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"English Auctions with Ensuing Risks and Heterogeneous Bidders"

by

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English Auctions with Ensuing Risks and Heterogeneous Bidders

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Abstract. We establish conditions under which an English auction for an indivisible risky asset has an efficient ex post equilibrium when the bidders are heterogeneous in both their exposures to, and their attitudes toward, the ensuing risk the asset will generate for the winning bidder. Each bidder's privately known type is unidimensional, but may affect both his risk attitude and the expected value of the asset's return to the winner. An ex post equilibrium in which the winning bidder has the largest willingness to pay for the asset exists if two conditions hold: each bidder's marginal utility of income is log-supermodular, and the vector-valued function mapping the type vector into the bidders' expected values for the asset satisfies a weighted average crossing condition. However, this equilibrium need not be efficient. We show that it is efficient if each bidder's expected value for the asset is nonincreasing in the types of the other bidders, or if the bidders exhibit nonincreasing absolute risk aversion, or if the asset is riskless.

Keywords. *English auction, ensuing risk, heterogeneous risk preferences, interdependent values, ex post equilibrium, ex post efficiency*

JEL Classification. D44, D82

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1 Introduction

A notable feature of an English auction is the open ascending-bid process that allows bidders to infer and use the private information of the others before the auction ends. A remarkable consequence is that, in a variety of settings, an English auction has an efficient ex post equilibrium. A number of recent studies have established this in settings with heterogeneous bidders and interdependent values: Maskin (1992), Krishna (2003), Dubra, Echenique and Manelli (2009), and Birulin and Izmalkov (2011). These are important results, given the detail-free nature and simplicity of English auctions, and the robust, belief-free nature of ex post equilibria.

In this paper we consider English auctions in environments with two additional features, both of which are prevalent in reality. First, the value the winning bidder will receive from the object may be uncertain when the auction is held. Specifically, we consider the case in which the object being sold is a risky asset that gives the winning bidder a random monetary return at some time after the auction. This return is the sum of its expected value, a function of all the bidders' types, and a random noise that is independent of the types which we refer to as an "ensuing risk." The second feature, without which the ensuing risk would be irrelevant, is that the bidders need not be risk neutral. Heterogeneity abounds: the bidders may have different utility functions for income, different expected value functions for the asset's return, and different ensuing risks. The model nests as special cases essentially all existing models of single-object auctions with unidimensional private information.

Ensuing risk is a feature of objects sold in many auctions. The value of fine wine, antiques, land, licenses, mineral rights, and takeover targets generally remains uncertain to the winner at the time an auction concludes. Eső and White (2004) provide the first analysis of auctions with ensuing risk, interdependent types, and risk averse bidders, focusing on the effects of ensuing risk on bidding levels in first-price and second-price auctions. Our environment generalizes theirs by allowing heterogeneity and nonmonotonic value functions, and our focus on the existence of efficient ex post equilibria in English auctions is quite different. An important assumption in both papers, however, is that the risk is noncontractible (uninsurable). This is the case when its realization is observed only by the winning bidder, or is unverifiable for some other reason to a third-party enforcer of contracts.[1]

The addition of ensuing risk and nonlinear utility significantly alters the analysis of an English auction. A bidder's optimal bidding behavior in any English auction is

[1] Contractible ensuing risks are the focus of the literature on designing auctions with contingent payments, as recently surveyed by Skrzypacz (2013).

determined largely by his willingness-to-pay function that maps the type vector into the maximal amount he would be willing to pay for the asset. In a standard quasilinear utility setting, a bidder's utility is his "value" for the object, a function of the bidders' types, less the price he pays. In this case his willingness-to-pay function is equal to his value function, and so trivially has the same assumed properties. In our environment, however, a bidder's willingness-to-pay function is determined by the nature of his utility function (for money), the asset's (expected) value function, and the distribution of his ensuing risk. Any desired properties of the endogenous willingness-to-pay functions must be derived from the properties of these underlying exogenous functions.

Income effects are an additional complication.[2] Without quasilinear utility, the maximum price a bidder is willing to pay for the asset is generally not equal to the minimum price he would be willing to sell it for after winning it in the auction. Consequently, even though the bidder with the largest willingness to pay wins the asset in the ex post equilibrium we identify, the resulting allocation need not be ex post efficient. The winner may be able to sell the asset to a losing bidder to their mutual benefit.

A bidder's privately known type in this paper is unidimensional,[3] but it plays two roles. In addition to possibly affecting the expected value of the asset to every bidder, a bidder's type may also affect his own risk attitude. Our first key assumption, A1, is that each bidder's marginal utility of income is log-supermodular in his type and income. This implies that a bidder's risk tolerance increases in his type. (We do not require the bidders to be risk averse; their utility functions may be concave, convex, or neither.[4])

In previous work on the existence of efficient ex post equilibria in English auctions, an important assumption is that the bidders' value functions satisfy some sort of single-crossing condition. Our second key assumption, A2, requires the bidders' (expected) value functions for the asset to satisfy a "weighted average crossing condition" that relates to a condition of Krishna (2003). However, what is actually important is that the willingness-to-pay functions, not the value functions, satisfy a single-crossing condition. This is where the log-supermodularity assumption is used: A1 and A2 together imply that the willingness-to-pay functions also satisfy our weighted average crossing condition.

This result allows us to prove that an English auction in our setting has an ex post equilibrium in which the asset is sold to a bidder with the highest willingness to pay

[2] Baisa (2013) considers the design of auctions in the presence of income effects.

[3] Unidimensionality seems to be a necessary restriction, since efficient ex post equilibria generically do not exist if types are multidimensional (Dasgupta and Maskin, 2000; Jehiel and Moldovanu, 2001).

[4] A bidder's utility for income could have convex parts because of, e.g., previously obtained financing that must be paid back, and limited liability or wealth constraints.

for it (Theorem 1 and Corollary 1).[5] The argument somewhat resembles those in the literature, especially that of Theorem 2 in Krishna (2003), applied to the willingness-to-pay functions instead of the value functions. However, in two ways our argument is novel. First, our formulation of a button auction model of an English auction allows a bidder to immediately drop out when he sees another bidder drop out first, so that he is able to drop out at the same price.[6] (This does not happen in the equilibrium we identify.) Second, instead of obtaining the (inverse) equilibrium strategies by applying an existence theorem to a system of ordinary differential equations, we construct them directly from the global inverse of the vector-valued willingness-to-pay function of (all) the bidders, which we show exists by applying a theorem of Gale and Nikaidô (1965).

Turning to the efficiency question, we ask whether the equilibrium allocations are Pareto efficient conditional on the realized types.[7] A symmetric, two-bidder example gives a strong negative answer: the equilibrium allocation in the example is not ex post efficient whenever the bidders' types are unequal. This inefficiency is due to two factors. First, the losing bidder's willingness to pay for the asset increases in the winning bidder's type. Since the price at which he drops out is based on an underestimate of the winning bidder's type, the losing bidder's willingness to pay is larger ex post. Second, an adverse income effect is caused by the winner's increasing absolute risk aversion. His positive information rent due to winning makes him more risk averse, and hence lowers the price at which he is willing to sell the asset. The result is that ex post, the loser is willing to pay more for the asset than the lowest price the winner is willing to sell it, generating gains from trade.

The equilibrium is, however, ex post efficient under relatively mild conditions. Our second main result (Theorem 2) is that the equilibrium shown to exist in Theorem 1 is ex post efficient if each bidder's expected value for the asset is nonincreasing in the types of the other bidders, or if the bidders exhibit nonincreasing absolute risk aversion, or if the asset is riskless.

The rest of the paper is organized as follows. The general environment is described in Section 2. Assumptions A1 and A2 are presented in Section 3, together with their implications. Our formulation of a button English auction is presented in Section 4.

[5] Existence *per se* is not an issue. In any setting in which some bidder wants to win at the reserve price for any realization of the types, the English auction has an ex post equilibrium in which that bidder never drops out and all the others drop out immediately, regardless of their types.

[6] This is done by a "clock stopping" convention similar to those used in Bulow and Klemperer (1994) and Ausubel (2004). The continuous time game issues so addressed are the subject of, e.g., Simon and Stinchcombe (1989).

[7] This is the usual notion of ex post effiiciency, but it does seem to presuppose that losing bidders learn the winning bidder's type. We discuss this more in Section 6.

Theorem 1 and Corollary 1 on existence appear in Section 5, and Theorem 2 on efficiency in Section 6. Properties of matrices that satisfy a dominant weighted average condition are derived in Appendix A, and proofs missing from the text are in Appendix B.

2 Environment

An indivisible asset is to be sold to one of several bidders using an English auction. The asset will generate a random return for the winning bidder after the auction has been held. Each bidder has a private type which can affect his risk preferences regarding the asset's ensuing risk, as well as the expected value of that risk.

The set of bidders is $N = \{1, ..., n\}$, where $n \geq 2$. The private type of bidder i, denoted as t_i, is an element of a compact interval $T_i = [\underline{t}_i, \bar{t}_i] \subset \mathbb{R}$. A type profile is denoted as $\mathbf{t} = (t_1, \ldots, t_n)$, an element of $\mathbf{T} := T_1 \times \cdots \times T_n$. The bidders may or may not have a common prior on the type space – we need not specify this because we are concerned with ex post equilibria.

If bidder i obtains the asset and the realized type vector is $\mathbf{t}$, the asset will generate for him a random income $v^i(\mathbf{t}) + \tilde{z}_i$. The random variables $\tilde{z}_1, ..., \tilde{z}$ have a commonly known distribution and may be correlated, but they are independent of $\mathbf{t}$ and satisfy $\mathbb{E}(\tilde{z}_i) = 0$.[8] We refer to $\tilde{z}_i$ as the *ensuing risk* of the asset to bidder i, and to $v^i(\mathbf{t})$ as the *expected value* of the asset to bidder i conditional on $\mathbf{t}$. Each v^i is assumed to be continuously differentiable on $\mathbf{T}$.[9]

Type t_i of bidder i has a Bernoulli utility function $u^i(\cdot, t_i)$ for money. We assume u^i is twice differentiable on $\mathbb{R} \times T_i$, and the first partial derivative u^i_1 is positive everywhere. We make no assumption about the sign of $u^i_{11}(\cdot, t_i)$, allowing each bidder to be locally risk loving, neutral, or averse at different levels of income.

If bidder i obtains the asset for price p, his payoff is $\mathbb{E}u^i\left(v^i(\mathbf{t}) + \tilde{z}_i - p, t_i\right)$. If he does not obtain the asset and pays nothing, his payoff is $u^i(0, t_i)$. We assume the former is weakly larger than the latter if $p = 0$. Thus, any bidder would always like to obtain the asset free of charge. We also assume a large enough price $\bar{p}$ exists such that the reverse strict preference holds for any $i \in N$, $\mathbf{t} \in \mathbf{T}$, and $p \geq \bar{p}$. Consequently, since $u^i_1 > 0$, for each $i \in N$ and $\mathbf{t} \in T$ a unique number $\pi^i(\mathbf{t}) \in [0, \bar{p})$ exists such that

$$\mathbb{E}u^i\left(v^i(\mathbf{t}) + \tilde{z}_i - \pi^i(\mathbf{t}), t_i\right) = u^i(0, t_i). \tag{1}$$

We refer to $\pi^i(\mathbf{t})$ as bidder i's *(maximum) willingness to pay* for the asset given the type

[8] Throughout, $\mathbb{E}(\cdot)$ denotes expectation with respect to the arguments marked with tildes.

[9] That is, v^i is continuous on $\mathbf{T}$, has partial derivatives that are continuous on the interior of $\mathbf{T}$, and those partial derivatives converge to finite quantities at boundary points of $\mathbf{T}$.

vector $\mathbf{t}$. Were he to know $\mathbf{t}$, bidder i would strictly prefer to purchase the asset at price p over not purchasing the asset if and only if $p < \pi^i(\mathbf{t})$. Much of the subsequent analysis focuses on the properties of these willingness-to-pay functions. Since each u^i is continuously differentiable and $u_1^i > 0$, π^i is continuously differentiable on $\mathbf{T}$.

A variety of special cases of this environment are of interest. Consider, for example, the following three.

(a) Types determine only risk preferences: $v^i(\mathbf{t}) := \bar{v}^i$. In this case we have a private value environment, since the type vector directly affects the payoff of bidder i only through his own type, in $u^i(\cdot, t_i)$. His willingness to pay is a function of only his own type, $\pi^i(t_i)$.

(b) Types determine only expected values: $u^i(x, t_i) = u^i(x)$. In this case t_i might be an informative signal about the asset's value to each bidder, causing the expected values $v^i(\mathbf{t})$ to depend on $\mathbf{t}$. Alternatively, $\mathbf{t}$ might determine the technology with which bidder i can use the asset, and hence its expected return. The symmetric model of Eső and White (2004) is a special case.

