ARROW'S EQUIVALENCY THEOREM IN A MODEL WITH NEOCLASSICAL FIRMS

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In this paper we consider a two-period general equilibrium model with uncertainty and real assets as financial instruments. The novelty of the analysis is that real assets are the stocks of neoclassical firms, so that both returns and yields depend on anticipated spot goods prices (and, of course, the yield matrix may change rank with prices). Assuming that financial markets are potentially complete, we establish generic existence of financial equilibrium and prove that there exists a dense set of economies such that financial equilibria are efficient.

Keywords: financial equilibrium, potentially complete financial markets.

1. Introduction

The fundamental feature of a general equilibrium model with real assets is that their yields depend on endogenous variables, such as anticipated spot goods prices, and the rank of the asset returns matrix may be less than full, and in general this rank may vary across financial equilibria. It is well known that existence and efficiency of financial equilibria crucially depend on whether asset markets are complete or not. In particular, Hart (1975) constructed an example of an economy and an incomplete asset structure for which no equilibrium existed, as well as an example of an economy for which one of the financial equilibrium allocations was inefficient. Thus, when the rank of the asset returns matrix is not fixed, one can at most prove generic existence and efficiency (when applicable) of financial equilibria.

Generic existence of financial equilibria with intrinsically incomplete real asset markets was shown in Duffie and Shafer (1985), Husseini et al. (1990), Geanakoplos and Shafer (1990), Hirsch et al. (1990). Generic existence of financial equilibria with potentially complete real asset markets was established in Duffie and Shafer (1986), Magill and Shafer (1990) and Cass and Rouzaud (2000). The main idea behind these proofs is to consider the Walrasian counterpart of the benchmark financial model and to show that generically, spot goods prices and firms' production decisions generate the asset returns matrix of full rank. Since in the case of complete asset markets Walrasian equilibria are

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allocation equivalent to financial equilibria, the result above implies generic existence of financial equilibria with potentially complete real asset markets.

To prove their results, Magill and Shafer (1990) consider point input-output technology, which is given exogenously, so the authors are free to impose a regularity condition on the asset structure. In Duffie and Shafer (1986) and Cass and Rouzaud (2000), firms make optimizing decisions but the economy is specified by households' and firms' endowments and all the generic results are obtained by varying these endowments which is equivalent to parallel shifts of transformation functions. In addition, both Duffie and Shafer (1986) and Cass and Rouzaud (2000) specify technology in terms of the transformation functions following the tradition of Balasko (1988). Though this tradition allows one to deal with the simplest specification of a smooth technology, it is not very meaningful economically: it does not exclude the case when output tomorrow is used as input today and allows firms to re-allocate their production across the states of nature after a state has been revealed.

In our view, it is more meaningful to use an alternative specification of technology: that is to assume that the firm chooses inputs today to produce state contingent outputs tomorrow. Such specification was used, for example, by Geanakoplos et al. (1990) in their proof of generic inefficiency of financial equilibria with intrinsically incomplete asset markets. However, these authors used translations of transformation functions to establish their results.

In this paper, the economy is characterized by households' endowments and firms' transformation functions, which specify technology described above. To establish generic results, we use perturbation of transformation functions. Even though the idea of characterizing an economy by a set of functions rather than by a set of endowments is not new in itself and goes back to Smale (1974), only perturbations of utility functions were considered so far (see, for example, Geanakoplos and Polemarchakis (1986), Cass and Citanna (1998), Citanna et al. (1998). Perturbation of transformation functions turns out to be more complicated because it requires that such important properties as boundedness of production sets hold uniformly for families of transformation functions. In addition, the perturbation technique of the aforementioned authors heavily depends on regularity of equilibria which is true for Walrasian equilibria or financial equilibria with nominal or numeraire assets, but it does not have to be true for financial equilibria with real assets. We use a different perturbation technique, which does not require regularity of equilibria, so we can apply it to show that there exists a dense set of parameters for which financial equilibria have complete asset markets.

It is necessary to notice that, in addition to all technical issues, a financial equilibrium model with neoclassical firms gives rise to a crucial conceptual problem, which is addressed, in Cass and Rouzaud (2000). This is a so-called "catch-22 of present value maximization". If in equilibrium, financial markets turn out to be incomplete, then neither the firm manager has a reliable way to infer how the market values future streams of profit, nor the firm's objective can be unambiguously defined. There are several ways of specification of the firm's objective, which make the firm's problem well defined (for details, see relevant discussion in the paper). We use present value maximization as the firm's objective with the same justification as in Duffie and Shafer (1986).

The rest of the paper is organized as follows. We provide problem specification in Section 2. In Section 3, we introduce Walrasian equilibria and show their existence. In Section 4, we prove that there exists an open dense set of parameters such that for all economies specified by parameters in this set, financial equilibria exist. In Section 5, the existence of a dense subset of parameters such that for all economies specified by this subset, financial equilibria are efficient, is established. Section 6 contains main conclusions. Technical results and proofs are delegated to Appendices.

2. Problem specification

2.1. Economic environment

We consider a competitive, two-period production and exchange economy with real assets and uncertainty. We assume that there are Ω, $\Omega > 1$, possible states of the world in the second period, and there are G, $G \geq 1$, commodities in each spot, labeled by $g = 1, 2, \dots, G$. We label each spot by $\omega = 0, 1, \dots, \Omega$, where $\omega = 0$ corresponds to the first period. There are H, $H > 1$, households labeled by $h = 1, 2, \dots, H$, and F firms, $F \geq 1$, labeled by $f = 1, 2, \dots, F$. Households own shares of firms, the initial shareholdings being s_{hf}. In the first period, firms choose inputs for production, commodities and assets are exchanged and first period consumption takes place. Then uncertainty is resolved, firms produce, and households fulfil their financial commitments and, finally, exchange and consume second-period commodities. Let $c_h^{\omega,g}$ be the consumption of commodity g in state ω by household h; similar notation is used for the endowments, $e_h^{\omega,g}$. Consumption is an element of $\mathbb{R}^L_{++}$ for each household, where $L \equiv G(\Omega + 1)$.

2.2 Preferences

Household h's preferences are represented by the utility function $u_h : \mathbb{R}^L_{++} \to \mathbb{R}$. Let

$$c_h^\omega \equiv (c_h^{\omega,g})_{g=1}^G; \quad c_h \equiv (c_h^\omega)_{\omega=0}^\Omega; \quad c \equiv (c_h)_{h=1}^H;$$

$$e_h^\omega \equiv (e_h^{\omega,g})_{g=1}^G; \quad e_h \equiv (e_h^\omega)_{\omega=0}^\Omega; \quad e \equiv (e_h)_{h=1}^H.$$

We assume that

A1 $u \in C^2(\mathbb{R}^L_{++})$
A2 $\forall h$ and $\forall c \in \mathbb{R}^L_{++}$ $Du_h(c) \gg 0$
A3 $\forall \Delta c \in \mathbb{R}^L$ such that $\Delta c \neq 0$ and $Du_h(c)\Delta c = 0$, $\Delta c^T D^2 u_h(c) \Delta c \ll 0$
A4 for each indifference surface of u_h, its closure is contained in $\mathbb{R}^L_{++}$
A5 $e_h \in E_h \equiv \mathbb{R}^L_{++}$;
A6 $\sum_h s_{hf} = 1 \ \ \forall\, f$.

The first five assumptions are standard assumptions of the smooth model of households' behavior, and the last one is a normalization of stock holdings.

2.3 Production

A typical firm is described by its production set $Y_f \subset \mathbb{R}^L$. The input vector is $y_f^0 \in \mathbb{R}^G_-$, and the output in each state is $y_f^\omega \in \mathbb{R}^G_+$. For simplicity, we assume that the firm may use all commodities in state 0 as inputs and produce all the variety of commodities in state ω. It is possible to show that these unrealistic assumptions can be relaxed and the firm can be described by the production set which includes only a certain subset of commodities chosen as inputs in state 0 and a certain subset of commodities produced in each state $\omega > 0$.

Equivalently, firms can be characterized by transformation functions $t_f^\omega : \mathbb{R}_-^G \times \mathbb{R}_+^G \to \mathbb{R}$. Let

$$y_f^\omega \equiv (y_f^{\omega,g})_{g=1}^G; \quad y_f \equiv (y_f^\omega)_{\omega=0}^\Omega; \quad y \equiv (y_f)_{f=1}^F.$$

The relationship between the production set and transformation functions is given by

$$Y_f = \{y_f \in \mathbb{R}^L \big|\ t_f^\omega(y_f^0, y_f^\omega) \geq 0,\ \forall\ \omega > 0\}.$$

To specify transformation functions, we introduce a class of functions $\mathcal{T}_0$ such that any $t \in \mathcal{T}_0$ satisfies the following conditions:

T1 $t \in C(\mathbb{R}_-^G \times \mathbb{R}_+^G) \cap C^2(\mathbb{R}_{--}^G \times \mathbb{R}_{++}^G)$, $D_v t \in C(\mathbb{R}_-^G \times \mathbb{R}_+^G)$ and $D_u t \in C(\mathbb{R}_{--}^G \times \mathbb{R}_+^G)$
T2 $Dt(u,v) \ll 0\ \forall\ (u,v) \in \mathbb{R}_{--}^G \times \mathbb{R}_{++}^G$ and $D_u t(u,0) \ll 0\ \forall\ u \in \mathbb{R}_{--}^G$
T3 $\forall (u,v) \in \mathbb{R}_{--}^G \times \mathbb{R}_{++}^G$, $\forall\ (0 \neq)\Delta w \in \mathbb{R}^G \times \mathbb{R}^G$, $\quad \Delta w^T D^2 t(u,v) \Delta w < 0$
T4 $t(u,0) = 0\ \forall\ u \in \partial\mathbb{R}_-^G$, and $t(u,v) < 0\ \forall\ (u,v) \in \partial\mathbb{R}_-^G \times \big(\mathbb{R}_+^G \setminus \{0\}\big)$
T5 $D_{v_j} t(u,v) = 0\ \forall\ (u,v) \in \mathbb{R}_-^G \times \mathbb{R}_+^G$ such that $v_j = 0$
T6 $||D_u t(u,v)|| \to \infty$ as $u \to 0$ uniformly in v such that $t(u,v) = 0$
T7 For any $\epsilon > 0$, there exists $C > 0$ such that

$$t(u,v) = 0\ \&\ (u,v) \in \mathbb{R}_-^G \times \mathbb{R}_+^G \ \Rightarrow\ ||v|| \leq C + \epsilon||u||.$$

The first two assumptions above are fairly standard assumptions concerning smoothness, monotonicity and concavity of transformation functions. Assumption **T4** is a positive input condition that we will use to rule out corner solutions to the firm's problem, and the last group of assumptions are generalized Inada conditions which will help us to establish boundedness and non-emptiness of a constraint set in the firm's problem.

We use as the leading example of a transformation function satisfying all the aforementioned assumptions the following function:

$$t(u,v) = -\sum_{g=1}^G a^g (v^g)^{\alpha_g} + \prod_{g=1}^G b^g (-u^g)^{\theta_g},$$

where

$$(u,v) \in \mathbb{R}_-^G \times \mathbb{R}_+^G; \quad \alpha_g > 1\ \forall\ g; \quad \theta_g > 0\ \forall\ g; \quad \sum_{g=1}^G \theta_g < 1; \quad a^g, b^g > 0\ \forall\ g.$$

This is a generalized Cobb-Douglas technology that makes it possible to produce a linear combination of outputs.

We endow $\mathcal{T}_0$ with the following topology: a sequence $\{t^\nu\}_{\nu=1}^\infty \subset \mathcal{T}_0$ converges to $t \in \mathcal{T}_0$ if and only if

(i) for any compact $K \subset \mathbb{R}_-^G \times \mathbb{R}_+^G$, $t^\nu\big|_K \to t\big|_K$ in $C(K)$ topology;

(ii) for any compact $K \subset \mathbb{R}_{--}^G \times \mathbb{R}_{++}^G$, $t^\nu\big|_K \to t\big|_K$ in $C^2(K)$ topology.

We identify t^ω, a function of (y^0, y^ω), with a function of $(y^0, y^1, \dots, y^\Omega)$, which does not depend on $y^{\omega'}$, $\omega' > 0$, $\omega' \neq \omega$. Let $t_f \equiv (t_f^\omega)_{\omega=1}^{\Omega}$ be a vector of transformation functions of firm f. We assume that

T8 For all f, $t_f \in (\mathcal{T}_0)^\Omega$, where $(\mathcal{T}_0)^\Omega$ is endowed with the product topology.

Introduce a set $\mathcal{T} \equiv (\mathcal{T}_0)^{F\Omega}$ and endow it with the product topology. Let a generic element of $\mathcal{T}$ be denoted by $T \equiv (t_f)_{f=1}^F$. Let $E \equiv \times_h E_h \equiv \mathbb{R}_{++}^{HL}$ be endowed with the usual topology. A generic element of E is denoted by e. So, an economy is characterized by $e \in E$ and $T \in \mathcal{T}$. The set of all economies is therefore $E \times \mathcal{T}$, which is endowed with the product topology.

Remarks on the properties of $\mathcal{T}_0$.

(1) The topology on $\mathcal{T}_0$ can be given by a countable system of seminorms. Thus there exists a metric on $\mathcal{T}_0$ such that the topology induced by this metric agrees with the one defined above. Hence, the spaces $\mathcal{T}$ and $E \times \mathcal{T}$ are also metrizable. The significance of this fact is that it allows us to formulate our topological arguments concerning continuity, closedness and density in terms of converging sequences rather than open sets.

(2) If $t \in \mathcal{T}_0$, from **T3** it follows by continuity of t that t is concave on $\mathbb{R}_-^G \times \mathbb{R}_+^G$. Indeed, if $y, y' \in \mathbb{R}_-^G \times \mathbb{R}_+^G$ and $\lambda \in (0,1)$, then for small enough $\epsilon > 0$, we have by **T3**

$$\frac{\lambda - \epsilon}{1 - 2\epsilon} t((1-\epsilon)y + \epsilon y') + \frac{1 - \epsilon - \lambda}{1 - 2\epsilon} t(\epsilon y + (1-\epsilon)y') < t(\lambda y + (1-\lambda)y'),$$

which, passing to the limit as $\epsilon \to 0$, gives

$$\lambda t(y) + (1-\lambda) t(y') \leq t(\lambda y + (1-\lambda)y'),$$

i.e. t is concave.

2.4 Financial instruments

For simplicity assume that the only financial instruments are stock shares, and for specificity, that financial markets are potentially complete, i.e. $F = \Omega$. Let z_h^f be the demand of asset f by household h, with $z_h \equiv (z_h^f)_{f=1}^F$, $z \equiv (z_h)_{h=1}^H$. Introduce q^f, the market value of firm f, $q \equiv (q^f)_{f=1}^F$, and $p'^g(\omega)$, the price of commodity g in spot ω, $p'(\omega) \equiv (p'^g(\omega))_{g=1}^G$ is the price vector in spot ω [2]. We will also use the following notation: $P' \equiv \text{diag}[p'(\omega)]_{\omega=0}^\Omega$. Let $Y(p', y) \equiv [p'(\omega) y_f^\omega]_{f,\omega=1}^\Omega$ be the matrix of all firms' anticipated future revenues. We will call $Y(p', y)$ the return matrix. Notice that both returns and yields depend on anticipated spot goods prices (and, of course, the return matrix may change rank with prices). It is also convenient to consider an $(\Omega+1) \times \Omega$ matrix $R'(p', y)$ given by

$$R'(p', y) \equiv \begin{bmatrix} -q \\ Y(p', y) \end{bmatrix}.$$

[2] All price vectors are row vectors.

A portfolio $z_h \in \mathbb{R}^\Omega$ has market value $q \cdot z_h$ and payoff $Y(p', y)z_h$ in $\mathbb{R}^\Omega$. We rule out arbitrage opportunities for z_h, all h. No arbitrage is equivalent to (see, for example, Duffie (1996)) existence of a state-price vector $\lambda \in \mathbb{R}^\Omega_{++}$ such that

$$\text{(NAC)} \qquad q_f = \sum_{\omega>0} \lambda(\omega) p'(\omega) y_f^\omega;$$

this condition implies that market values equal the expected future values.

2.5 Financial equilibrium

Now we are in a position to define a financial equilibrium (FE) for an economy (e, T) as a vector (p', c, y, q, z) such that

(F′): Firms maximize the present value, i.e., for all f, given p' and the vector of state prices
$\lambda = (\lambda(\omega))_{\omega=1}^\Omega \gg 0$, y_f is an optimal solution to

$$\max_{y_f} p'(0)y_f^0 + \sum_{\omega>0} \lambda(\omega)p'(\omega)y_f^\omega$$

$$\text{subj. to } t_f^\omega(y_f^0, y_f^\omega) \geq 0, \ \forall\, \omega > 0.$$

(H′): Households maximize utility, i.e., for all h, given p', λ, q and y, (c_h, z_h) is an optimal solution to

$$\max_{(c_h, z_h)} u_h(c_h)$$

$$\text{subj. to } c_h \in B_h(p', R'),$$

where

$$B_h(p', R') = \{c_h \in \mathbb{R}^L_{++} \,\big|\, P'(c_h - e_h - \sum_f s_{hf} y_f) = R'(p', y)z_h\}.$$

(M$^{\mathrm{F}}$): Spot goods and stock share markets clear, i.e.,

$$\sum_h (c_h - e_h) - \sum_f y_f = 0 \ \text{ and}$$

$$\sum_h z_h = 0.$$

(NAC): State prices satisfy no-arbitrage condition.

