CARESS Working Paper 00-08

REDUCING OVERLAPPING GENERATIONS ECONOMIES TO FINITE ECONOMIES

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June, 2000

ABSTRACT. This paper establishes in a general way the existence of a connection between the stationary equilibria of an infinite horizon economy and the equilibria of a naturally related finite economy.

More specifically, the connection is established first between the cycles of a stationary overlapping generations economy and the equilibria of a related finite economy with a cyclical structure. Then the connection is shown to hold also when extrinsic uncertainty (a sunspot) is introduced in the models. The connection holds in this case between a kind of sunspot equilibria called here sunspot cycles, and the correlated equilibria of the finite economy when there is asymmetric information about the extrinsic uncertainty. Incidentally, the sunspot cycles constitute a class of sunspot equilibria that are able to generate time series fluctuating in the recurrent but irregular way characteristic to some economic time series.

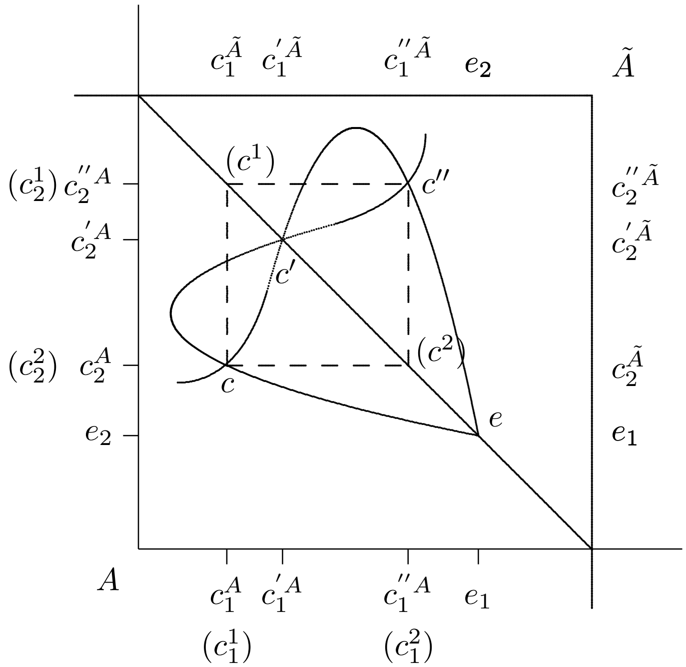
1. INTRODUCTION

Consider the simplest overlapping generations economy: a never-ending sequence of overlapping generations living for two periods, identical in preferences and endowments in the single commodity existing in each period, with no production and no starting date. Take two consecutive agents out of the never-ending chain of generations and, so to speak, close a loop with them. The resulting economy is an Edgeworth box where each agent faces his mirror image. Any equilibrium in this Edgeworth box corresponds to a cycle of period 2 of the original overlapping generations economy and conversely, including the steady state as a "degenerate" cycle of period 2 (See the Figure 1).[2]

[1]I thank the attendants to the Economic Theory seminar of the Department of Economics of the University of Pennsylvania for their comments on a previous version of this paper, as well as James Peck for his remarks on a previous work that paved the way to it.

[2]In Figure 1 the amounts between parentheses are the consumptions when young and old (1 and 2 in subscripts) of the odd/even-period born (1 and 2 in superscripts) representative agent A of an overlapping generations economy, and those without parentheses are equilibrium allocations of the Edgeworth box formed by A and his mirror image $\tilde{A}$. See also Balasko and Ghiglino (1995).

Figure 1



This simple example illustrates the connections that may exist between the equilibria of a dynamic economy and those of a static one. As a matter of fact, the previous thought experiment can actually be redone taking out any number n of consecutive generations and, again, closing a loop with them. As we shall see below, the resulting finite economy has now a very peculiar cyclical structure, which makes of any of its equilibria a cycle of a period (divisor of) n of the overlapping generations economy.[3]

The problem of clarifying the links between dynamic and static economies sharing some type of symmetry and between their equilibria is actually hidden in previous attempts to establish a connection between sunspot equilibria of overlapping generations economies and correlated equilibria of strategic market games (see, for instance, Maskin and Tirole (1987), Forgès and Peck (1995), Dávila (1999)). Actually, the link conjectured by Maskin and Tirole (1987) between, on the one hand, the correlated equilibria of their 2-agents, 2-commodities exchange economy with asymmetric information on an extrinsic uncertainty and, on the other hand, the sunspot equilibria of an overlapping generations economy constructed after that finite economy, hinted at a sort of extension of the connection between the cycles of the dynamic economy and the equilibria of the static one to a framework with (extrinsic) uncertainty.

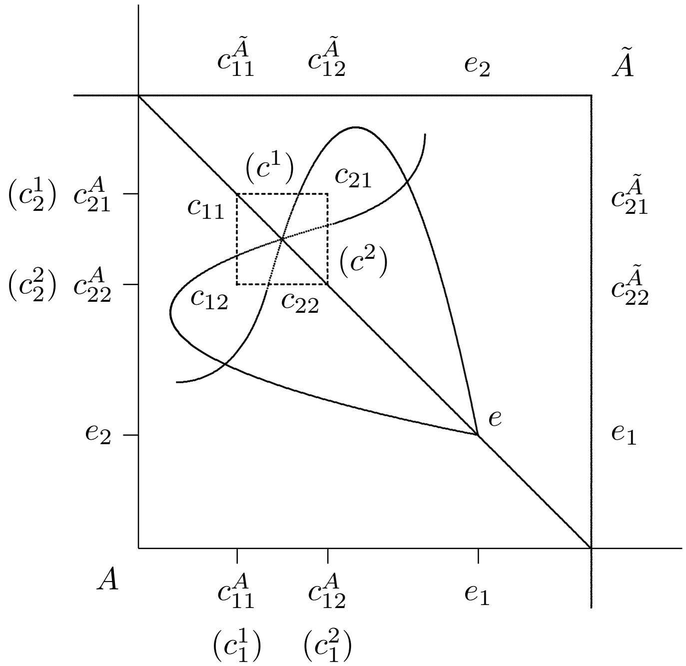
In effect, closing a loop with two consecutive agents of the overlapping generations economy makes a publicly observed sunspot become a distinct privately observed signal for each of the two agents.[4]. Thus, the possibility of a correlated equilibrium in which each agent uses his private signal as a randomizing device

[3]See Tuinstra and Weddepohl (1997) for a first approach to this idea, although not identical to the approach presented here.

[4]This is the effect of collapsing the line of time in one instant as the loop is closed. Nevertheless, the "publicly observed" sunspot was already private information for each cohort of young agents actually, since it is disclosed sequentially.

appears. Indeed, when considered in the Edgeworth box constructed with two consecutive agents, the sunspot equilibria of the original overlapping generations economy that follow a 2 states first order Markov process become, quite naturally, correlated equilibria of the market game underlying the 2 agents economy with asymmetric information[5] (see Figure 2).

Figure 2



These sunspot equilibria are similar to cycles of period 2, but for the fact that the fluctuations between the two states are random instead of deterministic. As a matter of fact, a cycle of period 2 can be considered an extreme (or degenerate) case of a sunspot equilibrium following a 2 states first order Markov process whose probabilitiy of changing the state is always 1.[6] Nevertheless, although in a first approximation such a finite state Markovian stationary sunspot equilibrium may seem the most natural extension of cycles to a framework with extrinsic uncertainty in order to obtain the connection between the equilibria of the dynamic and static frameworks (see Example 2 in Section 4.2), they are definitely not the right extension if we want such connection to hold in general (see Dávila (1999)). Rather, the correct counterparts are the equilibria of a more general and, at the same time, somewhat special class that I shall call sunspot cycles.[7] Intuitively, these sunspot

[5]As it will be shown below (see Example 1 in Section 4.1.3), the corners of the of the smaller box within the Edgeworth box in Figure 2 constitute the support (with parentheses) of a sunspot equilibrium of the overlapping generations economy with consumer A as representative agent. They constitute as well the allocation of resources (without parentheses)of a correlated equilibrium of the economy formed by consumer A and his mirror image, consumer $\tilde{A}$ (see the Example 2 in Section 4.2).

[6]This is not unrelated to the fact that their support lay at the "extremes" of the 8-shaped figure containing all the sunspot equilibria supports in Figure 2.

[7]Their seemingly contradictory character as simultaneously more general and particular than the usual sunspot equilibria is only apparent, since each statement correspond to different viewpoints; see the remarks in Section 4.2 for clearer insights on this.

cycles consist of superimposing a sunspot signal to a cycle, instead of looking at the cycle as a degenerate sunspot signal itself. More precisely, in a sunspot cycle the support from which the sunspot process selects randomly each period allocation fluctuates cyclically, instead of being the same support every period. Thus, contrarily to what happens when considering a cycle as a degenerate Markov process, in a sunspot cycle the periodicity of the underlying cycle and the number of values among which the sunspot fluctuates are completely unrelated.

Sunspot cycles have a subtle conceptual status with respect to the usual finite state Markovian stationary sunspot equilibria,[8] since the latter can be seen as particularly simple sunspot cycles whose support for the sunspot process "fluctuates" with a periodicity 1. Thus the finite Markovian stationary sunspot equilibria are to sunspot cycles what steady states are to cycles. Nonetheless, the simple sunspot cycles that I will consider mostly below are finite state Markovian stationary sunspot equilibria themselves, but with a high number of states (the periodicity with which the sunspot support fluctuates times the cardinality of the support itself) and a very special matrix of probabilities of transition exhibiting a lot of symmetries and zero entries.

The interest of sunspot cycles is twofold. Firstly, from a theoretical point of view, notice that simple sunspot processes (i.e. with low cardinal supports) can be combined with long period cyclical fluctuations of the support to result in a Markov process with a finite but very high number of states. This may overcome to some extent a usual criticism to the sunspot equilibrium concept as a positive concept for economic fluctuations: the need to assume the ability of the agents to coordinate spontaneously on complex sunspot signals in order to get realistic fluctuations and, hence, the unlikeliness of such a spontaneous coordination. Secondly, from a positive point of view, sunspot cycles blend their deterministic and stochastic ingredients in a manner able to reproduce, although in a very stylized way, the recurrent but irregular quasi-cyclical behavior characteristic to many economic time series (Figure 3 exibits a time series generated by a sunspot cycle of period 4 and order 3. See Section 4 for the definition).

(figure 3)

The rest of the paper is organized as follows. Section 2 introduces the two types of economies. In Section 3, I present a straightforward connection between the cycles of the overlapping generations economy and the equilibria of its associated cyclical economies.[9]. This is mainly done for the sake of completeness. Although strictly speaking everything in Section 3 is a special case of something in Section 4, where extrinsic uncertainty is introduced in the economies, keeping the uncertainty out of the stage for a while eases the exposition a lot. Section 4 extends the connection between the equilibria of these economies to the case where there is (extrinsic)

[8]Or k-SSE (for Stationary Sunspot Equilibria of order k) in Guesnerie (1986), Chiappori and Guesnerie (1989), Chiappori, Geoffard and Guesnerie (1992), Chiappori and Dávila (1996), and more generally Dávila (1997)

[9]Such a connection had been noticed for cycles of period 2 by Balasko and Ghiglino (1995). Tuinstra and Weddepohl (1997) extended this connection to cycles of any period, but with a static economy different from the one considered here and that allows me to extend the connection to the case with uncertainty as well.

uncertainty. Finally, Section 5 concludes and the Appendix collects proofs and lemmas.

2. A simple overlapping generations economy and its associated cyclical economies

Consider the simplest overlapping generations economy, i.e. an economy consisting of a never-ending sequence of generations dated by $t \in \mathbb{Z}$. All the members of a typical generation born at date t are identical to a representative agent who lives for two periods, is endowed with positive quantities[10] e^t_t and e^t_{t+1} of the single commodity of the economy at dates t and $t+1$, and has preferences over the consumption of this commodity along his lifetime which are represented by a utility function u^t depending on his consumption when young c^t_t and his consumption when old c^t_{t+1}, which is standard in the sense that it is continuous on $\mathbb{R}^2_+$, twice continuously differentiable on $\mathbb{R}^2_{++}$, strictly monotone,[11] strictly quasi-concave,[12] and well-behaved at the boundary.[13]. I shall refer to this overlapping generations economy as $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$.