(c) Bidders are risk neutral and types determine only expected values: this is the special case of (b) in which $u^i(x) = x$. In this case

$$\mathbb{E}\left[u^i(v^i(\mathbf{t}) + \tilde{z}_i - p, t_i)\right] = v^i(\mathbf{t}) - p,$$

and so the ensuing risks $\tilde{z}_i$ are irrelevant. The willingness-to-pay functions are equal to the expected value functions: $\pi^i(\mathbf{t}) = v^i(\mathbf{t})$.

Remark 1 *The previous literature on English auctions with asymmetric interdependent values focuses on case (c), e.g., Krishna (2003), Dubra et al. (2009), and Birulin and Izmalkov (2011). These papers also make one symmetry assumption, namely, that all bidders have the same expected value at the lowest type vector: $v^i(\underline{\mathbf{t}}) = v^j(\underline{\mathbf{t}})$ for all $i \neq j$.*[10] *This assumption provides a boundary condition for a system of differential equations that characterize an equilibrium. We dispense with this assumption partly to avoid assuming any symmetry, and partly because only in case (c) would it imply that all bidders have the same willingness to pay at $\underline{\mathbf{t}}$.*

3 Two Key Assumptions

In this section we present and discuss the two assumptions which together imply that the English auction has an ex post equilibrium in which the asset is always sold to a bidder with the highest willingness to pay.

[10] This common value at $\underline{\mathbf{t}}$ is normalized to equal 0.

3.1 Ordering Risk Tolerance

A function $g : X \times T \to \mathbb{R}$, where X and T are subsets of $\mathbb{R}$, is *log-supermodular* if

$$g(x,t)g(x',t') \geq g(x,t')g(x',t)$$

for all $x, x' \in \mathbb{R}$ and $t, t' \in T$ such that $x' > x$ and $t' > t$. The following assumption is that each bidder's *marginal* utility satisfies this property:

A1. u_1^i is log-supermodular on $\mathbb{R} \times T_i$ for all $i \in N$.

The relevant implication of A1 is that higher types of a bidder are more tolerant of risk than are lower types. This has been formalized in a variety of ways and contexts (e.g., Jewitt, 1987; Athey, 2002). For our purposes we provide the following result.

Lemma 1 *Let $X, T \subseteq \mathbb{R}$ be intervals, and suppose a function $u : X \times T \to \mathbb{R}$ has a derivative u_1 that is positive and log-supermodular on $X \times T$. Let $a \in X$ and $t, t' \in T$ with $t' > t$, and let $\tilde{x}$ be a random variable on X such that $\mathbb{E}u(\tilde{x}, t)$ and $\mathbb{E}u(\tilde{x}, t')$ exist. If*

$$\mathbb{E}u(\tilde{x}, t) = u(a, t),$$

then (i) $\mathbb{E}u(\tilde{x}, t') \geq u(a, t')$, and (ii) $\mathbb{E}u_2(\tilde{x}, t) \geq u_2(a, t)$ when $\mathbb{E}u_2(\tilde{x}, t)$ and $u_2(a, t)$ exist.

Lemma 1 (*i*) shows that log-supermodularity of the marginal utility function implies that if one type is indifferent between a gamble and a riskless amount, then a higher type weakly prefers the gamble. From this, familiar arguments (e.g., Pratt, 1964) show that the certainty equivalent for a fixed gamble is nondecreasing in the type, the utility function of a lower type is a concave transformation of that of any higher type, and, assuming the second derivative exists, the Arrow-Pratt measure of absolute risk aversion is nonincreasing in the type.

Lemma 1 (*ii*) yields a useful relationship between the derivatives of the willingness-to-pay and expected value functions. For all $i, j \in N$, differentiating (1) with respect to t_j and solving for the derivatives π_j^i yields

$$\pi_j^i(\mathbf{t}) = v_j^i(\mathbf{t}) \text{ for } j \neq i, \tag{2}$$

and

$$\pi_i^i(\mathbf{t}) = v_i^i(\mathbf{t}) + \frac{\mathbb{E}u_2^i\left(v^i(\mathbf{t}) + \tilde{z}_i - \pi^i(\mathbf{t}), t_i\right) - u_2^i(0, t_i)}{\mathbb{E}u_1^i\left(v^i(\mathbf{t}) + \tilde{z}_i - \pi^i(\mathbf{t}), t_i\right)}.$$

Given A1, (1) , and Lemma 1 (*ii*), the numerator on the right side of this equality is nonnegative. We thus have

$$\pi_i^i(\mathbf{t}) \geq v_i^i(\mathbf{t}). \tag{3}$$

We end this subsection with an example to show why an assumption like A1 is needed. It is a symmetric two-bidder example in which the expected value functions satisfy the single-crossing condition of Maskin (1992) (and our own upcoming A2). The auction would therefore have a symmetric (and efficient) ex post equilibrium if the bidders were risk neutral. But they are instead risk averse, with utility functions that violate A1, and a symmetric ex post equilibrium does not exist.

Example 1 *Each of two bidders, $i = 1, 2$, has CARA utility $u^i(x, t_i) = \left(1 - e^{-t_i x}\right)/t_i$, expected value $v^i(\mathbf{t}) = t_i + \frac{1}{2}t_{-i}$, and an ensuing risk $\tilde{z}_i$ that is normally distributed with mean zero and variance $\sigma^2 \in (2, 3)$. Each bidder's set of types is the same nondegenerate interval $[\underline{t}, \bar{t}]$. The willingness-to-pay functions take the familiar form of expected value less the cost of risk:*

$$\pi^i(\mathbf{t}) = v^i(\mathbf{t}) - \tfrac{1}{2}t_i\sigma^2 = \left(1 - \tfrac{1}{2}\sigma^2\right)t_i + \tfrac{1}{2}t_{-i}.$$

The partial derivatives are $\pi^i_i = 1 - \frac{1}{2}\sigma^2 < 0$ since $\sigma^2 > 2$, and $\pi^i_1 + \pi^i_2 = \frac{3}{2} - \frac{1}{2}\sigma^2 > 0$ since $\sigma^2 < 3$.

As usual, we model a two-bidder English auction as a second price auction. Suppose (β, β) is a symmetric ex post equilibrium, and that $\beta(t_1) \leq \beta(t_2)$ for some $\mathbf{t} = (t_1, t_2)$. Given this $\mathbf{t}$, bidder 2 wins with positive probability and pays $\beta(t_1)$ when he does, and bidder 1 loses with positive probability. Because it is an ex post equilibrium, β must satisfy two conditions. First, bidder 2 must not regret winning at $\mathbf{t}$, i.e., the amount he pays cannot exceed his willingness to pay:

$$\beta(t_1) \leq \pi^2(\mathbf{t}). \tag{4}$$

Second, bidder 1 must not regret losing at $\mathbf{t}$. Thus, since he could win and pay $\beta(t_2)$ by bidding high enough, this amount must exceed his willingness to pay:

$$\beta(t_2) \geq \pi^1(\mathbf{t}). \tag{5}$$

These inequalities hold for all $\mathbf{t}$ such that $\beta(t_1) \leq \beta(t_2)$, and in particular for $\mathbf{t} = (t, t)$. Thus, since $\pi^1(t, t) = \pi^2(t, t)$, for all $t \in [\underline{t}, \bar{t}]$ we have

$$\beta(t) = \pi^i(t, t).$$

Now, since $\beta' = \pi^i_1 + \pi^i_2 > 0$, β is a strictly increasing function. Hence, for all $t_1 < t_2$ we have $\beta(t_1) < \beta(t_2)$, and so (t_1, t_2) must satisfy (4) and (5). But this is impossible. As $\pi^1_1 < 0$ and $\pi^1(t_2, t_1) = \pi^2(t_1, t_2)$ ($\boldsymbol{\pi}$ is permutation symmetric), for $t_1 < t_2$ we have a contradiction of (4):

$$\beta(t_1) = \pi^1(t_1, t_1) > \pi^1(t_2, t_1) = \pi^2(t_1, t_2).$$

In this example a bidder's risk aversion increases with his type. It does so fast enough that the bidder's willingness to pay for the asset decreases in his type, even though his expected value for the asset increases in his type. (Inequality (3) fails to hold, as $\pi_i^i < 0 < v_i^i$.) However, because a bidder's expected value for the asset increases fast enough in the other bidder's type, any symmetric equilibrium bid function must still be increasing. The incompatibility of this with a willingness-to-pay function that decreases in his own type is why a symmetric ex post equilibrium does not exist.

3.2 Weighted Average Crossing

The single-crossing condition we use is related to the average crossing condition of Krishna (2003), as we discuss below. Our condition is based on the following property of square matrices.

Definition. For $n \geq 1$, an $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ satisfies the *dominant weighted average condition (DWAC)* if positive weights $\theta_1, \ldots, \theta_n$ exist that sum to one and satisfy, for all $i \neq j$,

$$\sum_{k=1}^{n} \theta_k a_{kj} > \max\{0, a_{ij}\}.$$

In words, a square matrix satisfies DWAC provided some weighted average of its rows is strictly positive, and each component of this average row is larger than each off-diagonal element in its column.[11] It will be useful to note now that each diagonal element of a DWAC matrix is positive and larger than every other element in its column:

$$a_{jj} > \max\{0, a_{ij}\} \text{ for all } i \neq j. \tag{6}$$

This and other results about DWAC matrices are proved in Appendix A.

If a function mapping $\mathbf{T}$ into $\mathbb{R}^n$ has a Jacobian matrix satisfying DWAC at each $\mathbf{t} \in \mathbf{T}$, we say it satisfies the *weighted average crossing condition.* Defining $\mathbf{v} : \mathbf{T} \to \mathbb{R}^n$ by $\mathbf{v} := (v^1, \ldots, v^n)$, we now make the following assumption:

A2. $\mathbf{v}$ satisfies the weighted average crossing condition.

This assumption requires that at each $\mathbf{t}$, a marginal increase in t_j should increase some weighted average of all the bidders' expected values, and it should increase that average more than it does the expected value of any bidder $i \neq j$. The weights used generally depend on $\mathbf{t}$. By (6), A2 implies $v_j^j(\mathbf{t}) > v_j^i(\mathbf{t})$ for all $i \neq j$ and $\mathbf{t} \in \mathbf{T}$, and hence it implies the pairwise single-crossing condition of Maskin (1992) and Dasgupta and Maskin

[11] If $n = 1$, the definition requires only that the one element of $\mathbf{A}$ be positive.

(2000). A prominent example of a value function satisfying A2 is one that is additively separable into common and private value components, $v^j(\mathbf{t}) = w(\mathbf{t}) + \hat{v}_j(t_j)$, with $\hat{v}_j' > 0$ and $\sum_{k=1}^n \max\{0, -w_k/\hat{v}_k'\} < 1$.[12] (A similar observation is made in Krishna, 2003.)

Remark 2 *A square matrix satisfies the* dominant average condition (DAC) *if it satisfies DWAC with $\theta_k = 1/n$ for each k. The* average crossing condition *of Krishna (2003) requires the Jacobian matrix of $\mathbf{v}$ to satisfy DAC at all type vectors at which two or more bidders have the maximal valuation. In comparison, our weighted average crossing condition is weaker in that the weights need not be equal or constant in $\mathbf{t}$, but stronger in that it requires the Jacobian matrix to satisfy DWAC at all $\mathbf{t}$. We require the latter because in our setting, two bidders who have the same expected value need not have the same willingness to pay.*[13]

Krishna (2003) also defines DWAC matrices (p. 286), but does not specify a corresponding single-crossing condition on $\mathbf{v}$. He does, however, say (p. 274) that in his setting the existence of an efficient ex post equilibrium can be shown if his average crossing condition is weakened to one involving weighted averages. As his setting is a special case of ours, the upcoming Theorems 1 and 2 show this is true.

The following lemma shows that A1 and A2 together imply that the willingness-to-pay function $\boldsymbol{\pi} := (\pi^1, \ldots, \pi^n)$ also satisfies the weighted average crossing condition.

Lemma 2 *Under* A1-A2, *for each $\mathbf{t} \in \mathbf{T}$ the matrix $[\pi_j^i(\mathbf{t})]$ satisfies DWAC.*

Proof. Fix $\mathbf{t} \in \mathbf{T}$. By A2, $\left[v_j^i(\mathbf{t})\right]$ satisfies DWAC, and so $v_j^j(\mathbf{t}) > 0$ by (6). By A1, (2) and (3) hold. Hence, $\left[\pi_j^i(\mathbf{t})\right]$ equals $\left[v_j^i(\mathbf{t})\right]$ plus a nonnegative diagonal matrix. This implies, since $\left[v_j^i(\mathbf{t})\right]$ satisfies DWAC, that $\left[\pi_j^i(\mathbf{t})\right]$ does too (using the same weights). ■

The most important result of Appendix A is Lemma A.3. It establishes that a DWAC matrix is a "P-matrix," which is a matrix for which the determinants of all principal submatrices (including itself) are positive. The importance of this result is that if the Jacobian matrix of a differentiable function from $\mathbf{T}$ onto a set $\mathbf{P} \subset \mathbb{R}^n$ is a P-matrix

[12] Given $\mathbf{t}$, for each k let $\theta_k = \max\{0, -w_k(\mathbf{t})/\hat{v}_k'(t_j)\} + \varepsilon_k$ for some $\varepsilon_k > 0$. Since $\sum \max\{0, -w_k/\hat{v}_k'\} < 1$, a strictly positive $(\varepsilon_1, \ldots, \varepsilon_n)$ can be found so that $\sum \theta_k = 1$. For $\mathbf{A} = [v_j^i(\mathbf{t})]$, we have $S(j) := \sum_k \theta_k a_{kj} = w_j + \theta_j \hat{v}_j' > w_j$. Obviously $S(j) > 0$ if $w_j > 0$, and also if $w_j \leq 0$ because then $S(j) = w_j + \left(-w_j/\hat{v}_j' + \varepsilon_j\right)\hat{v}_j' = \varepsilon_j \hat{v}_j' > 0$. This proves $\mathbf{A}$ is DWAC, and so A2 holds.