Notice that the above specification of FE contains a serious conceptual problem named in Cass and Rouzaud (2000) “catch-22 of present value maximization”: to calculate its present value and hence its optimal production plan, a firm has to infer the state prices, but to be able to do the latter, the firm already has to know its optimal production plan together with other firms’ optimal production plans, market values and spot goods prices. When financial markets are complete, it is possible to show that present value maximization is equivalent (up to re-scaling of prices) to the well-defined profit maximization. When markets are incomplete, the vector of state prices cannot be determined uniquely (any two shareholders will typically face different state prices) which implies that firms' objectives cannot be unambiguously defined. There are several ways of assigning a well-defined objective to each firm. One of them is to impose spanning conditions similar to those in Diamond (1967), Ekern and Wilson (1974), Radner (1974) and Leland (1974).However, these conditions prove to be rather restrictive. Also one may propose alternative firms' objectives like in Cass and Rouzaud (2000), or Grossman and Hart (1979), so that firms maximize present values weighted by firm specific coefficients, and the latter reflect beliefs of shareholders. However, as it is argued in Duffie and Shafer (1986), these objectives require even more information on the part of the firm's manager than inferring how the market values future payoffs. At the same time, present value maximization is consistent with existence of equilibrium. It can also be justified by a strong notion of competition, proposed by Makowski (1980), which means that shareholders take both prices and spans of markets as given. This argument seems the most appealing to us therefore we adopt present value maximization as the objective of the firm.

The (NAC) allows us to eliminate the asset prices from the budget constraints in (H$'$). We set $p(0) = p'(0)$, $p(\omega) = \lambda(\omega)p'(\omega)$, $\omega > 0$, then by no-arbitrage,

$$q_f = \sum_{\omega>0} p(\omega) y_f^{\omega}.$$

Introduce

$$\hat{P} \equiv \text{diag}[p(\omega)]_{\omega=1}^{\Omega}; \quad \hat{p} \equiv (p(\omega))_{\omega=1}^{\Omega};$$
$$\hat{c}_h \equiv (c_h^{\omega})_{\omega=1}^{\Omega}; \quad \hat{e}_h \equiv (e_h^{\omega})_{\omega=1}^{\Omega}; \quad \hat{y}_f \equiv (y_f^{\omega})_{\omega=1}^{\Omega};$$

then it is easy to show that

$$B_h(p', R') = B_h(p, Y) \equiv \Big\{ c_h \in \mathbb{R}^L_{++} \,\Big|\, p \cdot (c_h - e_h - \sum_f s_{hf} y_f) = 0 \text{ and}$$
$$\hat{P}(\hat{c}_h - \hat{e}_h - \sum_f s_{hf} \hat{y}_f) = Y(\hat{p}, \hat{y}) z_h \Big\}.$$

As a result, we can redefine FE as a vector (p, c, y, z) such that

(F$^{\text{F}}$): Given p, for all f, y_f is an optimal solution to

$$\max_{y_f} p \cdot y_f$$
$$\text{subj. to } t_f^{\omega}(y_f^0, y_f^{\omega}) \geq 0, \; \forall\, \omega > 0.$$

(H$^{\text{F}}$): For all h, given p and y, (c_h, z_h) is an optimal solution to

$$\max_{(c_h, z_h)} u_h(c_h)$$
$$\text{subj. to } c_h \in B_h(p, Y).$$

(M$^{\text{F}}$): The market clearing conditions hold.

It was shown by Hart (1975), that in the case of potentially complete markets, FE does not necessarily exist. Therefore, we can at most show that FE exists generically, i.e., on an open dense set in $E \times \mathcal{T}$. Notice that the conventional definition of ("strong") genericity means that some property holds on an open, full measure set. Here, we are able to talk only about a weaker notion of genericity because we treat transformation functions as exogenous variables. To prove generic existence of FE, we will first consider the Walrasian counterpart of the financial model described above and show that there is an open dense set in the space $E \times \mathcal{T}$ such that on this set, Walrasian model is allocation equivalent to financial model.

3. Walrasian equilibrium

3.1 Definition

We define Walrasian equilibrium (WE) for an economy (e, T) as a vector (p, c, y) such that

(F^{W}): Firms maximize profits, i.e., for all f, given p, y_f is an optimal solution to

$$\max_{y_f} p \cdot y_f$$
$$\text{subj. to } t_f^{\omega}(y_f^0, y_f^{\omega}) = 0, \ \forall\, \omega > 0$$

(H^{W}): Households maximize utility, i.e., for all h, given p and y, c_h is an optimal solution to

$$\max_{c_h} u_h(c_h)$$
$$\text{subj. to } c_h \in B_h(p),$$

where
$$B_h(p) = \{c_h \in \mathbb{R}^L_{++} \,\big|\, p \cdot (c_h - e_h - \sum_f s_{hf} y_f) = 0\}.$$

(M^{W}): Contingent commodity markets clear, i.e

$$\sum_h (c_h - e_h) - \sum_f y_f = 0.$$

The following lemma establishes the relationship between Walrasian and financial equilibria provided both exist.

Lemma 3.1: *(a) If (p, c, y) is WE and $Y(\hat{p}, \hat{y})$ has full rank, then there exists z such that (p, c, y, z) is FE.*
(b) If (p, c, y, z) is FE and $Y(\hat{p}, \hat{y})$ has full rank, then (p, c, y) is WE.
Proof. In Appendix A.

On the strength of Lemma 3.1(a), to prove generic existence of FE, it suffices to show that there is an open dense set in $E \times \mathcal{T}$ such that $Y(\hat{p}, \hat{y})$ has full rank.

3.2 Existence of WE

The existence of WE can be established by the following claims.

Claim 3.2: *Given $p \gg 0$, there exists a unique solution $y_f(p)$, all f, to* (F^{W}).

Claim 3.3: *Given* $p \gg 0$ *and* y, *there exists a unique solution* $c_h(p, y)$, *all* h, *to* $(\mathrm{H}^{\mathrm{W}})$.

We normalize $\sum_\omega \sum_g p^g(\omega) = 1$ and set excess demand

$$d(p) \equiv \sum_h (c_h(p) - e_h) - \sum_f y_f(p).$$

Thus, p belongs to the simplex

$$\mathcal{S}^{L-1} = \{p \in \mathbb{R}^L_+ \big| \sum_\omega \sum_g p^g(\omega) = 1\},$$

and d is a function on $\mathcal{S}^{L-1}$.

Claim 3.4: *For any sequence* $\{p^\nu\} \subset$ *int* $\mathcal{S}^{L-1}$ *converging to* $p \in \partial\mathcal{S}^{L-1}$,

$$\lim_{\nu \to \infty} ||d(p^\nu)|| = \infty.$$

Finally, we use the Brouwer's Fixed Point Theorem to prove the statement below.

Claim 3.5: *There exist market clearing prices* $p \in$ *int* $\mathcal{S}^{L-1}$.

The last 3 claims can be proven by standard arguments. The uniqueness result in Claim 3.2 follows from strict concavity of transformation functions, but existence is less trivial than for the case of smooth transformation functions. The idea of the proof is standard: we introduce a constraint set $\Phi(t, p) = \{y \in \mathbb{R}^G_- \times \mathbb{R}^{\Omega G}_+ \big| \; p \cdot y \geq 0, \; t(y) \geq 0\}$ and show that this set is non-empty and compact. After that, using the Maximum Value Theorem, we conclude that the profit function attains the maximum on $\Phi(t, p)$. Finally, it is possible to show that the first of the constraints defining $\Phi(t, p)$ is non-binding. It not difficult to argue compactness of the constraint set, but non-emptiness requires some elaboration due to the properties of transformation functions. We provide the complete proof of Claim 3.2 in Appendix A.

We need to establish one more auxiliary result before we characterize WE. The following lemma allows us to rule out corner solutions for the firms' optimization problems.

Lemma 3.6: *The optimal solution to* $(\mathrm{F}^{\mathrm{W}})$ *satisfies*

$$y_{\mathrm{opt}} \in \mathbb{R}^G_{--} \times (\mathbb{R}^G_{++})^\Omega.$$

Proof. In Appendix A.

3.3 Characterization

Existence of WE having been established, we proceed as follows. It is trivial to check that solutions to households' and firms' problems can be characterized by the Kuhn-Tucker conditions. Further, we can rule out corner solutions in the firms' problems on the strength of Lemma 3.6.

Let $\lambda_h > 0$ be a Lagrange multiplier for household h's problem, and $\mu_f \equiv (\mu_f^\omega)_{\omega=1}^\Omega \in \mathbb{R}^\Omega_{++}$ be a vector of Lagrange multipliers for firm f's problem. Define $\lambda \equiv (\lambda_h)_{h=1}^H \in \mathbb{R}^H_{++}$ and $\mu \equiv (\mu_f)_{f=1}^F \in \mathbb{R}^{F\Omega}_{++}$. The equilibrium system of equations can be written as follows:

$$Du_h(c_h) - \lambda_h p = 0 \ \ \forall\ h; \tag{3.1}$$

$$-\, p \cdot (c_h - e_h - \sum_f s_{hf} y_f) = 0 \ \ \forall\ h; \tag{3.2}$$

for all $\omega > 0$, f,

$$\begin{aligned} p(0) + \textstyle\sum_{\omega>0} \mu_f^\omega D_0 t_f^\omega(y_f^0, y_f^\omega) &= 0, \\ p(\omega) + \mu_f^\omega D_\omega t_f^\omega(y_f^0, y_f^\omega) &= 0; \end{aligned} \tag{3.3}$$

we use the notation D_0 for the derivative of $t_f^\omega(y_f^0, y_f^\omega)$ with respect to the first argument, and D_ω for the derivative with respect to the second argument;

$$t_f^\omega(y_f^0, y_f^\omega) = 0 \ \ \forall\ f,\ \forall\ \omega > 0; \tag{3.4}$$

$$\sum_h (c_h - e_h) - \sum_f y_f = 0. \tag{3.5}$$

A price vector, p, is defined up to a positive scalar factor. We choose it so that p lies on the unit sphere $S^{L-1} \subset \mathbb{R}^L$:

$$p \cdot p^T = 1,\ \ p \gg 0. \tag{3.6}$$

Denote by Θ' a set of $(e, T) \in E \times \mathcal{T}$ such that a system (3.1)–(3.6) has a solution satisfying $\text{rank} Y(\hat{p}, \hat{y}) < \Omega$. Set $\Theta = E \times \mathcal{T} \setminus \Theta'$. Our main goal is to show that Θ is open and dense in $E \times \mathcal{T}$, i.e. generically, matrix $Y(\hat{p}, \hat{y})$ has full rank.

4. Generic existence of Financial equilibriua

4.1 Openness

In this subsection, we will prove the following statement.

Theorem 4.1: *Θ is open in $E \times \mathcal{T}$.*

To establish openness of Θ, we may prove that Θ' is closed; since it is a subset of a metric space (see remarks at the end of subsection 2.3) it suffices to show that, given a sequence $\{(e^\nu, T^\nu)\}_{\nu=1}^\infty \subset \Theta'$ converging to $(e, T) \in E \times \mathcal{T}$, we have $(e, T) \in \Theta'$. We begin with several auxiliary results. First, we know that, given $(t, p) \in (\mathcal{T}_0)^\Omega \times \mathbb{R}^L_{++}$, the firm's optimization problem

$$\max_y p \cdot y \quad \text{subj. to} \ \ t^\omega(y^0, y^\omega) \geq 0 \ \ \forall\ \omega > 0 \tag{4.1}$$

has a unique solution $y = y(t, p)$. Moreover, we can prove the following result.

Lemma 4.2: *The function* $(t,p) \mapsto y(t,p)$ *is continuous.*

Proof. In Appendix A.

We also know that, given $(p,e,y) \in \mathcal{B} \equiv \mathbb{R}^L_{++} \times \mathbb{R}^L_{++} \times (\mathbb{R}^G_- \times \mathbb{R}^{\Omega G}_+)$, the household's optimization problem

$$\max_{c \in B(p)} u(c) \tag{4.2}$$

where

$$B(p) = \{c \in \mathbb{R}^L_{++} \,|\, p \cdot (c - e - \sum_f s_f y_f) = 0\}$$

has a unique solution $c = c(p,e,y)$. As above, we are able to show continuity of c.

Lemma 4.3: *The function* $(p,e,y) \mapsto c(p,e,y)$ *is continuous.*

Proof. In Appendix A.

Thus, we have established that given the price vector is strictly positive, consumption and production are continuous in prices, endowments and transformation functions. Therefore, when we show that for any sequence $\{(e^\nu, T^\nu)\}_{\nu=1}^\infty$ converging to (e,T), the sequence of price vectors converges to a strictly positive vector, then we will be able to prove that Θ' is closed. We will argue that prices converge to a positive vector by using the "boundary condition" for excess demand, but to be able to do this, we first need the following result.

Lemma 4.4: *Let* $\{t^\nu\} \subset \mathcal{T}_0$ *converge to* $t \in \mathcal{T}_0$ *and let* $(u^\nu, v^\nu) \in \mathbb{R}^G_- \times \mathbb{R}^G_+$ *be such that* $t^\nu(u^\nu, v^\nu) = 0$ *for all* ν. *Suppose that there exists* $C_1 > 0$ *such that* $||u^\nu|| \leq C_1$ *for all* ν. *Then there exists* $C_2 > 0$ *such that* $||v^\nu|| \leq C_2$ *for all* ν.

Proof. In Appendix A.

Now we prove the following analogue of Claim 3.4.

Lemma 4.5: *Let a sequence* $\{(e^\nu, T^\nu)\} \subset E \times \mathcal{T}$ *converge to* $(e,T) \in E \times \mathcal{T}$ *and a sequence* $\{p^\nu\} \subset S^{L-1} \cap \mathbb{R}^L_{++}$ *converge to* $p \in S^{L-1} \cap \partial \mathbb{R}^L_+$. *Let* $y(T^\nu, p^\nu)$ *and* $c(p^\nu, e^\nu, y(T^\nu, p^\nu))$ *denote the optimal solutions of the problems* $(\mathrm{F^W})$ *and* $(\mathrm{H^W})$, *respectively, and set excess demand*

$$d(p^\nu, e^\nu, T^\nu) = \sum_h (c_h(p^\nu, e^\nu, y(T^\nu, p^\nu)) - e^\nu_h) - \sum_f y_f(T^\nu, p^\nu).$$

Then

$$\lim_{\nu \to \infty} ||d(p^\nu, e^\nu, T^\nu)|| = \infty.$$

Proof. In Appendix A.

Finally, we can complete the proof of Theorem 4.1. If $\{(e^\nu, T^\nu)\} \subset \Theta'$ is a sequence converging to $(e,T) \in E \times \mathcal{T}$, choose a corresponding sequence $\{(y^\nu, p^\nu)\}$ such that $Y(p^\nu, y^\nu)$ has rank less than full, that is, $\det Y(p^\nu, y^\nu) = 0$. We claim that it has a subsequence converging to some $(y,p) \in \left(\mathbb{R}^G_- \times \mathbb{R}^{G\Omega}_+\right)^F \times \mathbb{R}^L_{++}$. Since the unit sphere S^{L-1} is compact, we may choose a subsequence of $\{p^\nu\}$, without loss of generality the sequence itself, converging to some $p \in S^{L-1}$; by Lemma 4.5, we have $p \gg 0$. Then by Lemma 4.2, $\{y^\nu\}$ also converges, and by Lemma 4.3, the corresponding sequence $\{c^\nu\}$ converges. An argument similar to the one used in Lemma 4.2 to prove that the mapping η is continuous shows that the sequences $\{D_0 t_f^{\omega\nu}(y_f^{0\nu}, y_f^{\omega\nu})\}, \{D_\omega t_f^{\omega\nu}(y_f^{0\nu}, y_f^{\omega\nu})\}$ converge. Now if $\{(\lambda^\nu, \mu^\nu)\}$ is the corresponding sequence of Lagrange multipliers solving (4.1)-(4.6), it is obvious that it also converges, hence the whole sequence of tuples of endogenous variables converges and the limit of the sequence is WE. Now by continuity, if $(y,p) = \lim_{\nu\to\infty}(y^\nu, p^\nu)$, then $\det Y(\hat{p}, \hat{y}) = 0$, so $(e,T) \in \Theta'$. *Q.E.D.*

4.2 Perturbation technique and density

In this section, we prove density of the set Θ in $E \times \mathcal{T}$. We are going to prove it by contradiction. Suppose that Θ is not dense, then there exists a point $(\bar{e}, \bar{T}) \in \Theta'$ and its open neighborhood $W(\bar{e}, \bar{T})$ such that $W(\bar{e}, \bar{T}) \subset \Theta'$. By definition of product topology, there exist open sets $U \subset E$, $V \subset \mathcal{T}$ such that $(\bar{e}, \bar{T}) \in U \times V \subset W(\bar{e}, \bar{T}) \subset \Theta'$. Starting with this assumption, we will eventually arrive at a contradiction. We are going to append to Kuhn-Tucker equations (3.1)–(3.6) additional equations which express the fact that the return matrix has less than full rank. Then, we will perturb all endogenous and exogenous variables in this extended system and write linearized extended system. If we show that its matrix has the highest possible rank, we will be able to conclude that there does not exist $W(\bar{e}, \bar{T}) \subset \Theta'$. The main problem here is that $(\bar{e}, \bar{T})$ is an element of an infinite dimensional set, so to be able to perform the procedure described above, we construct a finite dimensional family $\{T(A)\}_{A\in\mathcal{A}} \subset \mathcal{T}$, where $\mathcal{A} \subset \mathbb{R}^{FL}$ is an open set containing the origin, such that the map $T : \mathcal{A} \to \mathcal{T}$ is continuous, and $T(\mathbf{0}) = \bar{T}$. After that, we will consider the open set

$$M \equiv U \times T^{-1}(V) \subset E \times \mathcal{A}$$

parameterizing a set of WE for which the rank of the return matrix is not full. Notice that M is a smooth manifold, because it is an open subset of $E \times \mathcal{A} \subset \mathbb{R}^{HL}_{++} \times \mathbb{R}^{FL}$. We will show that there exists a smooth surjective map from a finite union of smooth manifolds of dimension less than $\dim M$ onto M, which will give a contradiction.