Now, to any given overlapping generations economy $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$, a date t_0 and a positive integer n, I associate an economy with n commodities and n consumers defined as follows: consumer i's preferences and endowments are[14]

$$(1) \qquad \begin{aligned} U^i(c^i_1, \ldots, c^i_n) &= u^{t_0+i}(c^i_i, c^i_{i+1}) \\ e^i_h &= \begin{cases} e^{t_0+i}_1 & \text{if } h = i \\ e^{t_0+i}_2 & \text{if } h = i+1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

where $i = 1, \ldots, n$, and with the understanding from now on that $i+1$ stands for 1 when $i = n$, in such a way that the utility function and endowments of the n-th consumer are $u^{t_0+n}(c^n_n, c^n_1)$ and $(e^{t_0+n}_2, 0, \ldots, 0, e^{t_0+n}_1)$. This economy is a sort of closed loop of the n consecutive agents, starting from generation t_0+1, of the overlapping generations economy $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$. Let us refer to it as the (t_0, n)-cyclical economy $(\{(u^t, e^t)\}_{t\in\mathbb{Z}}, t_0, n)$ associated to the overlapping generations $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$.

I will refer to an overlapping generations economy $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$ as a stationary economy whenever it repeats itself periodically, i.e. if $(u^t, e^t) = (u^{t'}, e^{t'})$ whenever $t' = t + r\bar{n}$ for some $\bar{n} \in \mathbb{N}$ and all $r \in \mathbb{Z}$. More specifically I will refer to any such economy as a cyclical economy of period $\bar{n}$. Notice that this includes as a particular case (when $\bar{n} = 1$) any overlapping generations economy with a representative agent, which will be referred to from now onwards as (u, e). Any two of its cyclical economies with the same number of agents $(\{(u^t, e^t)\}_{t\in\mathbb{Z}}, t_0, n)$ are identical and

[10] More precisely, non-negative but not simultaneously equal to zero. In what follows, subscripts refer to dated commodities, superscripts to generations.

[11] In the sense that $Du^t(c^t_1, c^t_2)$ is always in the strictly positive orthant.

[12] In the sense that $D^2u^t(c^t_1, c^t_2)$ is always negative definite in the subspace orthogonal to $Du^t(c^t_1, c^t_2)$.

[13] In the sense that either there is no intersection between the axes and the indifference curves or any such intersection is tangent.

[14] Again, superscripts refer to consumers, subscripts to commodities.

symmetric, i.e. invariant under a cyclical permutation of the indices,[15] and they will be denoted by (u, e, n).

3. The connection under certainty

3.1 The overlapping generations economy under certainty.

Consider first the problem of the generation t of the overlapping generations economy $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$, i.e.

$$\max_{0\leq c_t^t, c_{t+1}^t} u^t(c_t^t, c_{t+1}^t) \qquad (2)$$
$$p_t(c_t^t - e_1^t) + p_{t+1}(c_{t+1}^t - e_2^t) \leq 0.$$

Under the assumptions made on u^t and e^t, the unique solution of this problem is completely characterized by the corresponding first order conditions.

An equilibrium of the overlapping generations economy $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$ consists of an allocation of resources $\{(c_t^t, c_{t+1}^t)\}_{t\in\mathbb{Z}}$ and prices $\{p_t\}_{t\in\mathbb{Z}}$ such that (i) for all $t \in \mathbb{Z}$, (c_t^t, c_{t+1}^t) is the solution to (2), and (ii) the allocation of resources is feasible.

The next proposition gives a complete characterization of the equilibrium allocations of the overlapping generations economy $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$.

Proposition 1: (i) *If the allocation of resources $\{(c_t^t, c_{t+1}^t)\}_{t\in\mathbb{Z}}$ and prices $\{p_t\}_{t\in\mathbb{Z}}$ constitute an equilibrium of the overlapping generations economy $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$, then for all $t \in \mathbb{Z}$*

$$D_1u^t(c_t^t, c_{t+1}^t)(c_t^t - e_1^t) + D_2u^t(c_t^t, c_{t+1}^t)(c_{t+1}^t - e_2^t) = 0. \qquad (3)$$

(ii) *If the allocation of resources $\{(c_t^t, c_{t+1}^t)\}_{t\in\mathbb{Z}}$ satisfies* (3) *and the feasibility condition*

$$c_{t+1}^t + c_{t+1}^{t+1} = e_2^t + e_1^{t+1}, \qquad (4)$$

for all $t \in \mathbb{Z}$, then it is an equilibrium allocation of the overlapping generations economy $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$.

The condition (3) is nothing else than the individual rationality condition requiring the equalization of marginal rates of substitution and real interest rates supporting the allocation, expressed as the orthogonality of the utility gradient and excess demand. The proof of Proposition 1 is relegated to the Appendix.

Some equilibrium allocations of the overlapping generations economy $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$ among the uncountably many of them that typically exist, exhibit some sort of regularity that may make more likely an assumed spontaneous coordination of every agent of the economy on one of them.[16] For instance, an equilibrium may have an allocation that treats equally any two generations n periods apart from each other, and hence the following definition.

[15] That is to say, the permutation of the indices by any power of the $n \times n$ matrix ρ whose typical entry ρ_{ij} equals 1 whenever $j = i + 1$ (recall that $n + 1$ stands for 1) and is 0 otherwise.

[16] Although there may still be countably many of such recurrent equilibria: Grandmont(1985) establishes by means of a theorem by Sarkovskii(1964) on iterated maps on an interval of the real line that, should there be a cycle of period 3, then the economy would have cycles of any period.

DEFINITION 1: A *cyclical allocation* of period n of the overlapping generations economy $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$ is an allocation $\{(c^t_t, c^t_{t+1})\}_{t\in\mathbb{Z}}$ such that

$$(c^t_t, c^t_{t+1}) = (c^{t'}_{t'}, c^{t'}_{t'+1}) \tag{5}$$

for any t and t' such that $t' = t + rn$ for some integer $r \in \mathbb{Z}$.

The definition of a cycle follows straightforwardly.

DEFINITION 2: A *cycle* of period n of the overlapping generations economy $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$ is any of its equilibria whose allocation is a cyclical allocation of period n.

Notice that this defines a cyclical allocation of period n in a quite broad sense, since it allows seeing any such allocation as a cyclical allocation of any other period n' multiple of n. In particular, it allows for a steady state (i.e. an equilibrium which treats equally all the generations) to be considered a cyclical allocation of any period.

The allocation of a cycle of period n is, according to Proposition 1, completely characterized by at most n distinct consumption bundles $(c^1_1, c^1_2), \ldots, (c^n_1, c^n_2)$ in $\mathbb{R}^2_{++}$ such that, for some $t_0 \in \mathbb{Z}$, all $r \in \mathbb{Z}$, and all $i = 1, \ldots, n$

$$D_1 u^{t_0+rn+i}(c^i_1, c^i_2)(c^i_1 - e^{t_0+rn+i}_1) + D_2 u^{t_0+rn+i}(c^i_1, c^i_2)(c^i_2 - e^{t_0+rn+i}_2) = 0 \tag{6}$$

and

$$c^i_2 + c^{i+1}_1 = e^{t_0+rn+i}_2 + e^{t_0+rn+i+1}_1 \tag{7}$$

where, again, $i + 1$ stands for 1 if $i = n$.

If the overlapping generations economy $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$ happens to be cyclical itself with a period $\bar{n}$ divisor of n, i.e. such that $(u^t, e^t) = (u^{t'}, e^{t'})$ whenever $t' = t + r\bar{n}$ for some $r \in \mathbb{Z}$ (which includes every economy with a representative agent (u, e) as a particular case), then the allocation of a cycle of period n is characterized by a finite number of equations and consists of n points $(c^1_1, c^1_2), \ldots, (c^n_1, c^n_2)$ in $\mathbb{R}^2_{++}$ such that, for some $t_0 \in \mathbb{Z}$ and all $i = 1, \ldots, n$,

$$D_1 u^{t_0+i}(c^i_1, c^i_2)(c^i_1 - e^{t_0+i}_1) + D_2 u^{t_0+i}(c^i_1, c^i_2)(c^i_2 - e^{t_0+i}_2) = 0 \tag{8}$$

and

$$c^i_2 + c^{i+1}_1 = e^{t_0+i}_2 + e^{t_0+i+1}_1. \tag{9}$$

Notice that the previous definition of cycles does not take into account the equilibrium prices. As a consequence, in the case of a economy with a representative agent (u, e) it encompasses as a cycle of any period the autarky allocation $\{(e_1, e_2)\}_{t\in\mathbb{Z}}$ too, which would not qualify typically as a cycle if the same kind of regularity was required for the prices. In effect, for any cycle of period n distinct from the autarky, any p_t and $p_{t'}$ do coincide whenever they are n periods apart from each other as well: since, for all $\tau \in \mathbb{Z}$, it holds true that

$$p_\tau = -\frac{c^\tau_{\tau+1} - e_2}{c^\tau_\tau - e_1} p_{\tau+1} \tag{10}$$

because of the budget constraints; then, assuming $t < t'$ without loss of generality,

$$\begin{aligned} p_t &= \left(-\frac{c^t_{t+1}-e_2}{c^t_t-e_1}\right)\left(-\frac{c^{t+1}_{t+2}-e_2}{c^{t+1}_{t+1}-e_1}\right)\cdots\left(-\frac{c^{t'-1}_{t'}-e_2}{c^{t'-1}_{t'-1}-e_1}\right)p_{t'} \\ &= \left(-\frac{c^t_{t+1}-e_2}{c^{t+1}_{t+1}-e_1}\right)\left(-\frac{c^{t+1}_{t+2}-e_2}{c^{t+2}_{t+2}-e_1}\right)\cdots\left(-\frac{c^{t'-1}_{t'}-e_2}{c^t_t-e_1}\right)p_{t'} \\ &= \left(-\frac{c^{t'-1}_{t'}-e_2}{c^{t'}_{t'}-e_1}\right)p_{t'} \\ &= p_{t'}, \end{aligned} \tag{11}$$

after rearranging the denominators, substituting $c^{t'}_{t'}$ to c^t_t and noting that then each fraction becomes 1 because of the feasibility of the allocation of resources. As for the autarky allocation, any prices supporting it as an equilibrium must satisfy $p_t = \left(\frac{D_2u(e)}{D_1u(e)}\right)^{t-t'} \cdot p_{t'}$, which cannot show any cyclical regularity unless $\frac{D_2u(e)}{D_1u(e)} = 1$.

3.2 The cyclical economy under certainty.

Considering now the cyclical economy $(\{(u^t, e^t)\}_{t\in\mathbb{Z}}, t_0, n')$ formed by the n' generations of $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$ following t_0, its i-th consumer solves the problem[17]

$$\begin{aligned} &\max_{0\le c^i_i, c^i_{i+1}} \; u^{t_0+i}(c^i_i, c^i_{i+1}) \\ &p_i(c^i_i - e_1^{t_0+i}) + p_{i+1}(c^i_{i+1} - e_2^{t_0+i}) \le 0 \end{aligned} \tag{12}$$

whose unique solution is again completely characterized by the first order conditions.

An equilibrium of the cyclical economy $(\{(u^t, e^t)\}_{t\in\mathbb{Z}}, t_0, n')$ consists of an allocation of resources $\{(c^i_i, c^i_{i+1})\}_{i=1}^{n'}$ and prices $\{p_i\}_{i=1}^{n'}$ such that (i) for all $i = 1, \ldots, n'$, (c^i_i, c^i_{i+1}) is the solution to (12), and (ii) the allocation of resources is feasible.

The equilibrium allocations of the cyclical economy $(\{(u^t, e^t)\}_{t\in\mathbb{Z}}, t_0, n')$ are completely characterized by the following proposition.