[13] The alternative sufficient condition introduced in Krishna (2003), the cyclical crossing conditon, as well as those used in Dubra, Echenique and Manelli (2009) and Birulin and Izmalkov (2011), are less comparable to A2.

everywhere, then the function has an inverse that maps $\mathbf{P}$ onto $\mathbf{T}$.[14] Lemma A.3 also establishes that the inverse of a DWAC matrix maps the unit vector into a strictly positive vector. Putting these results together gives us the following result.

Lemma 3 *Under* A1-A2, $\boldsymbol{\pi}$ *has a continuously differentiable inverse function,* $\boldsymbol{\varphi} : \boldsymbol{\pi}(\mathbf{T}) \to \mathbf{T}$*. Its Jacobian matrix at any* $\mathbf{p} \in \boldsymbol{\pi}(\mathbf{T})$ *satisfies* $[\varphi^i_j(\mathbf{p})]\mathbf{e} \gg \mathbf{0}$.[15]

Proof. For each $\mathbf{t} \in \mathbf{T}$ the matrix $[\pi^i_j(\mathbf{t})]$ is DWAC by Lemma 2. Lemma A.3 thus implies it is a P-matrix at every $\mathbf{t} \in \mathbf{T}$. Hence, since $\mathbf{T}$ is a compact rectangle, $\boldsymbol{\pi}$ has an inverse, $\boldsymbol{\varphi} : \boldsymbol{\pi}(\mathbf{T}) \to \mathbf{T}$, by Theorem 4 in Gale and Nikaidô (1965). The continuity of $\boldsymbol{\pi}$ implies $\boldsymbol{\varphi}$ is continuous. As $\boldsymbol{\pi}$ is continuously differentiable and $\det[\pi^i_j(\mathbf{t})] > 0$ at any $\mathbf{t} \in \mathbf{T}$, the inverse function theorem implies $\boldsymbol{\varphi}$ is continuously differentiable at any $\mathbf{p}$ in the interior of $\boldsymbol{\pi}(\mathbf{T})$, and has the Jacobian matrix

$$\left[\varphi^i_j(\mathbf{p})\right] = \left[\pi^i_j(\boldsymbol{\varphi}(\mathbf{p}))\right]^{-1}.$$

Since $[\pi^i_j(\boldsymbol{\varphi}(\mathbf{p}))]$ is continuous in $\mathbf{p}$ and has a positive determinant at all $\mathbf{p} \in \boldsymbol{\pi}(\mathbf{T})$, we conclude that $\boldsymbol{\varphi}$ is continuously differentiable on $\boldsymbol{\pi}(\mathbf{T})$. By Lemma A.3, for any $\mathbf{p} \in \boldsymbol{\pi}(\mathbf{T})$ we have $[\pi^i_j(\boldsymbol{\varphi}(\mathbf{p}))]^{-1}\mathbf{e} \gg \mathbf{0}$, and so $[\varphi^i_j(\mathbf{p})]\mathbf{e} \gg \mathbf{0}$. ∎

We shall apply Lemmas 2 and 3 not only to $\boldsymbol{\pi}$, but also to any function obtained from $\boldsymbol{\pi}$ by holding the types of some bidders fixed. We can do this because, by Lemma A.2 in Appendix A, all principal submatrices of a DWAC matrix are also DWAC matrices. Thus, for any $A \subseteq N$ and $\mathbf{t}_{N\setminus A} \in \mathbf{T}_{N\setminus A}$, both lemmas apply under A1-A2 to the function $\hat{\boldsymbol{\pi}} : T_A \to \mathbb{R}^{|A|}$ defined by $\hat{\boldsymbol{\pi}}(\mathbf{t}_A) = \boldsymbol{\pi}^A(\mathbf{t}_A, \mathbf{t}_{N\setminus A})$.

Assumptions A1 and A2 are the basis, largely via Lemma 3, of all the upcoming results. For the rest of this paper they will be maintained background assumptions, not explicitly listed in the statements of results.[16]

4 The Button Auction Game

We model an English auction by the following version of a button auction game. The price of the asset steadily rises while all the active bidders hold down their buttons. The price

[14] The inverse function theorem does not imply this. A function may not have a global inverse even if its Jacobian matrix is continuous and nonsingular everywhere, and its determinant never changes sign. Examples are given, e.g., in Gale and Nikaidô (1965) and Rudin (1976, Exercise 9.17).

[15] Throughout, $\mathbf{e}$ and $\mathbf{0}$ denote the column vectors of 1's and 0's respectively, of length equal to the dimension of the associated matrix.

[16] All upcoming results remain true if instead of A1-A2, we simply assumed $\boldsymbol{\pi}$ satisfies the weighted average crossing condition. We prefer A1-A2 because they are assumptions about the exogenous functions from which $\boldsymbol{\pi}$ is derived.

stops rising the moment any bidder releases his button ("drops out" or "exits"). Some number of seconds later, say $\Delta > 0$, any of the remaining active bidders may exit too. If any do, then in another Δ seconds any of the remaining active bidders may again exit. This continues until no more bidders exit. Then, if at least two bidders are still active, the price resumes its steady rise until it reaches another price at which bidders exit. The process continues in this way until either only one bidder is active, in which case he is the winner and pays a price equal to the highest price at which the others dropped out, or all the active bidders simultaneously drop out, in which case they each win with equal probability and the winner pays the price at which they dropped out. All exits are publicly observed, both the exit prices and the identities of those who exit.

An *exit round* is a pair (p, D), where $p \geq 0$ and $\varnothing \neq D \subseteq N$. Its interpretation is that D is the set of bidders who simultaneously exit when the current price is p in this round. At any moment during the auction the public *history of exits* is either the *null history*, h_0, if no bidder has yet exited, or it is a finite sequence of exit rounds, $h = \{(p_k, D_k)\}_{k=1}^{K}$, for which $K \geq 1$, $p_1 \leq p_2 \leq \cdots \leq p_K$, and the sets D_k of exiting bidders are disjoint.[17] The rounds occur in temporal order, i.e., D_k exits before D_{k+1} exits. (If $p_k = p_{k+1}$, the two sets of bidders exit at the same price, but those in D_{k+1} exit after they have seen those in D_k exit.) The set of all histories is denoted as H.

Lastly, it will be useful to define the *subhistories,* $h_0, \ldots, h_K$, of an exit history $h = \{(p_k, D_k)\}_{k=1}^{K}$ by $h_k := \{(p_\ell, D_\ell)\}_{\ell=1}^{k}$ for $k \geq 0$. Note that $h_K = h$, and that the null history h_0 is a subhistory of every exit history.

For any history $h = \{(p_k, D_k)\}_{k=1}^{K}$, we denote the set of bidders who have dropped out as $D(h) := D_1 \cup \cdots \cup D_k$. (At the null history we have $D(h_0) := \varnothing$.) The set of *active bidders* after history h is $N \backslash D(h)$.

An *outcome* is a complete history, one that determines the set of winning bidders and the sale price. That is, it is a history $\{(p_k, D_k)\}_{k=1}^{K}$ for which $N \backslash D(h)$ either contains just one bidder, in which case he is the winner, or it is empty and D_K contains at least two bidders, all of whom win with equal probability. In both cases the sale price is p_K.

Let H_i be the set of exit histories at which bidder i is active and the game is not over, i.e., the histories h for which $i \notin D(h)$ and $|D(h)| \leq n - 2$. A *strategy* for bidder i is a function $\beta_i : T_i \times H_i \to \mathbb{R}_+$. When his type is t_i and the history is $h \in H_i$, strategy β_i requires the bidder to exit at price p if and only if $p \geq \beta_i(t_i, h)$. For each $\mathbf{t} \in \mathbf{T}$, the play of a strategy profile $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ gives rise to an outcome.

For each $\mathbf{t} \in \mathbf{T}$ the button auction protocol gives rise to a complete information game, which we denote as $\Gamma(\mathbf{t})$. In this game the vector of types $\mathbf{t}$ is common knowledge,

[17] For convenience, we take $h = \{(p_k, D_k)\}_{k=1}^{K}$ with $K = 0$ to be the null history h_0.

and an outcome $\{(p_k, D_k)\}_{k=1}^K$ yields commonly known payoffs, $\mathbb{E}u^i\left(v^i(\mathbf{t}) + \tilde{z}_i - p_K, t_i\right)$ for winners and $u^i(0, t_i)$ for losers. A strategy profile $\boldsymbol{\beta}$ of the button auction is an *ex post equilibrium* if, for each $\mathbf{t} \in \mathbf{T}$, $(\beta_1(t_1, \cdot), \ldots, \beta_n(t_n, \cdot))$ is a Nash equilibrium of $\Gamma(\mathbf{t})$.

5 Equilibrium Existence

We now show that under A1-A2, the button auction has an ex post equilibrium in which the winning bidder has the largest willingness to pay for the asset. The first step is to construct, for each bidder i and exit history $h = \{(p_k, D_k)\}_{k=1}^K$, a *minimum type function* $\tau_i(\cdot, h) : [p_K, \infty) \to T_i$. In equilibrium, when the exit history is $h \in H_i$, bidder i drops out if and only if the price rises to a level p at which his type is less than or equal to his minimum type $\tau_i(p, h)$. This will be an equilibrium strategy because, so long as his type exceeds his minimum type, the bidder's willingness to pay must exceed the current price even if all other active bidders immediately drop out. The type of an exiting bidder will be given by his minimum type function evaluated at the price at which he exited.

Let $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_n) = (\tau_i, \boldsymbol{\tau}_{-i})$. Given any history $h = \{(p_k, D_k)\}_{k=1}^K \in H_i$, the equilibrium strategy of bidder i when his type is t_i will be given by

$$\beta_i^*(t_i, h) := \min\left\{p \geq p_K : \pi^i\left(t_i, \boldsymbol{\tau}_{-i}(p, h)\right) \leq p\right\}. \tag{7}$$

According to this strategy, the bidder will exit once the price rises enough to equal his willingness to pay, calculated under the assumption that the other bidders' types are equal to their minimum types at that price and history. At lower (greater) prices this calculated willingness to pay is greater (lower) than the price, because $\boldsymbol{\tau}$ will satisfy the following single-crossing property:

P1. For all $i \in N$, $t_i \in T_i$, $h = \{(p_k, D_k)\}_{k=1}^K$, and $p' \geq p \geq p_K$:

(*i*) $\pi^i(t_i, \boldsymbol{\tau}_{-i}(p, h)) = p \Rightarrow t_i = \tau_i(p, h)$,

(*ii*) $\pi^i(t_i, \boldsymbol{\tau}_{-i}(p, h)) \leq p \Rightarrow \pi^i(t_i, \boldsymbol{\tau}_{-i}(p', h)) \leq p'$.

We now construct $\boldsymbol{\tau}$, establishing that it satisfies P1 and other useful properties along the way. The construction relies on the following lemma.[18]

[18] The properties of $\mathbf{m}$ in Lemma 4 are analogous to the "break-even conditions" of Krishna (2003), made more complicated here by the need to allow $\pi^i(\mathbf{m}(p)) > p$ when $m_i(p) = \underline{t}_i$ because we have $\boldsymbol{\pi}(\underline{\mathbf{t}}) \geq \mathbf{0}$ instead of $\boldsymbol{\pi}(\underline{\mathbf{t}}) = \mathbf{0}$. Our construction is consequently quite different. Krishna shows a solution to his break-even conditions exists by showing the existence of a solution to a set of differential equations. We instead prove Lemma 4 by a direct argument using the inverse of $\boldsymbol{\pi}$ shown to exist in Lemma 3.

