

Practice Problems for Probability and Statistics

1. (taken from Casella and Berger exercise 1.11)

Let S be a sample space.

- (a) Show that the collection $\mathcal{B} = \{\emptyset, S\}$ is a sigma algebra.
(b) Let $\mathcal{B} = \{\text{all subsets of } S, \text{ including } S \text{ itself}\}$. Show that \mathcal{B} is a sigma algebra.
(c) Show that the intersection of two sigma algebras is a sigma algebra.

2. (Flush vs. Straight)

Why in a 5-card poker game does a flush beat a straight?, i.e. show that a flush is probabilistically harder to draw than a straight.

3. (Color Blind)

Suppose that 5% of men and .25% of women are color blind. A person is chosen at random and that person is color-blind. What is the probability that the person is male? (Assume males and females to be in equal numbers.)

4. (Finish proof)

Show that if A and B are independent events, then so are A^C and B^C .

5. (Half-proof of Theorem 18)

Prove the necessity direction of Theorem 18 from the class notes.

6. (Cauchy distribution)

Let X have the uniform pdf $f_X(x) = \frac{1}{\pi}$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. Find the pdf of $Y = \tan(X)$. This is the pdf of a Cauchy distribution.

7. (Finish proof- properties of expectation)

Last name A-H, prove (b)
Last name I-Q, prove (c)
Last name R-Z, prove (d)

8. (Finish proof- existence of expectations)

Prove Theorem 42 from the class notes.

9. (Application of inequality)

A random variable X is defined by $Z = \log X$ where $EZ = 0$. Is EX greater than, less than, or equal to 1?

10. (Application of Delta Method)

Suppose we have a sequence of random variables $\{X_n\}$ s.t.

$$\sqrt{n} \left(\frac{1}{n} \sum_{i=1}^n X_i - \mu \right) \xrightarrow{d} N(0, \sigma^2) \text{ where } \mu = E[X_n]$$

Find the limiting distribution of $\sqrt{n} \left(\frac{1}{\frac{1}{n} \sum_{i=1}^n X_i} - \frac{1}{\mu} \right)$.

11. (Bernoulli into Binomial)

Show that the sum of n independent and identically distributed random variables $X_i \sim \text{Bernoulli}(p)$ has distribution $\sum_{i=1}^n X_i \sim \text{Binomial}(n, p)$.

12. (Poisson into Poisson)

Show that the sum of n independent and identically distributed random variables $X_i \sim \text{Poisson}(\lambda_i)$ has distribution $\sum_{i=1}^n X_i \sim \text{Poisson}(\sum_{i=1}^n \lambda_i)$.

13. (Normal as the limit of t)

Show that if $X \sim t_\nu$, $Y \sim N(0, 1)$, then $\lim_{\nu \rightarrow \infty} X = Y$.

Do this roughly by using the Table of Distributions:

$$P(X \leq x) \quad X \sim t_\nu$$

$$P(Y \leq y) \quad Y \sim N(0, 1)$$

14. (Find the MLE)

Suppose that the vector of random variables $X = (X_1, \dots, X_n)'$ are i.i.d. with distribution $X_i \sim \text{Pareto}(\alpha)$ (here $\theta = \alpha$) and the Pareto distribution is defined as $F_X(x; \alpha) = 1 - \left(\frac{1}{x}\right)^\alpha$ $1 \leq x < \infty$, $\alpha > 0$

Derive the MLE α^* .

15. (Find the Sufficient Statistic)

Let X_1, X_2, \dots, X_n denote random variables from a normal distribution, i.e. $X_i \sim N(0, \theta)$ $0 < \theta < \infty$

Show that $\sum_{i=1}^n X_i^2$ is a sufficient statistic for θ .