Proposition 2: (i) *If the allocation of resources* $\{(c^i_i, c^i_{i+1})\}_{i=1}^{n'}$ *and prices* $\{p_i\}_{i=1}^{n'}$ *constitute an equilibrium of the cyclical economy* $(\{(u^t, e^t)\}_{t\in\mathbb{Z}}, t_0, n')$ *then for all* $i = 1, \ldots, n'$

$$D_1u^{t_0+i}(c^i_i, c^i_{i+1})(c^i_i - e_1^{t_0+i}) + D_2u^{t_0+i}(c^i_i, c^i_{i+1})(c^i_{i+1} - e_2^{t_0+i}) = 0. \tag{13}$$

(ii) *If the allocation of resources* $\{(c^i_i, c^i_{i+1})\}_{i=1}^{n'}$ *satisfies* (13) *and the feasibility condition*

$$c^i_{i+1} + c^{i+1}_{i+1} = e_2^{t_0+i} + e_1^{t_0+i+1}, \tag{14}$$

for all $i = 1, \ldots, n'$, *then it is an equilibrium allocation of the cyclical economy* $(\{(u^t, e^t)\}_{t\in\mathbb{Z}}, t_0, n')$.

The proof of Proposition 2 is essentially identical to that of Proposition 1 characterizing the equilibrium allocations of the overlapping generations $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$.[18]

[17] Strictly speaking, the problem should be posed in $R^{n'}_+$ instead of R^2_+, but the cyclical structure of the economy makes this absolutely unnecessary. Recall also that, in what follows, $i+1$ stands for 1 whenever $i = n'$.

[18] Only substituting i to t wherever the latter appears and modifying the range of the indices from $t \in \mathbb{Z}$ to $i = 1, \ldots, n'$ accordingly is required.

3.3 The connection under certainty.

The next proposition establishes the connection between the cycles of a stationary overlapping generations economy $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$ and the equilibria of its cyclical economies $(\{(u^t, e^t)\}_{t\in\mathbb{Z}}, t_0, n')$.

PROPOSITION 3: *Any cycle of period n of a cyclical overlapping generations economy of period $\bar{n}$ divisor of n, $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$, can be translated into an equilibrium of any of its cyclical economies $(\{(u^t, e^t)\}_{t\in\mathbb{Z}}, t_0, n')$ with n' multiple of n, and conversely.*

The statement above is a straightforward consequence of comparing the equations characterizing the equilibrium allocations in each of the two frameworks, which happen to be the same up to a change of notation (see the proof in the Appendix). Proposition 3 applies obviously to any steady state also, considered as a degenerate cycle of period 1.

COROLLARY 1: *If $(\bar{c}_1, \bar{c}_2)$ is a steady state of a overlapping generations economy with a representative agent (u, e), then any n-replica of it $\{(\bar{c}_1, \bar{c}_2), \ldots, (\bar{c}_1, \bar{c}_2)\}$ constitutes an equilibrium allocation of the symmetric cyclical economy (u, e, n) and conversely, any symmetric equilibrium of a symmetric cyclical economy (u, e, n) is an n-replica of a steady state of the overlapping generations economy (u, e).*

4. THE CONNECTION UNDER UNCERTAINTY

4.1 The overlapping generations with sunspots.

Assume now that in the overlapping generations economy $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$ there is a sunspot signal[19] σ_t which is publicly observed at every period[20] and takes one of $k > 1$ values at random.[21]. Before getting into further details, the fact that the extrinsic uncertainty represented by this sunspot signal may end up having an influence on the outcome of the economy can be given the following rationale.

The agent born at t cares, in order to make his decision, about prices at t and $t+1$. In the presence of the (extrinsic) uncertainty introduced by the sunspot signal, he cannot exclude a priori a possible dependence of the prices on the values taken by the sunspot and therefore he is uncertain about the price he may face at $t+1$ (at date t the price is observed as well as the sunspot). Thus, the agent t has to

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[19]A signal with no influence on the fundamentals of the economy, i.e. a signal representing states of the world with respect to which the fundamentals (preferences, endowments, technology also if there were production) remain unchanged. It was first noticed in Shell (1977) that such signals could nevertheless have an influence on the outcome of the economy, leading to a so-called sunspot equilibrium. Later on Cass and Shell (1983) provided a characterization of the circumstances in which no sunspot equilibrium can exist (in few words, the Arrow-Debreu world), which amounts to a characterization by negation of the set-ups where they are likely to emerge. See Chiappori and Guesnerie (1991) and Guesnerie and Woodford (1996) for surveys on the subsequent literature that followed.

[20]Notice that, because of the demographic structure of the model, the signal publicly observed at each date, is actually private information of the generation currently making its consumption choices.

[21]As a matter of fact, if we let $k = 1$ in what follows in such a way that the stochastic process driving σ_t is a trivial one giving to the signal the same constant value at every period with probability 1, then all the claims and proofs still go through. Thus the results shown in Section 3 for the case under certainty are a particular case of those proved in this Section. They have nevertheless been presented separately in order to ease the exposition.

choose at t his current consumption and a plan of consumption at $t+1$ contingent to the price actually realized at $t+1$, bearing in mind some expectations about that price. These expectations may take, for instance, the form of a dependence of the price at $t+1$ on the value of the sunspot at that date. Thus the agent t will rather use, in the making of his decision, the information conveyed by the value of the sunspot at t about the probability distribution of the sunspot at $t+1$, i.e. about the probability distribution which applies to the prices at $t+1$ as well actually, according to his beliefs. The consumption choice at t will thus show a dependence on the sunspot observed at t, and it will moreover determine the price at t, resulting so in a dependence of the price at t on the current sunspot too. If this dependence ends up being the dependence believed to hold between the price $t+1$ and the sunspot at $t+1$, then such beliefs turn out to be self-fulfilling.[22].

In effect, consider the problem of a member of a generation t of the overlapping generations economy $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$ with a sunspot signal $\{\sigma_t\}_{t\in\mathbb{Z}}$. What he knows about the prevailing state of the world $s = (\dots, \sigma_{t-1}, \sigma_t, \sigma_{t+1}, \dots)$, from the continuum $S = \{1, \dots, k\}^{\mathbb{Z}}$ of possible states, is at most the history of sunspot values up to t, $s_t = (\dots, \sigma_{t-1}, \sigma_t)$,[23] as well as the price[24] p_{ts_t} for consumption $c^t_{ts_t}$ at t, and expects a price $p_{t+1s'_{t+1}}$ for consumption $c^t_{t+1s'_{t+1}}$ at $t+1$ if s'_{t+1} happens (where s'_{t+1} is necessarily such that[25] $s'_t = s_t$) with some probability $P(s'_{t+1}|s'_t = s_t)$. Then the member of a generation t born in a state of the world s faces the problem

$$(15) \qquad \max_{\substack{0\le c^t_{ts_t}, c^t_{t+1s'_{t+1}} \\ s'_{t+1}|s'_t=s_t}} \sum_{s'_{t+1}|s'_t=s_t} P(s'_{t+1}|s'_t = s_t)\; u^t(c^t_{ts_t}, c^t_{t+1s'_{t+1}})$$

$$p_{ts_t}(c^t_{ts_t} - e^t_1) + p_{t+1s'_{t+1}}(c^t_{t+1s'_{t+1}} - e^t_2) \le 0 \;,\; s'_{t+1}|s'_t = s_t.$$

The unique solution to this problem is completely characterized by its first order conditions.

An equilibrium of the overlapping generations economy $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$ with the sunspot signal $\{\sigma_t\}_{t\in\mathbb{Z}}$ consists of an allocation $\{(c^t_{ts_t}, (c^t_{t+1s'_{t+1}})_{s'_{t+1}|s'_t=s_t})\}_{t\in\mathbb{Z}, s\in S}$ and prices $\{p_{ts_t}\}_{t\in\mathbb{Z}, s\in S}$ such that (i) for all $s \in S$ and all $t \in \mathbb{Z}$, the contingent plan of consumption $(c^t_{ts_t}, (c^t_{t+1s'_{t+1}})_{s'_{t+1}|s'_t=s_t})$ is the solution to (15), and (ii) the allocation of resources is feasible.

The following proposition provides a complete characterization of the equilibrium allocations of the overlapping generations economy $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$ with the sunspot signal $\{\sigma_t\}_{t\in\mathbb{Z}}$.

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[22] More exactly, the expectations held after such beliefs turn out to be rational expectations actually.

[23] Consider a history as a mapping from the negative integers $-\mathbb{N}_0$, conveniently extended to include 0, to the set $\{1, \dots, k\}$. Thus the set of possible histories up to any given date is $\{1, \dots, k\}^{-N_0}$

[24] In what follows superscripts refer again to generations, while subscripts to dated, sunspot history-contingent commodities. The sequential unfolding of the allocation of resources of an overlapping generations economy prevents any generation to make consumption decisions contingent to information which will be disclosed after the consumption takes place. This is a distinctive feature of an overlapping generations economy under uncertainty that does not appear in a one-shot economy (like, for instance, an Arrow-Debreu economy), where consumption takes place after every uncertainty, if any, is resolved, even if the decisions may be made ex ante contingent to the realization of any uncertainty.

[25] Here s'_t denotes the truncation of the history s'_{t+1} up to t.

Proposition 4: (i) *If the allocation* $\{(c^t_{ts_t}, (c^t_{t+1s'_{t+1}})_{s'_{t+1}|s'_t=s_t})_{s_t}\}_{t\in\mathbb{Z},s\in S}$ *and prices* $\{p_{ts_t}\}_{t\in\mathbb{Z},s\in S}$ *constitute an equilibrium of the overlapping generations economy* $\{(u^t,e^t)\}_{t\in\mathbb{Z}}$ *with the sunspot signal* $\{\sigma_t\}_{t\in\mathbb{Z}}$*, then for all* $s\in S$ *and all* $t\in\mathbb{Z}$

$$\sum_{s'_{t+1}|s'_t=s_t} P(s'_{t+1}|s'_t=s_t)\big(D_1u^t(c^t_{ts_t},c^t_{t+1s'_{t+1}})(c^t_{ts_t}-e^t_1) + D_2u(c^t_{ts_t},c^t_{t+1s'_{t+1}})(c^t_{t+1s'_{t+1}}-e^t_2)\big)=0. \tag{16}$$

(ii) *If the allocation of resources* $\{(c^t_{ts_t}, (c^t_{t+1s'_{t+1}})_{s'_{t+1}|s'_t=s_t})\}_{t\in\mathbb{Z},s\in S}$ *satisfies* (16) *and the feasibility condition*

$$c^{t-1}_{ts_t}+c^t_{ts_t}=e^{t-1}_2+e^t_1 \tag{17}$$

for all $s\in S$ *and all* $t\in\mathbb{Z}$*, then it is an equilibrium allocation of the overlapping generations economy* $\{(u^t,e^t)\}_{t\in\mathbb{Z}}$ *with the sunspot signal* $\{\sigma_t\}_{t\in\mathbb{Z}}$*.*

Similarly to (3) in Proposition 1, condition (16) is the equalization, in mathematical expectation this time, of each agent's marginal rate of substitution and the real rate of interest supporting the allocation at every date, i.e. the individual rationality condition, while (17) is the feasibility of the allocation of resources, i.e. the market-clearing condition.

Any equilibrium of the overlapping generations economy $\{(u^t,e^t)\}_{t\in\mathbb{Z}}$ with the sunspot signal $\{\sigma_t\}_{t\in\mathbb{Z}}$ whose allocation of resources does actually depend (in a non trivial way) on the state s realized is known as a sunspot equilibrium.

