ECON897 Math Camp Part I. **Matthew Hoelle**
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Solutions to Probability and Statistics Practice Problems

1. For (a), we must verify the three properties of a sigma algebra:

   a. By definition $\emptyset \in \mathcal{B}$

   b. By definition, $\emptyset^C = S$ and both are in $\mathcal{B}$

   c. By defintion, $\emptyset \cup S = S \in \mathcal{B}$

   For (b), we must verify the three properties of a sigma algebra:

   a. By definition, $\emptyset$ is a subset of S and thus $\emptyset \in \mathcal{B}$

   b. If A is a subset of S, then so is A^C, thus $A \in \mathcal{B} \Rightarrow A^C \in \mathcal{B}$

   c. If $A_1, A_2, ...$ are subsets of S, then $\cup_{i=1}^{\infty} A_i$ is also a subset, thus $A_1, A_2, ... \in \mathcal{B} \Rightarrow \cup_{i=1}^{\infty} A_i \in \mathcal{B}$

   For (c), we must verify the three properties of a sigma algebra. Let $\mathcal{B}_1, \mathcal{B}_2$ be two sigma algebras and define $\mathcal{B}^* = \mathcal{B}_1 \cap \mathcal{B}_2$

   a. By definition, $\emptyset$ belongs to the intersection of any two collection of sets, so $\emptyset \in \mathcal{B}^*$

   b. If $A \in \mathcal{B}^*$, then we know that $A \in \mathcal{B}_1$ and $A \in \mathcal{B}_2$ and since $\mathcal{B}_1, \mathcal{B}_2$ are sigma algebras, then $A^C \in \mathcal{B}_1$ and $A^C \in \mathcal{B}_2$, which directly implies $A^C \in \mathcal{B}^*$

   c. If $A_1, A_2, ... \in \mathcal{B}^*$, then we know that $A_1, A_2, ... \in \mathcal{B}_1$ and $A_1, A_2, ... \in \mathcal{B}_2$ and since $\mathcal{B}_1, \mathcal{B}_2$ are sigma algebras, then $\cup_{i=1}^{\infty} A_i \in \mathcal{B}_1$ and $\cup_{i=1}^{\infty} A_i \in \mathcal{B}_2$, which directly implies $\cup_{i=1}^{\infty} A_i \in \mathcal{B}^*$

2. First, calculate the probability of drawing a flush (5 cards of the same suit).

   Thinking

   4 possible suits for the flush (clubs, diamonds, hearts, or spades)

   There are $\binom{13}{5}$ possible combinations of 5 cards given the 13 cards within each suit (unordered without replacement)

   We also have to eliminate the straight flushes. Since a straight flush can begin with an Ace, 2,3,...,10, there are 10 possible straight flushes within each suit. We must subtract this number from the numerator.

   Obviously, there are $\binom{52}{5}$ possible hands in poker (unordered without replacement), so

   $$P(\text{flush}) = \frac{4\binom{13}{5} - 40}{\binom{52}{5}} = 0.00197$$

   Second, calculate the probability of a straight (5 cards in a row, but not all of the same suit).

   Thinking

First, ignore the suits and just consider the number of possible straights. As before, a straight can begin with an Ace, 2,3,...,10, so there are 10 possible straights. Within each of these 10 possible straights, there are 5 cards, which can take on any suit except that we require that the suit does not match for all 5. Thus, we have $\left(4^5 - 4\right)$ ways to distribute the suits among the 5 cards already in straight order.

Obviously, there are $\binom{52}{5}$ possible hands in poker (unordered without replacement), so

$$P(\text{straight}) = \frac{10\times\left(4^5-4\right)}{\binom{52}{5}} = 0.00392$$

Thus, we conclude that a straight is nearly twice a likely to be drawn as a flush.

3. $P(\text{male} \mid \text{color blind}) = \frac{(.05)(.50)}{(.05)(.50)+(.0025)(.50)} = 0.9524$

4.
$$\begin{aligned} P(A^C \cap B^C) &= P(A^C) - P(A^C \cap B) \\ &= P(A^C) - [P(B) - P(A \cap B)] \\ &= P(A^C) - P(B) + P(A)P(B) \\ &= P(A^C) - P(B)(1 - P(A)) \\ &= 1 - P(A) - P(B)(1 - P(A)) \\ &= (1 - P(B))(1 - P(A)) \\ &= P(A^C)P(B^C) \end{aligned}$$

5. Will show that all three properites in Thereom 18 must hold by definition of the cdf.

a. As $x \to -\infty$, then $P_X(X \le x) = 0$ and as $x \to \infty$, $P_X(X \le x) = 1$

b. Take $x_1 > x_2$. Since $P_X(X \le x_2) \le P_X(X \le x_1)$, then $F_X(x_2) \le F_X(x_1)$ and thus $F_X(\cdot)$ is a nondecreasing function

c. As $x \downarrow x_0$ (this means approaches x_0 from above), then $x - x_0 < \epsilon$ for some $\epsilon > 0$. Then $F_X(x) \le P_X(X \le (x_0 + \epsilon))$. And as $\epsilon \to 0$ and since from (b), $F_X(x)$ is nondecreasing, then in the limit, $F_X(x) = F_X(x_0)$.

6. We know that $g(x) = \tan(x)$ is an increasing function over X, that $f_X(x)$ is a continuous function and that $g^{-1}(y) = \arctan(y)$ has a continuous derivative on Y. Thus, we will use the transformation theorem from the class notes in order to find the distribution of Y.

