0}^S$, $x \equiv (x_k)_{k=1}^K$, with the obvious meaning,
$u_k : \mathbb{R}_{++}^G \to \mathbb{R}$, the utility function of household h, $u \equiv (u_k)_{k=1}^K$,
$p^{s,c}$, the price of commodity c in spot s, $p \equiv (p^s)_{s=0}^S$, the commodity price vector,

b_k^j; the demand of asset j by agent k.

Price vectors and gradients are row vectors; all other vectors are column vectors.

Let Y be the $S \times J$ return matrix given by

$$Y \equiv \left[y^{s,j} \right]_{s,j}$$

For each asset price vector q, it is convenient to consider an $(S+1) \times I$ matrix R(q) given by

$$R(q) \equiv \begin{bmatrix} -q \\ a:Y \end{bmatrix}$$

where

$$a: = \begin{bmatrix} p^{1,1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & p^{1,S} \end{bmatrix}.$$

As before, $R^{\backslash j}(q^{\backslash j})$ represents R(q) without the $j-$th component (the $j-$th column).

We introduce the following standard assumptions on preferences.

A4 $u_k(x_k)$ is smooth, differentiably strictly increasing (i.e. $Du_k(x_k) \gg 0$), differentiably strictly quasi-concave and with the closure of the indifference surfaces contained in $\mathbb{R}_{++}^G$.

Therefore an economy is characterized by the endowment vector $e = (e_k)_{k=1}^K \in \mathbb{R}_{++}^{KG}$.

In order to justify our choice of the definition of equilibrium, we first introduce what could be seen as the natural candidate for the innovator's maximization problem.

For $j = 1, ..., J = S$, j solves, given $p, q, Y^{\backslash j}$;

$$\max_{x_j; b_j; y^j} \; u_j(x_j) \quad s.t.$$

$$p^0 \left(x_j^0 - e_j^0 \right) + q^{nj} b_j^{nj} + q^j b_j^j = 0$$
$$-\Pi^1 \left(x_j^1 - e_j^1 \right) + \Pi \cdot Y^{nj} b_j^{nj} + \Pi \cdot y^j b_j^j = 0$$
$$x_j \geq 0$$

where

$$\Pi^1 = \begin{bmatrix} p^1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & p^S \end{bmatrix}_{S \times S} .$$

and where $x_k^1 = (x_k^s)_{s \neq 0}$.

If q^j were unrelated to y^j, the innovator's problem would have no solution. For any given q^j, any y^j would give a lower utility than any $y^{j\prime} \leq y^j$. Loosely speaking, the household would like to push y_s^j to $-\infty$ for any s. In fact, q^j is related to y^j and each innovator has beliefs about the price of the asset she plan to introduce in the economy. More precisely, the innovator has beliefs about the way that price changes when the planned characteristics for the new asset change. Moreover, the price of the new asset has to satisfy the no arbitrage condition, i.e., it must be the case that $q^j = \delta_j \Pi \cdot y^j$ with δ_j such that

$$q^{nj} = \delta_j \Pi \cdot Y^{nj} \tag{2.1}$$

Therefore innovator j believes that if an asset with yield vector y^j is introduced, then its price q^j is going to be $\delta_j \Pi \cdot y^j$. δ_j^s can be interpreted as the belief of household j about the price of an s numéraire Arrow security, that is, an asset which pays 1 unit of the numéraire commodity in state s and nothing in any other state.

Different beliefs give different solutions to the innovator's maximization problem, even for the same (equilibrium) price of the newly introduced asset. We cannot therefore choose any arbitrary belief without loss of generality.[6] The choice set of each innovator changes when δ_j changes in $\Delta = \left\{ \delta_j' \in \mathbb{R}^S_{++} : q^{nj} = \delta_j' \Pi Y^{nj} \right\}$.