Notice that the allocation of an equilibrium of the overlapping generations economy $\{(u^t,e^t)\}_{t\in\mathbb{Z}}$ with the sunspot signal $\{\sigma_t\}_{t\in\mathbb{Z}}$ must specify in general not only how the resources are allocated at every date $t\in\mathbb{Z}$ (as in the case with certainty) but for every possible state $s\in S$ also. An equilibrium allocation must thus specify for every generation t what its typical member will receive for any possible history[26] s_t in which he may be born and any continuation σ_{t+1} of it in $t+1$. Nonetheless, such an allocation may exhibit some regularities, as in the case without uncertainty, that make it simpler and, hence, likelier to emerge from spontaneous coordination of the agents. Notice, however, that the regularities may arise now not only with respect to the dates $t\in\mathbb{Z}$, as in the cycles, but with respect to the histories s_t as well.

4.1.1 Regularities across time.

The following definition of a cyclical allocation in the framework with uncertainty, captures the same kind of regularity with respect to t that the concept of cyclical allocation in the framework without uncertainty did, i.e. that it treats equally any two generations n periods apart from each other.

Definition 3: A *cyclical allocation* of period n of an overlapping generations economy $\{(u^t,e^t)\}_{t\in\mathbb{Z}}$ with the sunspot signal $\{\sigma_t\}_{t\in\mathbb{Z}}$ is an allocation of resources $\{(c^t_{ts_t}, (c^t_{t+1s'_{t+1}})_{s'_{t+1}|s'_t=s_t})\}_{t\in\mathbb{Z},s\in S}$ such that

$$(c^t_{ts_t}, (c^t_{t+1s'_{t+1}})_{s'_{t+1}|s'_t=s_t}) = (c^{t'}_{t'\tilde{s}_{t'}}, (c^{t'}_{t'+1\tilde{s}'_{t'+1}})_{\tilde{s}'_{t'+1}|\tilde{s}'_{t'}=\tilde{s}_{t'}}) \tag{18}$$

[26] Notice that there are uncountably many possible histories $(\ldots,\sigma_{t-1},\sigma_t)\in\{1,\ldots,k\}^{-N}$.

for every $t, t' \in \mathbb{Z}$ such that $t' = t + rn$ for some $r \in \mathbb{Z}$, and every $s, \tilde{s} \in S$ such that $s_t = \tilde{s}_{t'}$.

Hence the concept of a sunspot cycle follows naturally.

DEFINITION 4: A *sunspot cycle* of period n and order[27] k is an equilibrium of the overlapping generations economy $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$ with the sunspot signal $\{\sigma_t\}_{t\in\mathbb{Z}}$ whose allocation is a cyclical allocation of period n.

Proposition 4 provides a characterization of the allocation of a sunspot cycle. In effect, the allocation of a sunspot cycle of period n and order k consists of at most n distinct sunspot-history contingent plans[28] $(c^i_{1s_1}, (c^i_{2s_2})_{s_2|s_2^{-1}=s_1})_{s_1\in\{1,\ldots,k\}^{-N}}$, for all $i = 1, \ldots, n$, such that for some date $t_0 \in \mathbb{Z}$, all $r \in \mathbb{Z}$, all $i = 1, \ldots, n$, and every history $s_1 \in \{1, \ldots, k\}^{-N_0}$,

$$\tag{19} \begin{aligned} \sum_{s_2|s_2^{-1}=s_1} P(s_2|s_2^{-1} = s_1)\big(D_1 u^{t_0+rn+i}(c^i_{1s_1}, c^i_{2s_2})(c^i_{1s_1} - e_1^{t_0+rn+i}) \\ + D_2 u^{t_0+rn+i}(c^i_{1s_1}, c^i_{2s_2})(c^i_{2s_2} - e_2^{t_0+rn+i})\big) = 0 \end{aligned}$$

and

$$\tag{20} c^i_{2s_1} + c^{i+1}_{1s_1} = e_2^{t_0+rn+i} + e_1^{t_0+rn+i+1}.$$

The simplest of sunspot cycles are obviously those for which $n = 1$ and, given their analogy with the steady states, I shall refer to them as sunspot steady states. Again, if the overlapping generations economy $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$ happens to be cyclical itself with a period $\bar{n}$ divisor of n (which includes economies with a representative agent (u, e) as a particular case), then the previous conditions simplify to requiring only that for some date t_0, all $i = 1, \ldots, n$, and every history $s_1 \in \{1, \ldots, k\}^{-N_0}$,

$$\tag{21} \begin{aligned} \sum_{s_2|s_2^{-1}=s_1} P(s_2|s_2^{-1} = s_1)\big(D_1 u^{t_0+i}(c^i_{1s_1}, c^i_{2s_2})(c^i_{1s_1} - e_1^{t_0+i}) \\ + D_2 u^{t_0+i}(c^i_{1s_1}, c^i_{2s_2})(c^i_{2s_2} - e_2^{t_0+i})\big) = 0 \end{aligned}$$

and

$$\tag{22} c^i_{2s_1} + c^{i+1}_{1s_1} = e_2^{t_0+i} + e_1^{t_0+i+1}.$$

Notice, nonetheless, that the dependence on the sunspot history makes of this a system of uncountably many equations. Some other kind of regularity, as for instance with respect to sunspot histories as well, will still be required in order to pin down simple equilibria characterized by a finite number of equations.

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[27] Following the literature on k-SSE, the order is the number of values taken by the sunspot.

[28] Here s_2^{-1} stands for the one-step forward shift of the history $s_2 = (\ldots, s_{23}, s_{22}, s_{21})$, i.e. the history $(\ldots, s_{24}, s_{23}, s_{22})$. With this notation the usual continuation condition $s'_{t+1}|s'_t = s_t$ would have become $s'_{t+1}|{s'_{t+1}}^{-1} = s_t$. The first one seems to be eloquent enough, while being less cumbersome, to justify this minor redundancy in notation.

4.1.2 Regularities across sunspot histories.

Considering regularities with respect to sunspot histories, an allocation may be such that, for instance, it allocates the resources in the same way at any two histories up to any given date $t \in \mathbb{Z}$ which coincide in the m (possibly all) last periods besides the current one. I shall refer to such an allocation as a m-memory allocation, and its formal definition follows.

DEFINITION 5: An *m-memory allocation* of an overlapping generations economy $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$ with the sunspot signal $\{\sigma_t\}_{t\in\mathbb{Z}}$ is an allocation such that

$$(c^t_{ts_t}, (c^t_{t+1s'_{t+1}})_{s'_{t+1}|s'_t=s_t}) = (c^t_{t\tilde{s}_t}, (c^t_{t+1\tilde{s}'_{t+1}})_{\tilde{s}'_{t+1}|\tilde{s}'_t=\tilde{s}_t}) \tag{23}$$

for all $t \in \mathbb{Z}$ and any $s, \tilde{s} \in S$ such that $\sigma_{t-i} = \tilde{\sigma}_{t-i}$, for all $i = 0, \ldots, m$.

The definition of an m-memory equilibrium of an overlapping generations economy follows straightforwardly.

DEFINITION 6: An *m-memory equilibrium* is an equilibrium of the overlapping generations economy $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$ with the sunspot signal $\{\sigma_t\}_{t\in\mathbb{Z}}$ whose allocation is an m-memory allocation.

According to Proposition 4, the conditions characterizing the allocation of an m-memory equilibrium, i.e. a[29] $(c^t_{ts_{t-m,t}}, (c^t_{t+1s_{t-m,t}\sigma_{t+1}})^k_{\sigma_{t+1}=1})$ for each $m+1$ most recent periods history $s_{t-m,t} = (\sigma_{t-m}, \ldots, \sigma_t) \in \{1, \ldots, k\}^{m+1}$ and each $t \in \mathbb{Z}$, are

$$\sum_{\sigma_{t+1}=1}^{k} m^{\sigma_t\sigma_{t+1}}_{s_{t-1}} \big(D_1u^t(c^t_{ts_{t-m,t}}, c^t_{t+1s_{t-m,t}\sigma_{t+1}})(c^t_{ts_{t-m,t}} - e^t_1) \\ + D_2u^t(c^t_{ts_{t-m,t}}, c^t_{t+1s_{t-m,t}\sigma_{t+1}})(c^t_{t+1s_{t-m,t}\sigma_{t+1}} - e^t_2)\big) = 0 \tag{24}$$

and

$$c^t_{t+1s_{t-m,t}\sigma_{t+1}} + c^{t+1}_{t+1s_{t+1-m,t+1}\sigma_{t+1}} = e^t_2 + e^{t+1}_1 \tag{25}$$

for all $t \in \mathbb{Z}$, all $s_{t-m,t} \in \{1, \ldots, k\}^{m+1}$, and all $\sigma_{t+1} = 1, \ldots, k$, where $m^{\sigma_t\sigma_{t+1}}_{s_{t-1}}$ in (24) stands for $P(s'_{t+1}|s'_t = s_t)$, i.e. the probability of transition from σ_t to each σ_{t+1} given the history s_{t-1} up to $t-1$.[30] Notice that because of the feasibility condition no $c^t_{t+1s_{t-m,t}\sigma_{t+1}}$ depends on σ_{t-m}.

The simplest of such equilibria are clearly those with $m = 0$, or 0-memory equilibria for which $c^t_{ts_t} = c^t_{t\tilde{s}_t}$ holds, whenever $s, \tilde{s} \in S$ and $t \in \mathbb{Z}$ are such that $\sigma_t = \tilde{\sigma}_t$. For such equilibria, the feasibility condition again implies necessarily $c^t_{t+1s_{t+1}} = c^t_{t+1\tilde{s}_{t+1}}$ as well, for any $s, \tilde{s} \in S$ and $t \in \mathbb{Z}$ such that $\sigma_{t+1} = \tilde{\sigma}_{t+1}$. Therefore, these are allocations whose consumptions depend on the current sunspot only, i.e. they are history independent allocations, and, for them, the history s_t

[29] From now on the concatenation $s\sigma$ of a history s, finite or infinite, and a sunspot value σ stands trivially for the history consisting of the continuation of s by σ.

[30] Notice that, as long as $k \geq 3$, the continuum of possible histories up to $t-1$ is not a problem for this dependence of the probabilities of transition on s_{t-1} because of the positive number of degrees of freedom for determining such probabilities. On the contrary, if $k = 2$, the condition (24) and $\Sigma m = 1$ determine unambiguously each row of the matrix of probabilities of transition and, hence, any such equilibrium has to be necessarily Markovian, i.e. such that the probabilities of transition do not depend on s_{t-1} actually (see below).

of sunspot values up to t can be identified to the current sunspot value σ_t. So a 0-memory equilibrium allocation is a $(c^t_{t\sigma_t}, (c^t_{t+1\sigma_{t+1}})^k_{\sigma_{t+1}=1})^k_{\sigma_t=1}$ such that

$$\sum_{\sigma_{t+1}=1}^{k} m^{\sigma_t\sigma_{t+1}}_{s_{t-1}} \big(D_1 u^t(c^t_{t\sigma_t}, c^t_{t+1\sigma_{t+1}})(c^t_{t\sigma_t} - e^t_1) + D_2 u^t(c^t_{t\sigma_t}, c^t_{t+1\sigma_{t+1}})(c^t_{t+1\sigma_{t+1}} - e^t_2)\big) = 0 \tag{26}$$

for all $t \in \mathbb{Z}$ and all $\sigma_t = 1, \ldots, k$, and

$$c^t_{t+1\sigma_{t+1}} + c^{t+1}_{t+1\sigma_{t+1}} = e^t_2 + e^{t+1}_1 \tag{27}$$

for all $t \in \mathbb{Z}$ and all $\sigma_{t+1} = 1, \ldots, k$.

4.1.3 Markovian sunspot cycles.

A particularly interesting class of equilibria is that of those equilibria for which the process followed by the sunspot signal is especially simple, i.e. such that the probabilities of transition $m^{\sigma_t\sigma_{t+1}}_{s_{t-1}}$ do not depend on the entire history of the signal up to $t-1$, s_{t-1}, but only on that of the most recent periods up to a finite number of them, maybe none, in which case the sunspot signal is Markovian. Such equilibria are specially appealing because, unlike the equilibria where $m^{\sigma_t\sigma_{t+1}}_{s_{t-1}}$ depends on the entire history s_{t-1}, they do not require the agents to be able to handle infinite sets of information, but rather a finite number of past values of σ_t along with a matrix of probabilities of transition.

