ECON897 Math Camp Part II. **Matthew Hoelle**
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Suggested Solutions for
Exercises Week II.

1. **Friday: Duality**

   Let $u : \mathbb{R}^G \to R$ be a strictly increasing and differentiable utility function. Let $I > 0$ be the fixed income of the household. Find the dual problem of

$$\begin{array}{cc} \max & u(x) \\ & p \cdot x \leq I \\ & x \geq 0 \end{array} \tag{P}$$

   and use the duality theorem to prove that the values of both problems are identical.

   Solution: First step:

   Show that if x^* is an optimal vector of (P), then x^* is also an optimal vector of (LP) :

$$\begin{array}{cc} \max & x^T \cdot [Du(x^*)]^T \\ & p \cdot x \leq I \\ & x \geq 0 \end{array} . \tag{LP}$$

   The convention is that derivatives, like $Du(x^*)$ are viewed as a row vector.

   Suppose that x^* is not an optimal solution to (LP) :

$$\begin{array}{rcl} Du(x^*)x & > & Du(x^*)x^* \\ px & \leq & I \\ x & \geq & 0 \end{array}$$

   but x^* is an optimal solution to (P). With $Du(x^*)(x - x^*) > 0$, define $h = x - x^*$. Then

$$\lim_{h \to 0} \frac{|u(x^* + h) - u(x^*) - Du(x^*)h|}{|h|} = 0$$

   implies $u(x^* + h) > u(x^*)$ or $u(x) > u(x^*)$. But this contradicts that x^* is an optimal solution to (P).

   Second step:

   The dual problem of (LP) is given by:

$$\begin{array}{cc} \min & zI \\ & zp \geq [Du(x^*)]^T \\ & z \geq 0 \end{array} . \tag{DLP}$$

   Details remain to be filled in.....

2. **Tuesday: Inverse Function Theorem**

   Suppose $T : V \to W$ is a linear map between finite-dimensional vector spaces. Use the Rank-Nullity Law to prove the following corollaries:

(a) T is a bijection $\implies \dim V = \dim W$.
Solution: T is injective $\implies \ker T = 0 \implies \dim\{\ker T\} = 0 \implies \dim V = \dim\{\operatorname{Im} T\}$.
T is surjective $\implies \operatorname{Im} T = W \implies \dim\{\operatorname{Im} T\} = \dim W \implies \dim V = \dim W$.

(b) Suppose $\dim V = \dim W$. Then T is surjective iff T is injective.
Solution: T is surjective $\iff \operatorname{Im} T = W \iff \dim\{\operatorname{Im} T\} = \dim W = \dim V \iff \dim\{\ker T\} = 0 \iff \ker T = 0T \iff$ is injective.

3. **Wednesday: Implicit Function Theorem**

(a) Firm Production
Let the output price p and the factor prices (r, w) be given by $(p, r, w) = (1, 2, 1)$. Suppose a firm exists that must use these parameters and pick an optimal capital and labor inputs (K, L) to maximize the firm's problem:

$$\max_{K,L} \pi = p \cdot F(K, L) - rK - wL.$$

The firm has access to the production function $F(K, L) = \ln(K) + \ln(L)$, which is strictly concave.

i. Calculate the optimal levels of capital and labor inputs (K^*, L^*) for the firm.
Solution: The optimal solution is found as the solutions to the first order conditions:

$$\begin{aligned} p\frac{1}{K^*} - r &= 0 \\ p\frac{1}{L^*} - w &= 0 \end{aligned}$$

which implies that $K^* = \frac{p}{r} = \frac{1}{2}$ and $L^* = \frac{p}{w} = 1$.

ii. How do the optimal levels of capital and labor change as the parameters (p, r, w) change. In particular, I am asking for the derivative of $K^*(p, r, w)$ and $L^*(p, r, w)$ as a function of (p, r, w).
Solution: To find this derivative, we will use the implicit function theorem. Using the matrix from class:

$$\begin{bmatrix} F_K(K, L) & -1 & 0 & pF^2_{KK} & pF^2_{KL} \\ F_L(K, L) & 0 & -1 & pF^2_{LK} & pF^2_{LL} \end{bmatrix} =$$

$$\begin{bmatrix} \frac{1}{K^*} & -1 & 0 & -p\frac{1}{(K^*)^2} & 0 \\ \frac{1}{L^*} & 0 & -1 & 0 & -p\frac{1}{(L^*)^2} \end{bmatrix}.$$