[6] This is in contrast with the existing literature. There it is possible to present definite solutions simply because it is decided to ignore the role of the rational beliefs δ and to fix them at some arbitrary value, or it is assumed that innovators know a lot more about the economy. See, for the first case, Allen and Gale [1], Pesendorfer [11], p.16 and 19, and, as a survey, Duffie and Rahi [8]. For the second case, Allen and Gale [2] and Bisin [5], p.9.

Instead of providing a proof of a general statement, we will content ourselves with illustrating this simple proposition by means of an example.

Consider the case with $J = S = 3$, and $\hat{Y}^{n1}, q^{n1}$ are the following:

$$\begin{bmatrix} -q^{n1} \\ \hat{Y}^{n1} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

Then $\Delta_1 = \left\{ \delta_1 \in \mathbb{R}^3_{++} . 1 = \delta_1^1 + \delta_1^2 + \delta_1^3 \right\}$. By assumption, it must be $q^1 = \delta_1 \hat{y}^1$ and therefore, in our example, we have

$$q^1 = \left[\delta_1^1, \ \delta_1^2, \ 1 - \left(\delta_1^1 + \delta_1^2\right) \right] \begin{bmatrix} \hat{y}^{11} \\ \hat{y}^{21} \\ \hat{y}^{31} \end{bmatrix} = \delta_1^1 \hat{y}^{11} + \delta_1^2 \hat{y}^{21} + \left[1 - \left(\delta_1^1 + \delta_1^2\right) \right] \hat{y}^{31}.$$

Therefore, for given q^1, the first innovator can choose y^1 so that y^{11} and y^{21} are arbitrary and

$$\hat{y}^{31} = \frac{q^1 - \left(\delta_1^1 \hat{y}^{11} + \delta_1^2 \hat{y}^{21}\right)}{1 - \left(\delta_1^1 + \delta_1^2\right)}$$

It is clear that the choice of $\hat{y}^{31}$ depends upon the beliefs about δ_1^1 and δ_1^2.

The requirement we impose on δ_j is to be one among those consistent with no arbitrage, and to be a constant function of all other variables known to the innovator. By choosing δ as above, we are consistently assuming that our innovators are rational in the sense explained in the Introduction.

On the basis of the above observations, we can rewrite the $j-$th innovator's maximization problem as:

given p, q^{nj}, Y^{nj} and δ_j satisfying (2.1),

$$\max_{x_j, b_j, y^j} \quad u_j(x_j) \qquad s.t.$$

$$p^0 \left(x_j^0 - e_j^0\right) + \delta_j \Psi . Y^{nj} b_j^{nj} + \delta_j \Psi . y^j b_j^j = 0$$
$$-\Psi_1 \left(x_j^1 - e_j^1\right) + \Psi . Y^{nj} b_j^{nj} + \Psi . y^j b_j^j = 0$$

We will transform the innovator's maximization problem to rule out solutions of the kind: $y^j \neq 0$, $b_j^j = 0$. If we interpret the introduction of a new asset as the design of an asset payoff together with the marketing of the product, this case corresponds to no innovation, since the asset creator is not trading the asset at all. If there is a small cost of producing the asset, this would not be in the innovator's interest. So we can safely assume that the asset cannot be traded by the households, and we modify the innovator's problem in the following way.

$$\max_{x_j; b_j^{nj}; y^j} \quad u_j(x_j) \qquad s.t.$$

$$p^0\left(x_j^0 - e_j^0\right) + \delta_j\, \alpha\cdot Y^{nj} b_j^{nj} + \delta_j\, \alpha\cdot y^j = 0$$
$$-\alpha^1\left(x_j^1 - e_j^1\right) + \alpha\cdot Y^{nj} b_j^{nj} + \alpha\cdot y^j = 0$$

where essentially $b_j^j = 1$; and this for all j. Of course, the asset market clearing equations are modified as well. Now we can give the definition of equilibrium we will use in this paper.