Thus, a 0-memory Markovian sunspot cycle of period n and order k, of a cyclical overlapping generations economy of period $\bar{n}$ divisor of n $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$ (which includes every economy with a representative agent (u, e) as a particular case) is characterized by at most n distinct sunspot-contingent consumption plans $(c^1_{1\sigma_1}, (c^1_{2\sigma_2})^k_{\sigma_2=1})^k_{\sigma_1=1}, \ldots, (c^n_{1\sigma_1}, (c^n_{2\sigma_2})^k_{\sigma_2=1})^k_{\sigma_1=1}$, such that, for some date $t_0 \in \mathbb{Z}$, all $i = 1, \ldots, n$, and all $\sigma_1 = \{1, \ldots, k\}$,

$$\sum_{\sigma_2=1}^{k} m^{\sigma_1\sigma_2} \big(D_1 u^{t_0+i}(c^i_{1\sigma_1}, c^i_{2\sigma_2})(c^i_{1\sigma_1} - e^{t_0+i}_1) + D_2 u^{t_0+i}(c^i_{1\sigma_1}, c^i_{2\sigma_2})(c^i_{2\sigma_2} - e^{t_0+i}_2)\big) = 0 \tag{28}$$

and

$$c^i_{2\sigma_1} + c^{i+1}_{1\sigma_1} = e^{t_0+i}_2 + e^{t_0+i+1}_1. \tag{29}$$

Once arrived to this particular class of sunspot cycles, some remarks are in order.[31]. Firstly, notice that the role of steady state in the deterministic framework (a cycle of period 1, indeed), is played now by the sunspot cycles of period 1, the sunspot steady states. Secondly, notice too that a 0-memory Markovian sunspot cycle of period 1 and order k of a stationary overlapping generations economy (u, e),

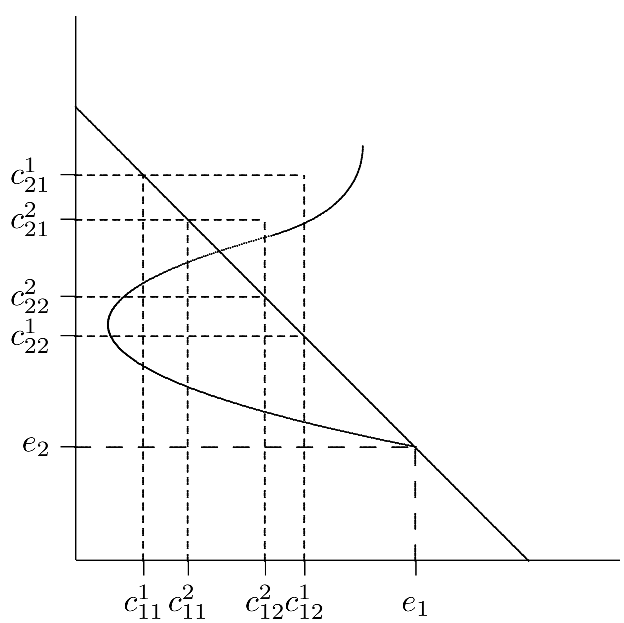
[31] Besides those already made on the notion of cycle, which apply straightforwardly to that of sunspot cycle, namely that any sunspot cycle of period n is also a sunspot cycle of any other period n' multiple of n.

turns out to be a sunspot equilibrium of the class k-SSE (for Stationary Sunspot Equilibrium of order k) studied in, for instance, Azariadis and Guesnerie (1986), Guesnerie (1986), Chiappori and Guesnerie (1989), Chiappori, Geoffard and Guesnerie (1992). As a matter of fact, the k-SSE are the simplest sunspot equilibria that can be produced in the simplest overlapping generations economies, namely those with a representative agent (u, e), and hence it is not surprising that they have been studied so thoroughly in the early literature on sunspot equilibria in the overlapping generations economies. The Example 1 below shows how the characterization of equilibrium allocations provided by Proposition 4 works for them.

Finally, notice that a 0-memory Markovian sunspot cycle with period n and order k, fluctuates typically between kn states. For instance, in the Figure 4 the support of a Markovian sunspot cycle of period 2 and order 2 as well is depicted.[32]. At such an equilibrium, the partition at each date of the economy resources between the contemporary young and old agents, fluctuates between the four points on the diagonal, although alternating (deterministically) the inner and the outer points for the (random) choice of the partition. Thus sunspot cycles are able to deliver considerably finer grids for fluctuations than the k-SSE, for a sunspot signal of a given cardinality k, i.e. assuming the same level of sophistication of the sunspot theory held by the agents or, put in other words, for the same cost in terms of their assumed ability to attain spontaneous coordination. Nonetheless, every sunspot cycle of this kind is actually a "high" order k-SSE with a Markov matrix with special symmetries. For instance, that of the sunspot cycle in Figure 4 may be of the form

$$\begin{pmatrix} 0 & 1-m^{12} & m^{12} & 0 \\ 1-m^{12} & 0 & 0 & m^{12} \\ m^{21} & 0 & 0 & 1-m^{21} \\ 0 & m^{21} & 1-m^{21} & 0 \end{pmatrix}. \tag{30}$$

Figure 4



[32]The gradients at the supporting points have to be such that the first order conditions are satisfied.

Figure 4b exhibits a time series generated by such a sunspot cycle when $m^{12} = 0.6$ and $m^{21} = 0.6$.

(figure 4b)

Example 1. A k-SSE of a overlapping generations economy with a representative agent (u, e) with a sunspot signal driven by a $k \times k$ Markov matrix (m^{ij}) is an equilibrium whose allocation of resources treats equally all the generations of the economy and the consumption when young depends only on the current sunspot signal and not on its entire history (the feasibility condition on the allocation of resources imposes then that the consumption when old depends on the current sunspot only as well). Let c_1^i denote in this example[33] the consumption when young at any t such that $\sigma_t = i$, and c_2^j the consumption when old at any t such that $\sigma_t = j$. Then the allocation of resources of a k-SSE has to satisfy (i) for all $i = 1, \dots, k$,

$$\sum_{j=1}^{k} m^{ij} \big(D_1 u(c_1^i, c_2^j)(c_1^i - e_1) + D_2 u(c_1^i, c_2^j)(c_2^j - e_2) \big) = 0 \tag{31}$$

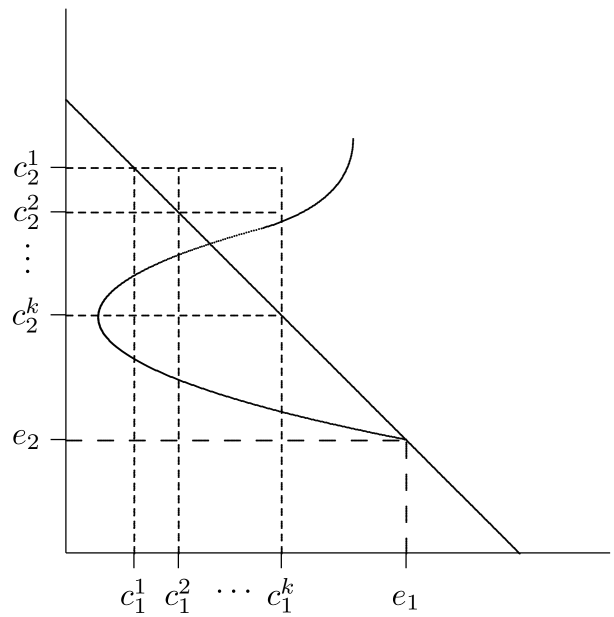
and (ii) for all $j = 1, \dots, k$,

$$c_2^j + c_1^j = e_1 + e_2. \tag{32}$$

Notice that the left-hand side of equation (31) is a convex linear combination of the inner products of the gradients of the utility function at each point (c_1^i, c_2^j) with its corresponding excess demand. A quite natural interpretation of equation (31) is therefore that, for $(c_1^i, (c_2^j)_{j=1}^k)$ to be the optimal current consumption and contingent plan of future consumption of a generation observing the i-th value of the sunspot, the orthogonality of the gradient with the corresponding excess demand that characterizes the optimal choice in the case with certainty, has to be satisfied in mean. As a consequence, not all the inner products in (31) can have the same sign and hence there must be at least two points (c_1^i, c_2^j) and $(c_1^i, c_2^{j'})$ which are separated by the offer curve (since the offer curve is characterized by making zero the inner product of gradients and excess demands). Assuming $c_1^1 < c_1^2 < \cdots < c_1^k$, without loss of generality, then $c_2^k < c_2^{k-1} < \cdots < c_2^1$ because of the feasibility condition and therefore, if the offer curve separates (c_1^i, c_2^k) and (c_1^i, c_2^1) for every $i = 1, \dots, k$ (see Figure 5), then $(c_1^i, (c_2^j)_{j=1}^k)_{i=1}^k$ is the support of a k-SSE.

[33] Instead of c_{1i} (resp. c_{2j}), as we should write according to the previous notation.

Figure 5



Thus, in order to exhibit such a k-SSE we only need to be able to produce a box with its top-left and bottom-right corners on the line going through the endowments e with slope -1,[34] and such that the top corners and bottom corners are separated by the offer curve. It may be clear now why, as established in Azariadis and Guesnerie (1986), Guesnerie (1986) and Woodford (1986), the indeterminacy of the steady state in the perfect foresight dynamics (i.e. that the slope of the offer curve at the non autarkic steady state is smaller than 1 in absolute value) is a sufficient condition for the existence of sunspot equilibria of this class (a continuum of them indeed), as well as why it is not by any means a necessary condition. It is nonetheless a necessary condition to be possible to produce such sunspot equilibria arbitrarily close to the steady state, i.e. the so-called local sunspot equilibria.[35]

4.2 The cyclical economy with private sunspots.

Consider now the cyclical economy $(\{(u^t, e^t)\}_{t\in\mathbb{Z}}, t_0, n)$ constructed with the n consecutive agents of the overlapping generations economy $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$ following t_0. Assume that each consumer i observes privately a different sunspot signal s_i which can take k values at random each. Thus the state of the world is an extrinsic random variable $s = (s_1, \ldots, s_n)$ taking values in $S = \{1, \ldots, k\}^n$. Let π^s denote the probability of s being the prevailing state and $\{(p_{is})_{i=1}^n\}_{s\in S}$ be the prices of each commodity i contingent to the state of the world s. Actually, as it will be seen below, for any equilibrium allocation the prices supporting it will be such that each p_{is} depends only on s_i.

[34] This condition takes care of the allocation of resources.

[35] See Chiappori, Geoffard and Guesnerie (1992) for a characterization of the existence of local k-SSE around the steady state of an abstract one-step forward looking dynamical system, Dávila (1997) for the existence of the same kind of local sunspot equilibria in similar systems but with predetermined variables or memory, and Woodford (1984) for a result showing that the indeterminacy of the steady state in the perfect foresight dynamics is a necessary and sufficient condition for the existence of more general local sunspot equilibria.

The problem that the i-th consumer observing s_i faces is[36][37]

$$\max_{\substack{0\leq c^i_{is_i}, c^i_{i+1s'} \\ \forall s'|s'_i=s_i}} \sum_{\forall s'|s'_i=s_i} \pi^{s'} u^{t_0+i}(c^i_{is_i}, c^i_{i+1s'}) \tag{33}$$

$$p_{is'}(c^i_{is_i} - e^{t_0+i}_1) + p_{i+1s'}(c^i_{i+1s'} - e^{t_0+i}_2)\ ,\ \forall s'|s'_i = s_i$$

where, as usual, $i+1$ stands for 1 when $i = n$, and the denominator has been dropped from the conditional probability for the sake of readability. An equilibrium of the cyclical economy $(\{(u^t, e^t)\}_{t\in\mathbb{Z}}, t_0, n)$ with private sunspot signals consists of an allocation of resources $\{(c^i_{is_i}, (c^i_{i+1s'})_{s'|s'_i=s_i})^k_{s_i=1}\}^n_{i=1}$ and prices $\{(p_{is})^n_{i=1}\}_{s\in S}$ such that (i) for all $i = 1, \ldots, n$ and all $s_i = 1, \ldots, k$, $(c^i_{is_i}, (c^i_{i+1s'})_{s'|s'_i=s_i})$ is the solution to the problem (33) above, and (ii) the allocation of resources is feasible.

