Exercises Week II.

1. **Friday: Duality**

   Let $u : \mathbb{R}^G \to R$ be a strictly increasing and differentiable utility function. Let $I > 0$ be the fixed income of the household. Find the dual problem of

$$\begin{aligned} \max \quad & u(x) \\ & p \cdot x \leq I \\ & x \geq 0 \end{aligned} \tag{P}$$

   and use the duality theorem to prove that the values of both problems are identical.

2. **Tuesday: Inverse Function Theorem**

   Suppose $T : V \to W$ is a linear map between finite-dimensional vector spaces. Use the Rank-Nullity Law to prove the following corollaries:

   (a) T is a bijection $\Longrightarrow \dim V = \dim W$.

   (b) Suppose $\dim V = \dim W$. Then T is surjective iff T is injective.

3. **Wednesday: Implicit Function Theorem**

   (a) Firm Production

   Let the output price p and the factor prices (r, w) be given by $(p, r, w) = (1, 2, 1)$. Suppose a firm exists that must use these parameters and pick an optimal capital and labor inputs (K, L) to maximize the firm's problem:

$$\max_{K,L} \pi = p \cdot F(K, L) - rK - wL.$$

   The firm has access to the production function $F(K, L) = \ln(K) + \ln(L)$, which is strictly concave.

   i. Calculate the optimal levels of capital and labor inputs (K^*, L^*) for the firm.

   ii. How do the optimal levels of capital and labor change as the parameters (p, r, w) change. In particular, I am asking for the derivative of $K^*(p, r, w)$ and $L^*(p, r, w)$ as a function of (p, r, w).

   (b) No Arbitrage Asset Pricing

   Let the payout matrix Y be given by:

$$Y = \begin{pmatrix} 1 & 4 & 4 \\ 1 & 1 & 5 \\ 1 & 6 & 2 \\ 1 & 3 & 1 \end{pmatrix}.$$

   Which of the following set of asset prices (if any) are No Arbitrage asset prices?

i. $q = (1, 2, 2)$
ii. $q = (1, 4, 2)$
iii. $q = (1, 2, 4)$

4. **Thursday: Integration in $\mathbb{R}^n$**

(a) Integration by Parts

i. Evaluate $\int_0^1 xe^x dx$.
ii. Evaluate $\int_{\pi/2}^{\pi} x \sin x dx$.

(b) Change of Variables

Let R be a $3-$dimensional space defined in Cartesian coordinates and let S be the same $3-$dimensional space defined in spherical coordinates. Recall that the Cartesian coordinates (x, y, z) are defined using the spherical coordinates (r, θ, γ) according to the transformation

$$(x, y, z) = T(r, \theta, \gamma) = (r \sin \gamma \cos \theta, r \sin \gamma \sin \theta, r \cos \gamma) .$$

Prove that

$$\iiint_R f(x, y, z) dx dy dz = \iiint_S f(r \sin \gamma \cos \theta, r \sin \gamma \sin \theta, r \cos \gamma) \left(r^2 \sin \gamma\right) dr d\theta d\gamma.$$

Recall that for spherical coordinates, we restrict $0 \leq \gamma \leq \pi$.