Quiz Week II
Suggested Solutions

1. **Duality** (10 points)

   Find the dual problem of the following maximization problem:

$$\begin{array}{ccc} \max & & x^T\beta \\ x \geq 0 & s.t. & x^T A \leq \gamma^T \end{array}.$$

   Solution: The dual problem of the above maximization problem is given by the minimization problem:

$$\begin{array}{ccc} \min & & y^T\gamma \\ y \geq 0 & s.t. & Ay \geq \beta \end{array}.$$

2. **Mean Value Inequality**

   (a) (15 points) Use the following lemma:
   "Suppose f is a continuous mapping of $[a,b]$ into $\mathbb{R}^k$ and f is differentiable in (a,b). Then $\exists x \in (a,b)$ s.t. $|f(b) - f(a)| \leq (b-a)\,|f'(x)|$."
   to prove the Mean Value Inequality (which is stated as
   "Suppose f maps a convex open set $E \subseteq \mathbb{R}^n$ into $\mathbb{R}^m$, f is differentiable in E, and $\exists M \in \mathbb{R}$ s.t. $\|f'(x)\| \leq M \quad \forall x \in E$. Then $|f(b) - f(a)| \leq M\,|b-a| \quad \forall a,b \in E$.").
   Solution: As in the class notes, fix $a,b \in E$. Define $\gamma(t) = (1-t)a + tb$. Since E is convex, then $\gamma(t) \in E \quad \forall t \in [0,1]$. Define $g(t) = f(\gamma(t))$. Then

$$g'(t) = f'(\gamma(t))\,\gamma'(t) = f'(\gamma(t))\,(b-a),$$

   so this implies

$$\left|g'(t)\right| \leq \left\|f'(\gamma(t))\right\| |b-a| \leq M\,|b-a| \quad \forall t \in [0,1].$$

   Applying the lemma, $|g(1) - g(0)| \leq (1-0)\,|g'(t)| \leq M\,|b-a|$ for some $t \in [0,1]$ and $g(1) = f(b)$ and $g(0) = f(a)$.

   (b) (10 points) Using the statement of the Mean Value Inequality as above, additionally suppose that g maps a convex open set $\hat{E} \subseteq f(E)$ into $\mathbb{R}^k$, g is differentiable in $\hat{E}$, and $\exists \hat{M} \in \mathbb{R}$ s.t. $\|g'(y)\| \leq \hat{M} \quad \forall y \in \hat{E}$. Define the mapping $h = g \circ f : E \to \mathbb{R}^k$. State and prove the Mean Value Inequality for the mapping h (we do NOT assume that $\exists M^* \in \mathbb{R}$ s.t. $\|h'(x)\| \leq M^* \quad \forall x \in E$).
   Solution: As it is a composition, the mapping h is differentiable in E. Since $\|f'(x)\| \leq M \quad \forall x \in E$ and $\|g'(y)\| \leq \hat{M} \quad \forall y \in \hat{E}$, then it must be that $\|h'(x)\| = \|g'(f(x))f'(x)\| \leq \|g'(y)\|\,\|f'(x)\| \leq \hat{M} \cdot M \quad \forall x \in E$. Thus, the statement of the Mean Value Inequality for the mapping h reads that $|h(b) - h(a)| \leq \left(M \cdot \hat{M}\right)|b-a|$ $\forall a,b \in E$. The Mean Value Inequality holds equally well for the mapping g as it does for the mapping f. Thus

$$|g(d) - g(c)| \leq \hat{M}\,|d-c| \;\; \forall c,d \in \hat{E}.$$

By definition, $|h(b) - h(a)| = |g(f(b)) - g(f(a))|$. Thus, using the inequalities $|g(d) - g(c)| \leq \hat{M}(d-c)$ and $|f(b) - f(a)| \leq M(b-a)$ implies

$$|g(f(b)) - g(f(a))| \leq \hat{M}\,|f(b) - f(a)| \leq \left(M \cdot \hat{M}\right)|b-a|$$

and this holds $\forall a, b \in E$.

3. **Implicit Function Theorem**

   (a) (5 points) Please state the Implicit Function Theorem.

