

# Proof of the Stack of Records Theorem

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August 11, 2009

## Abstract

The following is a proof of the Stack of Records theorem (as stated in class) which makes use of the Inverse Function Theorem (which was proved in exhaustive detail in class).

**Proof.** Take the regular value  $y'$ . By assumption,  $f^{-1}(y') \neq \emptyset$ . For each  $x_i \in f^{-1}(y')$ , then  $x_i$  is a regular point of  $f$ . Thus  $D_x f(x_i)$  has full rank  $n$  for all  $i \in \mathcal{I}$ . Applying the inverse function theorem, there exists open sets  $U_i \subseteq A$  and  $V_i \subseteq \mathbb{R}^n$  s.t.  $x_i \in U_i$ ,  $y \in V_i$ , and  $f$  is a bijection from  $U_i$  to  $V_i$  (and this holds  $\forall i \in \mathcal{I}$ ). Therefore,  $f^{-1}(y')$  is covered by

$$f^{-1}(y') \subseteq_{i \in \mathcal{I}} U_i, \text{ an open cover}$$

where  $U_i$  contains at least one element  $x_i \in f^{-1}(y')$ .

**Claim 1**  $U_i$  contains at most one element  $x_i \in f^{-1}(y')$ .

**Proof.** Suppose that  $U_i$  contains two elements (proof by contradiction). Call these elements  $x_i, x_j \in f^{-1}(y')$ . Then  $x_i \in U_i$  open and  $x_j \in U_j$  open. Wlog we can consider the sets s.t.  $V_i = V_j = V$ . Since  $f : U_i \rightarrow V$  is a bijection, then  $f^{-1}(V) = U_i$ . Likewise, since  $f : U_j \rightarrow V$  is a bijection, then  $f^{-1}(V) = U_j$ . Thus  $U_i = U_j$ . Letting the radius of the open sets  $U_i = U_j$  approach 0, then  $x_i = x_j$ . ■

$f^{-1}(y')$  is a closed subset of  $A$  since  $\{y'\}$  is closed (a singleton) and  $f$  is continuous. Since  $A$  is compact and  $f^{-1}(y')$  is a closed subset of  $A$ , then  $f^{-1}(y')$  is compact. Thus, for every open cover, there exists a finite subcover. Therefore for the open cover considered above,

$$f^{-1}(y') \subseteq_{i=1, \dots, n} U_i \text{ for } n < \infty.$$

Since  $U_i$  contains one and only one element  $x_i \in f^{-1}(y')$ , then  $f^{-1}(y') \subseteq \{x_1, \dots, x_n\}$  implying that  $\#f^{-1}(y') = n$ .

Finally, define  $V = \cap_{i=1}^n V_i$ , an open set since  $V_i$  are open sets. Then define  $\phi_i : V \rightarrow U_i$ . This is a  $C^1$  bijection  $\forall i = 1, \dots, n$ . Thus  $\#f^{-1}(y) = n \forall y \in V$ . ■