ECON897 Math Camp Part II. **Matthew Hoelle**
Summer 2009 **University of Pennsylvania**

Quiz Week I.
Suggested Solutions

1. **Convexity**

(a) (5 points) Let $A \subseteq \mathbb{R}^n$ be a convex set. Show that A is connected.
Solution: A is convex if $\forall x, y \in A$, then $\lambda x + (1-\lambda)y \in A \ \ \forall \lambda \in [0,1]$. Thus, A is path-connected. Use the proof from class that path-connected implies connected.

(b) (10 points) Let $A \subseteq \mathbb{R}^n$ be a convex set and $f : A \to \mathbb{R}$ be a continuous function. Show that $f(A)$ is convex (if you use the result that A connected implies $f(A)$ connected for f C^0, then you will need to prove this result).
Solution: A is connected, thus $f(A)$ is connected (proof to follow). In $\mathbb{R}$, connected and convex sets are just intervals.
Suppose that $f(A)$ is not connected. Then

$$f(A) = Y \cup Z$$

with $Y \neq \emptyset$, $Z \neq \emptyset$, and both $Y \cap \overline{Z} = \emptyset$ and $\overline{Y} \cap Z = \emptyset$. Then

$$\begin{aligned} A &= \left(A \cap f^{-1}(Y)\right) \cup \left(X \cap f^{-1}(Z)\right) \\ &= B \cup C \end{aligned}$$

for $B \neq \emptyset$ and $C \neq \emptyset$. Now, $B \subseteq f^{-1}(\overline{Y})$ and $f^{-1}(\overline{Y})$ is closed since f is C^0. Thus

$$\overline{B} \subseteq f^{-1}(\overline{Y}) \Longrightarrow f\left(\overline{B}\right) \subseteq \overline{Y}.$$

Therefore, $f\left(\overline{B} \cap C\right) \subseteq f\left(\overline{B}\right) \cap f(C) \subseteq \overline{Y} \cap Z = \emptyset$. It is thus obtained $\overline{B} \cap C = \emptyset$. Analogously, $B \cap \overline{C} = \emptyset$. Thus, A is not connected.

(c) (5 points) Prove the following result:
If $f : A \to A$ is continuous, $A \subseteq \mathbb{R}^1$ and A is both compact and convex, then $\exists a \in A$ s.t. $f(a) = a$ (if you cite the intermediate value theorem, you must prove this result).
Solution: Let $g : A \to \mathbb{R}$ be any continuous function defined on a convex domain A. Then $g(A)$ is a closed and bounded interval, thus $\forall \alpha \in \mathbb{R}$ s.t. $g(x) \leq \alpha \leq g(y)$ for some $x, y \in A$, the element α is in the image of g, $\alpha \in g(A)$. Define the mapping as $g(x) = f(x) - x$. Then $g(x) \leq 0 \leq g(y)$ for some $x, y \in A$. So, $\exists a \in A$ s.t. $f(a) = a$.

(d) (5 points) Define the sphere $S^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$. The sphere is a subset of $\mathbb{R}^2$ and is both compact and connected. If $f : S^1 \to S^1$ is continuous, is it necessarily the case that f has a fixed point (that is, an element $s \in S^1$ such that $f(s) = s$)?
Solution: f may not have a fixed point. Consider the function $f(x) = -x \ \ \forall x \in S^1$. We require that the set be convex in order for the fixed point theorem to hold in higher dimensions. S^1, though connected, is certainly not convex.

2. **Compact**

   (a) (15 points) Consider a continuous function $f : A \to B$ with $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$. Let the subset $C \subseteq A$ be bounded and additionally assume that $\overline{C} \subseteq A$. Prove that $f(C)$ is bounded (if you use the result that A compact implies $f(A)$ compact for f C^0, then you will need to prove this result).