Definition 2.1. $(x; Y; b; p; q; \delta) \in \mathbb{R}^{GK}_{++} \times \mathbb{R}^{S^2} \times \mathbb{R}^{HS+S(S-1)} \times \mathbb{R}^{G-1} \times \mathbb{R}^{S} \times \mathbb{R}^{S(S-1)}$ is a FEPI (Financial Equilibrium with Potential Innovation) at $e \in \mathbb{R}^{GK}_{++}$ if

for $j = 1, \ldots, S$, j solves, given p, q^{nj}, Y^{nj}, and δ_j satisfying (2.1);

$$\max_{x_j; b_j^{n}; y^j} \quad u_j(x_j) \qquad s.t.$$

$$-p^0\left(x_j^0 - e_j^0\right) + \delta_j\, \alpha\cdot Y^{nj} b_j + \delta_j\, \alpha\cdot y^j = 0$$
$$-\alpha^1\left(x_j^1 - e_j^1\right) + \alpha\cdot Y^{nj} b_j + \alpha\cdot y^j = 0$$

for $h = 1, \ldots, H$, h solves, given p, q, Y;

$$\max_{x_h; b_h} \quad u_h(x_h) \qquad s.t.$$

$$-p^0\left(x_h^0 - e_h^0\right) + q b_h = 0$$
$$-\alpha^1\left(x_h^1 - e_h^1\right) + \alpha\cdot Y b_h = 0$$

and

$$q = \delta_j\left(\alpha\cdot Y\right) \qquad \text{for } j=1,\ldots,S$$
$$\sum_{k=1}^{K}\left(x_k^{n01} - e_k^{n01}\right) = 0$$
$$\sum_{k\neq j} b_k^j + 1 = 0 \qquad \text{for } j=1,\ldots,S.$$

We can describe the expectation formation in this economy as follows. When time begins each innovator has beliefs about prices of goods p; prices and yields of the other innovators' assets (q^{nj}, Y^{nj}); she forms beliefs δ_j about the prices of the numéraire Arrow securities in a way consistent with the no arbitrage condition; she finally solves her maximization problem. Each pure household has beliefs about $(p; q; Y)$ and solves its maximization problem. Equilibrium prices are prices at which households maximize and markets clear, and which are consistent with individual beliefs. One can think of beliefs over $\left(p; q; Y^{nj}\right)$ as caused by the observation of those variables, and made possible by an "auctioneer", if one prefers to relate our concept of equilibrium to some institutional setting.

Remark 2.1. Observe that we normalized $p^{01} = 1$ and $\delta_1 = \underline{1}$; where now $\underline{1} = (1; 1; ::: ; 1)$; a vector of the appropriate dimension.

Remark 2.2. The budget set of the innovator's problem is not compact. In fact, the budget constraint can be rewritten as

$$\Psi(x_j - e_j) - R^{nj}(q^{nj}) b_j^{nj} - \begin{bmatrix} -\delta_j \\ I \end{bmatrix} \Psi \cdot y^j = 0.$$

But then it would be as if there were $(S + J - 1) + S$ assets for S states and $(S + J - 1)$ of them were redundant. Therefore for any choice of $(x_j; b_j; y^j)$ which satisfies the above budget constraint, for any $c^n \in \Re; c^n \to \infty$; $(x_j; b_j^1 + c^n; b_j^2; ::: ; b_j^{S+J-1}; y^j - y^1 c^n)$ still satisfies the budget constraint for any n. Nevertheless, since u_j is differentiably strictly quasi concave and therefore pseudo-concave, the constraints are linear, the derivative of the budget constraints with respect to x_j is Ψ -which has full rank- and $\lambda_j \gg 0$; necessary and sufficient conditions for Lagrange Theorem are satisfied. Therefore if a solution to Lagrange conditions exists, it is a solution to the maximization problem too.

Finally, for any given $p; q; Y$; the budget set of household h is compact. Therefore, from Assumption A4, we know that the demand map is a function. Since a change in Y may change the rank of the yield matrix faced by household h, the demand function is not continuous in Y. Still, the maximization problem is well known to satisfy necessary and sufficient conditions for the Lagrange theorem. In the next section, we will use this observation to show the existence of a FEPI.