The next proposition gives a complete characterization of the equilibrium allocation of the cyclical economy $(\{(u^t, e^t)\}_{t\in\mathbb{Z}}, t_0, n)$ with asymmetric information on the extrinsic uncertainty.

PROPOSITION 5: (i) *If the allocation of resources* $\{(c^i_{is_i}, (c^i_{i+1s'})_{s'|s'_i=s_i})^k_{s_i=1}\}^n_{i=1}$ *and the prices* $\{(p_{is})^n_{i=1}\}_{s\in S}$ *constitute an equilibrium of the cyclical economy* $(\{(u^t, e^t)\}_{t\in\mathbb{Z}}, t_0, n)$ *with* n *private sunspot signals with joint distribution* π*, then for all* $i = 1, \ldots, n$ *and all* $s_i = 1, \ldots, k$

$$\sum_{s'|s'_i=s_i} \pi^{s'} \Big(D_1 u^{t_0+i}(c^i_{is_i}, c^i_{i+1s'})(c^i_{is_i} - e^{t_0+i}_1) + D_2 u^{t_0+i}(c^i_{is_i}, c^i_{i+1s'})(c^i_{i+1s'} - e^{t_0+i}_1)\Big) = 0. \tag{34}$$

(ii) *If the allocation of resources* $\{(c^i_{is_i}, (c^i_{i+1s'})_{s'|s'_i=s_i})^k_{s_i=1}\}^n_{i=1}$ *satisfies* (34) *and the feasibility condition*

$$c^i_{i+1s'} + c^{i+1}_{i+1s_{i+1}} = e^{t_0+i}_2 + e^{t_0+i+1}_1 \tag{35}$$

for all $i = 1, \ldots, k$*, all* $s_{i+1} = 1, \ldots, k$ *and all* s' *such that* $s'_{i+1} = s_{i+1}$*, then it is an equilibrium allocation of the cyclical economy* $(\{(u^t, e^t)\}_{t\in\mathbb{Z}}, t_0, n)$.

---

[36] As previously, superscripts refer to consumers, while subscripts to sunspot contingent commodities. Notice that consumption by each consumer i of the "same-label" good i depends only on his private signal, while his consumption of commodity $i+1$ depends on the whole arrays of sunspot signals, i.e. on the state of the world. This amounts to say that consumption of the "same-label" commodity is decided as if it took place ex ante to the realization of the uncertainty while consumption of any other commodity takes place ex post, and therefore can be made contingent to it. Thus, this is not a one-shot economy, but it rather shares somehow the sequential character of the overlapping generations economy, while staying a finite horizon economy. This pattern of dependence of the consumption of different commodities on different information sets is imported here from Maskin and Tirole (1987), and is reminiscent of the interpretation of the "same-label" commodity as leisure which can be either consumed or used to produce a commodity to be traded with other consumers, very much as in the usual macroeconomics interpretation of the overlapping generations economy as an economy in which young agents produce, save their wage and consume it when old (see, for instance, Azariadis and Guesnerie (1986)).

[37] It may seem incongruous at first sight that the price $p_{is'}$ may depend on the entire array of sunspot signals while consumer i is not able to infer from its observation the state of the world realized. As a matter of fact, $p_{is'}$ will not depend at equilibrium on any other sunspot signal than s_i, as it has already remarked, in such a way that the absurdity is only apparent (see footnote 48 in the proof of the Proposition 5 in the Appendix).

As in propositions 1, 2 and 4, the first condition is the individual rationality condition requiring the orthogonality, in mathematical expectation, of the gradient of the utility and the excess demand, while the second is the feasibility condition.

Notice that an immediate consequence of the feasibility condition is that, at equilibrium, $c^i_{i+1s'} = c^i_{i+1s''}$ whenever $s'_{i+1} = s''_{i+1}$. Thus, if we denote by $c^i_{i+1s_{i+1}}$ the common value of c^i_{i+1s} for all the states $s \in S$ with the same s_{i+1}, then an equilibrium allocation of the cyclical economy $(\{(u^t, e^t)\}_{t\in\mathbb{Z}}, t_0, n)$ consists of $\{(c^i_{s_i}, (c^i_{i+1s_{i+1}})^k_{s_{i+1}=1})^k_{s_i=1}\}^n_{i=1}$ such that, for all $i = 1, \dots, n$ and all $s_i = 1, \dots, k$,

$$\sum_{s_{i+1}=1}^{k} \frac{\sum_{\substack{s'|s'_i=s_i \\ s'_{i+1}=s_{i+1}}} \pi^{s'}}{\sum_{s''|s''_i=s_i} \pi^{s''}} \big(D_1 u^{t_0+i}(c^i_{is_i}, c^i_{i+1s_{i+1}})(c^i_{is_i} - e^{t_0+i}_1) + D_2 u^{t_0+i}(c^i_{is_i}, c^i_{i+1s_{i+1}})(c^i_{i+1s_{i+1}} - e^{t_0+i}_2)\big) = 0 \tag{36}$$

and for all $i = 1, \dots, n$ and all $s_{i+1} = 1, \dots, k$,

$$c^i_{i+1s_{i+1}} + c^{i+1}_{i+1s_{i+1}} = e^{t_0+i}_2 + e^{t_0+i+1}_1. \tag{37}$$

In the next example an equilibrium is produced for the simple case of an economy with two agents and signals taking two values each.

Example 2. Consider the symmetric 2-cyclical economy depicted in Figure 2 where, for the sake of readability, the agents have been relabeled A and $\tilde{A}$ instead of 1 and 2. An equilibrium allocation of this economy consists of commodity 1 (respectively commodity 2) consumptions for agent A (resp. $\tilde{A}$) contingent his signal, c^A_{11}, c^A_{12} (resp. $c^{\tilde{A}}_{21}, c^{\tilde{A}}_{22}$), and commodity 2 (resp. 1) for A (resp. $\tilde{A}$) but contingent to agent $\tilde{A}$'s (resp. A's) signal c^A_{21}, c^A_{22} (resp. $c^{\tilde{A}}_{11}, c^{\tilde{A}}_{12}$), such that (36) and (37) above are satisfied.

The feasibility condition (37) asks for the four points (c^A_{11}, c^A_{22}), (c^A_{11}, c^A_{21}), (c^A_{12}, c^A_{21}) and (c^A_{12}, c^A_{22}) to form a box as shown in Figure 2, although in general this box needs not be neither square nor laying on the diagonal. Nonetheless, the allocation has to be necessarily such that there exist probabilities π^s for which (36) holds. This condition leads to a non-homogeneous system of five linear equations in the four π^s's that, in order to avoid overdeterminacy, requires the allocation to satisfy

$$D^1_{11} D^2_{12} D^2_{21} D^1_{22} = D^2_{22} D^1_{21} D^1_{12} D^2_{11} \tag{38}$$

where $D^i_{s_i s_{i+1}}$, for $i = 1, 2$, and $s_i, s_{i+1} = 1, 2$, stands for the inner product of consumer i's gradient with his excess demand, recalling that consumer 1 is A and consumer 2 is $\tilde{A}$. In the case of a symmetrical allocation of a symmetric cyclical economy as the one depicted in Figure 2, this condition is satisfied, as the existence of the solution exhibited below in (40) proves.

As in the Example 1 in Section 4.1.3 on k-SSE, the left-hand side of (36) can be interpreted as 0 being in the convex hull of the inner products of gradients and excess demands, the weights making 0 a convex linear combination of the latter being the probabilities of s conditional to the private information. Thus, a way to have the right signs for those inner products is that, in Figure 2, agent A's offer curve separates the top and bottom corners of the allocation box, while agent $\tilde{A}$'s offer curve separates the left and right corners. Yet for the allocation in

Figure 2 to be at equilibrium, it remains to be checked that there exists indeed a probability distribution for the two private signals such that (36) is satisfied. This is particularly easy to see in this example because, as we have seen in the Example 1, this allocation determines a 2-SSE of the overlapping generations economy with agent A as representative agent. In other words, there exists a Markov matrix (m^{ij}) whose rows are the weights which put 0 as convex linear combination of the inner products of gradients and excess demands for agent A, and hence for agent $\tilde{A}$ as well, because of the symmetry. The trick is now to see that there exists a joint distribution (π^s) for the two signals, inducing conditional distributions of each signal given the other which coincide with the Markov matrix of the 2-SSE, i.e. that there is a solution in π^{11}, π^{12}, π^{21} and π^{22} to

$$\pi^{11} + \pi^{12} + \pi^{21} + \pi^{22} = 1,$$

$$(39) \qquad \frac{\pi^{12}}{\pi^{11} + \pi^{12}} = m^{12}, \quad \frac{\pi^{21}}{\pi^{11} + \pi^{22}} = m^{21},$$

$$\frac{\pi^{21}}{\pi^{11} + \pi^{21}} = m^{12}, \quad \frac{\pi^{12}}{\pi^{12} + \pi^{22}} = m^{21}.$$

In effect, this system has as its unique solution

$$(40) \qquad \pi^{11} = \frac{(1 - m^{12})m^{21}}{m^{21} + m^{12}}, \quad \pi^{12} = \frac{m^{12}m^{21}}{m^{21} + m^{12}} = \pi^{21}, \quad \pi^{22} = \frac{m^{12}(1 - m^{21})}{m^{21} + m^{12}}.$$

This solution is actually the one provided in Forgès and Peck (1995) when the connection that they establish between the sunspot equilibria of an overlapping generations economy and the correlated equilibria of a market game mimicking it, is particularized to the case of 2-SSE, our agents A and $\tilde{A}$ being their odd and even generations.

The procedure followed in the previous example, to link the equilibria of a symmetric cyclical economy with asymmetric information to 2-SSE of the associated stationary overlapping generations economy, seems to be straightforward enough to expect it to go through for general k-SSE as well, as it was indeed conjectured in Maskin and Tirole (1987). Nevertheless, as showed in Dávila (1999), this is typically not the case. The reason is that the class of k-SSE is not the right choice to try extend the connection, but rather that of 0-memory Markovian sunspot cycles characterized above, as Proposition 6 in the next section shows.

4.3 The connection with extrinsic uncertainty.

Recalling the conditions (28,29) characterizing the allocation of a 0-memory Markovian sunspot cycle of period n and order k of an n-cyclical overlapping generations economy $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$, their similarity to the conditions (36,37) characterizing the equilibrium allocation of the corresponding cyclical economy $(\{(u^t, e^t)\}_{t\in\mathbb{Z}}, t_0, n)$ with private sunspot signals, hints at the following proposition, which establishes the connection between the 0-memory Markovian sunspot cycles of a cyclical overlapping generations economy $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$ and the equilibria of its associated cyclical economies.

