ECON897 Math Camp Part II. **Matthew Hoelle**
Summer 2009 **University of Pennsylvania**

Suggested Solutions for
Exercises Week I.

1. **Tuesday: Analysis in $\mathbb{R}^n$**

A topological property is defined as a property that is preserved under homeomorphism (a homeomorphism is simply a continuous, invertible function with a continuous inverse). Simple put, for any function $f : A \to B$ with $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$, then P is a topological property iff

$$A \text{ has property } P \Longrightarrow f(A) \text{ has property } P.$$

I have proven in class that both compactness and connectedness are topological properties.

(a) Show (by example) that closedness, completeness, and boundedness are NOT topological properties.

Simplest counter-examples:

Let $f : (0, 1] \to \mathbb{R}$ be defined s.t. $x \mapsto \frac{1}{x}$. Then f is continuous, the domain $(0, 1]$ is bounded, but the image $[1, \infty)$ is NOT bounded.

Let $f : [1, \infty) \to \mathbb{R}$ be defined s.t. $x \mapsto \frac{1}{x}$. Then f is continuous, the domain $[1, \infty)$ is both closed and complete (recall that complete implies closed), but the image $(0, 1]$ is NOT closed nor complete.

(b) Show that convexity is a topological property when $m = 1$, but not otherwise.

Claim: If $A \subseteq \mathbb{R}^n$ is convex, then A is connected.

Proof: Suppose A is not connected, then

$$A = X \cup Y$$

for $X \neq \emptyset, Y \neq \emptyset$, and both $X \cap \overline{Y} = \emptyset$ and $\overline{X} \cap Y = \emptyset$ (that is, A is the union of two separated sets). Thus, $\exists x \in X$ and $y \in Y$ s.t. for some $\lambda \in [0, 1]$

$$\lambda x + (1 - \lambda) y \notin X \cup Y.$$

Thus, A is not convex.$\blacksquare$

For $m = 1$, then saying $B \subseteq \mathbb{R}^1$ is convex is equivalent to saying B is an interval is equivalent to saying that B is connected. Thus, if A is convex (hence connected), if $f : A \to B$ is continuous, and $m = 1$, then $f(A)$ is connected (hence convex). Thus convexity is a topological property when $m = 1$.

Now, let $m > 1$. Let f be the continuous mapping with domain a disc (in $\mathbb{R}^2$) and image a 5-pointed star. Obviously, the disc is both convex and connected, but the 5-pointed star is only connected (not convex).

(c) Rigorously prove that density and path-connectedness are topological properties.

Claim: If $C \subset A$ is a dense subset of A and $f : A \to B$ is continuous, then $f(C)$ is a dense subset of $f(A)$.

Proof: I will first prove a lemma before proving the claim.

Lemma: If $f : A \to B$ is continuous, then $\forall$ subset $C \subset A$:

$$f(C' - C) \subseteq f(C)'.$$

Proof of lemma: Take $y \in f(C' - C)$. Then $\exists x \in C' - C$ s.t. $y = f(x)$. Since $x \notin C$, then $y \notin f(C)$. Suppose (for contradiction) that $y \notin f(C)'$. Then $\exists \epsilon > 0$ s.t. $N^*_\epsilon(y) \cap f(C) = \emptyset$ and since $y \notin f(C)$, then $N_\epsilon(y) \cap f(C) = \emptyset$. For this $\epsilon > 0$, by the definition of continuity, $\exists \delta > 0$ s.t. $z \in N_\delta(x) \cap C \implies f(z) \in N_\epsilon(y)$. Since there does not exist any $f(z) \in N_\epsilon(y) \cap f(C)$, then $\forall z \in C$, $|z - x| \geq \delta$. This implies $\forall z \in C$, $\exists \delta > 0$ s.t. $z \notin N^*_\delta(x)$ or that $\exists \delta > 0$ s.t. $N^*_\delta(x) \cap C = \emptyset$. Thus $x \notin C'$, a contradiction proving our result.

Returning to the proof of the claim, recall that C is dense in A iff $\forall x \in A$, $x \in C$ or $x \in C'$. Take anz $y \in f(A)$. Then $\exists x \in A$ s.t. $y = f(x)$. Then, it must be that $x \in C$ or $x \in C'$. If $x \in C$, then $y \in f(C)$. If $x \in C' - C$, then $y \in f(C' - C)$. From the lemma, $y \in f(C)'$. Thus, $\forall y \in f(A)$, either $y \in f(C)$ or $y \in f(C)'$. Thus, $f(C)$ is dense in the image $f(A)$.■

Claim: If A is path-connected and $f : A \to B$ is continuous, then $f(A)$ is path-connected.