Lemma 4 *Fix $\hat{p} \geq 0$, $D \subset N$, and $\hat{\mathbf{t}} \in \mathbf{T}$, and suppose that for all $i \notin D$,*

$$\pi^i(\hat{\mathbf{t}}) \begin{cases} \leq \hat{p} & \text{if } \hat{t}_i > \underline{t}_i \\ \geq \hat{p} & \text{if } \hat{t}_i < \bar{t}_i \end{cases} . \tag{8}$$

Then a continuous nondecreasing function $\mathbf{m} : [\hat{p}, \infty) \to \mathbf{T}$ exists that satisfies $\mathbf{m}(\hat{p}) = \hat{\mathbf{t}}$, $\mathbf{m}_D(\cdot) \equiv \hat{\mathbf{t}}_D$, and, for all $i \notin D$, $t_i \in T_i$, and $p' \geq p \geq \hat{p}$:

(i) $\pi^i(t_i, \mathbf{m}_{-i}(p)) = p \Rightarrow t_i = m_i(p)$,

(ii) $\pi^i(t_i, \mathbf{m}_{-i}(p)) \leq p \Rightarrow \pi^i(t_i, m_{-i}(p')) \leq p'$, *and*

(iii) $\pi^i(\mathbf{m}(p)) \begin{cases} \leq p & \text{if } m_i(p) > \underline{t}_i \\ \geq p & \text{if } m_i(p) < \bar{t}_i \end{cases}$.

The minimum type function $\boldsymbol{\tau}$ is defined to be the function $\mathbf{m}$ of Lemma 4 for various choices of the triple $(\hat{p}, D, \hat{\mathbf{t}})$. We start with the null history h_0. Letting $(\hat{p}, \hat{\mathbf{t}}) = (0, \underline{\mathbf{t}})$, we have $\pi^i(\hat{\mathbf{t}}) \geq \hat{p}$ and $\hat{t}_i < \bar{t}_i$ for all $i \in N$. So (8) is satisfied when $(\hat{p}, D, \hat{\mathbf{t}}) = (0, \varnothing, \underline{\mathbf{t}})$. We can thus define $\boldsymbol{\tau}(\cdot, h_0) : [0, \infty) \to \mathbf{T}$ to be the function $\mathbf{m}$ the lemma yields for this choice of $(\hat{p}, D, \hat{\mathbf{t}})$. Note that $\boldsymbol{\tau}(\cdot, h_0)$ is continuous, nondecreasing, and satisfies P1 (with $p_K = 0$).

Now consider a one-round history $h = \{(p_1, D_1)\}$, and let $(\hat{p}, D, \hat{\mathbf{t}}) = (p_1, D_1, \boldsymbol{\tau}(p_1, h_0))$. Then Lemma 4 *(iii)* and the definition of $\boldsymbol{\tau}(\cdot, h_0)$ in the previous paragraph together imply that (8) holds for all $i \notin D$. We can therefore again apply Lemma 4, defining $\boldsymbol{\tau}(\cdot, h) : [p_1, \infty) \to \mathbf{T}$ to be the $\mathbf{m}$ obtained for this choice of $(\hat{p}, D, \hat{\mathbf{t}})$. Lemma 4 implies $\boldsymbol{\tau}(\cdot, h)$ satisfies P1 and is continuous and nondecreasing.

An important property of the minimum type functions constructed so far is that they do not "jump" when bidders exit. That is, Lemma 4 and our choice of the boundary value $\hat{\mathbf{t}} = \tau(p_1, h_0)$ implies $\boldsymbol{\tau}(p_1, h_0) = \boldsymbol{\tau}(p_1, h_1)$ (where $h_1 = h$). We state this now as a general property that the minimum type functions will satisfy:

P2. For $1 \leq K < n$, $h = \{(p_k, D_k)\}_{k=1}^K$, and subhistories $h_0, \ldots, h_K$ of h,

$$\boldsymbol{\tau}(p_k, h_{k-1}) = \boldsymbol{\tau}(p_k, h_k) \text{ for all } k = 1, \ldots, K.$$

Having constructed the minimum type functions for the null history and all one-round histories, we use the same procedure to construct these functions for all two-round histories, then all three-round histories, and so on. The end result is a function $\boldsymbol{\tau}(p, h)$ mapping all histories $h = \{(p_k, D_k)\}_{k=1}^K$ and prices $p \geq p_K$ into $\mathbf{T}$ that satisfies P1-P2. For each $h = \{(p_k, D_k)\}_{k=1}^K$, the function $\boldsymbol{\tau}(\cdot, h)$ is continuous and nondecreasing on $[p_K, \infty)$.

Now that $\boldsymbol{\tau}$ has been defined, the strategies β_i^* are well defined by (7). The following lemma establishes an important property of these strategies: if bidder i uses β_i^* and drops out, the history and price at which he drops out perfectly reveal his type. This is true regardless of the strategies of the other bidders.

Lemma 5 *For any* $\mathbf{t} \in \mathbf{T}$, $i \in N$, *and strategies* $\boldsymbol{\beta}_{-i}$, *let* $h = \{(p_k, D_k)\}_{k=1}^K$ *be the outcome when* $(\beta_i^*, \boldsymbol{\beta}_{-i})$ *is played and* $\mathbf{t}$ *is the type vector. Then, letting* $h_0, \ldots, h_K$ *be the subhistories of* h, *for every* $i \in D_k$ *we have*

$$\pi^i\left(t_i,\, \boldsymbol{\tau}_{-i}(p_k, h_{k-1})\right) = p_k \quad \textit{and} \quad t_i = \tau_i(p_k, h_{k-1}).$$

Proof. Let $i \in D_k$. Then (7) implies $\pi^i\left(t_i,\, \boldsymbol{\tau}_{-i}(p_k, h_{k-1})\right) \leq p_k$. Assume this inequality is strict. Then $p_k > 0$, since $\pi^i(\cdot) \geq 0$. If $k = 1$, then $h_{k-1} = h_0$ (the null history), and the continuity of $\boldsymbol{\tau}_{-i}(\cdot, h_0)$ implies the existence of $p \in (0, p_1)$ such that $\pi^i\left(t_i,\, \boldsymbol{\tau}_{-i}(p, h_0)\right) < p$, and so β_i^* should have caused bidder i to drop out before the price reached p_1. This contradiction implies $k > 1$. A similar continuity argument implies that if $p_{k-1} < p_k$, bidder i should have dropped out before the price reached p_k. We thus have $p_{k-1} = p_k$. But then P2 implies

$$\begin{aligned}
\pi^i(t_i, \boldsymbol{\tau}_{-i}(p_{k-1}, h_{k-2})) &= \pi^i(t_i, \boldsymbol{\tau}_{-i}(p_{k-1}, h_{k-1})) \\
&= \pi^i(t_i, \boldsymbol{\tau}_{-i}(p_k, h_{k-1})) \\
&< p_k = p_{k-1}.
\end{aligned}$$

This and (7) imply that bidder i should have dropped out in a round before round k, i.e., in round $k-1$ if not earlier. As this contradicts $i \in D_k$, we conclude that $\pi^i\left(t_i,\, \boldsymbol{\tau}_{-i}(p_k, h_{k-1})\right) = p_k$. This and P1 imply $t_i = \tau_i(p_k, h_{k-1})$. ■

We are finally in position to prove the main result of this section.

Theorem 1 *The strategy profile* $\boldsymbol{\beta}^*$ *is an ex post equilibrium of the button auction.*

Proof. Fix $\mathbf{t} \in \mathbf{T}$, and let $h = \{(p_k, D_k)\}_{k=1}^K$ be the outcome when $\boldsymbol{\beta}^*$ is played in $\Gamma(\mathbf{t})$, with subhistories $h_0, \ldots, h_K = h$. Then p_K is the sale price. By Lemma 5, the imputed types of those who exit are equal to their true types: $\tau_i(p_k, h_{k-1}) = t_i$ for all $i \in D_k$. Hence, by P2 and the construction of τ_i, we have $\tau_i(p, h_\ell) = t_i$ for all $i \in D_k$, $\ell \geq k$, and $p \geq p_\ell$. Letting $i \in N$, we must prove β_i^* is a best reply to $\boldsymbol{\beta}_{-i}^*$ in the game $\Gamma(\mathbf{t})$. Without loss of generality we let $i = 1$. The proof breaks into two cases, depending on whether bidder 1 wins with probability one or less than one when $\boldsymbol{\beta}^*$ is played in $\Gamma(\mathbf{t})$.

Case 1: Bidder 1 wins with probability one. In this case $D(h) = N \backslash \{1\}$. Since bidder 1 does not drop out, (7) implies $\pi^1\left(t_1, \boldsymbol{\tau}_{-1}(p_K, h_{K-1})\right) > p_K$, and so P2 yields

$$\pi^1\left(t_1, \boldsymbol{\tau}_{-1}(p_K, h)\right) > p_K. \tag{9}$$

Since the other bidders do drop out, they have revealed their true types by the time the price reaches p_K, implying $\boldsymbol{\tau}_{-1}(p_K, h) = \mathbf{t}_{-i}$. Thus, (9) implies $\pi^1(\mathbf{t}) > p_K$, showing that bidder 1 prefers to win and pay p_K than to lose. Any deviation from β_1^* causes him either to lose or to win and pay p_K, as he cannot affect the price he pays when he wins. So β_1^* is indeed a best reply to $\boldsymbol{\beta}_{-1}^*$ in $\Gamma(\mathbf{t})$.

Case 2: Bidder 1 loses with positive probability. In this case $k \le K$ exists such that $1 \in D_k$. Hence, by Lemma 5,

$$\pi^1\left(t_1,\, \boldsymbol{\tau}_{-1}(p_k, h_{k-1})\right) = p_k. \tag{10}$$

As all deviations that cause bidder 1 to win generate the same sale price, we need only show that he weakly prefers in $\Gamma(\mathbf{t})$ to play β_1^* than to never exit. So let $h' = \{(p'_j, D'_j)\}_{j=1}^J$ be the outcome in $\Gamma(\mathbf{t})$ when bidder 1 never exits and the others play $\boldsymbol{\beta}_{-1}^*$. In this outcome bidder 1 wins and pays p'_J. Since $h'_{k-1} = h_{k-1}$, from (10) we obtain

$$\pi^1\left(t_1,\, \boldsymbol{\tau}_{-1}(p_k, h'_{k-1})\right) = p_k.$$

Hence, since $p_k \le p'_k \le p'_J$, P1-P2 imply

$$\pi^1\left(t_1,\, \boldsymbol{\tau}_{-1}(p'_J, h'_J)\right) \le p'_J. \tag{11}$$

Because all bidders $i > 1$ have dropped out by round J, we know $\boldsymbol{\tau}_{-1}(p'_J, h'_J) = \mathbf{t}_{-1}$. This and (11) yield $\pi^1(\mathbf{t}) \le p'_J$. We conclude that bidder 1 prefers losing to winning at price p'_J, and so β_1^* is indeed a best reply to $\boldsymbol{\beta}_{-1}^*$ in $\Gamma(\mathbf{t})$. ■

The following corollary establishes, in part, that the winning bidder has the largest willingness to pay when $\boldsymbol{\beta}^*$ is played. This is a key ingredient to showing that the equilibrium is ex post efficient. However, it is not a sufficient condition, as we shall see in the next section.

Corollary 1 *Let $\mathbf{t} \in \mathbf{T}$, and let $w \in N$ be a bidder who wins with positive probability when $\boldsymbol{\beta}^*$ is played in $\Gamma(\mathbf{t})$. Then, letting p be the resulting sale price, (i) $\pi^w(\mathbf{t}) \ge p$ and (ii) $\pi^w(\mathbf{t}) \ge \pi^\ell(\mathbf{t})$ for each $\ell \ne w$.*

Proof. Part (i) holds because $\boldsymbol{\beta}^*$ is an ex post equilibrium (in $\Gamma(\mathbf{t})$ bidder w would want to deviate by immediately dropping out if $\pi^w(\mathbf{t}) < p$ were true). To prove (ii), let $\ell \ne w$, and let $h = \{(p_k, D_k)\}_{k=1}^K$ be the outcome when $\boldsymbol{\beta}^*$ is played in $\Gamma(\mathbf{t})$. (So $p = p_K$.) Since bidder ℓ loses with positive probability, k exists such that $\ell \in D_k$. By Lemma 5 we have

$$\pi^\ell\left(t_\ell,\, \boldsymbol{\tau}_{-\ell}(p_k, h_{k-1})\right) = p_k. \tag{12}$$

Turning to bidder w, since he wins with positive probability, he never drops out or he drops out in round K. Hence, $\pi^w(t_w, \boldsymbol{\tau}_{-w}(p_K, h)) \geq p_K$. This implies, by continuity and the fact that $\pi^w(\cdot)$ is bounded above, that $\hat{p} \geq p_K$ exists such that

$$\pi^w(t_w, \boldsymbol{\tau}_{-w}(\hat{p}, h)) = \hat{p}. \tag{13}$$

By P1-P2, (12) implies $\pi^\ell(t_\ell, \boldsymbol{\tau}_{-\ell}(\hat{p}, h)) \leq \hat{p}$, and hence (13) yields

$$\pi^w(t_w, \boldsymbol{\tau}_{-w}(\hat{p}, h)) \geq \pi^\ell(t_\ell, \boldsymbol{\tau}_{-\ell}(\hat{p}, h)). \tag{14}$$

For $i \neq w$, bidder i drops out in round K or earlier, and so $\tau_i(\hat{p}, h) = t_i$. By P1 and (13) we have $\tau_w(\hat{p}, h) = t_w$. So (14) yields $\pi^w(\mathbf{t}) \geq \pi^\ell(\mathbf{t})$, proving (ii). ■

6 Equilibrium Efficiency

We now turn to the question of whether the equilibrium $\boldsymbol{\beta}^*$ is ex post efficient. By "ex post" we mean after the auction is held, but before the ensuing risk is realized. We start by characterizing this notion of efficiency in our environment.