We start with the description of the family $\{T(A)\}_{A\in\mathcal{A}}$. Let

$$A \equiv (A_f)_{f=1}^F, \quad A_f \equiv (\alpha_f^\omega)_{\omega=0}^\Omega,$$

where $\alpha_f^\omega \in \mathbb{R}^{2G}$ are such that $\alpha_f^{\omega j} > -1$ for all j. Drop indices f and ω for the time being. For a function $t \in \mathcal{T}_0$ and a vector $\alpha \in \mathbb{R}^{2G}$ ($\alpha^j > -1, \ \forall\ j$), introduce a function

$$t_\alpha : \mathbb{R}^G_- \times \mathbb{R}^G_+ \longrightarrow \mathbb{R}$$

$$y \mapsto t_\alpha(y) \equiv t(a \cdot y),$$

where a is the matrix $a \equiv I + \mathrm{diag}[\alpha^j]_{j=1}^{2G}$. Notice that $a_{ij} = 0$ if $i \neq j$ and $a_{ii} > 0, \forall\ i$; in particular, a is invertible. It is easy to see that such parameterization of transformation functions is equivalent to a change of units of input-output vector.

First of all, we have to verify that the family of functions described above belongs to the class of transformation functions $\mathcal{T}_0$.

Lemma 4.6: $t_\alpha \in \mathcal{T}_0$.

Proof. By the chain rule,

$$Dt_\alpha(y) = D_y t(a \cdot y) = Dt(a \cdot y) \cdot a \ll 0,$$

i.e. t_α is strictly decreasing. Next,

$$D^2 t_\alpha(y) = D_y(D_y t(a \cdot y)) = D_y(Dt(a \cdot y) \cdot a) = a^T \cdot D^2 t(a \cdot y) \cdot a.$$

Hence $D^2 t(a \cdot y) = a^{T^{-1}} D^2 t_\alpha(y) \cdot a^{-1}$. Since for every $0 \neq v \in \mathbb{R}^{2G}$, $v^T \cdot D^2 t(a \cdot y) \cdot v < 0$, $(a^{-1}v)^T \cdot D^2 t_\alpha(y) \cdot (a^{-1}v) < 0$.
Since $a^{-1} : \mathbb{R}^{2G} \to \mathbb{R}^{2G}$ is an isomorphism, we can write the last condition as

$$w^T \cdot D^2 t_\alpha(y) \cdot w < 0, \quad \forall\ 0 \neq w \in \mathbb{R}^{2G},$$

which means that t_α is differentiably strictly concave. It is straightforward to check that all other assumptions concerning the behavior of $t \in \mathcal{T}_0$ hold for t_α. *Q.E.D.*

Now we must check that $T : \mathcal{A} \to \mathcal{T}$ is continuous. In other words, we have to prove the following lemma.

Lemma 4.7: *Given a sequence* $\alpha^\nu \to \alpha$,
(i) for any compact $K \subset \mathbb{R}^G_- \times \mathbb{R}^G_+$, $t_{\alpha^\nu}\big|_K \to t_\alpha\big|_K$ *uniformly on* K *and*
(ii) for any compact $K \subset \mathbb{R}^G_{--} \times \mathbb{R}^G_{++}$, $t_{\alpha^\nu}\big|_K \to t_\alpha\big|_K$ *together with all the derivatives up to the second order uniformly on* K.
Proof. In Appendix C.

In order to prove density of Θ, we will have to work with linearized equilibrium equations. Therefore, we need the following result.

Lemma 4.8: *At any* $y \in \mathbb{R}^G_{--} \times \mathbb{R}^G_{++}$,

$$\begin{aligned} D_\alpha t_\alpha(y) \cdot \Delta\alpha &= Dt_\alpha(y)\left(a^{-1} \cdot y \Box \Delta\alpha\right); \\ D_\alpha D t_\alpha(y) \cdot \Delta\alpha &= \operatorname{diag}\left[\frac{1}{y^j}\partial_j t_\alpha(y)\right]_{j=1}^{2G} \left(a^{-1} \cdot y \Box \Delta\alpha\right) + D^2 t_\alpha(y)\left(a^{-1} \cdot y \Box \Delta\alpha\right). \end{aligned}$$

Here, $\Box$ denotes component-by-component product of vectors, and ∂_j stands for the partial derivative of $t_\alpha(y)$ with respect to the j-th argument.
Proof. Is obtained by direct computation with the help of the formulas derived in the proof of Lemma 4.6.

The finite dimensional parameterization of a subset of Θ' having been defined and its main properties having been established, we are ready to outline in more details the argument by which we arrive at a contradiction. If $\operatorname{rank} Y(\hat{p}, \hat{y}) < \Omega$ then there exists a vector $\beta \in \mathbb{R}^\Omega$ such that

$$\beta \cdot Y(\hat{p}, \hat{y}) = 0; \tag{4.3}$$

$$\beta \cdot \beta^T = 1. \tag{4.4}$$

Introduce

$$\Xi \equiv \mathbb{R}^{HL}_{++} \times \mathbb{R}^{L}_{++} \times \left(\mathbb{R}^{G}_{--} \times \mathbb{R}^{\Omega G}_{++}\right)^{F} \times \mathbb{R}^{H}_{++} \times \mathbb{R}^{F\Omega}_{++},$$

a manifold of endogenous variables with the typical element $\xi = (c, p, y, \lambda, \mu)$. Let $\tilde{\Xi} = \Xi \times \mathbb{R}^{\Omega}$, the manifold of extended endogenous variables with the typical element (ξ, β). Let

$$N = H(L+1) + F(L+\Omega) + L + 1 + \Omega + 1.$$

We append equations (4.3) and (4.4) to the system (3.1)–(3.6) and introduce

$$\phi : \tilde{\Xi} \times E \times \mathcal{T} \longrightarrow \mathbb{R}^{N},$$

a smooth map consisting of the left-hand sides (3.1)-(3.6), (4.3) and (4.4). Define

$$\Pi = \{(\xi, \beta, e, T) \big| \ \phi(\xi, \beta, e, T) = 0\} \subset \tilde{\Xi} \times E \times \mathcal{T},$$

or equivalently, $\Pi = \phi^{-1}(\mathbf{0})$. Obviously,

$$\Theta' = \tilde{\pi}(\Pi),$$

where $\tilde{\pi} : \tilde{\Xi} \times E \times \mathcal{T} \to E \times \mathcal{T}$ is the natural projection. Let M be a smooth manifold defined at the beginning of this subsection. Consider a smooth map

$$\psi : \tilde{\Xi} \times M \longrightarrow \mathbb{R}^{N},$$

given by the composition

$$(\xi, \beta, e, A) \longmapsto (\xi, \beta, e, T(A)) \longmapsto \phi(\xi, \beta, e, T(A))$$

Let $\Gamma = \psi^{-1}(0)$, which is the inverse image of Π in $\tilde{\Xi} \times M$, and $\pi_M : \tilde{\Xi} \times M \to M$ be the natural projection. By construction, $\pi_M(\Gamma) = M$. We will prove that Γ is contained in a finite union of submanifolds of $\tilde{\Xi} \times M$ of dimension less than $\dim M$. Once this is obtained, the equality $\pi_M(\Gamma) = M$ becomes impossible, and we are done.

For an element $\zeta = (\tilde{\xi}, e, A) \in \tilde{\Xi} \times M$, let $T_\zeta(\tilde{\Xi} \times M)$ denote the tangent space to $\tilde{\Xi} \times M$ at ζ. We identify the tangent space to $\mathbb{R}^N$ at the origin with $\mathbb{R}^N$ itself, then for every point $\zeta \in \Gamma$, we have the differential

$$D\psi\big|_{\zeta} : T_\zeta(\tilde{\Xi} \times M) \longrightarrow \mathbb{R}^{N}.$$

By the corollary to the Preimage Theorem (see Magill and Shafer (1990)), if we can show that $\mathrm{rank} D\psi\big|_{\zeta} \geq N-1$ for all $\zeta \in \Gamma$, then Γ is contained in a finite union of submanifolds of $\tilde{\Xi} \times M$ of dimension $\dim \tilde{\Xi} + \dim M - (N-1) = (N-2) + \dim M - (N-1) = \dim M - 1$. Fix a point $\zeta \in \Gamma$; we complete our argument with the following theorem.

Theorem 4.9: $\mathrm{rank}(D\psi\big|_{\zeta}) = N-1$, *which is the highest possible rank of the Jacobian.*

Here we will just outline the idea of the proof of the last theorem, and the complete proof is presented in Appendix C. We start with writing the system of equations $(D\psi\big|_{\zeta}) \cdot \Delta\zeta = \Delta b$, where $\Delta b \in \mathbb{R}^N$ is arbitrary. Notice that this system cannot be solved for any right-hand side since the original equilibrium equations (3.1)–(3.6) are not linearly independent because of the Walras' Law : $p \cdot \left(\sum_h (c_h - e_h) - \sum_f y_f\right) = 0$. This implies some condition on Δb which defines a hyperplane in $\mathbb{R}^N$. Therefore $\mathrm{rank}(D\psi\big|_{\zeta})$ cannot be higher than $N-1$. Below, in order to avoid introducing numerous new notation, we will put $**$ for arbitrary right-hand sides of all equations in this system:

$$D^2u_h(c_h)\cdot\Delta c_h-\lambda_h\Delta p-\Delta\lambda_h p^T=**\ \ \forall\ h; \tag{4.5}$$

$$\begin{aligned}&-p(\Delta c_h-\Delta e_h-\sum_f s_{hf}\Delta y_f)-\\&-(c_h-e_h-\sum_f s_{hf}y_f)^T\cdot\Delta p=**\ \ \forall\ h;\end{aligned} \tag{4.6}$$

$$\begin{aligned}&\Delta p(0)+\sum_{\omega>0}\Big[\Delta\mu_f^\omega D_0 t_{\alpha f}^{\omega^T}+\mu_f^\omega\Big(D_{00}^2 t_{\alpha f}^{\omega^T}\cdot\Delta y_f^0+\\&+D_{\omega 0}^2 t_{\alpha f}^\omega\Delta y_f^\omega+D_\alpha D_0 t_{\alpha f}^\omega\Delta\alpha_f\Big)\Big]=**\qquad\forall\ f;\end{aligned} \tag{4.7}$$

$$\begin{aligned}&\Delta p(\omega)+\Delta\mu_f^\omega D_\omega t_{\alpha f}^{\omega^T}+\mu_f^\omega\big(D_{0\omega}^2 t_{\alpha f}^\omega\Delta y_f^0+\\&+D_{\omega\omega}^2 t_{\alpha f}^\omega\Delta y_f^\omega+D_\alpha D_\omega t_{\alpha f}^\omega\Delta\alpha_f\big)=**\quad\forall\ f,\omega>0;\end{aligned} \tag{4.8}$$

$$D_0 t_{\alpha f}^\omega\Delta y_f^0+D_\omega t_{\alpha f}^\omega\Delta y_f^\omega+D_\alpha t_{\alpha f}^\omega\Delta\alpha_f=**\,\forall\ f,\omega>0. \tag{4.9}$$

Here we use the notation $D_0 t_{\alpha f}^\omega=D_{y_f^0}t_{\alpha f}^\omega(y_f^0,y_f^\omega)$, $D_\omega t_{\alpha f}^\omega=D_{y_f^\omega}t_{\alpha f}^\omega(y_f^0,y_f^\omega)$, and $D_{00}^2 t_{\alpha f}^\omega$, $D_{\omega 0}^2 t_{\alpha f}^\omega$, $D_{0\omega}^2 t_{\alpha f}^\omega$, $D_{\omega\omega}^2 t_{\alpha f}^\omega$ denote the corresponding Jacobians evaluated at (y_f^0,y_f^ω).

$$\sum_h(\Delta c_h-\Delta e_h)-\sum_f\Delta y_f\ =\ **\,; \tag{4.10}$$

$$p\cdot\Delta p\ =\ **\,; \tag{4.11}$$

$$\sum_{\omega>0}\Delta\beta^\omega p(\omega)y_f^\omega+\sum_{\omega>0}\beta^\omega y_f^{\omega^T}\Delta p(\omega)+\sum_{\omega>0}\beta^\omega p(\omega)\Delta y_f^\omega\ =\ **\ \forall\ f; \tag{4.12}$$

$$\beta\cdot\Delta\beta\ =\ **\ . \tag{4.13}$$

We prove the theorem by the following steps.

Step 1. Prove that the system (4.11)-(4.13) can always be solved for Δp, $\Delta\beta$ and Δy^ω.

Step 2. Set

$$\Delta\alpha_f\equiv\begin{pmatrix}\Delta\alpha_f^0\\\Delta\alpha_f^\omega\end{pmatrix};\quad\Delta\gamma_f\equiv\begin{pmatrix}\Delta\gamma_f^0\\\Delta\gamma_f^\omega\end{pmatrix}\equiv a^{-1}\cdot\begin{pmatrix}y_f^0\\y_f^\omega\end{pmatrix}\square\begin{pmatrix}\Delta\alpha_f^0\\\Delta\alpha_f^\omega\end{pmatrix},$$

then show that given Δp and Δy^ω, the system (4.7)-(4.9) is solvable for Δy^0 and $\Delta\gamma^\omega$.

Step 3. Given Δp, Δy and Δe_h for $h>1$, the system (4.5), (4.6), (4.10) is solvable for Δc, Δe_h^1 and $\Delta\lambda$ if the right-hand sides of these equations satisfy some condition (see the proof), which is derived from the Walras' Law.

These facts together imply that the whole system (4.5)-(4.13) is solvable provided its right-hand side satisfies a certain condition. This condition defines a hyperplane in $\mathbb{R}^N$. Thus,

$$\dim \mathrm{Im}(D\psi\big|_\zeta) = \mathrm{rank}(D\psi\big|_\zeta) = N - 1.$$

Q.E.D.

5. Generic efficiency

5.1 Preliminary results

In the previous section, we proved that there exists an open dense subset $\Theta \subset E \times \mathcal{T}$ such that for every $(e, T) \in \Theta$ and any WE (p, c, y), the return matrix $Y(\hat{p}, \hat{y})$ has full rank. Therefore, by Lemma 3.1, there exists z such that (p, c, y, z) is FE. Now we are going to show that there exists a dense set of economies $\Theta_\Omega \subset \Theta$ such that for any $(e, T) \in \Theta_\Omega$, FE (p, c, y, z) have complete financial markets, i.e., $\mathrm{rank} Y(\hat{p}, \hat{y}) = \Omega$. If this is the case, then again on the strength of Lemma 3.1, FE is allocation equivalent to WE, hence it is efficient.

Unfortunately, we cannot use the same argument as in Subsection 4.2. We can apply the perturbation technique, as before, but we cannot show that the corresponding Jacobian has the largest possible rank unless we know that the return matrix has full rank, but that is what we need to prove here. Therefore the proof of density will be more complicated. We are going to introduce pseudo-equilibria which are allocation equivalent to FE. After this, we will prove that there is a dense set $\Theta_\Omega \subset \Theta$ such that pseudo-equilibria with $\mathrm{rank} Y(\hat{p}, \hat{y}) < \Omega$ do not exist on Θ_Ω. This will imply that FE are efficient on Θ_Ω.

Introduce the following notation. If the return matrix has rank less than full, let $1 \leq \rho = \mathrm{rank} Y(\hat{p}, \hat{y}) \leq \Omega - 1$. (Notice that since $p(\omega) y_f^\omega > 0 \ \forall \ f, \omega > 0$, we have $\mathrm{rank} Y(\hat{p}, \hat{y}) \neq 0$.) Fix subsets $\mathfrak{I}^- = \{i_1, \dots, i_\rho\}$ and $\mathfrak{I}^+ = \{j_1, \dots, j_\rho\}$ of $\{1, \dots, \Omega\}$, where $\mathfrak{I}^-$ ($\mathfrak{I}^+$) denotes a set of linearly independent rows (columns) of $Y(\hat{p}, \hat{y})$. Let

$$\tilde{Y} \equiv [Y_{i_l, j_k}]_{l,k=1}^\rho; \quad Y^0 \equiv [Y_{i_l, j}]_{1 \leq l \leq \rho, \ j \notin \mathfrak{I}^+};$$

evidently, $\tilde{Y}$ has full rank.