   Solution: $\overline{C}$ is closed by definition. If C is bounded, then trivially $\overline{C}$ is bounded (elements $c \in C' - C$ differ only by $\epsilon > 0$ from any points $c' \in C$). Thus, $\overline{C}$ is compact and since $\overline{C} \subseteq A$, the function $f : A \to B$ is well-defined over the subset $\overline{C}$. Thus $f(\overline{C})$ is compact (proof to follow). Therefore, $f(C)$ is bounded (since $C \subseteq \overline{C}$, then $f(C) \subseteq f(\overline{C})$ and any subset of a bounded set is bounded [basically by definition]).

   Let $\mathcal{O}$ be an arbitrary open cover of $f(\overline{C})$. Since f is continuous, then $\forall$open set $V_i \in \mathcal{O}$, $f^{-1}(V_i)$ is open in $\overline{C}$. Thus $\overline{C} \subseteq \cup_i f^{-1}(V_i)$. Since $\overline{C}$ is compact, then $\exists$ finite subcover $\overline{C} \subseteq \cup_{i=1}^n f^{-1}(V_i)$. Thus, $f(\overline{C}) \subseteq \cup_{i=1}^n V_i$, so $f(\overline{C})$ is compact.

   (b) (10 points) Let $f : A \to B$ be a homeomorphism (a continuous bijection with a continuous inverse). If $f(X_1) \cap ... \cap f(X_k)$ is compact, show that $X_1 \cap ... \cap X_k$ is compact.

   Solution: $f(X_1) \cap ... \cap f(X_k) = f(X_1 \cap ... \cap X_k)$ since f is injective. The direction $f(X_1) \cap ... \cap f(X_k) \supseteq f(X_1 \cap ... \cap X_k)$ is trivial. To show that $f(X_1) \cap ... \cap f(X_k) \subseteq f(X_1 \cap ... \cap X_k)$, take any $x \in f(X_1) \cap ... \cap f(X_k)$. Then $\exists y_1 \in X_1, ..., y_k \in X_k$ s.t. $x = f(y_1) = ... = f(y_k)$. Since f is injective, then $y_1 = ... = y_k = y$. Thus $y \in X_1 \cap ... \cap X_k$ and so $x \in f(X_1 \cap ... \cap X_k)$.

   As f^{-1} is continuous, then using the theorem above, $f^{-1} f(X_1 \cap ... \cap X_k) = X_1 \cap ... \cap X_k$ is compact.

3. **Projections**

   Let $A \in M(m, n)$.

   (a) (5 points) Show that $A^T A = 0 \implies A = 0$.

   Solution: Define the matrix $B = A^T A$. Then $b_{i,j} = \sum_{k=1}^n a(k,i)a(k,j)$. Thus, if $A \neq 0$ (say $a(l,m) \neq 0$), then $b_{lm} \neq 0$ for $l = m$.

   (b) (5 points) If A has full column rank, show that $A^T A$ is invertible.

   Solution: $A^T A \in M(n)$ so it suffices to show that $A^T A$ has full column rank. Set $A^T A \Delta x = 0$. Then, obviously $\Delta x^T A^T A \Delta x = 0$. From (a), then $A \Delta x = 0$. Since A has full column rank, then $\Delta x = 0$.

   (c) (5 points) Define the projection matrix as

   $$P = A \left(A^T A\right)^{-1} A^T.$$

   Show that if A is invertible ($m = n$), then $P = I_n$. Show that $P^2 = P$.

   Solution: $\left(A^T A\right)^{-1} = A^{-1} \left(A^T\right)^{-1}$. Thus, $P = AA^{-1} \left(A^T\right)^{-1} A^T = I_n$.

   $P^2 = PP = A \left(A^T A\right)^{-1} A^T A \left(A^T A\right)^{-1} A^T = A \left(A^T A\right)^{-1} A^T = P.$

   (d) (5 points) Suppose that P is invertible. Calculate $\det(P)$.

   Solution: If P is invertible, then $\det(P) \neq 0$. $\det(P) = \det(P^2) = \det(P)\det(P)$ implies that $\det(P) = 1$.