The play of an English auction determines an allocation specifying which bidder wins and how much he pays; the losers pay zero. This allocation is efficient, given the preferences determined by a type vector $\mathbf{t} \in \mathbf{T}$, provided it is not Pareto dominated by any other feasible allocation. Upon reflection it is easy to see that an auction outcome is efficient at $\mathbf{t}$ if and only if there is no subsequent trade between the winning bidder and any losing bidder that makes them both better off, at least one of them strictly.[19] The most a losing bidder ℓ is willing to pay the winner for the asset is $\pi^\ell(\mathbf{t})$. Thus, an allocation is efficient at $\mathbf{t}$ if and only if the winner is better off keeping the asset than selling it to any other bidder ℓ for the price $\pi^\ell(\mathbf{t})$. Accordingly, we say that an auction allocation in which bidder w wins and pays price p is *efficient at* $\mathbf{t}$ if

$$\text{(EFF)} \quad \mathbb{E}u^w(v^w(\mathbf{t}) + \tilde{z}_w - p, t_w) \geq u^w(\pi^\ell(\mathbf{t}) - p, t_w) \text{ for all } \ell \neq w.^{20}$$

An auction allocation is *ex post efficient* if it is efficient at every $\mathbf{t} \in \mathbf{T}$.

Remark 3 *The preferences used in* EFF *are conditional on the entire vector* $\mathbf{t}$, *as though all the bidders' types are revealed when the auction ends. However, unless there is a tie*

[19] This is true under the assumption that the seller has zero value for the asset, which we now assume. Efficiency thus does not require the asset to be returned to the seller.

[20] When all bidders are risk neutral, the inequality in EFF is $v^w(\mathbf{t}) \geq v^\ell(\mathbf{t})$. This is just the familiar observation that in quasilinear utility settings, efficiency amounts to selling the asset to a bidder who values it the most, thereby maximizing the "surplus".

the winning bidder never drops out, and so does not reveal his type. In this case a losing bidder may not know $\pi^\ell(\mathbf{t})$. *Nonetheless, as a normative criterion, it seems reasonable for the preferences to be evaluated conditional on* $\mathbf{t}$ *being common knowledge, and this is standard practice in the literature. As a positive criterion to justify a prediction that no trade would occur after the auction is over, note that if EFF is commonly known to hold, the no-trade theorem (e.g., Milgrom and Stokey, 1982) implies that the equilibrium of any bargaining game between the winner and a loser results in no trade.*

An auction allocation is efficient at $\mathbf{t}$ if the willingness to pay for the asset of each losing bidder ℓ is less than the sale price p. This is because then Corollary 1 implies $\pi^\ell(\mathbf{t}) \leq p \leq \pi^w(\mathbf{t})$, and so EFF holds:

$$\begin{aligned}
\mathbb{E}u^w(v^w(\mathbf{t}) + \tilde{z}_w - p, t_w) &\geq \mathbb{E}u^w(v^w(\mathbf{t}) + \tilde{z}_w - \pi^w(\mathbf{t}), t_w) \\
&= u^w(0, t_w) \\
&\geq u^w(\pi^\ell(\mathbf{t}) - p, t_w).
\end{aligned}$$

However, $\pi^\ell(\mathbf{t}) > p$ is possible, especially if bidder ℓ is the last to drop out. In this case the equilibrium allocation may be ex post inefficient. Consider the following example.

Example 2 *Each of two bidders has the utility function,* $u^i(x, t_i) = u(x)$, *and exhibit increasing absolute risk aversion (IARA):* $-u''/u'$ *is an increasing function. The bidders also have the same value function,* $v^i(\mathbf{t}) = t_1 + t_2$.[21] *The ensuing risks are the same nondegenerate random variable,* $\tilde{z}_1 = \tilde{z}_2 = \tilde{z}$. *We now show that for any* $\mathbf{t} \in \mathbf{T}$ *for which* $t_1 \neq t_2$, *the allocation that results when* $\boldsymbol{\beta}^*$ *is played in* $\Gamma(\mathbf{t})$ *is not efficient at* $\mathbf{t}$.

For each i, *the willingness-to-pay* $\pi^i(\mathbf{t})$ *is determined by*

$$\mathbb{E}u(t_1 + t_2 + \tilde{z} - \pi^i(\mathbf{t})) = u(0), \tag{15}$$

and so $\pi^i(\mathbf{t}) = t_1 + t_2 + c =: \pi(\mathbf{t})$ *for some constant* c. *The equilibrium strategy of type* t_i *of bidder* i *is to exit at price* $\beta_i^*(t_i) = \pi(t_i, t_i)$. *The bidder with the highest type therefore wins. Suppose* $t_2 > t_1$. *Then bidder* 2 *wins and pays the price* $p = \pi(t_1, t_1)$. *Ex post, bidder* 1 *is willing to pay* $\pi(\mathbf{t}) > p$ *for the asset. Because of IARA, adding the positive amount* $\pi(\mathbf{t}) - p$ *to the arguments on both sides of (15) yields*

$$\mathbb{E}u(t_1 + t_2 + \tilde{z} - p) < u(\pi(\mathbf{t}) - p),$$

since the bidder is more risk averse at higher incomes. Thus, EFF fails to hold.

[21] This common value setting does not strictly satisfy assumption A2. However, the ex post inefficiency of this example also obtains if $v^i(\mathbf{t}) = t_i + (1 - \varepsilon)t_{-i}$ for sufficiently small $\varepsilon > 0$, which does satisfy A2.

The inefficiency in Example 2 is driven by three forces. First, a bidder's value increases in the other's type. This causes the losing bidder's willingness to pay for the asset once he learns the winner's type to be higher than his bid, and hence the sale price, because his bid is equal to his willingness to pay given an underestimate of the winner's type. Second, the winner's increasing risk aversion creates an adverse income effect. The winner's positive information rent, $\pi(\mathbf{t}) - p$ in monetary units, causes him to become more risk averse, and therefore willing to sell the asset for a price lower than what he was initially willing to pay for it. Third, there is ensuing risk, as without it the increasing risk aversion would play no role. Our final theorem shows that ex post efficiency obtains if any one of these three forces is absent.

Theorem 2 *Let $\mathbf{t} \in \mathbf{T}$, and let $w \in N$ be a bidder who wins with positive probability when $\boldsymbol{\beta}^*$ is played in $\Gamma(\mathbf{t})$. The resulting allocation is efficient at $\mathbf{t}$ if one of the following is true:*

(i) $v_w^\ell \leq 0$ *for all* $\ell \in N \backslash \{w\}$;

(ii) $-u_{11}^w(\cdot, t_w)/u_1^w(\cdot, t_w)$ *is nonincreasing for all* $t_w \in T_w$ *(NIARA); or*

(iii) $\tilde{z}^w = 0$ *almost surely.*

Proof. Let $h = \{(p_k, D_k)\}_{k=1}^K$ be the outcome when $\boldsymbol{\beta}^*$ is played in $\Gamma(\mathbf{t})$. (So $p = p_K$.)

(i) Let $\ell \in N \backslash \{w\}$. In this case we have $\pi_w^\ell = v_w^\ell \leq 0$, using (2). Because bidder ℓ loses with positive probability, $\ell \in D_k$ for some $k \leq K$. Hence, $\pi^\ell\left(t_\ell, \boldsymbol{\tau}_{-\ell}\left(p_k, h_{k-1}\right)\right) = p_k$. From this and P1-P2 we obtain

$$\pi^\ell\left(t_\ell, \, \boldsymbol{\tau}_{-\ell}\left(p_K, h\right)\right) \leq p_K. \tag{16}$$

All losing bidders have revealed their types by round K, and so

$$\left(t_\ell, \boldsymbol{\tau}_{-\ell}\left(p_K, h\right)\right) = \left(t_\ell, \tau_w(p_K, h), \mathbf{t}_{-\ell w}\right).$$

Thus, since $t_w \geq \tau_w\left(p_K, h\right)$, from (16) and $\pi_w^\ell \leq 0$ we obtain

$$\pi^\ell(\mathbf{t}) \leq \pi^\ell\left(t_\ell, \boldsymbol{\tau}_{-\ell}\left(p_K, h\right)\right) \leq p_K.$$

As noted in the text, this and Corollary 1 imply EFF.

(ii) We know EFF holds if $p \geq \pi^\ell(\mathbf{t})$ for all $\ell \neq w$. So suppose $p < \pi^\ell(\mathbf{t})$ for some $\ell \neq w$. Recall that $\pi^w(\mathbf{t})$ satisfies

$$\mathbb{E}u^w\left(v^w(\mathbf{t}) + \tilde{z}_w - \pi^w(\mathbf{t}), t_w\right) = u^w(0, t_w).$$

Hence, since $\pi^w(\mathbf{t}) \geq \pi^\ell(\mathbf{t})$ by Corollary 1, we have

$$\mathbb{E}u^w\left(v^w(\mathbf{t}) + \tilde{z}_w - \pi^\ell(\mathbf{t}), t_w\right) \geq u^w(0, t_w).$$

As u^w satisfies NIARA in this case, adding the positive amount $\pi^\ell(\mathbf{t}) - p$ to the first argument on both sides of this inequality yields EFF:

$$\mathbb{E}u^w\left(v^w(\mathbf{t}) + \tilde{z}_w - p, t_w\right) \geq u^w(\pi^\ell(\mathbf{t}) - p, t_w).$$

(iii) Since $\tilde{z}^w = 0$ a.e., (1) implies $\pi^w(\mathbf{t}) = v^w(\mathbf{t})$. We thus have $v^w(\mathbf{t}) \geq v^\ell(\mathbf{t})$ for all $\ell \neq w$ by Corollary 1. Hence, for all $\ell \neq w$ we have EFF:

$$\begin{aligned}\mathbb{E}u^w(v^w(\mathbf{t}) + \tilde{z}_w - p, t_w) &= u^w(v^w(\mathbf{t}) - p, t_w) \\ &\geq u^w(\pi^\ell(\mathbf{t}) - p, t_w). \blacksquare\end{aligned}$$

7 Conclusion

English auctions are often used to sell objects of uncertain post-auction value to buyers who are not risk neutral. We have sought sufficient conditions for such auctions to have efficient ex post equilibria in realistic settings with asymmetries and interdependencies. Previous work on this issue focused on environments with risk neutral bidders; in these environments any ensuing risk borne by a winning bidder is irrelevant. One contribution of this paper has been the formulation of sufficient conditions for a much larger class of environments in which bidders have heterogeneous ensuing risks and risk attitudes.

The essence of Theorem 1 is that if the bidders' willingness-to-pay functions satisfy our weighted average crossing condition, then an English auction has an ex post equilibrium in which the asset is always sold to a bidder who has the highest willingness-to-pay. The bidders' willingness-to-pay functions satisfy the weighted average crossing condition if their expected value functions for the asset satisfy it (A2), and if each bidder's risk tolerance is nondecreasing in his type (A1). Thus, A1-A2 are together sufficient for these ex post equilibria to exist.

Selling the asset to a bidder who has the highest willingness-to-pay may not result in an efficient allocation. We presented an example in which the equilibrium allocation is inefficient at all type vectors that do not give rise to a tie. However, Theorem 2 shows that an English auction has an efficient ex post equilibrium if, in addition to A1-A2, the bidders exhibit nonincreasing absolute risk aversion, or if each bidder's expected value for the asset is nonincreasing in the other bidders' types, or if the asset is riskless.

We have also made methodological contributions. While the English auction literature, especially Krishna (2003), has formulated versions of single crossing similar to our weighted average crossing condition, to the best of our knowledge the demonstration that these conditions imply the Jacobian matrix of the vector-valued willingness-to-pay function is a P-matrix is novel. This result implies that the willingness-to-pay function is globally invertible, a result that may be useful in other contexts.

Appendix A. DWAC Matrices

In this appendix we provide results about $n \times n$ matrices $\mathbf{A} = [a_{ij}]$ that satisfy the dominant weighted average condition:

(DWAC) Positive weights $\theta_1, \ldots, \theta_n$ exist that sum to one and satisfy, for all $i \neq j$,

$$\sum_{k=1}^{n} \theta_k a_{kj} > \max\{0, a_{ij}\}.$$

Lemma A.1 *If* $\mathbf{A}$ *is a DWAC matrix, then* $a_{jj} > \max\{0, a_{ij}\}$ *for all* $i \neq j$.