Let

$$\tilde{P} \equiv \mathrm{diag}[p(i_l)]_{l=1}^\rho; \quad \tilde{c}_h \equiv (c_h^{i_l})_{l=1}^\rho; \quad \tilde{e}_h \equiv (e_h^{i_l})_{l=1}^\rho; \quad \tilde{y}_f \equiv (y_f^{i_l})_{l=1}^\rho.$$

Clearly, for every $\omega \notin \mathfrak{I}^-$, there exists a vector $\beta^\omega \in \mathbb{R}^{\rho+1}$ such that

$$\beta^\omega \cdot \begin{bmatrix} \tilde{Y}, \ Y^0 \\ (p(\omega) y_f^\omega)_{f=1}^F \end{bmatrix} = 0, \quad \beta^\omega \cdot \beta^{\omega^T} = 1,$$

i.e.

$$\sum_{l=1}^\rho \beta_{i_l}^\omega p(i_l) y_f^{i_l} + \beta_\omega^\omega p(\omega) y_f^\omega = 0 \ \forall \ f \ \forall \ \omega \notin \mathfrak{I}^-.$$

Notice that since the matrix $\tilde{Y}$ is invertible by assumption, it follows that $\beta^\omega_\omega \neq 0$ for all ω. Recall that the following budget constraint defines the budget set $B_h(p, Y)$ in (H^{F}):

$$\hat{P}(\hat{c}_h - \hat{e}_h - \sum_f s_{hf}\hat{y}_f) = Y(\hat{p}, \hat{y})z_h. \tag{5.1}$$

Now we can rewrite it as

$$\tilde{P} \cdot (\tilde{c}_h - \tilde{e}_h - \sum_f s_{hf}\tilde{y}_f) \;=\; \begin{bmatrix} \tilde{Y} & Y^0 \end{bmatrix} z_h; \tag{5.2}$$

$$\beta^\omega \begin{bmatrix} \tilde{P} \cdot (\tilde{c}_h - \tilde{e}_h - \sum_f s_{hf}\tilde{y}_f) \\ p(\omega)(c^\omega_h - e^\omega_h - \sum_f s_{hf}y^\omega_f) \end{bmatrix} \;=\; 0 \;\forall\; \omega \notin \mathfrak{I}^-. \tag{5.3}$$

The last equation is equivalent to

$$\sum_{l=1}^{\rho} \beta^\omega_{i_l} p(i_l)(c^{i_l}_h - e^{i_l}_h - \sum_f s_{hf}y^{i_l}_f) + \beta^\omega_\omega p(\omega)(c^\omega_h - e^\omega_h - \sum_f s_{hf}y^\omega_f) = 0 \;\forall\; f \;\forall\; \omega \notin \mathfrak{I}^-.$$

5.2 Pseudo-equilibria

Introduce vectors $\alpha^\omega \in \mathbb{R}^\Omega$ indexed by $\omega \notin \mathfrak{I}^-$, defined as follows

$$\alpha^\omega_j = \begin{cases} \beta^\omega_j & \text{if } j \in \mathfrak{I}^- \\ \beta^\omega_\omega & \text{if } j = \omega \\ 0 & \text{otherwise} \end{cases} \tag{5.4}$$

Since $\beta^\omega_\omega \neq 0$ for all ω, the vectors α^ω are linearly independent; moreover, on the strength of (5.3),

$$\alpha^\omega \cdot \hat{P} \cdot (\hat{c}_h - \hat{e}_h - \sum_f s_{hf}\hat{y}_f) = 0 \;\forall\; \omega \notin \mathfrak{I}^-. \tag{5.5}$$

Re-label $\omega \notin \mathfrak{I}^-$ as $i = 2, \ldots, \Omega - \rho + 1$ and introduce new price vectors

$$p^1 \equiv p; \quad p^i = (\mathbf{1} + [0, \alpha^i])P,\; i \geq 2, \tag{5.6}$$

where $\mathbf{1} = (1, \ldots, 1)$, a vector of length $\Omega + 1$. Notice that since $||\beta^\omega|| = 1 \;\forall\; \omega \notin \mathfrak{I}^-$, we have $||\alpha^\omega|| = 1 \;\forall\; \omega \notin \mathfrak{I}^-$; moreover, since $\beta^\omega_\omega \neq 0$, and, clearly, at least one $\beta^\omega_{i_l} \neq 0$ (because $p(\omega)y^\omega_f > 0$), we have $|\alpha^\omega_j| < 1$ for all $j = 1, \ldots, \Omega$. In particular, $\alpha^\omega_\omega > -1$, so the vectors $\mathbf{1} + \alpha^i$ are linearly independent, whence so are the vectors p^i (because the matrix $\hat{P}$ has full row rank.) Also, $\mathbf{1} + \alpha^i \gg 0$, so $p^i \gg 0$ for all i. The next lemma allows us to reformulate the household h's problem.

Lemma 5.1: *If* $\mathrm{rank} Y(\hat{p}, \hat{y}) = \rho < \Omega$, *then the sets*

$$B'_h(p, Y) = \{c_h \in \mathbb{R}^L_{++} \,\big|\, p \cdot (c_h - e_h - \sum_f s_{hf}y_f) = 0$$

$$\text{and } \exists\; z_h :\; \hat{P}(\hat{c}_h - \hat{e}_h - \sum_f s_{hf}\hat{y}_f) = Y(\hat{p}, \hat{y})z_h\}$$

and

$$B_h(\{p^i\}) = \left\{ c_h \in \mathbb{R}^L_{++} \,\middle|\, p_i \cdot (c_h - e_h - \sum_f s_{hf} y_f) = 0, \ \forall \ i = 1, \dots, \Omega - \rho + 1 \right\}$$

are equal.

Proof. In Appendix C.

Define an $(\mathfrak{I}^+, \mathfrak{I}^-)$-equilibrium for an economy (e, T) as a vector (p, c, y) such that

(i): $\text{rank} Y(\hat{p}, \hat{y}) = \rho$ and $\mathfrak{I}^+$ (resp. $\mathfrak{I}^-$) is a set of linearly independent columns (resp. rows) of $\text{rank} Y(\hat{p}, \hat{y})$;

(ii): firms solve $(\mathrm{F}^{\mathrm{F}})$;

(iii): households solve

$$(\mathrm{H}^\rho) \qquad \max_{c_h} u_h(c_h) \ \text{ subj. to } c_h \in B_h(\{p^i\})$$

where p_i are defined by (5.3), (5.4) and (5.6);

(iv): contingent commodities markets clear, i.e.,

$$\sum_h (c_h - e_h) - \sum_f y_f = 0.$$

Now we are in a position to show that $(\mathfrak{I}^+, \mathfrak{I}^-)$-equilibria are allocation equivalent to FE with $\text{rank} Y(\hat{p}, \hat{y}) = \rho$.

Lemma 5.2: *(i) If (p, c, y, z) is FE such that* $\text{rank} Y(\hat{p}, \hat{y}) = \rho$ *and $\mathfrak{I}^+$ (resp. $\mathfrak{I}^-$) denotes the set of linearly independent columns (resp. rows) of $Y(\hat{p}, \hat{y})$, then (p, c, y) is an $(\mathfrak{I}^+, \mathfrak{I}^-)$-equilibrium.*
(ii) If (p, c, y) is an $(\mathfrak{I}^+, \mathfrak{I}^-)$-equilibrium, then there exists $z \in \mathbb{R}^{H\Omega}$ such that (p, c, y, z) is FE.

Proof. Part (i) is obvious. For part (ii), by Lemma 5.1, we know that there exists $z \in \mathbb{R}^{H\Omega}$ satisfying (5.1) (and hence (5.2)). We must prove that z can be chosen so that

$$\sum_h z_h = 0.$$

Since (p, c, y) is an $(\mathfrak{I}^+, \mathfrak{I}^-)$-equilibrium, equation (5.2) uniquely determines z_h^f for $f \in \mathfrak{I}^+$, while the rest of z_h^f remain undetermined. We can set them equal to zero, then by construction,

$$\sum_h z_h^f = 0 \ \ \forall \ f \notin \mathfrak{I}^+.$$

We can rewrite (5.2) as

$$\tilde{P} \cdot (\tilde{c}_h - \tilde{e}_h - \sum_f s_{hf} \tilde{y}_f) = \tilde{Y} \tilde{z}_h. \tag{5.7}$$

Summing up individual budget constraints over h and using goods market clearing, we derive

$$\tilde{Y}\cdot\sum_h \tilde{z}_h = 0 \quad\Rightarrow\quad \sum_h \tilde{z}_h = 0$$

because $\tilde{Y}$ is invertible by construction. *Q.E.D.*

5.3 Characterization of pseudo-equilibria

It is obvious that $(\mathfrak{I}^+,\mathfrak{I}^-)$-equilibria can be characterized by the Kuhn-Tucker conditions. Let $\lambda_h\in\mathbb{R}^{\Omega-\rho+1}_{++}$ be a vector of Lagrange multipliers for households' problem. Define $\lambda\equiv(\lambda_h)_{h=1}^H\in\mathbb{R}^{H(\Omega-\rho+1)}_{++}$, $\mathcal{P}\equiv(p^i)_{i=1}^{\Omega-\rho+1}\in\mathbb{R}^{L(\Omega-\rho+1)}_{++}$, $\beta\equiv(\beta^\omega)_{\omega\notin\mathfrak{I}^-}\in\mathbb{R}^{(\Omega-\rho)(\rho+1)}$. The rest of the notation remains the same as in the previous sections. The system of equations for a $(\mathfrak{I}^+,\mathfrak{I}^-)$-equilibrium can be written as

$$\text{(5.8)}\qquad Du_h(c_h)-\sum_{i=1}^{\Omega-\rho+1}\lambda_h^i p^i = 0 \ \ \forall\ h;$$

$$\text{(5.9)}\qquad p^i\cdot(c_h-e_h-\sum_f s_{hf}y_f) = 0 \ \ \forall\ h,\ (i=1,\dots,\Omega-\rho+1);$$

$$\text{(5.10)}\qquad \text{for } \imath\geq 2,\quad p^i(\omega) = \begin{cases}(1+\beta^i_\omega)p^1(\omega) & \text{if } \omega\in\mathfrak{I}^- \text{ or } \omega=i\\ p^1(\omega) & \text{otherwise}\end{cases};$$

$$\text{(5.11)}\qquad p^1(0)+\sum_{\omega>0}\mu_f^\omega D_0 t_f^\omega(y_f^0,y_f^\omega) = 0 \ \ \forall\ f;$$

$$\text{(5.12)}\qquad p^1(\omega)+\mu_f^\omega D_\omega t_f^\omega(y_f^0,y_f^\omega) = 0 \ \ \forall\ f,\omega>0;$$

$$\text{(5.13)}\qquad t_f^\omega(y_f^0,y_f^\omega) = 0 \ \ \forall\ f,\omega>0;$$

$$\text{(5.14)}\qquad \sum_h(c_h-e_h)-\sum_f y_f = 0;$$

$$\text{(5.15)}\qquad p^1\cdot p^{1^T} = 1;$$

$$\text{(5.16)}\qquad \sum_{\omega\in\mathfrak{I}^-}\beta^i_\omega p^1(\omega)y_f^\omega+\beta^i_i p^1(i)y_f^i = 0 \ \ i=2,\dots,\Omega-\rho+1,\ \ \forall\ f;$$

$$\text{(5.17)}\qquad \beta^i\cdot\beta^{i^T} = 1,\ \ i=2,\dots,\Omega-\rho+1.$$

5.4 Density argument

In this subsection we show that there exists a dense subset of Θ such that for any (e,T) in this subset, FE has the return matrix of full rank. Let $X=\mathbb{R}^L_{++}\times(\mathbb{R}^L_{++})^H\times(\mathbb{R}^G_{--}\times\mathbb{R}^{G\Omega}_{++})^F$ be the manifold of endogenous variables for FE with the typical element $x=(p,c,y)$. Introduce

$$\Sigma^F\equiv\{(x,e,T)\in X\times\Theta\big|\ x \text{ is FE}\}.$$

For $\rho=1,\dots,\Omega$, set

$$\Theta_\rho\equiv\{(e,T)\in\Theta\big|\ \exists\ (x,e,T)\in\Sigma^F:\ \text{rank}Y(\hat{p},\hat{y})=\rho$$

$$\text{and } \forall\ (x', e, T) \in \Sigma^F,\ \operatorname{rank} Y(\hat{p}', \hat{y}') \geq \rho\big\}\,.$$

Notice that for any $\rho' \neq \rho$, $\Theta_{\rho'} \cap \Theta_\rho = \emptyset$.

Thus Θ_Ω is the set of efficient economies, and $\Theta^c_\Omega = \Theta \setminus \Theta_\Omega = \bigcup_{\rho=1}^{\Omega-1} \Theta_\rho$ is the set of economies such that in some equilibria financial markets are incomplete. Notice that in the latter economies, there may exist equilibria with complete financial markets. On the other hand, an economy is efficient when all equilibria have complete financial markets.

Suppose that Θ_Ω is not dense, then there exists a point $(\bar{e}, \bar{T}) \in \Theta^c_\Omega$ and its open neighborhood $W(\bar{e}, \bar{T})$ such that $W(\bar{e}, \bar{T}) \subset \Theta^c_\Omega$. Let E_Θ and $\mathcal{T}_\Theta$ be the natural projections of Θ on E and $\mathcal{T}$, respectively. By definition of product topology, there exist open sets $U \subset E_\Theta$ and $V \subset \mathcal{T}_\Theta$ such that

$$(\bar{e}, \bar{T}) \in U \times V \subset W(\bar{e}, \bar{T}) \subset \Theta^c_\Omega.$$

Using the same finite dimensional family of transformation functions as before, we focus on the open set

$$M \equiv U \times T^{-1}(V) \subset E \times \mathcal{A}$$

parameterizing a subset of nonefficient economies. As in Subsection 4.2, we are going to show that there exists a smooth surjective map from a finite union of smooth manifolds of dimension less than $\dim M$ onto M which will give a contradiction. Introduce

$$\Xi^\rho \equiv \mathbb{R}^{HL}_{++} \times \mathbb{R}^{L(\Omega-\rho+1)}_{++} \times (\mathbb{R}^G_{--} \times \mathbb{R}^{\Omega G}_{++})^F \times \mathbb{R}^{H(\Omega-\rho+1)}_{++} \times \mathbb{R}^{F\Omega}_{++} \times \mathbb{R}^{(\Omega-\rho)(\rho+1)},$$

a manifold of endogenous variables for $(\mathfrak{I}^+, \mathfrak{I}^-)$-equilibria with typical element $\xi = (c, \mathcal{P}, y, \lambda, \mu, \beta)$. Let

$$N_\rho \equiv H(L + \Omega - \rho + 1) + F(L + \Omega) + L + L(\Omega - \rho) + 1 + (\Omega + 1)(\Omega - \rho).$$

Introduce

$$\phi_{(\mathfrak{I}^+,\mathfrak{I}^-)} : \Xi^\rho \times \Theta \longrightarrow \mathbb{R}^{N_\rho},$$

a smooth map consisting of the left-hand sides of equations (5.8)–(5.17). Define $\Pi_{(\mathfrak{I}^+,\mathfrak{I}^-)} \equiv \phi^{-1}_{(\mathfrak{I}^+,\mathfrak{I}^-)}(0)$. Obviously

$$\Theta^c_\Omega = \bigcup_{(\mathfrak{I}^+,\mathfrak{I}^-)} \pi(\Pi_{(\mathfrak{I}^+,\mathfrak{I}^-)}),$$

where π is the natural projection. Consider a smooth map

$$\psi_{(\mathfrak{I}^+,\mathfrak{I}^-)} : \Xi^\rho \times M \longrightarrow \mathbb{R}^{N_\rho}$$

given by the composition

$$(\xi, e, A) \longmapsto (\xi, e, T(A)) \longmapsto \phi(\xi, e, T(A))$$

Let $\Gamma_{(\mathfrak{I}^+,\mathfrak{I}^-)} = \psi^{-1}_{(\mathfrak{I}^+,\mathfrak{I}^-)}(0)$, which is the inverse image of $\Pi_{(\mathfrak{I}^+,\mathfrak{I}^-)}$ in $\Xi^\rho \times M$ and let $\pi_M : \Xi^\rho \times M \to M$ be the projection. By construction,

$$M = \bigcup_{(\mathfrak{I}^+,\mathfrak{I}^-)} \pi_M(\Gamma_{(\mathfrak{I}^+,\mathfrak{I}^-)}). \qquad (\dagger\dagger)$$

It is evident that the number of different pairs $(\mathfrak{I}^+, \mathfrak{I}^-)$ is finite. We will prove that each $\Gamma_{(\mathfrak{I}^+,\mathfrak{I}^-)}$ is contained in a finite union of submanifolds of $\Xi^\rho \times M$ of dimension less than $\dim M$. Once this is shown, the equality (††) becomes impossible and we are done.

For a point $\zeta = (\xi, e, A) \in \Xi^\rho \times M$, we have the differential

$$D\psi_{(\mathfrak{I}^+,\mathfrak{I}^-)}\big|_\zeta : T_\zeta(\Xi^\rho \times M) \longrightarrow \mathbb{R}^{N_\rho}.$$

If we can show that $\operatorname{rank} D\psi_{(\mathfrak{I}^+,\mathfrak{I}^-)}\big|_\zeta \geq \dim \Xi^\rho + 1$ for all $\zeta \in \Gamma_{(\mathfrak{I}^+,\mathfrak{I}^-)}$, then by the corollary to the Preimage Theorem, $\Gamma_{(\mathfrak{I}^+,\mathfrak{I}^-)}$ is contained in a finite union of submanifolds of $\Xi^\rho \times M$ of dimension $\dim M - 1$.

Fix a point $\zeta \in \Gamma_{(\mathfrak{I}^+,\mathfrak{I}^-)}$.

Theorem 5.3: $\operatorname{rank}\left(D\psi_{(\mathfrak{I}^+,\mathfrak{I}^-)}\big|_\zeta\right) = N_\rho - (\Omega - \rho + 1)$.

Proof. The proof basically repeats the main steps of the proof of Theorem 4.9 (see Appendix C).

In Subsection 4.2, the proof of Theorem 4.9 completed the density argument for WE. Here the proof of Theorem 5.3 is insufficient to establish density of Θ_Ω due to the following reason. We have

$$\dim \Xi^\rho = H(L+\Omega-\rho+1) + L(\Omega-\rho) + L + F(L+\Omega) + (\Omega-\rho)(\rho+1) = N_\rho - (\Omega-\rho)^2 - 1,$$

so by Theorem 5.3,

$$\begin{aligned} & \operatorname{rank}\left(D\psi\big|_\zeta\right) \geq \dim \Xi^\rho + 1 \\ \Leftrightarrow \quad & N_\rho - (\Omega - \rho + 1) \geq N_\rho - (\Omega - \rho)^2 \\ \Leftrightarrow \quad & (\Omega - \rho)(\Omega - \rho - 1) \geq 1. \end{aligned} \tag{5.18}$$

Obviously, (5.18) only holds if $\rho < \Omega - 1$, but at least we know that $\Gamma_{(\mathfrak{I}^+,\mathfrak{I}^-)}$ is contained in a finite union of submanifolds of $\Xi^\rho \times M$ of dimension less than $\dim M$ for all $\rho = \#(\mathfrak{I}^+) = \#(\mathfrak{I}^-) \leq \Omega - 1$. It remains to consider $\Gamma_{(\mathfrak{I}^+,\mathfrak{I}^-)}$ for $\#(\mathfrak{I}^+) = \#(\mathfrak{I}^-) = \Omega - 1$.

