Brouwer's Fixed Point Theorem

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Abstract

This note discusses the version of the Brouwer's Fixed Point Theorem as stated in economics and a generalization of it.

Theorem 1 *(Brouwer's Fixed Point Theorem)*
If $f : A \to A$ with $A \subseteq \mathbb{R}^n$ is continuous and A is nonempty, compact, and convex, then f has a fixed point, that is, $\exists a \in A$ s.t. $f(a) = a$.

Proof. You are not responsible for this proof, but it was seen in Kurt's session making use of the advanced topological result about continuous retractions. ■

I will now define what it means for 2 sets to be homeomorphic.

Definition 2 *The sets $A, B \subseteq \mathbb{R}^n$ are homeomorphic if $\exists$ homeomorphism $g : \mathbb{R}^n \to \mathbb{R}^n$ s.t. $g(A) = B$. Recall that a homeomorphis is a continuous bijection with continuous inverse.*

The most common example among mathematicians is that the coffee mug (with a handle) is homeomorphic to a donut with a hole in the middle (see the figure at http://en.wikipedia.org/wiki/Homeomorphism).

Theorem 3 *(The Garth Extension of Brouwer's Fixed Point Theorem)*
If $\gamma : B \to B$ is continuous and $B \subseteq \mathbb{R}^n$ is homeomorphic to the nonempty, compact, and convex set $A \subseteq \mathbb{R}^n$, then γ has a fixed point, that is, $\exists b \in B$ s.t. $\gamma(b) = b$.

Proof. This proof will make use of the definitions given and the Brouwer's Fixed Point Theorem. Since B and A are homeomorphic, then $\exists g : \mathbb{R}^n \to \mathbb{R}^n$ s.t. $g(A) = B$. Suppose that γ does NOT have a fixed point. Then, there exists some mapping $f : A \to A$ such that $\gamma = g \circ f \circ g^{-1} : B \to B$. Thus $\forall a \in A,\ \exists b \in B$ s.t. $a = g^{-1}(b)$. Since f has a fixed point (all continuous mappings $f : A \to A$ do), then there exists some $a \in A$ s.t. $f(a) = a$. Thus, $g \circ f(a) = g(a) = b$. This contradicts that γ does NOT have a fixed point since $b \in B$ s.t. $\gamma(b) = b$ exists by construction. ■