Proof: If A is path connected, then $\forall x_1, x_2 \in A$, there exists a continuous $\gamma : [0, 1] \to A$ s.t. $\gamma(0) = x_1$ and $\gamma(1) = x_2$. Now, take any two points $y_1, y_2 \in f(A)$, the image. Thus, $\exists x_1, x_2 \in A$ s.t. $y_1 = f(x_1)$ and $y_2 = f(x_2)$. By construction, there exists a continuous path $\gamma^* : [0, 1] \to f(A)$ defined as $\gamma^* = f \circ \gamma$ s.t. $\gamma^*(0) = f(x_1) = y_1$ and $\gamma(1) = f(x_2) = y_2$.■

2. **Wednesday: Linear Algebra Half 1**

   (a) Recall from lecture that the following condition must be met for the matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

to be invertible:

$$a(ei - fh) \neq bid - chd - bfg + ceg.$$

Are the following 3x3 matrices invertible? If so, please find the inverse.

i. $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Solution: This matrix is invertible. The inverse is given by $A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

ii. $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

Solution: This matrix is not invertible. Notice that

$$\begin{aligned} a(ei - fh) &= 1 \\ bid - chd - bfg + ceg &= 1. \end{aligned}$$

(b) Consider any invertible, nonnegative matrix $A \in M(n)$. A matrix is nonnegative if all the elements of the matrix are nonnegative.

i. For $n-$dimensional vectors $e_i = (0, .., 0, 1, 0, .., 0)$ and $\overrightarrow{1} = (1, .., 1)^T$, show that

$$e_i \cdot A^{-1} \cdot \overrightarrow{1} \geq 0 \text{ for some } i.$$

To prove this result, start with the initialization case of $n = 2$ and try to use an induction argument.

Solution: To start the induction argument, let $n = 2$. With $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $a, b, c, d \geq 0$, then $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ and so

$$A^{-1} \cdot \overrightarrow{1} = \begin{pmatrix} \frac{d-b}{ad-bc} \\ \frac{a-c}{ad-bc} \end{pmatrix}.$$

Suppose that both $\frac{d-b}{ad-bc} < 0$ and $\frac{a-c}{ad-bc} < 0$. If $ad - bc > 0$, this implies that both $0 \leq d < b$ and $0 \leq a < c$. These inequalities contradict $ad - bc > 0$. Symmetrically, if $ad - bc < 0$, this implies that both $0 \leq b < d$ and $0 \leq c < a$. But this contradicts that $ad - bc < 0$.

Let us extend the argument by induction. The induction hypothesis is that $e_i \cdot A^{-1} \cdot \overrightarrow{1} \geq 0$ for some i when $A \in M(n)$. Now, consider a matrix $A^* \in M(n+1)$ that is both invertible and nonnegative and for which

$$A^* = \begin{bmatrix} a & \overrightarrow{b} \\ \overrightarrow{c} & A \end{bmatrix}.$$

If $\overrightarrow{c} = 0$, then $A^{*-1} = \begin{bmatrix} \cdots & \cdots \\ 0 & A^{-1} \end{bmatrix}$ and if $e_i \cdot A^{-1} \cdot \overrightarrow{1} \geq 0$ for some i, then $e_{i+1} \cdot A^{*-1} \cdot \overrightarrow{1} \geq 0$.

If $\overrightarrow{c} \neq 0, ...$(can verify with examples on Excel).

ii. Assume that $\forall i$:

$$a_{i,i} \geq a_{i+1,i} \geq ... \geq a_{n,i} \geq a_{1,i} \geq ... \geq a_{i-1,i}$$

where the sequence of inequalities contains at least one strict inequality. Prove that

$$A^{-1} \cdot \overrightarrow{1} >> 0$$

or written equivalently that

$$e_i \cdot A^{-1} \cdot \overrightarrow{1} > 0 \text{ for every } i.$$

To prove this result, start with the initialization case of $n = 2$ and try to use an induction argument.

Solution: To start the induction argument, let $n = 2$. As above

$$A^{-1} \cdot \overrightarrow{1} = \begin{pmatrix} \frac{d-b}{ad-bc} \\ \frac{a-c}{ad-bc} \end{pmatrix}.$$

With $d > b$ and $a > c$ by the assumption above, then obviously $A^{-1} \cdot \overrightarrow{1} >> 0$.

Let us extend the argument by induction. The induction hypothesis is that $A^{-1} \cdot \overrightarrow{1} >> 0$ when $A \in M(n)$. Now, consider a matrix $A^* \in M(n+1)$ that is both invertible and nonnegative and for which

$$A^* = \begin{bmatrix} a & \overrightarrow{b} \\ \overrightarrow{c} & A \end{bmatrix}.$$

If $\overrightarrow{c} = 0$ and $\overrightarrow{b} = 0$, then $A^{*-1} = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & A^{-1} \end{bmatrix}$ and with $a \neq 0$, $A^{*-1} \cdot \overrightarrow{1} >> 0$

If $\overrightarrow{c} \neq 0$ or $\overrightarrow{b} \neq 0$, ...(can verify with examples on Excel).