(e) (5 points) Define the span of the matrix A as

$$< A >= \{x \in \mathbb{R}^m : \exists y \in \mathbb{R}^n \text{ s.t. } x = Ay\}.$$

Show that $Pz \in < A > \ \forall z \in \mathbb{R}^m$.

Solution: Trivially, $\exists y = \left(A^T A\right)^{-1} A^T z$ s.t. $Pz = Ay$.

4. **Rank**

(a) (5 points) Let $A \in M(m,n)$ and $D \in M(o,p)$ have full row rank. Show that the matrix $\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$ has full row rank.

Solution: Set $\left(\Delta x^T, \Delta y^T\right)\begin{bmatrix} A & B \\ 0 & D \end{bmatrix} = 0$. $\Delta x^T A = 0$ implies $\Delta x^T = 0$. Then $\Delta y^T D = 0$ implies $\Delta y^T = 0$.

(b) (10 points) Let $A \in M(n)$ and $D \in M(n)$ have full row rank. Let $I_n - AD$ have full row rank. Show that $\begin{bmatrix} A & I_n \\ I_n & D \end{bmatrix}$ has full row rank.

Solution: Set $\left(\Delta x^T, \Delta y^T\right)\begin{bmatrix} A & I_n \\ I_n & D \end{bmatrix} = 0$. $\Delta x^T A + \Delta y^T = 0$ and $\Delta x^T + \Delta y^T D = 0$. Postmultiply the first equality by $D:$ $\Delta x^T AD + \Delta y^T D = 0$. Then, $\Delta x^T = \Delta x^T AD$. Therefore, $\Delta x^T\left(I_n - AD\right) = 0$ and since $I_n - AD$ has full row rank, then $\Delta x^T = 0$. From the second equality, $\Delta y^T = 0$.

(c) (10 points) Find the det $\begin{bmatrix} A & I_n \\ I_n & D \end{bmatrix}$ in terms of $\det(A)$, $\det(D)$, and $\det(I_n - (AD)^{-1})$. Use this to show that $I_n - (AD)^{-1}$ has full row rank.

Solution: Rewrite $A^* = \begin{bmatrix} A & I_n \\ I_n & D \end{bmatrix} = \begin{bmatrix} A & I_n \\ 0 & D \end{bmatrix}\begin{bmatrix} I_n - A^{-1}D^{-1} & 0 \\ D^{-1} & I_n \end{bmatrix}$. Then $\det(A^*) = \det(A)\cdot\det(D)\cdot\det(I_n - (AD)^{-1})$. Since $\det(A^*) \neq 0$, then $\det(I_n - (AD)^{-1}) \neq 0$. Thus, $I_n - (AD)^{-1}$ has full row rank.

Consider the matrix $A^* = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$. This matrix is in the form $A^* = \begin{bmatrix} A & I_n \\ I_n & D \end{bmatrix}$ with $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Thus $\det(A) = 4$, $\det(D) = 1$, $AD = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$, $I_n - AD = \begin{bmatrix} -1 & -3 \\ 0 & -1 \end{bmatrix}$, $\det\left(I_n - AD\right) = 1$, $(AD)^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{3}{4} \\ 0 & \frac{1}{2} \end{bmatrix}$, $I_n - (AD)^{-1} = \begin{bmatrix} -\frac{1}{2} & -\frac{3}{4} \\ 0 & -\frac{1}{2} \end{bmatrix}$, and finally $\det\left(I_n - (AD)^{-1}\right) = \frac{1}{4}$.

Thus, $\det(A)\cdot\det(D)\cdot\det(I_n - (AD)^{-1}) = 4 \cdot 1 \cdot \frac{1}{4} = 1$. Using Excel, it is easy to check that $\det(A^*) = 1$.

Note importantly that it is NOT TRUE that $\det(A^*) = \det(A)\det(D) - \det(I_n)\det(I_n)$ as this would only give $\det(A^*) = 3$.