Proof. A weighted average of a column cannot be larger than the column's largest element. Hence, the DWAC inequalities imply that each diagonal element of the matrix must be the largest in its column, and hence positive. ■

Lemma A.2 *Each principal submatrix of a DWAC matrix is also a DWAC matrix.*

Proof. Let $\mathbf{A} = [a_{ij}]$ be an $n \times n$ DWAC matrix. If $n = 1$, the only submatrix is $\mathbf{A}$ itself, and it is DWAC by hypothesis. So assume $n \geq 2$. We show that the principal submatrix $\mathbf{A}_n$ of $\mathbf{A}$ obtained by deleting the n^{th} row and column satisfies DWAC. Applying the same argument to the other size $n-1$ principal submatrices, and then repeated applications of the argument to smaller and smaller principal submatrices, then shows that all the principal submatrices satisfy DWAC.

Let $\theta_1, \ldots, \theta_n$ be the weights that with $\mathbf{A}$ satisfy the DWAC inequalities. Define $\hat{\theta}_1, \ldots, \hat{\theta}_{n-1}$ by

$$\hat{\theta}_k := \frac{\theta_k}{1 - \theta_n}.$$

These new weights are positive and sum to one. Let $\hat{W}_j := \sum_{k=1}^{n-1} \hat{\theta}_k a_{kj}$ for every $j = 1, ..., n-1$, and $W_j := \sum_{k=1}^{n} \theta_k a_{kj}$ for $j = 1, \ldots, n$. Because $\mathbf{A}$ is DWAC, for any $j < n$ we have $a_{nj} < W_j$. This and $\theta_n > 0$ imply

$$\begin{aligned} W_j &= (1 - \theta_n)\hat{W}_j + \theta_n a_{nj} \\ &< (1 - \theta_n)\hat{W}_j + \theta_n W_j. \end{aligned}$$

Thus, since $\theta_n < 1$, we have $\hat{W}_j > W_j$ for all $j < n$. This and the fact that $W_j > \max\{0, a_{ij}\}$ for all $i \neq j$ imply

$$\hat{W}_j > \max\{0, a_{ij}\} \text{ for all } i, j < n, \ i \neq j.$$

So $\mathbf{A}_n$ indeed satisfies DWAC. ■

Say that a square matrix is a *P-matrix* if all its principal minors, i.e., determinants of its principal submatrices, are positive. The following lemma shows that any DWAC matrix is a P-matrix,[22] together with a useful result on the inverse of the matrix.

Lemma A.3 *Any DWAC matrix* $\mathbf{A} = [a_{ij}]$ *is a P-matrix and satisfies* $\mathbf{A}^{-1}\mathbf{e} \gg \mathbf{0}$.

Proof. We prove this by induction on n, starting with $n = 2$.[23] In this case the principal minors of $\mathbf{A}$ are a_{11} and a_{22}, and (6) implies both are positive. So $\mathbf{A}$ is a P-matrix if its determinant, $|\mathbf{A}| = a_{11}a_{22} - a_{21}a_{12}$, is positive. Since $a_{11}a_{22} > 0$, we have $|\mathbf{A}| > 0$ if $a_{12}a_{21} \leq 0$. If instead both a_{12} and a_{21} are positive, then $|\mathbf{A}| > 0$ because (6) implies $a_{11} > a_{21}$ and $a_{22} > a_{12}$. Lastly, if both a_{12} and a_{21} are negative, then DWAC implies $\theta_1 a_{11} > \theta_2 |a_{21}|$ and $\theta_2 a_{22} > \theta_1 |a_{12}|$, where θ_1 and θ_2 are the positive weights in DWAC. Hence, in this case we have

$$(\theta_1 a_{11})(\theta_2 a_{22}) > (\theta_2 |a_{21}|)(\theta_1 |a_{12}|) \quad \Rightarrow \quad |\mathbf{A}| > 0.$$

Thus, $\mathbf{A}$ is a P-matrix. This induction step is completed by noting that

$$\begin{aligned}\mathbf{A}^{-1}\mathbf{e} &= |\mathbf{A}|^{-1} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= |\mathbf{A}|^{-1} \begin{pmatrix} a_{22} - a_{12} \\ a_{11} - a_{21} \end{pmatrix} \gg \begin{pmatrix} 0 \\ 0 \end{pmatrix}.\end{aligned}$$

Now suppose the conclusions of the lemma hold for all DWAC matrices of size $n-1$ or less, and let $\mathbf{A} = [a_{ij}]$ be an $n \times n$ DWAC matrix. By Lemma A.1, the principal submatrices of $\mathbf{A}$ satisfy DWAC. The induction hypothesis thus implies that all the principal submatrices other than $\mathbf{A}$ itself are P-matrices, and their inverses map vectors of 1's into strictly positive vectors. We must show only that $|\mathbf{A}| > 0$ (and so $\mathbf{A}$ is a P-matrix) and $\mathbf{A}^{-1}\mathbf{e} \gg \mathbf{0}$.

Let $\theta_1, \ldots, \theta_n$ be the weights that with $\mathbf{A}$ satisfy the DWAC inequalities, and define $W_j := \sum_{k=1}^{n} \theta_k a_{kj}$ for $j = 1, \ldots, n$. Define also new weights

$$\hat{\theta}_k := \frac{\theta_k}{1 - \theta_n} \text{ for } k = 1, ..., n-1,$$

and weighted averages

$$\hat{W}_j := \sum_{k=1}^{n-1} \hat{\theta}_k a_{kj} \text{ for } j = 1, ..., n.$$

[22]McKenzie (1960) shows that a matrix satisfying a (generalized) diagonal dominance condition with positive diagonal entries is a P-matrix. However, this condition is not implied by DWAC, and so McKenzie's result does not imply that a DWAC matrix is a P-matrix.

[23]The lemma is vacuously true for $n = 1$, since by definition $\mathbf{A} = [a_{11}]$ is both DWAC and a P-matrix if and only if $a_{11} > 0$.

An alternative way of writing $\hat{W}_j$ is

$$\hat{W}_j = \frac{1}{1-\theta_n}\left(W_j - \theta_n a_{nj}\right). \tag{17}$$

Since $\mathbf{A}$ is DWAC, we have $W_n < a_{nn}$ and $W_j > a_{nj}$ for $j < n$. These inequalities together with (17) imply $\hat{W}_n < a_{nn}$ and $\hat{W}_j > a_{nj}$ for $j < n$.

We shall prove $\mathbf{A}^{-1}\mathbf{e} \gg \mathbf{0}$ by showing that the solution to $A\mathbf{x} = \mathbf{e}$ satisfies $\mathbf{x} \gg \mathbf{0}$. Let $(\mathbf{A}|\mathbf{e})$ be the augmented matrix of this system of equations. We apply the following row transformations. The first one consists of subtracting, for each $k < n$, a multiple $\hat{\theta}_k$ of the k^{th} row from the n^{th} row. This transforms the n^{th} row of $(\mathbf{A}|\mathbf{e})$ to

$$(a_{n1} - \hat{W}_1, ..., a_{nn} - \hat{W}_n|0).$$

Then, for each $k < n$, subtract a multiple $a_{kn}/\left(a_{nn} - \hat{W}_n\right)$ of this transformed n^{th} row from the k^{th} row. This transforms $(\mathbf{A}|\mathbf{e})$ to

$$\left(\begin{array}{cccc|c} \hat{a}_{11} & & \hat{a}_{1,n-1} & 0 & 1 \\ & \ddots & & \vdots & \vdots \\ \hat{a}_{n-1,1} & & \hat{a}_{n-1,n-1} & 0 & 1 \\ a_{n1} - \hat{W}_1 & \cdots & a_{n,n-1} - \hat{W}_{n-1} & a_{nn} - \hat{W}_n & 0 \end{array}\right), \tag{18}$$

where

$$\hat{a}_{kj} = a_{kj} + a_{kn}\frac{\hat{W}_j - a_{nj}}{a_{nn} - \hat{W}_n} \text{ for } k < n. \tag{19}$$

Let $\hat{\mathbf{A}}$ denote the top left $(n-1)\times(n-1)$ matrix $[\hat{a}_{ij}]$ in (18). As these row transformations preserved the value of the determinant $|\mathbf{A}|$, we see from (18) that $|\mathbf{A}| = \left(a_{nn} - \hat{W}_n\right)\left|\hat{\mathbf{A}}\right|$. Thus, if $\hat{\mathbf{A}}$ is DWAC, the induction hypothesis implies it is a P-matrix, and we have $|\mathbf{A}| > 0$ because $a_{nn} > \hat{W}_n$. Furthermore, from (18) we see that the system $\mathbf{Ax} = \mathbf{e}$ reduces to $\hat{\mathbf{A}}\hat{\mathbf{x}} = \hat{\mathbf{e}}$, where $\hat{\mathbf{x}} = (x_1, \ldots, x_{n-1})^T$ and $\hat{\mathbf{e}}$ is the $(n-1)$-vector of 1's, and that x_n satisfies

$$x_n = \sum_{j=1}^{n-1}\left(\frac{\hat{W}_j - a_{nj}}{a_{nn} - \hat{W}_n}\right)x_j. \tag{20}$$

If $\hat{\mathbf{A}}$ is DWAC, the induction hypothesis and $\hat{\mathbf{A}}\hat{\mathbf{x}} = \hat{\mathbf{e}}$ imply $\hat{\mathbf{x}} \gg \mathbf{0}$. Also in this case, as each coefficient in (20) is positive, we have $x_n > 0$. The lemma is therefore proved by showing that $\hat{\mathbf{A}}$ is DWAC.

Multiply both sides of (19) by $\hat{\theta}_k$ and sum over k to obtain

$$\sum_{k=1}^{n-1}\hat{\theta}_k\hat{a}_{kj} = \hat{W}_j + \hat{W}_n\frac{\hat{W}_j - a_{nj}}{a_{nn} - \hat{W}_n}.$$

Using (17) to substitute out $\hat{W}_j$ and $\hat{W}_n$ yields, after some algebra,

$$\sum_{k=1}^{n-1} \hat{\theta}_k \hat{a}_{kj} = W_j + W_n \frac{W_j - a_{nj}}{a_{nn} - W_n}.$$

This expression is positive, since $\mathbf{A}$ satisfying DWAC implies that W_j, W_n, $W_j - a_{nj}$, and $a_{nn} - W_n$ are all positive. Now, let $j < n$ and $i \notin \{j, n\}$, and note from (19) and (17) that

$$\begin{aligned} \hat{a}_{ij} &= a_{ij} + a_{in} \frac{W_j - \theta_n a_{nj} - (1 - \theta_n) a_{nj}}{(1 - \theta_n) a_{nn} - (W_n - \theta_n a_{nn})} \\ &= a_{ij} + a_{in} \frac{W_j - a_{nj}}{a_{nn} - W_n}. \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{k=1}^{n-1} \hat{\theta}_k \hat{a}_{kj} - \hat{a}_{ij} &= W_j + W_n \frac{W_j - a_{nj}}{a_{nn} - W_n} - a_{ij} - a_{in} \frac{W_j - a_{nj}}{a_{nn} - W_n} \\ &= (W_j - a_{ij}) + (W_n - a_{in}) \frac{W_j - a_{nj}}{a_{nn} - W_n}, \end{aligned}$$

which is also positive because $\mathbf{A}$ satisfies DWAC. We conclude that for all $i, j < n$, $i \neq j$,

$$\sum_{k=1}^{n-1} \hat{\theta}_k \hat{a}_{kj} > \max\{0, \hat{a}_{ij}\},$$

and so $\hat{\mathbf{A}}$ indeed satisfies DWAC. ■

Appendix B. Proofs not in the Text

Proof of Lemma 1. Let $t' \in T$ satisfy $t' > t$, and assume the expectation $\mathbb{E}u(\tilde{x}, t')$ exists. A function $\varphi : X \to \mathbb{R}$ is well-defined by

$$\varphi(x) := \frac{u(x, t')}{u_1(a, t')} - \frac{u(x, t)}{u_1(a, t)},$$

since $u_1 > 0$ on $X \times T$. As u_1 is log-supermodular, for any $x \in X$ we have

$$(x - a) \varphi'(x) = (x - a) \left(\frac{u_1(x, t')}{u_1(a, t')} - \frac{u_1(x, t)}{u_1(a, t)} \right) \geq 0.$$

Integrating φ' on $[a, x]$ thus yields $\varphi(x) \geq \varphi(a)$. This inequality rearranges to

$$\frac{u(x, t') - u(a, t')}{u_1(a, t')} \geq \frac{u(x, t) - u(a, t)}{u_1(a, t)}.$$

Now replace x by $\tilde{x}$, take expectations, and use $\mathbb{E}u(\tilde{x}, t) = u(a, t)$ to obtain

$$\frac{\mathbb{E}u(\tilde{x}, t') - u(a, t')}{u_1(a, t')} \geq 0.$$

Since $u_1(a, t') > 0$, this proves (i). Part (ii) follows immediately from (i). $\blacksquare$

We now turn to Lemma 4. A key part of its proof are the following two preliminary lemmas.