(c) Using Gaussian elimination, find the inverse of the following $4x4$ matrices.

i. $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Solution: The inverse of this matrix is given by $A^{-1} = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

ii. $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

Solution: The inverse of this matrix is given by $A^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & -\frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$.

iii. Is it true that matrices (as above) that are upper triangular are always invertible (yes or no)?
Solution: Upper triangular matrices are always invertible if all the diagonal elements are nonzero.

3. **Thursday: Linear Algebra Half 2**

(a) Find the derivatives for the following matrices:

i. $A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 1 & 2 \\ 5 & 2 & 6 \end{bmatrix}$
Solution: $\det(A) = -2$.

ii. $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & 3 \\ 2 & 0 & 0 & 4 \end{bmatrix}$
Solution: $\det(A) = -4$.

(b) Use Cramer's rule to solve for the prices $p \in \mathbb{R}^3$ such that the budget constraints

$$\sum_i p_i \cdot c_i^h = w^h$$

holds for all $h = 1, 2, 3$ where:

$$\begin{aligned} c^1 &= (1,1,0), \quad c^2 = (0,1,1), \quad c^3 = (1,0,1) \\ w^1 &= 2, \quad w^2 = 3, \quad w^4 = 4. \end{aligned}$$

Solution: $p = (\frac{3}{2}, \frac{1}{2}, \frac{5}{2})$.

(c) Find the derivative of the following matrix with respect to x :

$$A = \begin{bmatrix} e^x & \sin(x) \\ \ln(x) & 3x^2 + 2x + 1 \end{bmatrix}.$$

Solution: $dA = \begin{bmatrix} e^x & \cos(x) \\ \frac{1}{x} & 6x + 2 \end{bmatrix}$.

(d) Matrix addition is defined element-wise. That is, for

$$A = \begin{bmatrix} a_{11} & .. & a_{1n} \\ : & & : \\ a_{m1} & .. & a_{mn} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & .. & b_{1n} \\ : & & : \\ b_{m1} & .. & b_{mn} \end{bmatrix},$$

then $C = A + B$ has elements

$$C = \begin{bmatrix} a_{11} + b_{11} & .. & a_{1n} + b_{1n} \\ : & & : \\ a_{m1} + b_{m1} & .. & a_{mn} + b_{mn} \end{bmatrix} \quad .$$

Prove or verify the following result for matrices $A, B \in M(m, n)$:

$$\det\left(I_m + A \cdot B^T\right) = \det\left(I_n + B^T \cdot A\right).$$

Proof: The proof will be conducted with two claims.

Claim I: $\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A)\det(D - CA^{-1}B) = \det(D)\det(A - BD^{-1}C)$ for $A \in M(m)$ and $D \in M(n)$.

Notice first that

$$\begin{aligned} \begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \begin{pmatrix} A & 0 \\ C & I_n \end{pmatrix} \begin{pmatrix} I_m & A^{-1}B \\ 0 & D - CA^{-1}B \end{pmatrix} \\ &= \begin{pmatrix} I_m & B \\ 0 & D \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ D^{-1}C & I_n \end{pmatrix}. \end{aligned}$$

The determinant of block diagonal matrices $\begin{pmatrix} A & 0 \\ C & I_n \end{pmatrix}$ is given by $\det \begin{pmatrix} A & 0 \\ C & I_n \end{pmatrix} = \det(A)\det(I_n) = \det(A)$ and likewise for all other block-diagonal matrices. Thus,

$$\begin{aligned} \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \det(A)\det\left(D - CA^{-1}B\right) \\ &= \det(D)\det\left(A - BD^{-1}C\right). \end{aligned}$$

Claim II: $\det \begin{pmatrix} I_m & -A \\ B^T & I_n \end{pmatrix} = \det\left(I_n + B^T \cdot A\right) = \det\left(I_m + A \cdot B^T\right)$.

From claim I, $\det \begin{pmatrix} I_m & -A \\ B^T & I_n \end{pmatrix} = \det(I_m)\det\left(I_n + B^T \cdot I_m \cdot A\right) = \det\left(I_n + B^T \cdot A\right)$ and $\det \begin{pmatrix} I_m & -A \\ B^T & I_n \end{pmatrix} = \det(I_n)\det\left(\left(I_m + A \cdot I_n \cdot B^T\right)\right) = \det\left(I_m + A \cdot B^T\right)$.

This finishes the proof.