Thus, $(DK^*(p, r, w), DL^*(p, r, w)) = \begin{bmatrix} -p\frac{1}{(K^*)^2} & 0 \\ 0 & -p\frac{1}{(L^*)^2} \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{K^*} & -1 & 0 \\ \frac{1}{L^*} & 0 & -1 \end{bmatrix} =$

$$\begin{bmatrix} -\frac{(K^*)^2}{p} & 0 \\ 0 & -\frac{(L^*)^2}{p} \end{bmatrix} \begin{bmatrix} \frac{1}{K^*} & -1 & 0 \\ \frac{1}{L^*} & 0 & -1 \end{bmatrix}.$$

Plugging in values yields:

$$(DK^*(p, r, w), DL^*(p, r, w)) = \begin{bmatrix} -\frac{1}{4} & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{4} & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

(b) No Arbitrage Asset Pricing

Let the payout matrix Y be given by:

$$Y = \begin{pmatrix} 1 & 4 & 4 \\ 1 & 1 & 5 \\ 1 & 6 & 2 \\ 1 & 3 & 1 \end{pmatrix}.$$

Which of the following set of asset prices (if any) are No Arbitrage asset prices?

i. $q = (1, 2, 2)$
ii. $q = (1, 4, 2)$
iii. $q = (1, 2, 4)$

Solution: As in class, the asset prices are No Arbitrage asset prices iff $\exists \alpha \in \mathbb{R}^5_{++}$ s.t. $\alpha^T \begin{pmatrix} -q \\ Y \end{pmatrix} = 0$. The equation can be rewritten as:

$$(\alpha_2, \alpha_3, \alpha_4) \begin{bmatrix} 1 & 4 & 4 \\ 1 & 1 & 5 \\ 1 & 6 & 2 \end{bmatrix} + (\alpha_1, \alpha_5) \begin{pmatrix} & -q & \\ 1 & 3 & 1 \end{pmatrix} = 0.$$

Solving for $(\alpha_2, \alpha_3, \alpha_4)$ using the fact that the inverse of the matrix $\begin{bmatrix} 1 & 4 & 4 \\ 1 & 1 & 5 \\ 1 & 6 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -7 & 4 & 4 \\ 0.75 & -0.5 & -0.25 \\ 1.25 & -0.5 & -0.75 \end{bmatrix}$ yields:

$$(\alpha_2, \alpha_3, \alpha_4) = -(\alpha_1, \alpha_5) \begin{pmatrix} & -q & \\ 1 & 3 & 1 \end{pmatrix} \begin{bmatrix} -7 & 4 & 4 \\ 0.75 & -0.5 & -0.25 \\ 1.25 & -0.5 & -0.75 \end{bmatrix}.$$

- Plugging in the values of q for (i) yields:

$$(\alpha_2, \alpha_3, \alpha_4) = (-3\alpha_1 + 3.5\alpha_5, 2\alpha_1 - 2\alpha_5, 2\alpha_1 - 2.5\alpha_5).$$

  From these equations, if both $\alpha_2 > 0$ and $\alpha_4 > 0$, then $\alpha_5 > \frac{6}{7}\alpha_1$ and $\alpha_1 > \frac{5}{4}\alpha_5$. Taken together, this implies the contradiction $\alpha_1 > \frac{5}{4}\frac{6}{7}\alpha_1$. Thus the prices $q = (1, 2, 2)$ are not No Arbitrage asset prices.
- Plugging in the values of q for (ii) yields:

$$(\alpha_2, \alpha_3, \alpha_4) = (-1.5\alpha_1 + 3.5\alpha_5, \alpha_1 - 2\alpha_5, 1.5\alpha_1 - 2.5\alpha_5).$$

  Values such as $\alpha_1 = 2$ and $\alpha_5 = 0.9$ are such that $\alpha \in \mathbb{R}^5_{++}$. Thus, the prices $q = (1, 4, 2)$ are No Arbitrage asset prices.
- Plugging in the values of q for (iii) yields:

$$(\alpha_2, \alpha_3, \alpha_4) = (-0.5\alpha_1 + 3.5\alpha_5, \alpha_1 - 2\alpha_5, 0.5\alpha_1 - 2.5\alpha_5).$$

  Values such as $\alpha_1 = 2$ and $\alpha_5 = \frac{1}{3}$ are such that $\alpha \in \mathbb{R}^5_{++}$. Thus, the prices $q = (1, 2, 4)$ are No Arbitrage asset prices.