Proposition 6: *Any 0-memory Markovian sunspot cycle of period n of a cyclical overlapping generations economy of period $\bar{n}$ divisor of n, $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$, can be* *translated into an equilibrium of any of its cyclical economies* $(\{(u^t,e^t)\}_{t\in\mathbb{Z}},t_0,n')$ *with private sunspot signals and with* n' *multiple of* n*, and conversely.*[38]

PROOF: Eliminating $c^i_{2\sigma_2}$ from (28,29) and $c^i_{i+1s_{i+1}}$ from (36,37), and letting[39] $c^i_{1\sigma_1}=c^i_{is_i}$ whenever $\sigma_1=s_i$, the two sets of equations turn out to be the same one, but for the number of equations (kn equations for the sunspot cycle and kn' for the finite economy), if it happened to be the case that, for any $\sigma_1,\sigma_2=1,\dots,k$ and all $i=1,\dots,n'$,

$$\frac{\displaystyle\sum_{\substack{s'|s'_i=s_i\\ s'_{i+1}=s_{i+1}}}\pi^{s'}}{\displaystyle\sum_{s''|s''_i=s_i}\pi^{s''}}=m^{\sigma_1\sigma_2} \tag{41}$$

whenever $(\sigma_1,\sigma_2)=(s_i,s_{i+1})$. Leaving aside for a moment whether (41) can actually hold true and assuming it can, since n' is a multiple of n, a solution to (28,29) becomes a solution to (36,37) by replication and, the other way round, a solution to (36,37) is at least a solution to (28,29) for the case $n=n'$, while it may well be also a replication of a solution to (28,29) for some n divisor of n'.[40]

Now let us see that, whichever is the $k\times k$ Markov matrix (m^{ij}) driving the sunspot signal in the overlapping generations economy $\{(u^t,e^t)\}_{t\in\mathbb{Z}}$, there is a probability distribution (π^s) for $s\in\{1,\dots,k\}^{n'}$ in the cyclical economy $(\{(u^t,e^t)\}_{t\in\mathbb{Z}},t_0,n')$, such that (41) holds. In effect, the equations (41) together with the necessary condition

$$\sum_{s\in\{1,\dots,k\}^{n'}}\pi^s=1 \tag{42}$$

form a system of $n'(k^2-k)+1$ linear equations (for each agent $i=1,\dots,n'$, there is one condition for each off-diagonal entry of the Markov matrix and, moreover, all the probabilities have to add up to one), in $k^{n'}$ variables (the probabilities π^s, for all $s\in\{1,\dots,k\}^{n'}$). The number of degrees of freedom of the system is then $k^{n'}-n'(k^2-k)-1$, which is positive for any $n'>2$ and $k\geq 2$.[41] This implies that

---

[38]As a matter of fact, in the case $n'=2$ the statement actually holds true if $k=2$ (see Example 2 in Section 4.2), but not for $k\geq 3$ typically (see Dávila (1999)).

[39]Again, as in the certainty case, a more careful notation has been sacrificed for the sake of the readability of the argument and thus remarks similar to those in footnote 44 in the proof of the proposition 3 apply. Namely, in the case $i=n$, $c^n_{1s_1}=c^n_{n\sigma_n}$, with $\sigma_n=s_1$, does not refer to any constraint on the choice of the n-th consumer of the cyclical economy, but to the identification of his consumption of commodity n when observing σ_n $(=s_1)$ to the n-th consumption when young if s_1 $(=\sigma_n)$ is observed in the sunspot cycle of the overlapping generations economy. As in the certainty case too, if $n<n'$, then for any right-hand side i exceeding n, $c^i_{i\sigma_i}$ is to be identified to $c^{i \bmod k}_{1s_1}$ whenever $\sigma_i=s_1$.

[40]See footnote 45 in the proof of Proposition 3 in the Appendix.

[41]It can easily be checked that 1 is, for every integer n', both a root and a critical point of the polynomial in k defining the number of degrees of freedom $f_{n'}(k)=k^{n'}-n'(k^2-k)-1$. Moreover its curvature is non-negative at $k=1$ and strictly increasing at every positive k, for every $n'\geq 3$. Therefore $f_{n'}(k)$ is strictly convex at every $k\geq 1$, for every $n'\geq 3$ and, thus, $f_{n'}(k)$ has no root bigger than 1, which guarantees $f_{n'}(k)>0$ for all $k\geq 2$ and $n'>2$. As for the case $n'=2$, see the footnote 38.

the linear subspace of solutions to (41) has positive dimension and, therefore, in order to show that the system of equations has a solution, it suffices to show that this linear subspace meets the interior of the positive orthant, because in that case it necessarily meets the affine space (42) as well.

In effect, each subspace associated to an equation in (41) meets the strictly positive orthant, otherwise it would be possible to separate these two convex sets by a hyperplane. However, this hyperplane can only be the subspace itself. But then, necessarily, a normal vector of such hyperplane would have to be in the strictly positive orthant too. Nevertheless, it can be easily seen that the normal vector of each subspace in (41) has positive (of the form $1 - m^{\sigma_1\sigma_2}$) as well as negative (of the form $-m^{\sigma_1\sigma_2}$) coordinates. So the separation is not possible and thus the non-trivial intersection subspace determined by the system (41) does meet the positive orthant indeed. Hence it meets the affine space (42) and a solution to the system exists.[42] *Q.E.D.*

5. Concluding remarks

This paper establishes the existence of a connection between the stationary equilibria of a class of overlapping generations economies and the equilibria of naturally related finite economies. The connection is first established in a framework without uncertainty and then it is extended to the case in which there are sunspots.

This results clarify definitively the way in which the conjecture made in Maskin and Tirole (1987), about the existence of such a connection, has to be modified to make it hold. In effect, Maskin and Tirole (1987) conjectured the possibility of establishing an identification of any finite Markovian stationary sunspot equilibrium of a simple overlapping generations economy and a correlated equilibrium of the economy with two agents and two commodities that they study, when this economy is symmetric. Nevertheless, Dávila (1999) shows how this identification does not hold typically for any sunspot equilibrium whose sunspot signal takes more than two values. Thus the conjecture needs to be qualified in order to be true in general.

The Proposition 6 in Section 4 establishes that the right framework to make the connection hold in general is that of a cyclical finite economy. In effect, any finite Markovian stationary sunspot equilibrium of an overlapping generations economy with a representative agent can be identified with a correlated equilibrium of a cyclical finite economy associated to the overlapping generations economy, as long as the number of agents of the finite economy is at least three. With just two agents, as in Maskin and Tirole (1987), only the 2-state Markovian stationary sunspot equilibria of the overlapping generations can be captured.

As a matter of fact, with finite cyclical economies of more than two agents the identification has been established in the previous section for a larger class of sunspot equilibria, namely the so-called here sunspot cycles of a certain class (0-memory Markovian sunspot cycles of finite order), and a larger class of stationary overlapping generations economies as well (the cyclical overlapping generations economies). Incidentally, the introduction of these sunspot equilibria extends in an interesting way the kinds of behavior that sunspot-driven time series can exhibit.

[42] As a matter of fact, there is a continuum of them whenever the number of degrees of freedom of the system is bigger than 1, which is always the case but for $k = 2$ and $n' = 3$.

In particular, stylized recurrent but irregular fluctuations characteristic of some economic time series can be generated by these sunspot cycles.

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APPENDIX

Proof of Proposition 1: (i) If $\{(c_t^t, c_{t+1}^t)\}_{t\in\mathbb{Z}}$ and $\{p_t\}_{t\in\mathbb{Z}}$ is an equilibrium of the overlapping generations economy $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$, then for every $t \in \mathbb{Z}$, (c_t^t, c_{t+1}^t) is a solution to (2) i.e. there exists a positive multiplier λ^t such that

$$\begin{aligned} D_1u^t(c_t^t, c_{t+1}^t) - \lambda^t p_t &= 0 \\ D_2u^t(c_t^t, c_{t+1}^t) - \lambda^t p_{t+1} &= 0 \\ p_t(c_t^t - e_1^t) + p_{t+1}(c_{t+1}^t - e_2^t) &= 0. \end{aligned} \tag{43}$$

Multiplying the first equation by $(c_t^t - e_1^t)$, the second by $(c_{t+1}^t - e_2^t)$ and adding them up taking into account the budget constraint in the third equation, the condition (3) follows.

(ii) In order to produce prices $\{p_t\}_{t\in\mathbb{Z}}$ supporting $\{(c_t^t, c_{t+1}^t)\}_{t\in\mathbb{Z}}$ as an equilibrium allocation, let p_1 be any positive price and define for each $t \in \mathbb{Z}$, the price

$$p_t = -\frac{c_1^1 - e_1^1}{c_t^{t-1} - e_2^{t-1}} p_1 \tag{44}$$

and the multiplier[43]

$$\lambda^t = -D_2u^t(c_t^t, c_{t+1}^t)\frac{c_{t+1}^t - e_2^t}{c_1^1 - e_1^1}\frac{1}{p_1}. \tag{45}$$

Then, for all $t \in \mathbb{Z}$, the allocation of resources is feasible by assumption and, moreover, the first order conditions (43) are satisfied since, firstly, the budget constraint holds

$$\frac{p_t}{p_{t+1}} = \frac{-\frac{c_1^1 - e_1^1}{c_t^{t-1} - e_2^{t-1}} p_1}{-\frac{c_1^1 - e_1^1}{c_{t+1}^t - e_2^t} p_1} = \frac{c_{t+1}^t - e_2^t}{c_t^{t-1} - e_2^{t-1}} = -\frac{c_{t+1}^t - e_2^t}{c_t^t - e_1^t}; \tag{46}$$

[43] Notice that, since $Du^t(c^t) \in R^2_{++}$ guarantees that $(c_t^t - e_1^t)(c_{t+1}^t - e_2^t) \le 0$ because of (3), and since $(c_{t+1}^t - e_2^t) = -(c_{t+1}^{t+1} - e_1^{t+1})$, then $(c_t^t - e_1^t)(c_{t+1}^{t+1} - e_1^{t+1}) \ge 0$ and hence there is no two generations whose excess demands have opposite signs. Therefore, if $c_1^1 - e_1 < 0$ (> 0), then $c_t^t - e_1 < 0$ (> 0), i.e. $c_t^{t-1} - e_2 > 0$ (< 0), for all $t \in \mathbb{Z}$ and thus p_t and λ^t are always positive. If the allocation of resources is the autarky, as it should be if for some generation an excess demand is zero, then the first order conditions will be satisfied by any positive price p_1, $p_t = \prod_{i=1}^{t-1}(D_2u^i(e^i)/D_1u^i(e^i))^{t-1}p_1$ if $t > 1$ and $p_t = \prod_{i=t}^{0}(D_1u^i(e^i)/D_2u^i(e^i))^{t-1}p_1$ if $t < 1$, and $\lambda^t = D_2u^t(e^t)\prod_{i=1}^{t}(D_1u^i(e^i)/D_2u^i(e^i))^t \cdot 1/p_1$ if $t > 0$, $\lambda^t = D_2u^t(e^t) \cdot 1/p_t$ if $t = 0$, and $\lambda^t = D_2u^t(e^t)\prod_{i=t+1}^{0}(D_2u^i(e^i)/D_1u^i(e^i))^t \cdot 1/p_1$ if $t < 0$, for each $t \in \mathbb{Z}$.

secondly, the partial derivative of the lagrangian with respect to c^t_{t+1} is zero because of the very definition of λ^t; and, finally, the partial derivative with respect to c^t_t is also zero since

$$
\begin{aligned}
&D_1u^t(c^t_t, c^t_{t+1}) - \lambda^t p_t = \\
&D_1u^t(c^t_t, c^t_{t+1}) + D_2u^t(c^t_t, c^t_{t+1})\frac{c^t_{t+1}-e^t_2}{c^1_1-e^1_1}\frac{1}{p_1}\cdot - \frac{c^1_1-e^1_1}{c^{t-1}_t - e^{t-1}_2}p_1 = \\
&\frac{1}{c^t_t - e^t_1}\left(D_1u^t(c^t_t, c^t_{t+1})(c^t_t - e^t_1) + D_2u^t(c^t_t, c^t_{t+1})(c^t_{t+1} - e^t_2)\right) = 0
\end{aligned} \tag{47}
$$

by mere substitutions, recalling (4) and (3). *Q.E.D.*

Proof of Proposition 3: In effect, eliminating c^i_2 in (8,9) and c^i_{i+1} in (13,14) and letting[44] $c^i_1 = c^i_i$, the two sets of equations turn out to be the same but for the number of equations: n in (8) while n' in (13). Since n' is a multiple of n, a solution to (8) becomes a solution to (13) by replication and, the other way round, a solution to (13) is at least a solution to (8) for the case $n = n'$, while it may well be also a replication of a solution to (8) for some n divisor of n'.[45] *Q.E.D.*