3. Existence of FEPI.

Let

$$\Xi = \Re^{HG}_{++} \times \Re^{H(S+1)}_{++} \times \Re^{HS} \times \Re^{SG}_{++} \times \Re^{S(S+1)}_{++} \times \Re^{S^2}_{++} \times \Re^{S^2} \times \Re^{G-(S+1)}_{++} \times \Re^{S}$$

be the space of endogenous variables with typical element

$$\xi = \left((x_h; \lambda_h; b_h)_{h \in H}; (x_j; \lambda_j; b_j; y^j; \delta_j)_{j \in J}; p; q \right):$$

Define the equilibrium system in the model with potential innovation through the function

$$F_{eq} : \Xi \times E \to \Re^{\dim \Xi};$$

$$F_{eq}(\xi; e) = \begin{pmatrix} \cdots \\ Du_j(x_j) - \lambda_j \Psi(p) \\ -\Psi(p)(x_j - e_j) + R^{\setminus j}(q^{\setminus j}) b_j^{\setminus j} + \begin{bmatrix} -\delta_j \\ I \end{bmatrix} \Psi \cdot y^j \\ \lambda_j \left\langle \left[R^{\setminus j}(q^{\setminus j}) \right] \right\rangle \\ \lambda_j \begin{bmatrix} -\delta_j \\ I \end{bmatrix} \\ \cdots \\ Du_h(x_h) - \lambda_h \Psi(p) \\ -\Psi(p)(x_h - e_h) + R(q) b_h \\ \lambda_h [R(q)] \\ \cdots \\ q - \delta_j \Psi \cdot Y \\ \sum_k x_k^n - e_k^n; \\ \sum_{k \neq j} b_k^j + 1; \text{ for any } j \in J: \end{pmatrix} \tag{3.1}$$

To prove existence of a FEPI, we cannot apply the general, standard strategy used in models of smooth economies using function (3.1). More precisely, we cannot apply either a fixed point or a degree argument to solve the system of aggregate excess demand smooth functions (which may or may not be continuous, or even well-defined), and function (3.1) may not cut out a manifold, when some assets become redundant. It is nevertheless possible and sufficient to our purpose to construct equilibria via brute force upon equilibria of related and simpler economies. We write the equilibrium system in the standard model of incomplete markets, with a fixed payoff matrix $\hat{Y}$. Let

$$\Xi = \mathbb{R}^{KG}_{++} \times \mathbb{R}^{K(S+1)}_{++} \times \mathbb{R}^{G-(S+1)}_{++} \times \mathbb{R}^{KI} \times Q$$

be the space of endogenous variables with typical element $\xi = (x, \lambda, p, z, q)$, where

$$Q = \left\{ q \in \mathbb{R}^{I} : \text{there exists } \delta \in \mathbb{R}^{S}_{++} \text{ for which } q = \delta \Box \hat{Y} \right\}.$$

Define

$$F^{I}_{eq} : \Xi \times E \to \mathbb{R}^{\dim \Xi},$$

$$F^{I}_{eq}(\xi; e; \hat{Y}) = \begin{pmatrix} \vdots \\ Du_k(x_k) - \lambda_k \Box(p) \\ -\Box(p)(x_k - e_k) + R(q) z_k \\ \lambda_k R(q) \\ \vdots \\ \sum_k x^{n}_k - e^{n}_k \\ \sum_k z_k \end{pmatrix}. \tag{3.2}$$

with normalization $p^{0,1} = 1$ and $\lambda^{s}_{1} / \lambda^{0}_{1} = 1$; for all $s > 0$.

Theorem 3.1. For any $I = 0, 1, \ldots, S$, there exists an open and full measure set E^{*}_{I} in E and for all $e \in E^{*}_{I}$, there exists an open and full measure subset $\Theta^{*}_{e,I}$ of $S \times I$ − dimensional matrices with full rank, such that when $Y \in \Theta^{*}_{e,I}$, a FEPI exists where I linearly independent assets are introduced, corresponding to the matrix Y. If $I = 0$, then $E^{*}_{I} = E$.