   (b) (20 points) Let $f : \mathbb{R}^m \to \mathbb{R}$ be a C^2 mapping that is strictly concave. Let's use the Implicit Function Theorem to prove that Df is injective. Let $x, y \in \mathbb{R}^m$. Then define the C^1 mapping $F : \mathbb{R}^{2m} \to \mathbb{R}^m$ such that

$$F(x,y) = Df(x) - Df(y) = \left[D_{x_1}f(x) - D_{y_1}f(y), ..., D_{x_m}f(x) - D_{y_m}f(y)\right].$$

Obviously the mapping $F(x,y) = 0$ when $x = y$. Suppose that you take any $(x + \Delta x, x + \Delta y) \in N_\epsilon(x,x)$ for some $\epsilon > 0$. Then, is it true that given a marginal change in x (denoted by Δx), the marginal change in y (denoted by Δy) such that $Df(x + \Delta x) = Df(x + \Delta y)$ (or such that $F(x + \Delta x, x + \Delta y) = 0$) must be the same (that is, is $\Delta x = \Delta y$ always)?

Hint: $\Delta x = \Delta y$ iff for some appropriately defined implicit function $g : \mathbb{R}^m \to \mathbb{R}^m$ with $x = g(y)$, the derivative matrix $g'(y) = I_m$.

Solution: Obviously, $F(x,y) = 0$ for some $x = y$. I want to show that $D_x F(x,y)$ has full rank. The matrix

$$D_x F(x,y) = D^2 f(x) = \begin{bmatrix} D^2_{11}f(x) & ... & D^2_{m1}f(x) \\ ... & ... & ... \\ D^2_{1m}f(x) & ... & D^2_{mm}f(x) \end{bmatrix}$$

has full rank since f is strictly concave (by definition, f strictly concave implies that $D^2 f(x)$ is negative definite so that setting $\Delta x^T D^2 f(x) = 0$ implies $\Delta x^T D^2 f(x) \Delta x = 0$ implies $\Delta x^T = 0$). Therefore, applying the implicit function theorem, I can find a C^1 mapping $g : \mathbb{R}^m \to \mathbb{R}^m$ with $x = g(y)$ and the derivative condition is given by:

$$g'(y) = -\left[D_x F(x,y)\right]^{-1} D_y F(x,y).$$

However, since $D_y F(x,y) = -D_x F(x,y)$ by the setup, then $g'(y) = I_m$. The only way that $F(x,y) = 0$ (the only way that $Df(x) = Df(y)$) is if $x = y$. Thus, we have proven that Df is injective.

4. **Inverse Functions**

   (a) (5 points) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible C^2 mapping. Use the identity $f^{-1} \circ f(x) = x$ and the chain rule to find $Df^{-1}(y)$ in terms of $Df(x)$ (where $y = f(x)$).

Solution: From the chain rule applied to the equality $f^{-1} \circ f(x) = x$:

$$\begin{aligned} Df^{-1}(y) \cdot Df(x) &= I_n \\ Df^{-1}(y) &= [Df(x)]^{-1}. \end{aligned}$$

(b) (10 points) Now, let $n = 1$ and find $D^2 f^{-1}(y)$ in terms of $Df(x)$ and $D^2 f(x)$.
Solution: Taking $Df^{-1}(f(x)) \cdot Df(x) = 1$ as fact, apply the chain rule again:

$$D^2 f^{-1}(f(x))Df(x) \cdot Df(x) + Df^{-1}(f(x))D^2 f(x) = 0.$$

After rearranging terms:

$$\begin{aligned} D^2 f^{-1}(y)\,(Df(x))^2 &= -D^2 f(x)\,(Df(x))^{-1} \\ D^2 f^{-1}(y) &= -\frac{D^2 f(x)}{(Df(x))^3}. \end{aligned}$$

(c) (10 points) Now, let $n > 1$. Define the new mapping $G : \mathbb{R}^n \to \mathbb{R}$ by $G(y) = (1, ..., 1) f^{-1}(y)$. Then find $D^2 G(y)$ in terms of $D^2 f(x)$, $DG(y)$, and $Df(x)$.
Hint: The matrix multiplication $Df^{-1}(y) \cdot Df(x)$ can be written as $\sum_k D_k f^{-1}(y) \cdot Df_k(x)$ where $f_k : \mathbb{R}^n \to \mathbb{R}$ is a real-valued function. Then $D^2 f(x)$ can be viewed as the vector of matrices $\left\{D^2 f_1(x), ..., D^2 f_n(x)\right\}$.
Solution: Taking $Df^{-1}(f(x)) \cdot Df(x) = I_n$ as fact, then premultiplying by $(1, ..., 1)$ yields:

$$(1, ..., 1) Df^{-1}(f(x)) \cdot Df(x) = (1, ..., 1).$$

Writing in terms of a summation yields:

$$\sum_j Df_j^{-1}(f(x)) \cdot Df(x) = (1, ..., 1).$$

Taking a second derivative yields (via the chain rule) and substituting y for $f(x)$ yields:

$$\sum_j D^2 f_j^{-1}(y) Df(x) Df(x) + \sum_j \sum_k D_k f_j^{-1}(y) D^2 f_k(x) = 0 \in M(n). \tag{1}$$