So far, we have only used the fact that if $\operatorname{rank} Y(\hat{p}, \hat{y}) < \Omega$, then the rows of the return matrix are not linearly independent, and we have never used linear dependency of the columns. Assume that $\operatorname{rank} Y(\hat{p}, \hat{y}) = \Omega - 1$ and re-label the states and assets so that the first $\Omega - 1$ rows and columns of $Y(\hat{p}, \hat{y})$ are linearly independent. There exists a vector $\gamma \in \mathbb{R}^\Omega$ such that

$$\sum_{f=1}^{\Omega-1} \gamma_f p^1(\omega) y_f^\omega + \gamma_\Omega p^1(\omega) y_F^\omega \;=\; 0 \;\; \forall\, \omega; \tag{5.19}$$

$$\gamma \cdot \gamma^T \;=\; 1. \tag{5.20}$$

Append equations (5.19) and (5.20) to equations (5.8)-(5.17) written for the case $\rho = \Omega - 1$. Let

$$\tilde{\Xi} = \mathbb{R}^{HL}_{++} \times \mathbb{R}^{2L}_{++} \times \left(\mathbb{R}^{G}_{--} \times \mathbb{R}^{\Omega G}_{++}\right)^F \times \mathbb{R}^{2H}_{++} \times \mathbb{R}^{F\Omega}_{++} \times \mathbb{R}^{\Omega} \times \mathbb{R}^{\Omega}$$

be a manifold of extended endogenous variables with the typical element $\tilde{\xi} = (\xi, \gamma)$. Let $\tilde{N} = N_{\Omega-1} + \Omega + 1$, and let

$$\tilde{\phi} : \tilde{\Xi} \times \Theta \longrightarrow \mathbb{R}^{\tilde{N}}$$

be the smooth map consisting of the left-hand sides of the extended system of equations.

Also, as before, we introduce

$$\tilde{\psi} : \tilde{\Xi} \times M \longrightarrow \mathbb{R}^{\tilde{N}},$$

and $\tilde{\Gamma} = \tilde{\psi}^{-1}(0)$. Fix $\tilde{\zeta} \in \tilde{\Gamma}$. We want to show that

$$\operatorname{rank}\left(D\tilde{\psi}\big|_{\tilde{\zeta}}\right) \geq \dim \tilde{\Xi} + 1 = \tilde{N} - 2.$$

If we prove this, then, using the corollary to the Preimage Theorem, we may conclude that $\tilde{\Gamma}$ is contained in a finite union of submanifolds of $\tilde{\Xi} \times M$, each of dimension less than $\dim M$.

Lemma 5.4:

$$\operatorname{rank}\left(D\tilde{\psi}\big|_{\tilde{\zeta}}\right) = \tilde{N} - 2.$$

Proof. In Appendix C.

6. Conclusion

We have examined the general equilibrium model incorporating households' and firms' optimizing behavior, uncertainty and potentially complete real asset markets. The two-period economy is specified by households' endowments and firms' transformation functions. The transformation functions belong to a sufficiently wide family of functions; the technology is state dependent and goods in period 0 are used as inputs to produce state dependent outputs in period 1. The main motivation for this analysis was to provide justification for complete asset markets models used in theoretical Finance. In those models, it is common to take stock prices as given and just to assume completeness of financial markets.

In the first part of the paper, we proved, in the most general setting allowed by a two-period general equilibrium model, that generically the model with neoclassical firms and stock shares as financial instruments has an equilibrium. We had to use the weak notion of genericity in terms of an open dense set of parameters, but if we consider parametric transformation functions instead of more general functions, we will be able to establish the same result on an open full measure set of parameters, i.e. strong genericity applies. In the second part of the paper, we showed that FE are efficient on Θ_Ω which is a dense set of parameters. Similar result was obtained in Cass and Rouzaud (2000) for the relevant set of parameters specifying the economy and family of transformation functions. Unfortunately, so far we managed neither to prove that this set was open, nor to find a counterexample for this fact. In our opinion, there is no particular reason for this set to be always open, or equivalently for the complement to this set to be always closed. To explain the last statement, we reproduce here one of the constraints defining a household's budget set in FE:

$$\hat{P}(\hat{c}_h - \hat{e}_h - \sum_f s_{hf}\hat{y}_f) = Y(\hat{p}, \hat{y})z_h \ \forall\ h \tag{6.1}$$

Let a sequence $\{(e^\nu, T^\nu)\}_{\nu=1}^\infty \subset \Theta_\Omega^c$ converging to $(e,T) \in E \times \mathcal{T}$ be such that in FE specified by the sequence, $\text{rank}Y(\hat{p}^\nu, \hat{y}^\nu) = \rho < \Omega$ and

$$\hat{P}^\nu(\hat{c}_h^\nu - \hat{e}_h^\nu - \sum_f s_{hf}\hat{y}_f^\nu) \in \text{Im}Y(\hat{p}^\nu, \hat{y}^\nu) \ \forall\ h$$

or equivalently, there exist z_h^ν, $\forall\ h$ such that

$$\hat{P}^\nu(\hat{c}_h^\nu - \hat{e}_h^\nu - \sum_f s_{hf}\hat{y}_f^\nu) = Y(\hat{p}^\nu, \hat{y}^\nu)z_h^\nu \ \forall\ h.$$

Suppose that $\text{rank}Y(\hat{p}, \hat{y}) < \rho$, then it is very unlikely that $\hat{P}(\hat{c}_h - \hat{e}_h - \sum_f s_{hf}\hat{y}_f) \in \text{Im}Y(\hat{p}, \hat{y})$, and if this is the case, then there is at least one h such that z_h satisfying (6.1) does not exist, or equivalently, FE with incomplete markets disappear in the limit. Now, unless $(e,T) \in \Theta'$ (the set of parameters for which FE do not exist), on the strength of density of efficient economies, by small perturbations, we may be able to find $\{(e^\nu, T^\nu)\}_{\nu=1}^\infty$ enjoying all the above properties such that $(e,T) \in \Theta_\Omega$, which will imply that Θ_Ω^c is not closed. If this result can be obtained, it is rather a bad news for the researchers who assume ad hoc complete real asset markets in their models. It will also imply that Arrow's Equivalency Theorem does not necessarily apply to real asset models of FE. More precisely, it applies in the following sense: generically WE constitute a subset of FE, and there is a dense set of economies such that FE are allocation equivalent to WE.

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Appendix A.

A.1. Proof of Lemma 3.1

First of all, notice that after re-formulation of FE, the firm's problem and spot goods market clearing conditions are the same for WE and FE. Existence of z is obvious since $\text{rank}Y(\hat{p},\hat{y}) = \Omega$ implies that (6.1) has a solution z for any left-hand side. Hence if $c_h \in B_h(p)$, then there is z_h such that $c_h \in B_h(p, Y)$. It remains to show that asset markets clear. To this end, sum up (6.1) over h to get

$$\hat{P}\left(\sum_h (\hat{c}_h - \hat{e}_h) - \sum_f \hat{y}_f\right) = Y(\hat{p},\hat{y})\sum_h z_h.$$

From commodities market clearing, the left-hand side of the last equation equals zero, and since $Y(\hat{p},\hat{y})$ has full rank, we obtain $\sum_h z_h = 0$.
(b) Is obvious. *Q.E.D.*

A.2. Proof of Claim 3.2

In the firm's problem (F^{W}), drop subscript f and define $\Phi(t,p) = \{y \in \mathbb{R}^G_- \times \mathbb{R}^{\Omega G}_+ \,|\, p\cdot y \geq 0,\ t(y) \geq 0\}$.

We show that:

(i) $\Phi(t,p)$ is bounded, and, since it is evidently closed, it is compact.

(ii) The set $\Psi(t,p) = \{y \in \mathbb{R}^G_- \times \mathbb{R}^{\Omega G}_+ \,|\, p\cdot y > 0,\ t(y) \gg 0\}$ is nonempty, so that $\Phi(t,p) \supset \Psi(t,p)$ is nonempty as well.

We argue boundedness of $\Phi(t,p)$ as follows. Let $y = (y^0, w)$ and $p = (p(0), p_w)$. By property **T7** and equivalence of norms in $\mathbb{R}^G$, for any $\epsilon > 0$, there exists c_ϵ such that

$$\sum_{\omega>0}\sum_g w^{\omega g} \leq c_\epsilon - \epsilon\sum_g y^{0g}$$

for any $(y^0, w) \in \Phi(t,p)$. Also $p(0)y^0 + p_w w \geq 0$. Take $\epsilon > 0$ such that $2(\epsilon, \dots, \epsilon) \ll p(0)$, then for $y^0 \ll 0$, $p(0)y^0 \leq 2\epsilon\sum_g y^{0g}$ and hence, since $p_w \ll 1$,

$$0 \leq p(0)y^0 + p_w w \leq 2\epsilon\sum_g y^{0g} + \sum_{\omega>0}\sum_g w^{\omega g} \leq \epsilon\sum_g y^{0g} + c_\epsilon.$$

Hence $-\sum_g y^{0g} \leq c_\epsilon/\epsilon$ and $\sum_{\omega>0}\sum_g w^{\omega g} \leq 2c_\epsilon$. Therefore $\Phi(t,p)$ is bounded.

To establish non-emptiness we need to prove two auxiliary lemmas.

Lemma A.1: *For any $\omega > 0$ and $u \in \mathbb{R}^G_{--}$, $t^\omega(u,0) > 0$.*

Proof. By the mean value theorem, there exists $\theta \in (0,1)$ such that $t^\omega(u,0) - t^\omega(0,0) = Dt^\omega(\theta u, 0)u$. But $\theta u \ll 0$ and **T2** imply that $Dt^\omega(\theta u,0)u > 0$, and since from **T4**, $t^\omega(0,0) = 0$, we are done. *Q.E.D.*

Lemma A.2: *For all $p \gg 0$, there exists $\epsilon > 0$ such that $\forall\ \delta \in (-\epsilon, 0)$, $\exists\ \sigma > 0$ such that $\forall\ \omega \geq 1$,*

$$(u, v) \equiv (\delta, \dots, \delta, \sigma, \dots, \sigma) \in \Psi(t^\omega, p),$$

where

$$\Psi(t^\omega, p) = \{(u, v) \in \mathbb{R}^G_- \times \mathbb{R}^G_+ \big|\ t^\omega(u, v) > 0,\ p(0)u + p(\omega)v > 0\}.$$

Proof. From **T1**, $D_v t^\omega(u, v)$ is bounded in a neighborhood of $(u, v) = (0, 0)$, hence from **T6**,

$$||D_v t^\omega(u, v)|| = o(||D_u t(u, v)||) \text{ as } (u, v) \to 0.$$

On $\mathbb{R}_{--} \times \mathbb{R}_{++}$, define

$$\tilde{t}^\omega(r, \rho) = t^\omega(r, \dots, r, \rho, \dots, \rho).$$

Then

$$|D_\rho \tilde{t}^\omega(r, \rho)| = o(|D_r \tilde{t}^\omega(r, \rho)|) \text{ as } (r, \rho) \to 0,$$

and from **T2**, $D_r \tilde{t}^\omega < 0$, $D_\rho \tilde{t}^\omega < 0$ on $\mathbb{R}_{--} \times \mathbb{R}_{++}$. By applying the Implicit Function theorem, an equation $\tilde{t}^\omega(r, \rho) = 0$ defines ρ as a function of r with

$$\frac{d\rho}{dr} = -\frac{\partial \tilde{t}^\omega / \partial r}{\partial \tilde{t}^\omega / \partial \rho} < 0, \frac{d\rho}{dr} \to -\infty \text{ as } r \to -0.$$

From **T1** and **T3**, $\rho(-0) = 0$, and by applying to $\rho = \rho(r)$ the Mean Value Theorem on $(-r, 0)$, we obtain

$$\rho(r)/r \to -\infty \text{ as } r \to -0.$$

Hence, for any $C > 0$, there exists $\delta > 0$ such that if $r \in (-\delta, 0)$,

$$\rho(r) \geq -2Cr.$$

It follows that for $\sigma = \rho(r)/2$, $\tilde{t}^\omega(r, \sigma) > 0$, and if C is chosen large enough (for a given p) then

$$-r \cdot p(0) + \sigma \cdot p(\omega) > 0,$$

completing the proof. *Q.E.D.*

Now by the Maximum Value Theorem, there exists an optimal solution $y(p)$ to

$$(\mathrm{F}'') \qquad \max_y p \cdot y \quad \text{subj. to } \ y \in \Phi(t, p).$$

Since $\Psi(t, p) \neq \emptyset$, and by the strict monotonicity of the objective function in (F″), we may conclude that the constraint $p \cdot y \geq 0$ in the definition of $\Phi(t, p)$ is nonbinding, i.e., any $y_*(p)$ solving (F″) also solves ($\mathrm{F^W}$). Uniqueness of an optimal $y(p)$ follows from strict concavity of the transformation function. This completes the proof of Claim 3.2. *Q.E.D.*

A.3. Proof of Lemma 3.6

(1) $y^0_{\text{opt}} \ll 0$ since otherwise $v = 0$ from **T4**, and then $p \cdot y \leq 0$. But from Claim 3.2, $p \cdot y_{\text{opt}} > 0$.
(2) Let $y^0 = y^0_{\text{opt}} \ll 0$. Then a firm finds y^ω, $\omega \geq 1$, by maximizing $p^\omega \cdot v$ on the constraint set

$$\Phi(t^\omega, p, y^0) = \{v > 0 \big| \; t^\omega(y^0, v) \geq 0, \; p^\omega \cdot v \geq 0\}.$$

Since $y^0 \ll 0$, from Lemma A.1, $t^\omega(y_0, v) > 0$ in a neighborhood of 0; from **T2** and **T3**,

$$D_v t^\omega(y^0, v) \ll 0, \; D^2_v t^\omega(y^0, v) < 0,$$

so the set $\{v > 0 \big| \; t^\omega(y^0, v) \geq 0\}$ is convex. From **T5**, $D_v t^\omega(y^0, v)$ is orthogonal to coordinate axes and planes where $\{v > 0 \big| \; \tilde{t}(v) = 0\}$ intersects them, and since $p \gg 0$, an optimal v cannot be at the boundary of $\mathbb{R}^G_+$. *Q.E.D.*

A.4. Proof of Lemma 4.2

Recall our previous notation:

$$\Phi(t, p) = \{y \in \mathbb{R}^G_- \times \mathbb{R}^{\Omega G}_+ \big| \; p \cdot y \geq 0, \; t(y) \geq 0\};$$

$$\Psi(t, p) = \{y \in \mathbb{R}^G_- \times \mathbb{R}^{\Omega G}_+ \big| \; p \cdot y > 0, \; t(y) \gg 0\}.$$

We know that the problem (4.1) is equivalent to

$$\max_{y \in \Phi(t,p)} p \cdot y,$$

so it suffices to verify that the correspondence Φ satisfies the assumptions of Theorem B.1 (b) (Appendix B). On the strength of **T3**, Φ is convex-valued, and it is nonempty and bounded-valued by the proof of Claim 3.2. Hence we must check that it is continuous, in the sense of definition in Appendix B. So let $\{(t^\nu, p^\nu)\}_{\nu=1}^\infty \subset (\mathcal{T}_0)^\Omega \times \mathbb{R}^L_{++}$ be a sequence converging to $(t, p) \in (\mathcal{T}_0)^\Omega \times \mathbb{R}^L_{++}$. We have to prove that

(i) if a sequence $\{y^\nu \in \Phi(t^\nu, p^\nu)\}_{\nu=1}^\infty$ converges to some $y \in \mathbb{R}^G_- \times \mathbb{R}^{G\Omega}_+$, then $y \in \Phi(t, p)$;
(ii) if $y \in \Phi(t, p)$, then there exists a sequence $\{y^\nu\} \subset \mathbb{R}^G_- \times \mathbb{R}^{G\Omega}_+$ converging to y and a number $N \in \mathbf{N}$ such that for any $\nu \geq N$, $y^\nu \in \Phi(t^\nu, p^\nu)$.

Introduce $\mathcal{R} = (\mathcal{T}_l)^\Omega \times \mathbb{R}^{\mathcal{L}}_{++} \times (\mathbb{R}^{\mathcal{G}}_- \times \mathbb{R}^{\mathcal{G}\Omega}_+)$. First we show that the map

$$\eta : \mathcal{R} \longrightarrow \mathbb{R}^{\Omega+1}$$

$$(t, p, y) \mapsto (t(y), p \cdot y)$$

is continuous. Let $\{(t^\nu, p^\nu, y^\nu)\} \subset \mathcal{R}$ be a sequence converging to $(t, p, y) \in \mathcal{R}$. Clearly, $\lim_{\nu \to \infty} p^\nu \cdot y^\nu = p \cdot y$, hence we must show that $\lim_{\nu \to \infty} t^\nu(y^\nu) = t(y)$. Let $\epsilon > 0$ be given. Since the set $K = \{y^\nu\} \cup \{y\}$ is compact, $t^\nu \to t$ uniformly on K (by definition of topology on $\mathcal{T}_0$), so there is $N_1 \in \mathbf{N}$ such that

$$\forall \; \nu \geq N_1, \; \forall \; x \in K, \;\; ||t(x) - t^\nu(x)|| < \frac{\epsilon}{2}.$$

Since t is continuous, there is $N_2 \in \mathbf{N}$ such that for all $\nu \geq N_2$, $||t(y) - t(y^\nu)|| < \frac{\epsilon}{2}$. Then for all $\nu \geq \max\{N_1, N_2\}$, we have

$$||t(y) - t^\nu(y^\nu)|| \leq ||t(y) - t(y^\nu)|| + ||t(y^\nu) - t^\nu(y^\nu)|| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Now (i) is obvious. For (ii), we first consider a point $y \in \Psi(p,t)$, that is, $\eta(t,p,y) \in \mathbb{R}^{\Omega+1}_{++}$. Since η is continuous, the set $\eta^{-1}(\mathbb{R}^{\Omega+1}_{++})$ is open; by definition of product topology, there are open sets $U_t \subset (\mathcal{T}_0)^\Omega$, $U_p \subset \mathbb{R}^L_{++}$, and $U_y \subset \mathbb{R}^G_- \times \mathbb{R}^{G\Omega}_+$ such that

$$(t,p,y) \in U_t \times U_p \times U_y \subset \eta^{-1}(\mathbb{R}^{\Omega+1}_{++}).$$

Since $t^\nu \to t$ and $p^\nu \to p$, there exists $N = N_y$ such that for all $\nu \geq N_y$, $t^\nu \in U_t$ and $p^\nu \in U_p$. Then for all $\nu \geq N_y$, $\eta(t^\nu, p^\nu, y) \in \mathbb{R}^{\Omega+1}_{++}$, that is, $y \in \Psi(t^\nu, p^\nu)$.