Proof.

First, consider $I \neq 0$.

It is well-known that there exists an open and full measure set D^{*}_{I} in $E \times \mathbb{R}^{SI}$ such that if $(e, \hat{Y}) \in D^{*}_{I}$, then $b^{j}_{k} \neq 0$ for any j and k, in any equilibrium of the corresponding numéraire asset market model in which $\hat{Y}$ is exogenously given. Projecting D^{*}_{I} onto E, we get an open and full-measure subset E^{*}_{I} of E, such that for all $e \in E^{*}_{I}$ there exists $\Theta^{*}_{e,I} \subset \mathbb{R}^{SI}$, open and full measure where, for all $\hat{Y} \in \Theta^{*}_{e,I}$, all the numéraire asset model equilibria corresponding to $(e, \hat{Y})$ have the property $b^{j}_{k} \neq 0$ for any j and k. It is also well-known that an equilibrium exists in the standard model, for any e and any matrix $\hat{Y}$.[7] Take any $S \times I$ full rank matrix $\hat{Y} \in \Psi^*_{e;I}$. Consider the extended system defining a standard equilibrium in this economy, i.e., system (3.2). The proof of the desired result can be obtained just comparing systems (3.1) and (3.2). Define $I^* \equiv \{1, ..., I\}$ and $S^n \equiv \{I+1, ..., S\}$. We can show that

$$F^I_{eq}(\hat{\xi}; e; \hat{Y}) = 0 \implies F_{eq}(\xi; e) = 0;$$

where ξ is chosen as follows:

$$\begin{aligned} x_j &= \hat{x}_j;\ \lambda_j = \hat{\lambda}_j;\ \text{all } j; \\ x_h &= \hat{x}_h;\ \lambda_h = \hat{\lambda}_h;\ \text{all } h; \\ p &= \hat{p}; \end{aligned}$$

with portfolios

$$b^j_{j'} = \begin{cases} \dfrac{\hat{b}^j_{j'}}{\hat{b}^j_j} & j \in I^* \\ \dfrac{-1}{K-1} & j \in S^n \end{cases}; \text{ for } j' \in I^*$$

$$b^j_{j'} = \begin{cases} \dfrac{\hat{b}^j_{j'}}{\hat{b}^j_j} & j \in I^* \\ \dfrac{-1}{K-1} & j \in S^n \setminus \{j'\} \\ 1 & j = j' \end{cases} \text{ for } j' \in S^n$$

$$b^j_h = \begin{cases} \dfrac{\hat{b}^j_h}{\hat{b}^j_j} & j \in I^* \\ \dfrac{-1}{K-1} & j \in S^n \end{cases} \text{ for all } h.$$

Moreover, beliefs will be chosen as

$$\delta_j = \left(\frac{\hat{\lambda}^s_j}{\hat{\lambda}^0_j} \right)_{s=1}^S$$

for all j; and asset payoffs as

$$y^j = \begin{cases} \hat{y}^{j'} \hat{b}^{j'}_{j'} & j \in I^* \\ 0 & j \in S^n \end{cases}$$

7 See Geanakoplos and Polemarchakis [10].

Finally,

$$q^j = \underline{1}y^j \text{ for all } j$$

The case in which $I = 0$ is similarly treated, the only difference here being that the set I^* is now empty, and therefore we don't need any generic property of the portfolio holdings.

■

A converse of the implication used in the theorem holds too, and completely characterizes the FEPI's.

