ECON897 Math Camp Part II. **Matthew Hoelle**
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Exercises Week I.

1. **Tuesday: Analysis in $\mathbb{R}^n$**

   A topological property is defined as a property that is preserved under homeomorphism (a homeomorphism is simply a continuous, invertible function with a continuous inverse). Simple put, for any function $f : A \to B$ with $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$, then P is a topological property iff

   $$A \text{ has property } P \Longrightarrow f(A) \text{ has property } P.$$

   I have proven in class that both compactness and connectedness are topological properties.

   (a) Show (by example) that closedness, completeness, and boundedness are NOT topological properties.

   (b) Show that convexity is a topological property when $m = 1$, but not otherwise.

   (c) Rigorously prove that density and path-connectedness are topological properties.

2. **Wednesday: Linear Algebra Half 1**

   (a) Recall from lecture that the following condition must be met for the matrix

   $$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

   to be invertible:

   $$a(ei - fh) \neq bid - chd - bfg + ceg.$$

   Are the following $3x3$ matrices invertible? If so, please find the inverse.

   i. $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

   ii. $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

   (b) Consider any invertible, nonnegative matrix $A \in M(n)$. A matrix is nonnegative if all the elements of the matrix are nonnegative.

   i. For $n-$dimensional vectors $e_i = (0, .., 0, 1, 0, .., 0)$ and $\overrightarrow{1} = (1, .., 1)^T$, show that

   $$e_i \cdot A^{-1} \cdot \overrightarrow{1} \geq 0 \text{ for some } i.$$

   To prove this result, start with the initialization case of $n = 2$ and try to use an induction argument.

ii. Assume that $\forall i$:

$$a_{i,i} \geq a_{i+1,i} \geq ... \geq a_{n,i} \geq a_{1,i} \geq ... \geq a_{i-1,i}$$

where the sequence of inequalities contains at least one strict inequality. Prove that

$$A^{-1} \cdot \overrightarrow{1} >> 0$$

or written equivalently that

$$e_i \cdot A^{-1} \cdot \overrightarrow{1} > 0 \text{ for every } i.$$

To prove this result, start with the initialization case of $n = 2$ and try to use an induction argument.

(c) Using Gaussian elimination, find the inverse of the following $4x4$ matrices.

i. $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

ii. $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

iii. Is it true that matrices (as above) that are upper triangular are always invertible (yes or no)?

3. **Thursday: Linear Algebra Half 2**

(a) Find the determinants for the following matrices:

i. $A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 1 & 2 \\ 5 & 2 & 6 \end{bmatrix}$

ii. $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & 3 \\ 2 & 0 & 0 & 4 \end{bmatrix}$

(b) Use Cramer's rule to solve for the prices $p \in \mathbb{R}^3$ such that the budget constraints

$$\sum\nolimits_i p_i \cdot c_i^h = w^h$$

holds for all $h = 1, 2, 3$ where:

$$\begin{aligned} c^1 &= (1,1,0), \quad c^2 = (0,1,1), \quad c^3 = (1,0,1) \\ w^1 &= 2, \quad w^2 = 3, \quad w^4 = 4. \end{aligned}$$

(c) Find the derivative of the following matrix with respect to x :

$$A = \begin{bmatrix} e^x & \sin(x) \\ \ln(x) & 3x^2 + 2x + 1 \end{bmatrix}.$$

(d) Matrix addition is defined element-wise. That is, for

$$A = \begin{bmatrix} a_{11} & .. & a_{1n} \\ : & & : \\ a_{m1} & .. & a_{mn} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & .. & b_{1n} \\ : & & : \\ b_{m1} & .. & b_{mn} \end{bmatrix},$$

then $C = A + B$ has elements

$$C = \begin{bmatrix} a_{11} + b_{11} & .. & a_{1n} + b_{1n} \\ : & & : \\ a_{m1} + b_{m1} & .. & a_{mn} + b_{mn} \end{bmatrix} \quad .$$

Prove or verify the following result for matrices $A, B \in M(m, n)$:

$$\det\left(I_m + A \cdot B^T\right) = \det\left(I_n + B^T \cdot A\right).$$