Lemma B.1. *Let $\hat{p} \geq 0$ and $\mathbf{m} : [\hat{p}, \infty) \to \mathbf{T}$ be any continuous function with a right-side derivative at all $p \geq \hat{p}$ satisfying $m_i'(p) \geq 0$ for all $i \in N$. Letting $\frac{d}{dp}\pi^i(\mathbf{m}(p))$ denote the right-side derivative of $\pi^i(\mathbf{m}(\cdot))$ at p, suppose that for all $i \in N$,*

$$m_i'(p) = 0 \text{ or } \tfrac{d}{dp}\pi^i(\mathbf{m}(p)) = 1.$$

Then, for all $i \in N$,

$$m_i'(p) = 0 \quad \Rightarrow \quad \tfrac{d}{dp}\pi^i(\mathbf{m}(p)) < 1.$$

Proof. For the given p we write $\mathbf{m} = \mathbf{m}(p)$ and $\mathbf{m}' = \mathbf{m}'(p)$. Let A be the set of all $i \in N$ such that $m_i' \neq 0$. Hence, $m_i' = 0$ if and only if $i \notin A$, and $\frac{d}{dp}\pi^i(\mathbf{m}) = 1$ for all $i \in A$. Re-indexing if necessary, we can assume $A = \{1, ..., a\}$. The lemma holds trivially if $A \in \{\varnothing, N\}$, so we can assume $1 \leq a < n$. We have

$$\frac{d}{dp}\pi^i(\mathbf{m}) = \pi_1^i(\mathbf{m})m_1' + ... + \pi_a^i(\mathbf{m})m_a' = 1, \quad i \in A, \tag{21}$$

$$\frac{d}{dp}\pi^\ell(\mathbf{m}) = \pi_1^\ell(\mathbf{m})m_1' + ... + \pi_a^\ell(\mathbf{m})m_a' =: c_\ell, \quad \ell \notin A. \tag{22}$$

Letting $\ell \notin A$, we must show that $c_\ell < 1$. Lemma 2 implies $\left[\pi_j^i(\mathbf{m})\right]$ is DWAC, and so Lemma A.2 in Appendix A implies the principal submatrix $\left[\pi_j^i(\mathbf{m})\right]_{i,j \in A \cup \{\ell\}}$ is DWAC. Letting $(\theta_1, \ldots, \theta_a, \theta_\ell) \gg \mathbf{0}$ be the weights the definition of DWAC for the principal submatrix requires, it is easy to show that for $\hat{\theta}_i := \theta_i / (1 - \theta_\ell)$ we have $\sum_{i=1}^a \hat{\theta}_i = 1$ and, for all $j \neq \ell$,

$$\pi_j^\ell(\mathbf{m}) < \hat{W}_j := \sum_{i=1}^a \hat{\theta}_i \pi_j^i(\mathbf{m}).$$

Multiply the i^{th} equation in (21) by $\hat{\theta}_i$ and then subtract them all from the ℓ^{th} equation in (22) to obtain

$$\sum_{j=1}^a \left(\pi_j^\ell(\mathbf{m}) - \hat{W}_j\right) m_j' = c_\ell - 1. \tag{23}$$

Because $\mathbf{m}' \geq \mathbf{0}$, (21) implies $m_j' > 0$ for some $j \in A$. The left side of (23) is therefore negative. Hence, $c_\ell < 1$. $\blacksquare$

The bulk of Lemma 4 is established in the following preliminary lemma.

Lemma B.2. *Let $\hat{p} \geq 0$, $D \subset N$, $\hat{\mathbf{t}} \in \mathbf{T}$, and suppose*

$$\pi^i(\hat{\mathbf{t}}) \begin{cases} \leq \hat{p} & \text{if } \hat{t}_i > \underline{t}_i \\ \geq \hat{p} & \text{if } \hat{t}_i < \bar{t}_i \end{cases} \quad \textit{for all } i \notin D. \tag{24}$$

Then a continuous, right differentiable, and nondecreasing $\mathbf{m} : [\hat{p}, \infty) \to \mathbf{T}$ exists that satisfies $\mathbf{m}(\hat{p}) = \hat{\mathbf{t}}$, $\mathbf{m}_D(\cdot) \equiv \hat{\mathbf{t}}_D$, and

$$\pi^i(\mathbf{m}(p)) \begin{cases} \leq p & \text{if } m_i(p) > \underline{t}_i \\ \geq p & \text{if } m_i(p) < \bar{t}_i \end{cases} \quad \text{for all } i \notin D \text{ and } p \geq \hat{p}. \tag{25}$$

As the construction of $\mathbf{m}$ in the proof of Lemma B.2 is rather involved, we illustrate it informally by an example before presenting the proof.

Example B.1 This is a two-bidder example, with $\mathbf{T} = [0,1]^2$, $\pi^1(\mathbf{t}) = 2 + 2t_1 - t_2$, and $\pi^2(\mathbf{t}) = 1 + t_2$. The image $\boldsymbol{\pi}(\mathbf{T})$ of $\boldsymbol{\pi}$ is the parallelogram in Figure B.1. We illustrate the construction of $\mathbf{m}$ for the case $(\hat{p}, D, \hat{\mathbf{t}}) = (0, \varnothing, \mathbf{0})$. The construction progresses from small to large values of p, as shown in the figure.[24]

(a) $p = 0$. The desired initial condition, $\mathbf{m}(\hat{p}) = \hat{\mathbf{t}}$, yields $\mathbf{m}(0) := \mathbf{0}$.

(b) $0 < p \leq 1$. If $p < 1$, each bidder's willingness-to-pay exceeds p : $\boldsymbol{\pi}(\mathbf{t}) \gg p\mathbf{e}$ for all $\mathbf{t} \in \mathbf{T}$, where $e = (1,1)$. Thus, the only way to satisfy (25) with a continuous function is to define it on $(0,1]$ as $\mathbf{m}(p) := \underline{\mathbf{t}} = \mathbf{0}$.

(c) $1 < p < \frac{3}{2}$. To make $\mathbf{m}$ continuous at 1, we must have $\mathbf{m}(p) \approx \mathbf{0}$ for $p \approx 1$. We cannot have $\mathbf{m}(p) = \mathbf{0}$, since then $\pi^2(\mathbf{m}(p)) = 1 < p$, and (25) could not hold. We also cannot have $\boldsymbol{\pi}(\mathbf{m}(p)) = p\mathbf{e}$, since $p < \frac{3}{2}$ implies $p\mathbf{e} \notin \boldsymbol{\pi}(\mathbf{T})$. The remaining possibility is $m_1(p) = 0$ and $m_2(p) > 0$, so that $\pi^1(\mathbf{m}(p)) > p$ and $\pi^2(\mathbf{m}(p)) = p$. This implies $\mathbf{m}(p) := (0,\, p - 1)$, and it satisfies (25).

(d) $\frac{3}{2} \leq p \leq 2$. In this case $p\mathbf{e} \in \boldsymbol{\pi}(\mathbf{T})$. By Lemma 3, $\boldsymbol{\pi}$ has an inverse defined on $\boldsymbol{\pi}(\mathbf{T})$. Hence, we can define $\mathbf{m}(p) := \boldsymbol{\pi}^{-1}(p\mathbf{e})$, which in this example is

$$\mathbf{m}(p) := (p - \tfrac{3}{2},\; p - 1).$$

This choice satisfies (25) trivially because $\pi^i(\mathbf{m}(p)) = p$ for each i.

---

[24] The minimum type function for the null history, $\boldsymbol{\tau}(\cdot, h_0)$, is this $\mathbf{m}$. The corresponding equilibrium bidding functions in this example can be obtained from it using (7), and are $\beta_1(t_1) = t_1 + \frac{3}{2}$ if $t_1 \leq \frac{1}{2}$, $\beta_1(t_1) = 2t_1 + 1$ if $t_1 > \frac{1}{2}$, and $\beta_2(t_2) = 1 + t_2$. Other, outcome-equivalent ex post equilibria are obtained by replacing β_1 on $(\frac{1}{2}, 1]$ by any function $\hat{\beta}_1$ satisfying $\hat{\beta}_1(\cdot) \geq 2$.

(e) $2 < p \leq 3$. Since $m_2(2) = 1 = \bar{t}_2$ and m_2 must be nondecreasing, we define $m_2(p) := 1$ for $p > 2$. Then, defining $m_1(p) := \frac{1}{2}(p-1)$ maintains $\pi^1(\mathbf{m}(p)) = p$, and hence (25).

(f) $3 < p$. As $\mathbf{m}(3) = \bar{\mathbf{t}}$, we must define $\mathbf{m}(p) := \bar{\mathbf{t}}$ for all $p > 3$.

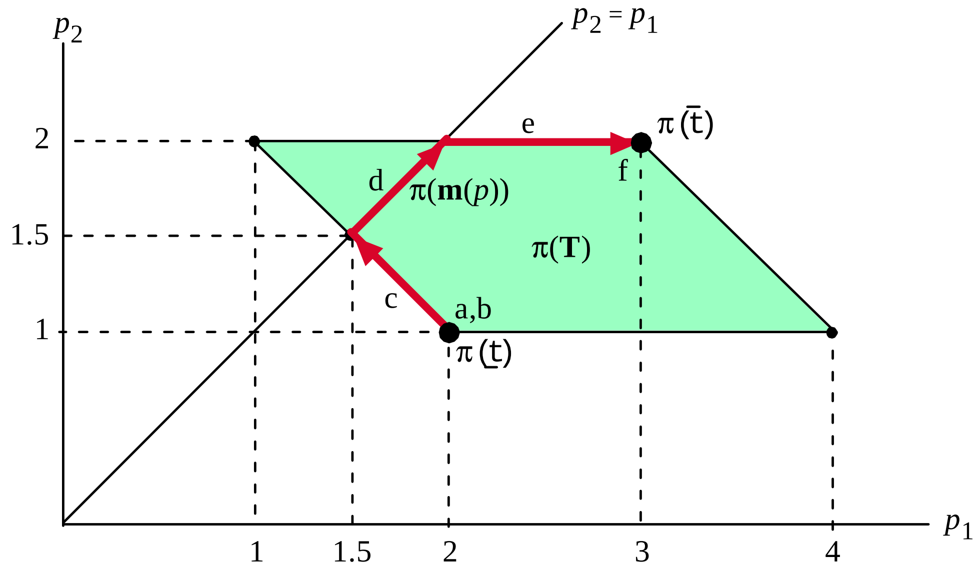

Figure B.1

Proof of Lemma B.2. Set $\mathbf{m}_D(p) := \hat{\mathbf{t}}_D$ for all $p \geq \hat{p}$. Letting $A := N \backslash D$ and $\mathbf{m} = (\mathbf{m}_A, \mathbf{m}_D)$, it remains to find $\mathbf{m}_A$ so that $\mathbf{m}$ satisfies the desired properties. We do so in a finite number of steps, all of which are iterations of the following:

Step 0. Show the existence of $q > \hat{p}$ and a function $\mathbf{m}_A : [\hat{p}, q] \to \mathbf{T}_A$ that is continuous nondecreasing, right differentiable on $[\hat{p}, q)$, satisfies $\mathbf{m}_A(\hat{p}) = \hat{\mathbf{t}}_A$, and yields a function $\mathbf{m} = (\mathbf{m}_A, \mathbf{m}_D)$ that satisfies (25) on $[\hat{p}, q]$ and, if $q < \infty$, the following property:

$$I(q) \subseteq I(\hat{p}), \text{ and } M(q) \subset M(\hat{p}) \text{ if } I(q) = I(\hat{p}), \tag{*}$$

where for any $p \in [\hat{p}, q]$, $M(p)$ and $I(p)$ are the sets

$$M(p) := \left\{ i \in A : \pi^i(\mathbf{m}(p)) = p, \ m_i(p) < \bar{t}_i \right\} \text{ (marginal bidders)},$$

$$I(p) := \left\{ i \in A : \pi^i(\mathbf{m}(p)) > p \right\} \text{ (inframarginal bidders)}.$$

Accomplishing Step 0 proves the lemma if $q = \infty$. If $q < \infty$, then $(\mathbf{m}(q), q)$ satisfies (25), and so (24) holds with $(\mathbf{m}(q), q)$ replacing $(\hat{\mathbf{t}}, \hat{p})$. This allows us to make Step 1 a repeat of Step 0, with this replacement of $(\hat{\mathbf{t}}, \hat{p})$ by $(\mathbf{m}(q), q)$, yielding $q^1 > q$ and a function $\mathbf{m}_A : [q, q^1] \to \mathbf{T}_A$ satisfying the desired properties on $[q, q^1]$, as well as (*) with (q, q^1) replacing $(\hat{p}, q)$ if $q^1 < \infty$. Gluing this function to the one found in Step 0 results in the desired $\mathbf{m}_A$ defined on the larger interval $[\hat{p}, q^1]$. This proves the lemma if $q^1 = \infty$. Otherwise, since (*) will then hold at (q, q^1), we can repeat Step 0 yet again, with $(\mathbf{m}(q^1), q^1)$ replacing $(\hat{\mathbf{t}}, \hat{p})$, to obtain $q^2 > q^1$. We continue to iterate until we obtain $q^k = \infty$ for some $k < \infty$, at which point we are done. The iteration ends in a finite number of steps because of property (*). To see why, note that repeated applications of (*) shows that the sequence of integer pairs,

$$\left\{(i_k, m_k) : i_k = \left|I(q^k)\right|, \, j_k = \left|M(q^k)\right|, \, q^k < \infty\right\},$$

is strictly decreasing in the lexicographic ordering.[25] If this is an infinite sequence, it would reach the lower bound $(0, 0)$ in a finite number of steps, say k. But then the next iteration would yield $q^{k+1} = \infty$, because (*) cannot hold when $(q, \hat{p})$ is replaced by (q^k, q^{k+1}), for any finite q^{k+1}, given that $I(q^k) = M(q^k) = \varnothing$. This contradiction shows that the sequence is finite. Hence, $q^k = \infty$ for some $k < \infty$.