Proposition 3.2. If ξ is a FEPI for an economy $e \in \Re^{GK}_{++}$; then

$$F_{eq}(\xi; e) = 0 \Rightarrow F^I_{eq}(\hat{\xi}; e; \hat{Y}) = 0$$

for some $\hat{\xi}; \hat{Y}$ and some I:

Proof. Given ξ such that $F_{eq}(\xi; e) = 0$; without loss of generality, assume that rank of $\left[y^1; \ldots; y^I\right]$ be equal to I: Then there exists an $I \times (S - I)$ matrix A such that $Y = [y_1; \ldots; y_I][I_I \mid A] \equiv Y^*[I_I \mid A]$: Partition b_k as $\begin{bmatrix} b^*_k \\ b^0_k \end{bmatrix}$ with $b^*_k \in \Re^I$ and $b^0_k \in \Re^{S-I}$; and observe that $Y b_k = Y^*[I_I \mid A] \begin{bmatrix} b^*_k \\ b^0_k \end{bmatrix} = Y^*(b^*_k + Ab^0_k)$: Define $\hat{Y} = Y^*$ and $\hat{b}_k = b^*_k + Ab^0_k$: Then,

$$(F_{eq}(\xi; e) = 0) \Rightarrow F^I_{eq}(\hat{\xi}; e; \hat{Y}) = 0;$$

where $\hat{\xi}$ is such that only I nonredundant financial assets are introduced and $\hat{\xi} = \left(x; \lambda; p; \hat{b}; \underline{1}\hat{Y}\right)$:

■

All Theorem 3.1 and Proposition 3.2 are saying is that incomplete markets models can be interpreted as the result of a (particular) financial innovation process, and that market incompleteness can be explained as a self-fulfilling belief that innovators hold in equilibrium.

4. Innovation leads to incomplete markets.

In this section we are concerned with the properties of allocations and beliefs when there is innovation. The second issue is particularly important in this context. Since Theorem 3.1 establishes that almost any matrix can be interpreted as endogenous in our model, one could ask how likely is the outcome corresponding to an incomplete markets matrix. In this context, likelihood should be interpreted as the relative size of equilibrium beliefs leading to incomplete markets matrices. It should be clear by now that different beliefs have an impact on allocations, as much as in the standard model when one allows for the return matrix to be endogenous. Indeed, the following result is an extension of a well-known result in economies with nominal assets. Let $E^* = \cap E^*_l$; an open and full measure subset of endowments.

Theorem 4.1. Let $l^* = \arg\max_{0 \leq l \leq S} l(S - l)$. Let $e \in E^*$ be an economy with potential innovation. The set X_e of equilibrium allocations contains a smooth manifold of dimension $l^*(S - l^*)$.

Proof. From Theorem 5.3 in Balasko and Cass [4], we know that, given l^*, and any $e \in E^*$; the space of equilibrium allocations X_e of a nominal-asset economy associated with e contains a smooth manifold of dimension $l^*(S - l^*)$; when the payoff matrix Y is endogenous. It is a straightforward exercise to show that the equilibrium set with nominal asset is isomorphic to the equilibrium set with numéraire assets, modulo S normalizations. Applying Theorem 3.1 and Proposition 3.2 gives the result.

■

The same logic of the proof of Theorem 4.1 can be applied to show that the equilibrium allocations space X_e contains the union of l smooth manifolds, each of dimension $l(S - l)$; for $0 < l \leq S$.

It is also clear that the space of equilibrium beliefs leading to l newly introduced assets contains a manifold of dimension $l(S - l)$; as we now formally show. Fix an $e \in E^*$. Let $\xi = (\xi^0; \delta)$; a partition of the previously defined vector ξ; after rearrangement, and let

$$\Delta^l_e = \left\{ \delta \in \mathbb{R}^{S(S-1)}_{++} : F_{eq}(\xi^0; \delta; e) = 0; \text{ for some } \xi^0; \text{ with rank} Y = l \right\}$$

Theorem 4.2. Let $e \in E^*$. Take $l > 0$. Then Δ^l_e contains an $l(S - l)$-dimensional smooth manifold.