Now let $y \in \Phi(t,p)$ be arbitrary. By convexity of $\Phi(t,p)$, non-emptiness of $\Psi(t,p)$ (Lemmas A.1 and A.2) and strict concavity of t on $\mathbb{R}^G_{--} \times \mathbb{R}^{G\Omega}_{++}$, there is a sequence $\{y_n\} \subset \Psi(t,p)$ which converges to y. We have just shown that for each $n \in \mathbf{N}$, there exists $N_n \in \mathbf{N}$ such that for all $\nu \geq N_n$, $y_n \in \Psi(p^\nu, t^\nu)$. We will construct a new sequence $\{y^\nu\} \subset \mathbb{R}^G_- \times \mathbb{R}^{\Omega G}_+$ which converges to y such that for $\nu \geq N_1$, $y^\nu \in \Psi(t^\nu, p^\nu)$, as follows. First, define a sequence $\{n^\nu\} \subset \mathbf{N}$ by the following recursive formula: $n^1 = n^2 = \ldots = n^{N_1} = 1$ and for $\nu \geq N_1$,

$$n^{\nu+1} = \begin{cases} n^\nu & \text{if } \nu + 1 < N_{n^\nu+1} \\ n^\nu + 1 & \text{otherwise} \end{cases}$$

Then let $y^\nu = y_{n^\nu}$. By construction, $\nu \geq N_{n^\nu}$ for all $\nu \geq N_1$, hence $y^\nu \in \Psi(t^\nu, p^\nu)$ for $\nu \geq N_1$. It is also clear that the sequence $\{n^\nu\}$ is (nonstrictly) increasing and is not eventually constant, thus $y^\nu \to y$. This completes the proof of (ii) and with it of Lemma 4.2. *Q.E.D.*

A.5. Proof of Lemma 4.3

The argument is entirely similar to the proof of Lemma 4.2.

$$\bar{c}^j(p,e,y) = \frac{p \cdot (e + \sum_f s_f y_f)}{2 \sum_g \sum_\omega p^g(\omega)} \quad \forall\, j.$$

We know that $p \cdot e > 0$ and $p \cdot y_f \geq 0$, thus $\bar{c}(p,e,y) \gg 0$. Since u is strictly increasing, problem (4.2) is equivalent to the following problem

$$\max_{c \in \tilde{\Phi}(p,e,y)} u(c)$$

where

$$\tilde{\Phi}(p,e,y) = \{c \in \mathbb{R}^L_{++} \,|\, -p \cdot (c - e - \sum_f s_f y_f) \geq 0,\ u(c) - u(\bar{c}(p,e,y)) \geq 0\}.$$

Introduce

$$\tilde{\Psi}(p,e,y) = \{c \in \mathbb{R}^L_{++} \,|\, -p \cdot (c - e - \sum_f s_f y_f) > 0,\ u(c) - u(\bar{c}(p,e,y)) > 0\}.$$

Since u is strictly increasing, $\tilde{\Psi}(p,e,y)$ (and hence $\tilde{\Phi}(p,e,y)$) is nonempty; for example, $\frac{3}{2}\cdot\bar{c}(p,e,y)\in\tilde{\Psi}(p,e,y)$. It is clear that $\tilde{\Phi}$ is convex-valued and bounded-valued. Thus, as in the proof of Lemma 4.2 ,

(i) if a sequence $\{c^\nu\in\tilde{\Phi}(p^\nu,e^\nu,y^\nu)\}$ converges to some $c\in\mathbb{R}^L$, then $c\in\tilde{\Phi}(p,e,y)$;

(ii) if $c\in\tilde{\Phi}(p,e,y)$, there exists a sequence $\{c^\nu\}\subset\mathbb{R}^L_{++}$ converging to c and a number $N\in\mathbf{N}$ such that for any $\nu\geq N$, $c^\nu\in\tilde{\Phi}(p^\nu,e^\nu,y^\nu)$.

This time, however, our task is a little easier since it is evident that the mapping

$$\tilde{\eta}:\mathcal{B}\times\mathbb{R}^L_{++}\longrightarrow\mathbb{R}^2$$
$$(p,e,y,c)\mapsto(-p\cdot(c-e-\sum_f s_f y_f),u(c)-u(\bar{c}(p,e,y)))$$

is continuous. The proof of (ii) is exactly the same as in the proof of Lemma 4.2; therefore, we will not present it here. For (i), the only difficulty is that we have to allow the limit point $c\in\mathbb{R}^L$, whereas $\tilde{\eta}$ is only defined on $\mathcal{B}\times\mathbb{R}^L_{++}$. This problem can be avoided as follows. Since $\bar{c}(p^\nu,e^\nu,y^\nu)\to\bar{c}(p,e,y)\in\mathbb{R}^L_{++}$, there exists $\epsilon>0$ such that for all $\nu\in\mathbf{N}$, $\bar{c}^j(p^\nu,e^\nu,y^\nu)>\epsilon$. Since u is strictly increasing, we have $u(\bar{c}(p^\nu,e^\nu,y^\nu))>u(\epsilon,\dots,\epsilon)$ for all $\nu\in\mathbf{N}$, whence $u(c^\nu)>u(\epsilon,\dots,\epsilon)$ for all $\nu\in\mathbf{N}$. But the closures of indifference surfaces of u are contained in $\mathbb{R}^L_{++}$, so $c\in\mathbb{R}^L_{++}$. After this, the argument proceeds as in the proof of Lemma 4.2. *Q.E.D.*

A.6. Proof of Lemma 4.4

We claim that there exists $r>0$ such that $t(u,v)<0$ for all $(u,v)\in\mathbb{R}^G_-\times\mathbb{R}^G_+$ with $||u||\leq C_1$ and $||v||=r$. Indeed, if this is false, then for all $r>0$, there exists a pair $(u,v)\in\mathbb{R}^G_-\times\mathbb{R}^G_+$ such that $||u||\leq C_1$, $||v||=r$ and $t(u,v)\geq 0$. Since t is strictly decreasing, by applying **T7** with $\epsilon=1$, we find $C>0$ such that $||v||\leq C+||u||$ for these u and v. As r was arbitrary, we obtain $r\leq C+C_1$ for all $r>0$, which is a contradiction. Now for the r described above, the set

$$\mathcal{B}_r=\{y=(u,v)\in\mathbb{R}^G_-\times\mathbb{R}^G_+\big|\ ||u||\leq C_1,\ ||v||=r\}$$

is a compact, so since t is continuous, there exists $c>0$ such that for all $(u,v)\in\mathcal{B}_r$, $t(u,v)<-2c$. By definition of topology on $\mathcal{T}_0$, $t^\nu\to t$ uniformly on $\mathcal{B}_r$, so there exists $N\in\mathbf{N}$ such that for all $\nu\geq N$ and all $y\in\mathcal{B}_r$, $|t(y)-t^\nu(y)|<c$. Then for all $y\in\mathcal{B}_r$ and all ν, $t^\nu(y)<-c$. Since each t^ν is decreasing, we also have $t^\nu(y)<-c$ for all $y=(u,v)$ such that $||u||\leq C_1$ and $||v||\geq r$.

Similarly, for all $\nu=1,2,\dots,N-1$, there exists $r_\nu>0$ such that for all $(u,v)\in\mathcal{B}_{r_\nu}$, $t^\nu(u,v)<0$. Now by construction, if $||u||\leq C_1$ and $||v||\geq\max\{r,r_1,r_2,\dots,r_{N-1}\}$, we have $t^\nu(u,v)<0$ for all ν. Hence in the statement of the lemma, it suffices to take $C_2=\max\{r,r_1,r_2,\dots,r_{N-1}\}$. *Q.E.D.*

A.7. Proof of Lemma 4.5

To simplify notation, write $y^\nu=y(T^\nu,p^\nu)$, $c^\nu=c(p^\nu,e^\nu,y^\nu)$ and $d^\nu=d(p^\nu,e^\nu,T^\nu)$.

(1) If $y^{0\nu}_f\to-\infty$ for some f, then $||d^\nu||\to\infty$ because $c^\nu\gg 0$ and $\{e^\nu\}$ converges.

(2) Suppose there exists $C_1 > 0$ such that $||y_f^{0\nu}|| \leq C_1$ for all f and ν. By Lemma 4.4, there exists $C_2 > 0$ such that for all f, ω and ν, $||y_f^{\omega\nu}|| \leq C_2$. Since e^ν converges, there exists $C_3 > 0$ such that for all ν,

$$||\sum_h e_h^\nu + \sum_f y_f^\nu|| \leq C_3.$$

By the boundary condition for utility functions, $||c_h^\nu|| \to \infty$ for all h. Therefore $||d^\nu|| \to \infty$.

Q.E.D.

Appendix B

In this appendix, we state and prove a version of the Maximum Theorem which is used in the proof of Lemmas 4.2 and 4.3.

Let A be a metric space and $X \subset \mathbb{R}^n$ a subset. Let $\Phi : A \to X$ be a correspondence and $f : X \times A \to \mathbb{R}$ a continuous function. We are interested in problems of the form

$$\max_{x \in \Phi(\alpha)} f(x, \alpha).$$

For $\alpha \in A$, denote by $\sigma(\alpha)$ the set of optimal solutions of the problem above.

We say that Φ is

(i) *upper hemi-continuous* (uhc) if, given a sequence $\{\alpha^\nu\} \subset A$ converging to $\alpha \in A$ and a sequence $x^\nu \in \Phi(\alpha^\nu)$ converging to $x \in \mathbb{R}^n$, $x \in \Phi(\alpha)$;

(ii) *lower hemi-continuous* (lhc) if, given a sequence $\{\alpha^\nu\} \subset A$ converging to $\alpha \in A$ and an element $x \in \Phi(\alpha)$, there exists a sequence $\{x^\nu\} \subset X$ converging to x and a number $N \in \mathbf{N}$ such that for all $\nu \geq N$, $x^\nu \in \Phi(\alpha^\nu)$;

(iii) *continuous* if it is both uhc and lhc;

(iv) *bounded-valued* if for all $\alpha \in A$, $\Phi(\alpha)$ is bounded;

(v) *convex-valued* if for all $\alpha \in A$, $\Phi(\alpha)$ is convex.

Note that our definition of upper hemi-continuity is weaker than the conventional one (see, for example, Stokey et al. (1989)).

The main result is as follows.

Theorem B.1: *(a) If Φ is continuous, bounded-valued and nonempty, then the correspondence $\sigma : A \to X$ is nonempty and uhc;*

(b) If, in addition, Φ is convex valued and σ is single-valued, then σ is continuous as a function.

Proof. (a) It is obvious that if Φ is uhc, then $\Phi(\alpha)$ is closed for all $\alpha \in A$. If Φ is also bounded-valued, then $\Phi(\alpha)$ is compact for all $\alpha \in A$; thus σ is nonempty. To prove upper hemi-continuity, consider a sequence $\alpha^\nu \subset A$ converging to $\alpha \in A$ and a sequence $x^\nu \in \sigma(\alpha^\nu)$ converging to $x \in \mathbb{R}^n$. Since Φ is uhc, $x \in \Phi(\alpha)$. To prove that $x \in \sigma(\alpha)$, it suffices to show that for any $x' \in \Phi(\alpha)$, $f(x, \alpha) \geq f(x', \alpha)$. Choose such an x', then by lower hemi-continuity of Φ, there is a sequence $\{x'^\nu\} \subset X$ converging to x' and a number $N \in \mathbf{N}$ such that for all $\nu \geq N$, $x'^{\nu} \in \Phi(\alpha^{\nu})$. By definition of σ, for all $\nu \geq N$, $f(x^{\nu}, \alpha^{\nu}) \geq f(x'^{\nu}, \alpha^{\nu})$. Taking the limit as $\nu \to \infty$ yields $f(x, \alpha) \geq f(x', \alpha)$.

(b) We claim that if a sequence $\{\alpha^{\nu}\} \subset A$ is converging, then the set $\sigma(\{\alpha^{\nu}\})$ is bounded. First we show how the claim implies the theorem. Let $\{\alpha^{\nu}\} \subset A$ and $\alpha \in A$ be such that $\lim_{\nu\to\infty} \alpha^{\nu} = \alpha$. By the claim, the sequence $\{\sigma(\alpha^{\nu})\}$ is bounded, thus it contains a converging subsequence $\{\sigma(\alpha^{\nu_j})\}_{j=1}^{\infty}$. By upper hemi-continuity and single-valuedness of σ,

$$\lim_{j\to\infty} \sigma(\alpha^{\nu_j}) = \sigma(\alpha).$$

Now if the whole sequence $\{\sigma(\alpha^{\nu})\}$ does not converge to $\sigma(\alpha)$, there is $\delta > 0$ and a subsequence $\{\sigma(\alpha^{\nu_i})\}_{i=1}^{\infty}$ such that

$$||\sigma(\alpha) - \sigma(\alpha^{\nu_i})|| \geq \delta \ \ \forall\, i \in \mathbf{N}, \tag{B.1}$$

but the sequence $\{\sigma(\alpha^{\nu_i})\}_{i=1}^{\infty}$ is itself bounded, thus by the argument above, it contains a subsequence converging to $\sigma(\alpha)$. This contradicts (B.1). Hence $\lim_{\nu\to\infty} \sigma(\alpha^{\nu}) = \sigma(\alpha)$, and σ is a continuous function.

To prove the claim, assume that the sequence $\{\sigma(\alpha^{\nu})\}$ is unbounded, then there is a subsequence (without loss of generality the sequence itself) such that

$$\lim_{\nu\to\infty} ||\sigma(\alpha^{\nu})|| = \infty. \tag{B.2}$$

Let $\{\alpha^{\nu}\}$ converge to $\alpha \in A$. Choose a point $x \in \Phi(\alpha)$ (this is possible since Φ is nonempty). By lower hemi-continuity of Φ, there is a sequence $\{x^{\nu}\} \subset X$ converging to x and a number $N \in \mathbf{N}$ such that for all $\nu \geq N$, $x^{\nu} \in \Phi(\alpha^{\nu})$. Choose $R \in \mathbb{R}_+$. We will prove that there is a point $y \in \Phi(\alpha)$ such that $||x - y|| = R$; this will contradict the boundedness of $\Phi(\alpha)$. Since $x^{\nu} \to x$, there exists a number $N_1 \in \mathbf{N}$ such that for all $\nu \geq N_1$, $||x^{\nu} - x|| < 1$. By (B.2), there exists $N_2 \in \mathbf{N}$ such that for all $\nu \geq N_2$,

$$||\sigma(\alpha^{\nu})|| \geq ||x|| + R + 1.$$

Then for all $\nu \geq M = \max\{N, N_1, N_2\}$, we have $x^{\nu} \in \Phi(\alpha^{\nu})$, $\sigma(\alpha^{\nu}) \in \Phi(\alpha^{\nu})$ and

$$||\sigma(\alpha^{\nu}) - x^{\nu}|| = ||\sigma(\alpha^{\nu}) - x - (x^{\nu} - x)|| \geq$$

$$||\sigma(\alpha^{\nu})|| - ||x|| - ||x^{\nu} - x|| > ||x|| + R + 1 - ||x|| - 1 = R.$$

Since Φ is convex-valued, for $\nu \geq M$, the whole line segment between x^{ν} and $\sigma(\alpha^{\nu})$ lies in $\Phi(\alpha^{\nu})$ and has the length greater than R by the calculation above. Thus we may choose a point $y^{\nu} \in \Phi(\alpha^{\nu})$ on this segment so that $||y^{\nu} - x^{\nu}|| = R$. For $\nu \geq M$, we have

$$||y^{\nu} - x|| \leq ||y^{\nu} - x^{\nu}|| + ||x^{\nu} - x|| < R + 1,$$

thus the sequence $\{y^{\nu}\}_{\nu=M}^{\infty}$ is bounded and hence it contains a converging subsequence $\{y^{\nu_j}\}_{j=1}^{\infty}$. Let $y = \lim_{j\to\infty} y^{\nu_j}$. By upper hemi-continuity of Φ, $y \in \Phi(\alpha)$. Passing to the limit as $j \to \infty$ in the equality $||y^{\nu_j} - x^{\nu_j}|| = R$ gives $||y - x|| = R$. *Q.E.D.*

Appendix C

C.1. Proof of Lemma 4.7

First, we show that if $\alpha^\nu \to \alpha$ (equivalently, $a^\nu \to a$), then for every compact $K \subset \mathbb{R}^G_{--} \times \mathbb{R}^G_{++}$, $D^2 t_{\alpha^\nu}(y) \to D^2 t_\alpha(y)$ uniformly on K.