By this iterative logic, we need only accomplish Step 0. Start by setting $\mathbf{m}_A(\hat{p}) := \hat{\mathbf{t}}_A$. By (24), $\mathbf{m}(\hat{p})$ satisfies (25) for $p = \hat{p}$. We accomplish the rest of the step separately in three cases.

Case 1: $I(\hat{p}) = \varnothing$, $M(\hat{p}) = \varnothing$. Fix $i \in A$. Because $\mathbf{m}_A(\hat{p}) = \hat{\mathbf{t}}_A$, we obtain $\pi^i(\hat{\mathbf{t}}) \leq \hat{p}$ from (24) and $I(\hat{p}) = \varnothing$. This and $M(\hat{p}) = \varnothing$ imply $\hat{t}_i = \bar{t}_i$ or $\pi^i(\hat{\mathbf{t}}) < \hat{p}$. In turn, this and (24) yield $\hat{t}_i = \bar{t}_i$. Now define $q := \infty$ and $\mathbf{m}_A(p) := \bar{\mathbf{t}}_A$ for $p > \hat{p}$. As a constant function, $\mathbf{m}$ is trivially continuous, nondecreasing, and right differentiable on $[\hat{p}, \infty)$. For any $i \in A$, $\mathbf{m}$ satisfies (25) at all $p \geq \hat{p}$ because

$$\pi^i(\mathbf{m}(p)) = \pi^i(\hat{\mathbf{t}}) \leq \hat{p} \leq p$$

and $m_i(p) = \bar{t}_i > \underline{t}_i$. This accomplishes Step 0 in this case.[26]

Case 2: $I(\hat{p}) \neq \varnothing$, $M(\hat{p}) = \varnothing$. Now let $q = \min_{i \in I(\hat{p})} \pi^i(\hat{\mathbf{t}})$, and note that $q > \hat{p}$. Define $\mathbf{m}_A(p) := \hat{\mathbf{t}}_A$ for $p \in (\hat{p}, q]$. Again, this trivially yields a continuous nondecreasing $\mathbf{m}$ on $[\hat{p}, q]$ that satisfies the boundary condition $\mathbf{m}(\hat{p}) = \hat{\mathbf{t}}$ and is right differentiable on $[\hat{p}, q)$. Fix $i \in A$. If $i \notin I(\hat{p})$, then, since $i \notin M(\hat{p})$, $\mathbf{m}$ satisfies (25) on $[\hat{p}, q]$ for this i by the argument used in Case 1. So suppose $i \in I(\hat{p})$. Then $\pi^i(\hat{\mathbf{t}}) > \hat{p}$ and, by (24), $\hat{t}_i < \bar{t}_i$. Thus, by the definition of q, we have $\pi^i(\hat{\mathbf{t}}) \geq p$ for all $p \in [\hat{p}, q]$. This, together with $\hat{t}_i < \bar{t}_i$, implies that for this i, $\mathbf{m}$ satisfies (25) on $[\hat{p}, q]$. Lastly, (*) holds because $I(q) \subset I(\hat{p})$.

[25] The strict lexicographic ordering on integer pairs, $<_L$, is defined by $(i, j) <_L (i', j')$ if and only if either $i < i'$ or $i = i'$ and $j < j'$.

[26] Step 0 proves the lemma in this case, since it yields $q = \infty$.

Case 3: $M(\hat{p}) \neq \varnothing$. Simplify notation by letting $M = M(\hat{p})$. Consider the function $\hat{\boldsymbol{\pi}} := \boldsymbol{\pi}^M(\cdot, \hat{\mathbf{t}}_{N\setminus M})$ that maps T_M into $\mathbb{R}_+^{|M|}$. By Lemmas 3 and A.2, $\hat{\boldsymbol{\pi}}$ has a C^1 inverse, $\boldsymbol{\varphi} : \hat{\boldsymbol{\pi}}(T_M) \to T_M$, satisfying

$$[\varphi_j^i(\mathbf{p})]\mathbf{e} \gg \mathbf{0} \text{ for any } \mathbf{p} \in \hat{\boldsymbol{\pi}}(\mathbf{T_M}). \tag{26}$$

We can assume, by the Whitney Extension Theorem, that $\boldsymbol{\varphi}$ is actually a C^1 function defined on $\mathbb{R}^{|M|}$. This allows us to define a C^1 function $\boldsymbol{\alpha} : \mathbb{R} \to \mathbb{R}^{|M|}$ by $\boldsymbol{\alpha}(p) := \boldsymbol{\varphi}(p\mathbf{e})$. The chain rule yields $\boldsymbol{\alpha}'(p) = [\varphi_j^i(p\mathbf{e})]\mathbf{e}$ for all $p \in \mathbb{R}$. Thus, since $\hat{p}\mathbf{e} = \hat{\boldsymbol{\pi}}(\hat{\mathbf{t}}_M) \in \hat{\boldsymbol{\pi}}(\mathbf{T_M})$, from (26) we obtain $\boldsymbol{\alpha}'(\hat{p}) \gg \mathbf{0}$. We also have $\boldsymbol{\alpha}(\hat{p}) = \hat{\mathbf{t}}_M$ and, by the definition of M, $\hat{\mathbf{t}}_M \ll \bar{\mathbf{t}}_M$. We conclude that $\hat{q} > \hat{p}$ exists such that $\boldsymbol{\alpha}(\hat{q}) \ll \bar{\mathbf{t}}_M$ and $\boldsymbol{\alpha}$ is strictly increasing on $[\hat{p}, \hat{q})$. We define $\mathbf{m}_A : [\hat{p}, \hat{q}) \to \mathbf{T}_A$ by $\mathbf{m}_A(p) := \left(\boldsymbol{\alpha}(p), \hat{\mathbf{t}}_{A\setminus M}\right)$. Hence,

$$\boldsymbol{\pi}^M(\mathbf{m}(p)) = p\mathbf{e} \text{ for } p \in [\hat{p}, \hat{q}), \tag{27}$$

and $\mathbf{m}$ is continuous, right-differentiable, and nondecreasing on $[\hat{p}, \hat{q})$. We also have

(a) $\underline{t}_i \leq m_i(p) < \bar{t}_i$ for all $i \in M, p \in [\hat{p}, \hat{q})$.

Furthermore, we can take $\hat{q} > \hat{p}$ sufficiently small so that

(b) $\pi^i(\mathbf{m}(p)) > p$ for all $i \in I(\hat{p}), p \in [\hat{p}, \hat{q})$.

Now let q be the supremum of all such $\hat{q}$ that satisfy (a)-(b). We thus have $\mathbf{m} : [\hat{p}, q) \to \mathbf{T}$, which we extend to q so that it is continuous: $\mathbf{m}(q) := \lim_{p \uparrow q} \mathbf{m}(p)$. It remains to show that (25) and (*) hold.

To show that (25) holds, fix $i \in A$ and $p \in [\hat{p}, q]$. If $i \in M$, then $\pi^i(\mathbf{m}(p)) = p$ by (27) (and continuity if $p = q$), and so (25) holds. Assume now that $i \notin M$. Then $\hat{t}_i \in \{\underline{t}_i, \bar{t}_i\}$, since otherwise (24) would imply $i \in M$. If $\hat{t}_i = \underline{t}_i$, then $\pi^i(\hat{\mathbf{t}}) \geq \hat{p}$ by (24), and so $\pi^i(\hat{\mathbf{t}}) > \hat{p}$ because $i \notin M$. Hence $i \in I(\hat{p})$, and so (b) yields $\pi^i(\mathbf{m}(p)) \geq p$. This and $m_i(p) = \hat{t}_i = \underline{t}_i$ imply (25). Lastly, suppose $\hat{t}_i = \bar{t}_i$. Then $\pi^i(\hat{\mathbf{t}}) \leq \hat{p}$ by (24). Because $\mathbf{m}$ is nondecreasing and m_j is constant on $[\hat{p}, q]$ for all $j \in N/M$ (including $j = i$), (27) and Lemma B.1 imply $m_i'(p') < 1$ for all $p' \in [\hat{p}, q)$. Thus, since $\mathbf{m}(\hat{p}) = \hat{\mathbf{t}}$ and $\pi^i(\hat{\mathbf{t}}) \leq \hat{p}$, we have $\pi^i(\mathbf{m}(p)) \leq p$. This and $m_i(p) = \hat{t}_i = \bar{t}_i$ yield (25).

It remains only to prove (*). To prove $I(q) \subseteq I(\hat{p})$, consider some $i \notin I(\hat{p})$. Then $\pi^i(\mathbf{m}(\hat{p})) \leq \hat{p}$. If $i \in M$, then $\pi^i(\mathbf{m}(q)) = q$ by (27), and so $i \notin I(q)$. If instead $i \notin M$, then $m_i(q) = m_i(p)$, and so

$$\pi^i(m_i(q), \mathbf{m}_{-i}(\hat{p})) = \pi^i(\mathbf{m}(\hat{p})) \leq \hat{p}.$$

As $q > \hat{p}$, Lemma B.1 now implies $\pi^i(\mathbf{m}(q)) < q$, and so again $i \notin I(q)$. This proves $I(q) \subseteq I(\hat{p})$.

Now suppose $I(q) = I(\hat{p})$. Let $i \in M(q)$. Then $\pi^i(\mathbf{m}(q)) = q$ and $m_i(\hat{p}) \leq m_i(q) < \bar{t}_i$. If $i \notin M$, then $m_i(\hat{p}) = m_i(q)$, and so $\pi^i(m_i(\hat{p}), \mathbf{m}_{-i}(q)) = q$. This and Lemma B.1 imply $\pi^i(\mathbf{m}(\hat{p})) > \hat{p}$, and so $i \in I(\hat{p})$. But then $i \in I(q)$, since $I(q) = I(\hat{p})$, contrary to $i \in M(q)$. We thus have $M(q) \subseteq M(\hat{p})$. Now, as $\mathbf{m}$ cannot be extended above q without violating (a) or (b), we have

(a′) $m_i(q) = \bar{t}_i$ for some $i \in M$, or

(b′) $\pi^i(\mathbf{m}(q)) = q$ for some $i \in I(\hat{p})$.

If (b′) holds, then $I(q) \neq I(\hat{p})$. So (a′) must hold. As the i in (a′) cannot be in $M(q)$, we conclude that $M(q) \subset M(\hat{p})$. This completes the proof of (*), and thus of Step 0 in this case. ■

Proof of Lemma 4. Let $\mathbf{m} : [\hat{p}, \infty) \to \mathbf{T}$ be the function obtained by applying Lemma B.2 to $(\hat{p}, D, \hat{\mathbf{t}})$. It is immediate that $\mathbf{m}$ satisfies all the properties to be proved except (i) and (ii). As $\mathbf{m}$ is right differentiable, Lemma B.1 applies to $\hat{\mathbf{m}} := (t_i, \mathbf{m}_{-i})$, and so implies (ii). To show (i), suppose that for some $i \notin D$ and $t_i \in T_i$, we have $\pi^i(t_i, \mathbf{m}_{-i}(p)) = p$. If $t_i > m_i(p)$, then because $\pi^i_i > 0$, we have

$$\pi^i(\mathbf{m}(p) < \pi^i(t_i, \mathbf{m}_{-i}(p)) = p.$$

But then, by Lemma B.2, we have the contradiction $m_i(p) = \bar{t}_i \geq t_i$. Similarly, if $t_i < m_i(p)$, then

$$\pi^i(\mathbf{m}(p) > \pi^i(t_i, \mathbf{m}_{-i}(p)) = p,$$

leading by Lemma B.2 to the contradiction $m_i(p) = \underline{t}_i \leq t_i$. Hence $t_i = m_i(p)$, proving (i). ■

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