By definition, $\left(\sum_j D^2 f_j^{-1}(y)\right) = D^2 G(y)$. Also, $\sum_j D_k f_j^{-1}(y) = D_k G(y)$, a scalar. Then equation (1) can be rewritten as:

$$D^2 G(y)\,(Df(x))^2 + \sum_k D_k G(y) D^2 f_k(x) = 0. \tag{2}$$

Solving (2) for $D^2 G(y)$ yields:

$$D^2 G(y) = -\left(\sum_k D_k G(y) D^2 f_k(x)\right) (Df(x))^{-2}.$$ [1]

5. **Change of Variables** (15 points)

   State the change of variables equation (in $\mathbb{R}^k$ for $k > 1$). Write down the integral of $\iiint_R f(x, y, z)dxdydz$ in spherical coordinates where the transformation from spherical coordinates to Cartesian coordinates is given by

$$(x, y, z) = T(r, \theta, \gamma) = (r \sin\gamma \cos\theta, r \sin\gamma \sin\theta, r \cos\gamma)$$

   where $0 \leq \gamma \leq \pi$.

[1] Though this can be equivalently written as the tensor product $D^2 G(y) = -DG(y) D^2 f(x)\,(Df(x))^{-2}$, without having introduced tensors and without a clear idea about how to go about multiplying them, the listed solution above is clearly preferable.

Solution: As in the Exercises, with the transformation defined as

$$(x, y, z) = T(r, \theta, \gamma) = (r \sin\gamma \cos\theta, r \sin\gamma \sin\theta, r \cos\gamma),$$

then I have to find the determinant matrix:

$$DT(r, \theta, \gamma) = \begin{bmatrix} \sin\gamma\cos\theta & -r\sin\gamma\sin\theta & r\cos\gamma\cos\theta \\ \sin\gamma\sin\theta & r\sin\gamma\cos\theta & r\cos\gamma\sin\theta \\ \cos\gamma & 0 & -r\sin\gamma \end{bmatrix}.$$

The determinant of the matrix $DT(r, \theta, \gamma)$ is calculated as:

$$\begin{aligned} & \sin\gamma\cos\theta\left(-r^2\sin^2\gamma\cos\theta\right) + r\sin\gamma\sin\theta\left(-r\sin^2\gamma\sin\theta - r\cos^2\gamma\sin\theta\right) + \\ & r\cos\gamma\cos\theta\left(-r\sin\gamma\cos\gamma\cos\theta\right) \\ = \; & -r^2\left(\sin^2\gamma\sin\gamma\cos^2\theta\right) - r^2\sin\gamma\sin\theta\left(\sin\theta\right) - r^2\left(\sin\gamma\cos^2\theta\cos^2\gamma\right) \end{aligned}$$

after using the identity that $\sin^2\gamma + \cos^2\gamma = 1$. Simplifying further yields:

$$\begin{aligned} & -r^2\sin\gamma\left(\cos^2\theta\sin^2\gamma + \sin^2\theta + \cos^2\theta\cos^2\gamma\right) \\ = \; & -r^2\sin\gamma\left(\cos^2\theta\left[\sin^2\gamma + \cos^2\gamma\right] + \sin^2\theta\right) \\ = \; & -r^2\sin\gamma\left(\cos^2\theta + \sin^2\theta\right) \\ = \; & -r^2\sin\gamma \end{aligned}$$

after using the identity as above twice. Thus, the absolute value of the derivative is given by $r^2\sin\gamma$ where $\sin\gamma \geq 0$ for $0 \leq \gamma \leq \pi$. Plugging into our change of variables formula from class, then the result

$$\iiint_R f(x, y, z)\,dxdydz = \iiint_S f(r\sin\gamma\cos\theta, r\sin\gamma\sin\theta, r\cos\gamma)\left(r^2\sin\gamma\right)drd\theta d\gamma$$

is obtained.