Let $\tilde{K} \equiv \left\{ a \cdot y \middle| \; y \in K \right\}$ and $\tilde{a}^\nu \equiv a^\nu \cdot a^{-1} \equiv I + \text{diag}[\tilde{\alpha}^j]_{j=1}^{2G}$, then $\tilde{K} \subset \mathbb{R}^G_{--} \times \mathbb{R}^G_{++}$ is also a compact and

$$D^2 t_{\alpha^\nu}(y) \to D^2 t_\alpha(y) \text{ uniformly on } K \;\Leftrightarrow\; D^2 t_{\tilde{a}^\nu}(y) \to D^2 t(y) \text{ uniformly on } \tilde{K}.$$

Hence we may assume without loss of generality that $a = I$ is the identity matrix. Since $a^\nu \to I$, there exists a compact $K' \subset \mathbb{R}^G_{--} \times \mathbb{R}^G_{++}$ such that $K \subset K'$ and $a^\nu \times K \subset K'$ for all ν. Since t is C^2, there exists a constant $M > 0$ such that for all $y' \in K'$, $||D^2 t(y')|| \leq M$. Then for all $y \in K$ and all ν,

$$||D^2 t_{\alpha^\nu}(y) - D^2 t(y)|| = ||(a^\nu)^T D^2 t(a^\nu y) a^\nu - D^2 t(y)||$$
$$\leq ||(a^\nu)^T D^2 t(a^\nu y) a^\nu - D^2 t(a^\nu y)|| + ||D^2 t(a^\nu y) - D^2 t(y)||,$$

and

$$||(a^\nu)^T D^2 t(a^\nu y) a^\nu - D^2 t(a^\nu y)|| \leq C \cdot ||a^\nu - I||^2 \cdot ||D^2 t(a^\nu y)|| \leq C \cdot M \cdot ||a^\nu - I||^2$$

for some constant $C > 0$. Since $||a^\nu - I||^2 \to 0$ as $\nu \to \infty$ independently of y, it suffices to show that $D^2 t(a^\nu y) \to D^2 t(y)$ as $\nu \to \infty$ uniformly for $y \in K$. Choose $\epsilon > 0$. By definition of continuity, every point $y_0 \in K'$ has an open neighborhood $V_{y_0} \subset \mathbb{R}^G_{--} \times \mathbb{R}^G_{++}$ such that for all $y' \in V_{y_0}$, $||D^2 t(y') - D^2 t(y_0)|| < \epsilon/2$. By the triangle inequality, for all $y', y'' \in V_{y_0}$, $||D^2 t(y') - D^2 t(y'')|| < \epsilon$. The open set $\{V_{y_0} \big| \; y_0 \in K'\}$ obviously form an open cover of K'. By the Lebesgue Lemma (see for example, Bredon (1993)), there exists $\delta > 0$ such that if $y', y'' \in K'$ satisfy $||y' - y''|| < \delta$, then there is y_0 such that $y', y'' \in V_{y_0}$. Since K is compact, there exists a constant $N > 0$ such that $||y|| \leq N$ for all $y \in K$. Then for large enough ν, we have $||a^\nu - I|| < \delta/N$ (because $a^\nu \to I$), and then for all $y \in K$ (and large ν),

$$||a^\nu y - y|| \leq ||a^\nu - I|| \cdot ||y|| < \frac{\delta}{N} \cdot N = \delta.$$

This means that if ν is large enough, then for each $y \in K$, there is $y_0 \in K'$ such that $y, a^\nu y \in V_{y_0}$, and then by construction, $||D^2 t(a^\nu y) - D^2 t(y)|| < \epsilon$.

Since there exists a compact such that if $y \in K$, then $a^\nu \cdot y, a \cdot y \in \tilde{K}$. It is well-known that every continuously differentiable function satisfies the Lipschitz condition: there exists $C > 0$ such that for all $y, y' \in \tilde{K}$, $|t(y') - t(y)| \leq C \cdot ||y' - y||$. Now for all $y \in K$, we have

$$|t_{\alpha^\nu}(y) - t_{\alpha^\nu}(y)| = |t(a \cdot y) - t(a^\nu \cdot y)| \leq C \cdot ||(a - a^\nu) y|| \leq C \cdot ||a - a^\nu|| \cdot ||y||,$$

where $||a - a^\nu||$ is the norm of the matrix $a - a^\nu$. Since K is compact, there exists $M = \sup_{y \in K} ||y|| < \infty$. Since $\alpha^\nu \to \alpha$, we have $a^\nu \to a$, that is, given, $\epsilon > 0$, there is $N \in \mathbf{N}$ such that for all $\nu \geq N$,

$$||a^\nu - a|| < \frac{\epsilon}{M \cdot C}.$$

Then for all $\nu \geq N$ and all $y \in K$, $|t_\alpha(y) - t_{\alpha^\nu}(y)| < \epsilon$, as was to be shown. Note that we only used the fact that t is continuously differentiable. Hence, in view of **T1**, the same argument shows that the partial derivatives of order up to 2 of t_{α^ν} converge to those of t_α uniformly on K.

Note that we cannot use exactly the same argument for (i), since we cannot even prove the Lipschitz condition for t on $\mathbb{R}^G_- \times \mathbb{R}^G_+$. Instead, we will prove directly that, given $\epsilon > 0$, there is $N \in \mathbf{N}$ such that for all $\nu \geq N$ and all $y \in K$,

$$|t(a^\nu \cdot y) - t(a \cdot y)| < \epsilon. \qquad (*)$$

First, as before, choose a compact $\tilde{K} \subset \mathbb{R}^G_- \times \mathbb{R}^G_+$ such that for all ν and all $y \in K$, $a^\nu \cdot y \in \tilde{K}$. Since t is continuous and $t = 0$ on $\partial\mathbb{R}^G_- \times \{0\}$, there exists $\tilde{r} > 0$ such that if $\tilde{y} \in \tilde{K}$ and
$\text{dist}(\tilde{y}, \partial\mathbb{R}^G_- \times \{0\}) < \tilde{r}$, then $|t(\tilde{y})| < \frac{\epsilon}{2}$. By the triangle inequality, it follows that $(*)$ holds for all ν and all y such that $\text{dist}(y, \partial\mathbb{R}^G_- \times \{0\}) < r$, where we pick $r > 0$ such that $||a^\nu|| \cdot r < \tilde{r}$ for all ν; such an r exists because $a^\nu \to a$. Now it remains to check that there is $N \in \mathbf{N}$ such that $(*)$ holds for all $\nu \geq N$ and all $y \in K$ with $\text{dist}(y, \partial\mathbb{R}^G_- \times \{0\}) \geq r$. But the set of all these y is again a compact, and now, by **T1**, t is continuously differentiable on this compact, so the argument above (using Lipschitz condition) works to complete the proof. *Q.E.D.*

C.2. Proof of Theorem 4.9

Step 1. It is obvious that since $p \gg 0$ and $\beta \neq 0$, equations (4.11) and (4.13) can be solved for Δp and $\Delta\beta$ for any right-hand sides. Consider equations (4.12) as equations in Δy^ω_f. Since $p(\omega) \gg 0$ and there is at least one $\omega > 0$, without loss of generality $\omega = 1$, such that $\beta^1 \neq 0$, we can solve (4.12) say for Δy^{11}_f. Thus, if $\mathfrak{L}'$ denotes the linear operator corresponding to the left-hand sides of equations (4.11)-(4.13), such that

$$\mathfrak{L}' : \mathbb{R}^L \times \left(\mathbb{R}^{G\Omega}\right)^F \times \mathbb{R}^\Omega \longrightarrow \mathbb{R}^{\Omega+2},$$

we have shown that $\mathfrak{L}'$ is surjective.

Step 2. First, notice that by Lemma 4.8,

$$\text{(C.1)} \quad D_\alpha t^\omega_{\alpha f} \cdot \Delta\alpha = D_0 t^\omega_{\alpha f} \Delta\gamma^0_f + D_\omega t^\omega_{\alpha f} \Delta\gamma^\omega_f;$$

$$\text{(C.2)} \quad D_\alpha D_0 t^\omega_{\alpha f} \cdot \Delta\alpha = \text{diag}\left[\frac{1}{y^{g0}} \partial_g t_{\alpha f}(y)\right]_{g=1}^G \cdot \Delta\gamma^0_f + D^2_{00} t^\omega_{\alpha f} \Delta\gamma^0_f + D^2_{\omega 0} t^\omega_{\alpha f} \Delta\gamma^\omega_f$$

$$\text{(C.3)} \quad D_\alpha D_\omega t^\omega_{\alpha f} \cdot \Delta\alpha = \text{diag}\left[\frac{1}{y^{g\omega}} \partial_g t_{\alpha f}(y)\right]_{g=1}^G \cdot \Delta\gamma^\omega_f + D^2_{0\omega} t^\omega_{\alpha f} \Delta\gamma^0_f + D^2_{\omega\omega} t^\omega_{\alpha f} \Delta\gamma^\omega_f.$$

For a given f, let

$$\mathfrak{L}_f : \mathbb{R}^G \times \mathbb{R}^{G\Omega} \times \mathbb{R}^\Omega \longrightarrow \mathbb{R}^{\Omega+G(\Omega+1)}$$

be the linear operator corresponding to the left-hand sides of equations (4.7)–(4.9). We claim that these equations have a unique solution $(\Delta y_f^0, \Delta\gamma_f^\omega, \Delta\mu_f^\omega, \omega > 0)$ for any right-hand sides or, equivalently (by the rank-nullity theorem), $\mathrm{Ker}\mathfrak{L}_f = \{0\}$, i.e.

$$\mathfrak{L}_f \cdot \begin{pmatrix} \Delta y_f^0 \\ \Delta\gamma_f^\omega \\ \Delta\mu_f^\omega \end{pmatrix} = 0 \quad \Leftrightarrow \quad \Delta y_f^0 = \Delta\gamma_f^\omega = 0,\ \Delta\mu_f^\omega = 0.$$

To show this, rewrite equations (4.7)-(4.9) as equations with respect to $(\Delta y_f^0, \Delta\gamma_f^\omega, \Delta\mu_f^\omega, \omega > 0)$ with zero right-hand sides, using (C.1)–(C.3):

$$\text{(C.4)} \qquad \sum_{\omega>0} \Delta\mu_f^\omega D_0 t_{\alpha f}^{\omega^T} + \sum_{\omega>0} \mu_f^\omega \left[D_{00}^2 t_{\alpha f}^{\omega^T}; D_{\omega 0}^2 t_{\alpha f}^\omega\right] \begin{pmatrix} \Delta y_f^0 \\ \Delta\gamma_f^\omega \end{pmatrix} = 0;$$

$$\text{(C.5)} \qquad \begin{aligned} &\Delta\mu_f^\omega D_\omega t_{\alpha f}^{\omega^T} + \mu_f^\omega \left[D_{0\omega}^2 t_{\alpha f}^\omega; D_{\omega\omega}^2 t_{\alpha f}^\omega\right] \begin{pmatrix} \Delta y_f^0 \\ \Delta\gamma_f^\omega \end{pmatrix} + \\ &+ \mu_f^\omega \mathrm{diag}\left[\frac{1}{y^{g\omega}} \partial_g t_{\alpha f}(y)\right]_{g=1}^G \Delta\gamma_f^\omega = 0; \end{aligned}$$

$$\text{(C.6)} \qquad \left[\Delta y_f^{0^T}; \Delta\gamma_f^{\omega^T}\right] \begin{pmatrix} D_0 t_{\alpha f}^\omega \\ D_\omega t_{\alpha f}^\omega \end{pmatrix} = 0.$$

Premultiply (C.4) by $\Delta y_f^{0^T}$ and each of the equations (C.5) for $\omega > 0$ by $\Delta\gamma_f^{\omega^T}$ and add these equations, using (C.6). As a result, we get the following equation

$$\Delta y_f^{0^T} \sum_{\omega>0} \mu_f^\omega \left(D_{00}^2 t_{\alpha f}^\omega + D_{\omega 0}^2 t_{\alpha f}^\omega\right) \Delta y_f^0 +$$

$$+\Delta\gamma_f^{\omega^T} \sum_{\omega>0} \mu_f^\omega \left(D_{0\omega}^2 t_{\alpha f}^\omega + D_{\omega\omega}^2 t_{\alpha f}^\omega + \mathrm{diag}\left[\frac{1}{y^{g\omega}} \partial_g t_{\alpha f}(y)\right]_{g=1}^G\right) \Delta\gamma_f^\omega = 0$$

$$\text{(C.7)} \qquad \Leftrightarrow \quad \sum_{\omega>0} \mu_f^\omega \left[\begin{array}{cc} \Delta y_f^{0^T} & \Delta\gamma_f^{\omega^T} \end{array}\right] \left(D^2 t_{\alpha f}^\omega + \mathsf{R}_\mathsf{f}^\omega\right) \left[\begin{array}{c} \Delta y_f^0 \\ \Delta\gamma_f^\omega \end{array}\right] = 0,$$

where $\mathsf{R}_\mathsf{f}^\omega$ is a $2G \times 2G$ negative semi-definite matrix:

$$\mathsf{R}_\mathsf{f}^\omega \equiv \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathrm{diag}\left[\frac{1}{y^{g\omega}} \partial_g t_{\alpha f}(y)\right]_{g=1}^G \end{pmatrix}$$

By strict concavity of the functions t_f^ω, (C.7) is possible only if

$$\Delta y_f^0 = \Delta\gamma_f^\omega = 0 \ \forall\ \omega > 0.$$

Now from equations (C.5) it follows that

$$\Delta\mu_f^\omega = 0 \ \forall\ \omega > 0.$$

Summing up, we have established that the linear operator

$$\mathfrak{L} \equiv (\mathfrak{L}_f)_{f=1}^F,$$

consisting of the left-hand sides of equations (4.7)-(4.9) for all f, is surjective.

Step 3. It remains to consider equations (4.5), (4.6) and (4.10). Notice that these equations cannot be solved for any right-hand sides because due to the Walras' Law, they are not linearly independent. Rewrite equations (4.5), (4.6) and (4.10) as a system of equations in the unknowns Δc_h, $\Delta\lambda_h$ $\forall$ h and Δe_1 (we can choose any other Δe_h):

$$D^2u_h(c_h)\Delta c_h - \Delta\lambda_h p^T = \Delta x_h \;\forall\; h; \tag{C.8}$$

$$-p(\Delta c_1 - \Delta e_1) = \Delta w_1; \tag{C.9}$$

$$-p\Delta c_h = \Delta\; w_h \;\forall\; h > 1; \tag{C.10}$$

$$\sum_h \Delta c_h - \Delta e_1 = \Delta d, \tag{C.11}$$

where $\Delta x_h, \Delta w_h$ $\forall$ h and Δd stand for arbitrary right-hand sides. From (3.2), (4.6) and (4.10) it follows that if the system (C.8)–(C.11) is solvable, the right-hand sides have to satisfy $\sum_f \Delta\tilde{w}_h = -p \cdot \Delta\tilde{d}$.

Let

$$\mathfrak{L}'' : \mathbb{R}^{LH} \times \mathbb{R}^H \times \mathbb{R}^L \longrightarrow \mathbb{R}^{H(L+1)+L}$$

be the linear operator consisting of left-hand sides of equations (C.8)-(C.11). As we have just remarked, $\dim \mathrm{Im}\mathfrak{L}'' \leq H(L+1)+L-1$. We claim that, in fact, equality holds. For this, it suffices to check that $\dim \mathrm{Im}\mathfrak{L}'' \geq H(L+1)+L-1$. By the rank-nullity theorem, this is equivalent to showing that $\dim \mathrm{Ker}\mathfrak{L}'' \leq 1$. To describe the kernel of $\mathfrak{L}''$, we rewrite the system (C.8)-(C.11) with zero right-hand sides:

$$D^2u_h(c_h) - \Delta\lambda_h p^T = 0 \;\forall\; h; \tag{C.12}$$

$$-\,p(\Delta c_1 - \Delta e_1) = 0; \tag{C.13}$$

$$-\,p\Delta c_h = 0 \;\forall\; h > 1; \tag{C.14}$$

$$\sum_h \Delta c_h - \Delta e_1 = 0. \tag{C.15}$$

On the strength of (3.1), $Du_h(c_h) = \lambda_h p$, hence from (C.12) and (C.14) it follows that, for all $h > 1$,

$$Du_h(c_h)\Delta c_h = 0 \;\text{ and }\; \Delta c_h^T D^2u_h(c_h)\Delta c_h = 0.$$

Due to the strict quasi-concavity of u_h, these equations imply that $\Delta c_h = 0$ for all $h > 1$. Now from (C.12), $\Delta\lambda_h = 0$ for $h > 1$, and from (C.15), $\Delta c_1 = \Delta e_1$, and (C.13) is trivially satisfied. Thus we are left with

$$D^2u_1(c_1)\Delta c_1 = \Delta\lambda_1 \cdot p^T. \tag{C.16}$$

Now strict quasi-concavity of u_1 implies that

$$(\mathrm{Ker}D^2u_1(c_1)) \cap (Du_1(c_1))^\perp = \{0\},$$

or, equivalently,

$$(\mathrm{Ker}D^2u_1(c_1)) \cap (p^\perp) = \{0\}, \tag{C.17}$$

where, for a vector v, $v^{\perp}$ denotes the set of all vectors w such that $w^T \cdot v = 0$. Since $\dim(p^{\perp}) = L-1$, we have

$$\dim \mathrm{Ker} D^2 u_1(c_1) \leq 1. \tag{C.18}$$

We have two cases. If $\mathrm{Ker} D^2 u_1(c_1) = \{0\}$, then (C.16) shows that Δc_1 is uniquely determined by $\Delta\lambda_1$, so the whole tuple $(\Delta c_h, \Delta\lambda_h, \Delta e_1) \in \mathrm{Ker}\mathfrak{L}''$ is uniquely determined by $\Delta\lambda_1$ and therefore $\dim \mathrm{Ker}\mathfrak{L}'' \leq 1$. Now let $\dim \mathrm{Ker} D^2 u_1(c_1) = 1$. From (C.17), there exists $\Delta c_1' \in \mathrm{Ker} D^2 u_1(c_1)$ such that $p \cdot \Delta c_1' = 0$. Premultplying (C.16) by $\Delta {c_1'}^T$, we get

$$\Delta {c_1'}^T D^2 u_1(c_1) \Delta c_1 = \Delta\lambda_1 \cdot \Delta {c_1'}^T \cdot p^T,$$

so

$$0 = (D^2 u_1(c_1) \Delta c_1')^T \cdot \Delta c_1 = \Delta\lambda_1 \cdot (p \cdot \Delta c_1')^T.$$

Since $p \cdot \Delta c_1' \neq 0$ by assumption, we get $\Delta\lambda_1 = 0$, and then by (C.16),

$$D^2 u_1(c_1) \cdot \Delta c_1 = 0.$$

Now (C.18) implies that the set of all Δc_1 satisfying the last relation is at most 1 dimensional.

Thus we have proved that $\dim \mathrm{Ker}\mathfrak{L}'' \leq 1$, so that $\dim \mathrm{Im}\mathfrak{L}'' = H(L+1) + L - 1$.

Summary. Consider again the system (4.5)-(4.13). In steps 1-3, we have shown that

- the system (4.11)-(4.13) can always be solved for Δp, $\Delta\beta$ and Δy^{ω}.
- Given Δp and Δy^{ω}, the system (4.7)-(4.9) is solvable for Δy^0 and $\Delta\gamma^{\omega}$.
- Given Δp, Δy and Δe_h for $h > 1$, the system (C.12)–(C.15) (equivalently, the system (4.5), (4.6), (4.10) under a corresponding condition on the right-hand sides) is solvable for Δc and $\Delta\lambda$ under a condition $\sum_h \Delta w_h = -p \cdot \Delta d$. Note that the last condition defines a hyperplane in $\mathbb{R}^N$.