4. **Thursday: Integration in $\mathbb{R}^n$**

(a) Integration by Parts

i. Evaluate $\int_0^1 xe^x dx$.

Solution: Using integration by parts, set $g(x) = e^x$ and $F(x) = x$. Then $G(x) = e^x$ and $f(x) = 1$. Therefore

$$\begin{aligned}\int_0^1 xe^x dx &= [xe^x]_0^1 - \int_0^1 e^x dx \\ &= e^x - (e^x - 1) = 1.\end{aligned}$$

ii. Evaluate $\int_{\pi/2}^{\pi} x \sin x dx$.

Solution: Using integration by parts, set $g(x) = \sin(x)$ and $F(x) = x$. Then $G(x) = -\cos(x)$ and $f(x) = 1$. Therefore

$$\begin{aligned}\int_{\pi/2}^{\pi} x \sin x dx &= [-x\cos(x)]_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi} -\cos x \\ &= \pi - (\sin(\pi) - \sin(\pi/2)) = \pi + 1.\end{aligned}$$

(b) Change of Variables

Let R be a 3−dimensional space defined in Cartesian coordinates and let S be the same 3−dimensional space defined in spherical coordinates. Recall that the Cartesian coordinates (x, y, z) are defined using the spherical coordinates (r, θ, γ) according to the transformation

$$(x, y, z) = T(r, \theta, \gamma) = (r \sin\gamma \cos\theta, r \sin\gamma \sin\theta, r \cos\gamma).$$

Prove that

$$\iiint_R f(x, y, z) dxdydz = \iiint_S f(r \sin\gamma \cos\theta, r \sin\gamma \sin\theta, r \cos\gamma) \left(r^2 \sin\gamma\right) drd\theta d\gamma.$$

Recall that for spherical coordinates, we restrict $0 \leq \gamma \leq \pi$.

Solution: With the transformation defined as

$$(x, y, z) = T(r, \theta, \gamma) = (r \sin\gamma \cos\theta, r \sin\gamma \sin\theta, r \cos\gamma),$$

then I have to find the determinant matrix:

$$DT(r, \theta, \gamma) = \begin{bmatrix} \sin\gamma\cos\theta & -r\sin\gamma\sin\theta & r\cos\gamma\cos\theta \\ \sin\gamma\sin\theta & r\sin\gamma\cos\theta & r\cos\gamma\sin\theta \\ \cos\gamma & 0 & -r\sin\gamma \end{bmatrix}.$$

The determinant of the matrix $DT(r, \theta, \gamma)$ is calculated as:

$$\begin{aligned} & \sin\gamma\cos\theta\left(-r^2\sin^2\gamma\cos\theta\right) + r\sin\gamma\sin\theta\left(-r\sin^2\gamma\sin\theta - r\cos^2\gamma\sin\theta\right) + \\ & r\cos\gamma\cos\theta\left(-r\sin\gamma\cos\gamma\cos\theta\right) \\ = \; & -r^2\left(\sin^2\gamma\sin\gamma\cos^2\theta\right) - r^2\sin\gamma\sin\theta\left(\sin\theta\right) - r^2\left(\sin\gamma\cos^2\theta\cos^2\gamma\right)\end{aligned}$$

after using the identity that $\sin^2\gamma + \cos^2\gamma = 1$. Simplifying further yields:

$$\begin{aligned} & -r^2\sin\gamma\left(\cos^2\theta\sin^2\gamma + \sin^2\theta + \cos^2\theta\cos^2\gamma\right) \\ = \; & -r^2\sin\gamma\left(\cos^2\theta\left[\sin^2\gamma + \cos^2\gamma\right] + \sin^2\theta\right) \\ = \; & -r^2\sin\gamma\left(\cos^2\theta + \sin^2\theta\right) \\ = \; & -r^2\sin\gamma\end{aligned}$$

after using the identity as above twice. Thus, the absolute value of the derivative is given by $r^2 \sin\gamma$ where $\sin\gamma \geq 0$ for $0 \leq \gamma \leq \pi$. Plugging into our change of variables formula from class, then the result

$$\iiint_R f(x,y,z)dxdydz = \iiint_S f(r\sin\gamma\cos\theta, r\sin\gamma\sin\theta, r\cos\gamma)\left(r^2\sin\gamma\right) drd\theta d\gamma$$

is obtained.