Proof of Proposition 4: (i) If $\{(c^t_{ts_t}, (c^t_{t+1s'_{t+1}})_{s'_{t+1}|s'_t=s_t})\}_{t\in\mathbb{Z}, s\in S}$ and prices $\{p_{ts_t}\}_{t\in\mathbb{Z},s\in S}$ constitute an equilibrium, then, for any $s \in S$ and any $t \in \mathbb{Z}$, $(c^t_{ts_t}, (c^t_{t+1s'_{t+1}})_{s'_{t+1}|s'_t=s_t})$ is the solution to (15), i.e. there exist positive multipliers $\lambda^t_{s'_{t+1}}$ for all s'_{t+1} such that $s'_t = s_t$, for which

$$
\begin{aligned}
&\sum_{s'_{t+1}|s'_t=s_t} P(s'_{t+1}|s'_t = s_t)D_1u^t(c^t_{ts_t}, c^t_{t+1s'_{t+1}}) - \sum_{s'_{t+1}|s'_t=s_t} \lambda^t_{s'_{t+1}} p_{ts_t} = 0 \\
&P(s'_{t+1}|s'_t = s_t)D_2u^t(c^t_{ts_t}, c^t_{t+1s'_{t+1}}) - \lambda^t_{s'_{t+1}} p_{t+1s'_{t+1}} = 0 \ , \ s'_{t+1}|s'_t = s_t \\
&p_{ts_t}(c^t_{ts_t} - e^t_1) + p_{t+1s'_{t+1}}(c^t_{t+1s'_{t+1}} - e^t_2) = 0 \ , \ s'_{t+1}|s'_t = s_t .
\end{aligned} \tag{48}
$$

Multiplying the first equation by $(c^t_{ts_t} - e^t_1)$, each of the equations in the second line by the corresponding $(c^t_{t+1s'_{t+1}} - e^t_2)$ and adding all them up taking into account the budget constraints in the third line, then the condition (16) follows. (ii) In order to produce prices supporting $\{(c^t_{ts_t}, (c^t_{t+1s'_{t+1}})_{s'_{t+1}|s'_t=s_t})\}_{t\in\mathbb{Z},s\in S}$ as an equilibrium allocation, let p_{1s_1}, for every[46] $s \in S$, be any positive price such that, for any other

[44] Two minor points about notation are in order here. Firstly, it would have been preferable to use different letters to denote consumption in each of the economies, should we have wanted to avoid now the somewhat confusing condition $c^n_1 = c^n_n$ which appears when $i = n$. Such a condition does not refer by any means to a constraint on the consumption bundle of the n-th consumer of the cyclical economy, but to the identification of his consumption of commodity n (the right-hand side) to the n-th consumption when young in the cycle of the overlapping generations economy (the left-hand side). A more rigorous notation has been discarded for the sake of the readability of the paper. Secondly, another source of confusion may originate in the fact that the left-hand side index i runs from 1 through n, while the right-hand side index i runs from 1 through n', multiple of n. Whenever $n < n'$ it goes without saying that, for any right-hand side i exceeding n, c^i_i is to be identified to $c^{i \bmod n}_1$.

[45] If it is invariant to the n-th power of the $n' \times n'$ matrix ρ whose typical entry ρ_{ij} equals 1 whenever $j = i + 1$ (recall that $n' + 1$ stands for 1) and is 0 otherwise.

[46] Actually, just one p_{1s_1} needs to be fixed arbitrarily, all the other prices at date 1 and at every state of the world s being then determined by the normalization (49).

$s' \in S$,

$$\frac{p_{1s_1}}{p_{1s'_1}} = \frac{c^1_{1s'_1} - e^1_1}{c^1_{1s_1} - e^1_1}. \tag{49}$$

Then define, for all $s \in S$ and all $t \in \mathbb{Z}$, the prices

$$p_{ts_t} = -\frac{c^1_{1s_1} - e^1_1}{c^{t-1}_{ts_t} - e^{t-1}_2} p_{1s_1}, \tag{50}$$

and for all $s' \in S$ such that $s'_t = s_t$, the multipliers

$$\lambda^t_{s'_{t+1}} = -P(s'_{t+1}|s'_t = s_t) D_2 u^t(c^t_{ts_t}, c^t_{t+1s'_{t+1}}) \frac{c^t_{t+1s'_{t+1}} - e^t_2}{c^1_{1s_1} - e^1_1} \frac{1}{p_{1s_1}}. \tag{51}$$

Then, for all $t \in \mathbb{Z}$ and all $s \in S$, the feasibility constraint is satisfied by assumption and the first order conditions (48) are satisfied too: firstly, the budget constraints are satisfied since, for all s'_{t+1} such that $s'_t = s_t$,

$$\frac{p_{ts_t}}{p_{t+1s'_{t+1}}} = \frac{-\frac{c^1_{1s_1} - e^1_1}{c^{t-1}_{ts_t} - e^{t-1}_2} p_{1s_1}}{-\frac{c^1_{1s'_1} - e^1_1}{c^t_{t+1s'_{t+1}} - e^t_2} p_{1s'_1}} = \frac{c^t_{t+1s'_{t+1}} - e^t_2}{c^{t-1}_{ts_t} - e^{t-1}_2} = -\frac{c^t_{t+1s'_{t+1}} - e^t_2}{c^t_{ts_t} - e^t_1}, \tag{52}$$

where the second equality results from the normalization (49) adopted; secondly, the partial derivatives of the lagrangian with respect to $c^t_{t+1s'_{t+1}}$ are satisfied by the very definition of the multipliers $\lambda^t_{s'_{t+1}}$, and finally, the partial derivative with respect to $c^t_{ts_t}$ is again satisfied by mere substitutions, recalling (17) and (16). *Q.E.D.*

Proof of Proposition 5: (i) If $\{(c^i_{is_i}, (c^i_{i+1s'})_{s'|s'_i=s_i})^k_{s_i=1}\}^n_{i=1}$ and $\{(p_{is})^n_{i=1}\}_{s\in S}$ constitute an equilibrium of the cyclical economy $(\{(u^t, e^t)\}_{t\in\mathbb{Z}}, t_0, n)$ then, for all $i = 1\ldots, n$ and all $s_i = 1, \ldots, k$, $(c^i_{is_i}, (c^i_{i+1s'})_{s'|s'_i=s_i})$ is the solution to the problem (33) above, i.e. there exist positive multipliers $\lambda^{is_i}_{s'}$, one for each s' such that $s'_i = s_i$, such that

$$\begin{aligned} \sum_{s'|s'_i=s_i} \pi^{s'} D_1 u^{t_0+i}(c^i_{is_i}, c^i_{i+1s'}) - \sum_{s'|s'_i=s_i} \lambda^{is_i}_{s'} p_{is'} = 0 \\ \pi^{s'} D_2 u^{t_0+i}(c^i_{is_i}, c^i_{i+1s'}) - \lambda^{is_i}_{s'} p_{i+1s'} = 0 \ , \ \forall s'|s'_i = s_i \\ p_{is'}(c^i_{is_i} - e^{t_0+i}_1) + p_{i+1s'}(c^i_{i+1s'} - e^{t_0+i}_2) = 0 \ , \ \forall s'|s'_i = s_i. \end{aligned} \tag{53}$$

Multiplying the first equation by $(c^i_{is_i} - e^{t_0+i}_1)$, each equation in the second line by the corresponding $(c^i_{i+1s'} - e^{t_0+i}_2)$ and adding all them up taking into account the budget constraints in the third line, the condition (34) follows. (ii) In order to produce a set of prices p_{is}, one for each commodity $i = 1, \ldots, n$ in each state of the world $s \in \{1, , \ldots, k\}^n$, supporting the allocation $\{(c^i_{is_i}, (c^i_{i+1s'})_{s'|s'_i=s_i})^k_{s_i=1}\}^n_{i=1}$ as an equilibrium, let p_{1s}, for each state of the world $s \in \{1, \ldots, k\}^n$, be any positive price in such a way that, for all[47] $s, s' \in S$,

$$\frac{p_{1s}}{p_{1s'}} = \frac{c^1_{1s'_1} - e_1^{t_0+1}}{c^1_{1s_1} - e_1^{t_0+1}}. \tag{54}$$

Then define,[48] for every $i = 1, \ldots, n$ and $s \in S$,

$$p_{is} = -\frac{c^1_{1s_1} - e_1^{t_0+1}}{c^{i-1}_{is} - e_2^{t_0+i-1}} p_{1s} \tag{55}$$

and let moreover, for all $i = 1, \ldots, n$, all $s_i = 1, \ldots, k$ and all s',

$$\lambda^{is_i}_{s'} = -\pi^{s'} D_2 u^{t_0+i}(c^i_{is_i}, c^i_{i+1s'}) \frac{c^i_{i+1s'} - e_2^{t_0+i}}{c^1_{1s'_1} - e_1^{t_0+1}} \frac{1}{p_{1s'}}. \tag{56}$$

Then the allocation of resources satisfies the feasibility condition by assumption, as well as the first order conditions: the budget constraints are satisfied since, for any given $s \in S$ and any s' such that $s'_i = s_i$,

$$\frac{p_{is'}}{p_{i+1s'}} = \frac{-\frac{c^1_{1s'_1} - e_1^{t_0+1}}{c^{i-1}_{is'} - e_2^{t_0+i-1}} p_{1s'}}{-\frac{c^1_{1s'_1} - e_1^{t_0+1}}{c^i_{i+1s'} - e_2^{t_0+i}} p_{1s'}} = \frac{c^i_{i+1s'} - e_2^{t_0+i}}{c^{i-1}_{is'} - e_2^{t_0+i-1}} = -\frac{c^i_{i+1s'} - e_2^{t_0+i}}{c^i_{is_i} - e_1^{t_0+i}} \tag{57}$$

where the last equality follows from the feasibility condition; the partial derivatives of the lagrangian with respect to $c^i_{i+1s'}$, for all $s' \in S$ such that $s'_i = s_i$, are all zero by the very definition of the multipliers $\lambda^{is_i}_{s'}$, and the partial derivative with respect to $c^i_{is_i}$ are zeroed too since

$$\begin{aligned} &\sum_{s'|s'_i=s_i} \pi^{s'} D_1 u^{t_0+i}(c^i_{is_i}, c^i_{i+1s'}) - \sum_{s'|s'_i=s_i} \lambda^{is_i}_{s'} p_{is'} = \\ &\sum_{s'|s'_i=s_i} \pi^{s'} \big(D_1 u^{t_0+i}(c^i_{is_i}, c^i_{i+1s'}) + D_2 u^{t_0+i}(c^i_{is_i}, c^i_{i+1s'}) \cdot \\ &\qquad \frac{c^i_{i+1s'} - e_2^{t_0+i}}{c^1_{1s'_1} - e_1^{t_0+1}} \frac{1}{p_{1s'}} \cdot \Big(-\frac{c^1_{1s'_1} - e_1^{t_0+1}}{c^{i-1}_{is'} - e_2^{t_0+i-1}}\Big) p_{1s'}\big) = \\ &\frac{1}{c^i_{is_i} - e_1^{t_0+i}} \sum_{s'|s'_i=s_i} \pi^{s'} \big(D_1 u^{t_0+i}(c^i_{is_i}, c^i_{i+1s'})(c^i_{is_i} - e_1^{t_0+i}) + \\ &\qquad D_2 u^{t_0+i}(c^i_{is_i}, c^i_{i+1s'})(c^i_{i+1s'} - e_2^{t_0+i})\big) = 0 \end{aligned} \tag{58}$$

where the last two equalities follow from the feasibility condition (35) and the condition (34) respectively. *Q.E.D.*

[47] Actually just one p_{1s} needs to be fixed arbitrarily, all the other prices of commodity 1 at each state of the world being determined by the normalization (54).

[48] Notice that since, firstly, from the feasibility condition (35), c^{i-1}_{is} cannot depend at equilibrium on any other sunspot signal that s_i and, secondly, $(c^1_{1s_1} - e_1^{t_0+1})p_{1s}$ does not depend on s according to the normalization (54), then p_{is} cannot depend on any other sunspot signal than s_i and, thus, equilibrium prices do not convey any information on the signal observed by anybody else, as we had already anticipated in footnote 37.

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