These facts together imply that the whole system (4.5)-(4.13) is solvable provided its right-hand side satisfies a certain condition which defines a hyperplane in $\mathbb{R}^N$. Thus,

$$\dim \mathrm{Im}(D\psi\big|_{\varsigma}) = N - 1.$$

This completes the proof of Theorem 4.9. *Q.E.D.*

C.3. Proof of Lemma 5.1

Let

$$\tilde{B}_h(p, Y) = \left\{ c_h \in \mathbb{R}^L_{++} \,\middle|\, \exists\ z_h \ :\ \hat{P} \cdot (\hat{c}_h - \hat{e}_h - \sum_f s_{hf} \hat{y}_f) = Y(\hat{p}, \hat{y}) z_h \right\}.$$

We have

$$\begin{aligned} c_h \in \tilde{B}_h(p, Y) \quad &\Leftrightarrow \quad \hat{P} \cdot (\hat{c}_h - \hat{e}_h - \sum_f s_{hf} \hat{y}_f) \in \mathrm{Im} Y(\hat{p}, \hat{y}) \\ &\Leftrightarrow \quad \hat{P} \cdot (\hat{c}_h - \hat{e}_h - \sum_f s_{hf} \hat{y}_f) \in (\mathrm{Ker} Y(\hat{p}, \hat{y})^T)^{\perp} \\ &\Leftrightarrow \quad \forall\ \lambda^T \in \mathrm{Ker} Y(\hat{p}, \hat{y})^T, \quad \lambda \cdot \hat{P} \cdot (\hat{c}_h - \hat{e}_h - \sum_f s_{hf} \hat{y}_f) = 0. \end{aligned}$$

We have $\dim \mathrm{Ker} Y(\hat{p}, \hat{y})^T = \Omega - \rho$, hence $\mathrm{Ker} Y(\hat{p}, \hat{y})^T$ is spanned by $\Omega - \rho$ linearly independent vectors. There is no loss of generality in choosing them equal to the α^ω defined earlier. Hence instead of the constraint (5.1), we can write equations (5.5). Adding to each of the equations (5.5) the Walrasian budget constraint $p \cdot (c_h - e_h - \sum_f s_{hf} y_f) = 0$ which defines $B_h(p, Y)$ together with (5.1), and using (5.6) gives us

$$p^i \cdot (c_h - e_h - \sum_f s_{hf} y_f) = 0, \quad i = 1, \ldots, \Omega - \rho + 1.$$

Q.E.D.

C.4. Proof of Theorem 5.3

As in the proof of Theorem 4.9, we start with writing the system of equations

$$\left(D\psi_{(\mathfrak{I}^+, \mathfrak{I}^-)}\big|_\zeta \right) \cdot \Delta\zeta = \Delta b.$$

Consider the linearized equations (5.10) and (5.15):

$$\begin{aligned} \Delta p^i(\omega) - (1 + \beta^i_\omega)\Delta p^1(\omega) - \Delta\beta^i_\omega p^1(\omega) &= \Delta^i_p(\omega) \ \text{ if } \omega \in \mathfrak{I}^- \text{ or } \omega = i \ \ (i \geq 2); \\ \Delta p^i(\omega) - \Delta p^1(\omega) &= \Delta^i_p(\omega) \ \text{ otherwise } \ (i \geq 2); \\ p^1 \cdot \Delta p^1 &= \Delta_p. \end{aligned}$$

Obviously, this group of equations can be solved for Δp^i $(i \geq 1)$. Hence, the linear operator

$$\mathfrak{L}' : \mathbb{R}^{L(\Omega - \rho + 1)} \longrightarrow \mathbb{R}^{L(\Omega - \rho) + 1}$$

which sends $(\Delta p^i)_{i=1}^{\Omega - \rho + 1}$ to the left-hand sides of these equations is surjective (for all values of other variables).

Consider linearized equations (5.16) and (5.17):

$$\sum_{\omega \in \mathfrak{I}^-} \left[\Delta\beta^i_\omega p^1(\omega) y^\omega_f + (y^\omega_f)^T \Delta p^1(\omega) + p^1(\omega) \Delta y^\omega_f \right] +$$

$$\begin{aligned} \text{(C.19)} \quad +\Delta\beta^i_i p^1(i) y^i_f + \beta^i_i (y^i_f)^T + \beta^i_i p^1(i) \Delta y^i_f &= \Delta^i_f \ \forall\, f, \ \forall\, i = 2, \ldots, \Omega - \rho + 1; \\ \text{(C.20)} \quad \beta^i \cdot \Delta\beta^i &= \Delta^i_\beta, \quad i = 2, \ldots, \Omega - \rho + 1. \end{aligned}$$

We can solve equations (C.20) for $\Delta\beta^i$ since $\beta^i \neq 0$. After that, we can view (C.19) as equations in Δy^ω_f $(\omega > 0$, all $f)$. For example, we can solve (C.19) for Δy^{1i}_f $(\forall\, f, \ \forall\, i = 2, \ldots, \Omega - \rho + 1)$ because $\beta^i_i \neq 0$ and $p^1(i) \gg 0$. Therefore, the linear operator

$$\mathfrak{L}''' : (\mathbb{R}^{G\Omega})^F \times \mathbb{R}^{(\Omega - \rho)(\rho + 1)} \longrightarrow \mathbb{R}^{(\Omega + 1)(\Omega - \rho)}$$

which sends $\left((\Delta y^\omega_f)^\Omega_{\omega, f = 1}, (\Delta\beta^i)_{i=2}^{\Omega - \rho + 1} \right)$ to the left-hand sides of equations (C.19) and (C.20) is surjective (for all values of other variables).

Notice that equations (5.11)–(5.13) are exactly the same as equations (3.3)–(3.4), hence the linearized equations will be the same as well, and by *Step 2* of the proof of Theorem 4.9 we know that the linear operator

$$\mathfrak{L} : (\mathbb{R}^G \times \mathbb{R}^{G\Omega} \times \mathbb{R}^\Omega)^F \longrightarrow \mathbb{R}^{F(\Omega + L)}$$

defined there is surjective.

Finally, consider linearized equations (5.8), (5.9) and (5.14) with arbitrary right-hand sides.

$$D^2u_h(c_h)\Delta c_h - \sum_{i=1}^{\Omega-\rho+1} \Delta\lambda_h^i p^i - \sum_{i=1}^{\Omega-\rho+1} \lambda_h \Delta p^{i^T} = \Delta x_h \quad \forall\ h; \tag{C.21}$$

$$\begin{aligned} & - p^i(\Delta c_h - \Delta e_h - \sum_f s_{hf}\Delta y_f) - \\ & - (c_h - e_h - \sum_f s_{hf} y_f)^T \Delta p^i = \Delta w_h^i \quad i \leq \Omega - \rho + 1,\ \forall\ h; \end{aligned} \tag{C.22}$$

$$\sum_h (\Delta c_h - \Delta e_h) - \sum_f \Delta y_f = \Delta d. \tag{C.23}$$

Notice that these equations cannot be solved for any right-hand sides because from (5.9), (C.22) and (C.23), it follows that if (C.21), (C.22) and (C.23) are satisfied, then $\sum_h \Delta w_h^i = -p^i \cdot \Delta d$ $(i = 1, \ldots, \Omega-\rho+1)$. Rewrite (C.21)-(C.23) as a system of equations in the unknowns Δc_h, $\Delta\lambda_h^i$, $i = 1, \ldots, \Omega - \rho + 1$, $\forall\ h$, and Δe_1.

$$D^2u_h(c_h)\Delta c_h - \sum_{i=1}^{\Omega-\rho+1} \Delta\lambda_h^i p^{i^T} = \Delta\tilde{x}_h \quad \forall\ h; \tag{C.24}$$

$$-p^i \cdot (\Delta c_1 - \Delta e_1) = \Delta\tilde{w}_1^i; \tag{C.25}$$

$$-p^i \cdot \Delta c_h = \Delta\tilde{w}_h^i, \quad h \geq 2; \tag{C.26}$$

$$\sum_{h\geq 2} \Delta c_h + \Delta c_1 - \Delta e_1 = \Delta\tilde{d}. \tag{C.27}$$

Define a linear operator

$$\mathfrak{L}'^v : \mathbb{R}^{LH} \times \mathbb{R}^{H(\Omega-\rho+1)} \times \mathbb{R}^L \longrightarrow \mathbb{R}^{H(L+\Omega-\rho+1)+L}$$

which corresponds to the left-hand sides of equations (C.24)–(C.27). As we have just remarked, $\dim \mathrm{Im}\mathfrak{L}'^v \leq H(L+\Omega-\rho+1)+L-(\Omega-\rho+1)$. We claim that, in fact, equality holds. By the rank-nullity theorem, this is equivalent to showing that $\dim \mathrm{Ker}\mathfrak{L}'^v \leq \Omega - \rho + 1$. To describe the kernel of $\mathfrak{L}'^v$, we rewrite the system (C.24)–(C.27) with zero right-hand sides:

$$D^2u_h(c_h)\Delta c_h - \sum_i \Delta\lambda_h^i \cdot p^{i^T} = 0; \tag{C.28}$$

$$p^i \cdot (\Delta c_1 - \Delta e_1) = 0; \tag{C.29}$$

$$p^i \cdot \Delta c_h = 0, \quad h \geq 2; \tag{C.30}$$

$$\sum_{h\geq 2} \Delta c_h + \Delta c_1 - \Delta e_1 = 0. \tag{C.31}$$

On the strength of (5.8), $Du_h(c_h) = \sum_i \lambda_h^i p^i$, hence from (C.28) and (C.30) it follows that for all $h \geq 2$,

$$Du_h(c_h)\Delta c_h = 0 \quad \text{and} \quad \Delta c_h^T D^2 u_h(c_h)\Delta c_h = 0.$$

By strict quasi-concavity of u_h, $\Delta c_h = 0$ for $h \geq 2$. Now due to linear independence of p^i, from (C.28) we get

$$\Delta\lambda_h^i = 0, \quad h \geq 2, \; i = 1, \ldots, \Omega - \rho + 1.$$

From (C.31) it follows that $\Delta e_1 = \Delta c_1$. For $h = 1$, rewrite (C.28) as

$$D^2 u_1(c_1)\Delta c_1 = \sum_i \Delta\lambda_1^i p^{i^T}. \tag{C.32}$$

Strict quasi-concavity of u_1 implies that

$$\left(\mathrm{Ker} D^2 u_1(c_1)\right) \cap (Du_1(c_1))^\perp = \{0\},$$

or, equivalently,

$$\left(\mathrm{Ker} D^2 u_1(c_1)\right) \cap \left(\sum_i \lambda_1^i p^{i^T}\right)^\perp = \{0\}. \tag{C.33}$$

Since $\dim \left(\sum_i \lambda_1^i p^{i^T}\right)^\perp = L - 1$, (C.33) implies that $\dim \mathrm{Ker} D^2 u_1(c_1) \leq 1$. Consider two cases. If $\mathrm{Ker} D^2 u_1(c_1) = \{0\}$, then (C.32) shows that Δc_1 is uniquely determined by $\{\Delta\lambda_1^i\}_{i=1}^{\Omega-\rho+1}$, so the whole tuple $(\Delta c_h, \Delta\lambda_h^i, \Delta e_1) \in \mathrm{Ker}\mathfrak{L}'^v$ is uniquely determined by $\{\Delta\lambda_1^i\}_{i=1}^{\Omega-\rho+1}$, and therefore $\dim \mathrm{Ker}\mathfrak{L}'^v \leq \Omega - \rho + 1$.
Now we consider the case $\dim \mathrm{Ker} D^2 u_1(c_1) = 1$. By (C.33), there exists $\Delta c_1' \in \mathrm{Ker} D^2 u_1(c_1)$ such that $\sum_i \lambda_1^i p^i \cdot \Delta c_1' \neq 0$. Premultiplying (C.32) by $\Delta c_1'^T$, we obtain

$$0 = \left(D^2 u_1(c_1)\Delta c_1'\right)^T \cdot \Delta c_1 = \sum_i \Delta\lambda_1^i (p^i \cdot \Delta c_1')^T,$$

which shows that (C.32) cannot hold if $\Delta\lambda_1^i = \lambda_1^i \;\forall\, i$. Consider a linear operator

$$\tilde{\mathfrak{L}} : \mathbb{R}^{L+\Omega-\rho+1} \longrightarrow \mathbb{R}^L$$
$$(\Delta c_1, \Delta\lambda_1) \longmapsto D^2 u_1(c_1)\Delta c_1 - \sum_i \Delta\lambda_1^i p^{i^T}$$

where $\Delta\lambda_1 \equiv (\Delta\lambda_1^i)_{i=1}^{\Omega-\rho+1}$. We have shown above that $\dim \mathrm{Ker}\mathfrak{L}'^v = \dim \mathrm{Ker}\tilde{\mathfrak{L}}$ and that

$$(\Delta c_1, \lambda_1) \notin \mathrm{Ker}\tilde{\mathfrak{L}} \;\; \forall\; \Delta c_1 \in \mathbb{R}^L.$$

Consider the projection

$$\tilde{\pi} : \mathbb{R}^{L+\Omega-\rho+1} \longrightarrow \mathbb{R}^{\Omega-\rho+1}$$
$$(\Delta c_1, \Delta\lambda_1) \longmapsto \Delta\lambda_1$$

We have $\lambda_1 \notin \tilde{\pi}\left(\mathrm{Ker}\tilde{\mathfrak{L}}\right)$, whence

$$\dim \tilde{\pi}\left(\mathrm{Ker}\tilde{\mathfrak{L}}\right) \leq \Omega - \rho. \tag{C.34}$$

Also, equation (C.32) shows that Δc_1 is determined by $\Delta\lambda_1$ up to $\mathrm{Ker} D^2 u_1(c_1)$, which is one-dimensional, so

$$\dim \mathrm{Ker}\left(\tilde{\pi}\big|_{\mathrm{Ker}\tilde{\mathfrak{L}}}\right) \le 1. \tag{C.35}$$

Combining equations (C.34) and (C.35) with the rank-nullity theorem gives $\dim \mathrm{Ker}\tilde{\mathfrak{L}} \le \Omega - \rho + 1$, completing the proof. *Q.E.D.*

C.5. Proof of Lemma 5.4

Let $\mathfrak{L}$, $\mathfrak{L}'''$, $\mathfrak{L}'^{v}$ be the same linear operators as in the proof of Theorem 5.3. We know that

$$\mathrm{rank}\mathfrak{L} + \mathrm{rank}\mathfrak{L}''' + \mathrm{rank}\mathfrak{L}'^{v} = F(L+\Omega) + L + 1 + H(L+2) + L - 2 =$$
$$N_{\Omega-1} - 2 - (\Omega+1) = \tilde{N} - 2 - 2(\Omega+1).$$

Let $\mathfrak{L}^{v}$ be the linear operator corresponding to the linearized equations (5.16), (5.17), (5.19), (5.20). If we prove that $\dim \mathrm{Im}\mathfrak{L}^{v} = 2(\Omega+1)$, then we are done because we obtain

$$\mathrm{rank}\left(D\tilde{\psi}\big|_{\tilde{\zeta}}\right) = \tilde{N} - 2.$$

Consider the linearized system (5.16), (5.17), (5.19), (5.20) with arbitrary right-hand sides:

$$\sum_{\omega=1}^{\Omega-1} \left(\Delta\beta_\omega p^1(\omega) y_f^\omega + \beta_\omega (y_f^\omega)^T \Delta p^1(\omega) + \beta_\omega p^1(\omega) \Delta y_f^\omega\right) +$$
$$+ \Delta\beta_\Omega p^1(\Omega) y_f^\Omega + \beta_\Omega (y_f^\Omega)^T \Delta p^1(\Omega) + \beta_\Omega p^1(\Omega) = ** \ \forall\ f; \tag{C.36}$$

$$\beta \cdot \Delta\beta = **; \tag{C.37}$$

$$\sum_{f=1}^{\Omega-1} \left(\Delta\gamma_f p^1(\omega) y_f^\omega + \gamma_f (y_f^\omega)^T \Delta p^1(\omega) + \gamma_f p^1(\omega) \Delta y_f^\omega\right) +$$
$$+ \Delta\gamma_\Omega p^1(\omega) y_\Omega^\omega + \gamma_\Omega (y_\Omega^\omega)^T \cdot \Delta p^1(\omega) + \gamma_\Omega p^1(\omega) \Delta y_\Omega^\omega = ** \ \forall\ \omega; \tag{C.38}$$

$$\gamma \cdot \Delta\gamma = ** . \tag{C.39}$$

It suffices to show that we can always solve this system for $\Delta\beta$, $\Delta\gamma$ and Δy. It is clear that for $f < \Omega$, we can solve equation (C.36) for Δy_f^Ω because $p^1(\Omega) \gg 0$ and $\beta_\Omega \neq 0$. Similarly, if $\omega < \Omega$, we can solve (C.38) for Δy_Ω^ω. It remains to see that if $f = \Omega$, we can solve the system (C.36)–(C) for $\Delta\beta$, and if $\omega = \Omega$, we can solve the system (C.38)–(C.39) for $\Delta\gamma$. Now with respect to $\Delta\beta$, the system (C.36)–(C) has the following form:

$$\begin{bmatrix} p^1(1) y_\Omega^1 & \dots & p^1(\Omega) y_\Omega^\Omega \\ \beta_1 & \dots & \beta_\Omega \end{bmatrix} \begin{pmatrix} \Delta\beta_1 \\ \cdot \\ \cdot \\ \cdot \\ \Delta\beta_\Omega \end{pmatrix} = \Delta_B. \tag{C.40}$$

By construction,

$$\sum_{\omega=1}^{\Omega} \beta_\omega p^1(\omega) y_\Omega^\omega = 0,$$

so the matrix on the left-hand side of equation (C.40) has orthogonal rows, which are also nonzero, so the matrix has full row rank. Thus (C.40) can always be solved for $\Delta\beta$. The argument for the system (C.38)–
(C.39) is similar. *Q.E.D.*

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