Proof. Let $\hat{\xi} = (\hat{\xi}^0; \hat{\lambda})$; a partition of $\hat{\xi}$ (similar to that of ξ); and let

$$\hat{\Delta}^l_e = \left\{ \hat{\lambda} \in \mathbb{R}^{S(S-1)}_{++} : F_{eq}(\hat{\xi}^0; \hat{\lambda}; e; \hat{Y}) = 0; \text{ for some } \hat{\xi}^0; \text{ some } \hat{Y} \right\}$$

be the set of "normalized" multipliers, i.e., with typical element $\hat{\lambda}_j = \lambda^s_j / \lambda^0_j$; for $j = 2, \ldots, S$; and $\hat{\lambda}_1 = 1$. From Theorem 3.1 and Proposition 3.2, we know that we can answer the question about Δ^I_e and $\hat{\Lambda}^I_e$ indifferently. Choose a $\hat{Y} \in \Lambda_{e,I}$; the subset of Theorem 3.1, so that the economy $\left(e; \hat{Y}\right)$ is regular. Fix this economy and construct a neighborhood $V_{\hat{Y}}$ of $\hat{Y}$ in the appropriate Grassmannian. From regularity, we know that, locally, $\hat{\lambda}_j : V_{\hat{Y}} \to \Re^{S(S-1)}_{++}$ (defined as $\hat{\lambda}_j = \hat{\lambda}_j(Y)$; for all $Y \in V_{\hat{Y}}$) is a smooth function. We want to show that the inverse is also a function, and smooth. This in turn would imply that it is a (local) diffeomorphism between $V_{\hat{Y}}$ and $\hat{\Lambda}^I_e$; or, equivalently, Δ^I_e; thereby proving the result.

In order to do this, we might have to restrict our attention to the set of economies $(e; Y)$ such that rank of $\left(\underline{1} - \hat{\lambda}_2; \ldots; \underline{1} - \hat{\lambda}_S\right)$ is equal to $S - I$; that is, its maximum. This is a strongly generic property in a standard nominal (or numéraire) asset economy. Looking at the equations

$$\lambda_j R(q) = 0$$

for $j > 1$; we observe that, given the restriction on the first innovator's multiplier, these can be written as

$$\left(\underline{1} - \hat{\lambda}_j\right) a{:}\hat{Y} = 0 \tag{4.1}$$

Without loss of generality, we can rewrite the matrix $a{:}\hat{Y}$ as

$$\begin{bmatrix} A' \\ A'' \end{bmatrix}$$

where A', a $I \times I$ submatrix, has full rank I. Now we can postmultiply (4:1) by A'^{-1} to get

$$\left(\underline{1} - \hat{\lambda}_j\right) \begin{bmatrix} I \\ B \end{bmatrix} = 0 \tag{4.2}$$

Given $\hat{\lambda} \in \hat{\Lambda}^I_e \cap \hat{\lambda}\left(V_{\hat{Y}}\right)$ after stacking all equations (4:2) with respect to j, we can partition the resulting matrix of multipliers in four (the hat is hereafter omitted):

$$\Lambda = \begin{bmatrix} \Lambda'_{S-I \times I} & \Lambda''_{S-I \times S-I} \\ \Lambda'''_{I-1 \times I} & \Lambda^{iv}_{I-1 \times S-I} \end{bmatrix}$$

After explicitly multiplying the first block of $S - I$ equations, we get from (4:2)

$$\Lambda' + \Lambda'' B = 0$$

which can be rearranged to give

$$B = -\alpha''^{-1}\alpha'$$

that is, B as a function of λ: Indeed, B uniquely defines the equivalence class of matrices Y giving rise to the same linear subspace of dimension I in $\Re^S$; i.e., B is uniquely defined in the neighborhood V_{γ}: A routine argument shows how to recover all the other endogenous variables, once this subspace is fixed, in particular, prices in ª; and conclude the argument.

■

To conclude, we have shown that the dimension of the space of beliefs generating FEPI with I new assets is at least $I(S - I)$: This in particular means that it is more likely to observe market incompleteness even when financial innovation is explicitly modeled, only because of indeterminacy of the innovators' beliefs.

The indeterminacy of expectations translates into real indeterminacy. This essentially supports the work by Balasko and Cass and gives, at least to us, a more plausible explanation of what it means for the asset payoff matrix to be endogenously determined.

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