From the table of derivatives, $\frac{d}{dy}\arctan(y) = \frac{1}{1+y^2}$ and notice this value is always positive.

Thus, $f_Y(y) = \frac{1}{\pi}\frac{1}{1+y^2} \qquad -\infty < y < \infty$ (as mentioned, this is the Cauchy distribution)

7. (b) $Eg_1(X) = \int_{-\infty}^{\infty} g_1(x) f_X(x) dx$

since $g_1(x)$ and $f_X(x) \ge 0 \quad \forall x$, then since integral is a linear operator, $Eg_1(X) \ge 0$

(c) $Eg_1(X) = \int_{-\infty}^{\infty} g_1(x) f_X(x) dx \qquad Eg_2(X) = \int_{-\infty}^{\infty} g_2(x) f_X(x) dx$

since $g_1(x) \ge g_2(x) \quad \forall x$

$Eg_1(X) - Eg_2(x) = \int_{-\infty}^{\infty} (g_1(x) - g_2(x)) f_X(x)dx$ by linearity of integral

then $(g_1(x) - g_2(x)) \geq 0$, so $Eg_1(X) - Eg_2(x) \geq 0$

(d) $Eg_1(X) = \int_{-\infty}^{\infty} g_1(x) f_X(x)dx$

Define $g_3(X) = a \qquad g_4(X) = b$

then $Eg_3(X) = \int_{-\infty}^{\infty} a f_X(x)dx = a$

$Eg_4(X) = \int_{-\infty}^{\infty} b f_X(x)dx = b$

applying Property (c) twice, we know that $g_3(X) \leq g_1(X) \leq g_4(X)$

$\Rightarrow Eg_3(X) \leq Eg_1(X) \leq Eg_4(X)$

8. $E\left|X^k\right| = \int_{-\infty}^{\infty} |x|^k f(x)dx = \int_{|x|\leq 1} |x|^k f(x)dx + \int_{|x|>1} |x|^k f(x)dx$

$\leq \int_{|x|\leq 1} f(x)dx + \int_{|x|>1} |x|^m f(x)dx$ since $k \leq m$

$\leq \int_{-\infty}^{\infty} f(x)dx + \int_{-\infty}^{\infty} |x|^m f(x)dx$

$\leq 1 + E|X^m| < \infty$

9. Define $X = \phi(Z) = \exp Z$. We know that ϕ is a convex function. Then from the Jensen's inequality, we obtain:

$EX = E\phi(Z) \geq \phi[EZ] = \phi(0) = \exp(0) = 1$

thus, $EX \geq 1$.

10. Define $a(\mu) = \frac{1}{\mu}$, then $A(\mu) = -\frac{1}{\mu^2}$

so from the Delta method, $\sqrt{n}\left(\frac{1}{\frac{1}{n}\sum_{i=1}^{n} X_i} - \frac{1}{\mu}\right) \underset{d}{\rightarrow} -\frac{1}{\mu^2} N\left(0, \sigma^2\right) = N\left(0, \frac{\sigma^2}{\mu^4}\right)$

11. We will use the moment generating function to prove this claim. We know that the mgf uniquely characterizes a distribution for a random variable. Further, we know that the mgf of the sum of i.i.d. random variables is equal to the product of their mgfs. Thus, if we can show that the product of a Bernoulli mgf is a Binomial mgf, then we have proven our claim.

For X ~Bernoulli (p) $\qquad M_X(t) = (1-p) + pe^t$

For Y ~Binomial (n, p) $\qquad M_Y(t) = \left[(1-p) + pe^t\right]^n$

obviously, $(M_X(t))^n = M_Y(t)$

12. We will again use the moment generating function with the reasoning as in Exercise 11 above to prove this claim.

For X_i ~Poisson (λ_i) $\qquad M_{X_i}(t) = e^{\lambda_i(e^t-1)}$

Then, $M_Y(t) = \prod_{i=1}^{n} M_{X_i}(t) = e^{(e^t-1)\cdot\sum_{i=1}^{n}\lambda_i}$

which implies $Y = \sum_{i=1}^{n} X_i$ ~Poisson $\left(\sum_{i=1}^{n}\lambda_i\right)$

13. This can be verified easily from the following table taken from any Table of Distributions (in this case for $P(X \leq x) = 0.975$):

$P(X \leq x) = 0.975$	ν	x
t_ν	1	12.706
·	3	3.182
·	5	2.571
·	9	2.262
·	60	2.000
·	120	1.980
$N(0,1)$		1.960

14. $f_X(x;\alpha) = \frac{\partial F_X(x;\alpha)}{\partial x} = -\alpha \left(\frac{1}{x}\right)^{\alpha-1} \left(-\frac{1}{x^2}\right) = \alpha \left(\frac{1}{x}\right)^{\alpha+1}$

$$L(\theta; X) = \prod_{i=1}^{n} \left(\alpha \left(\frac{1}{x_i}\right)^{\alpha+1} \right)$$

$$\log L(\theta; X) = \sum_{i=1}^{n} \log(\alpha) - (\alpha+1)\log(x_i)$$

$$= n\log(\alpha) - (\alpha+1)\sum_{i=1}^{n} \log(x_i)$$

$$\frac{\partial \log L(\theta;X)}{\partial \alpha} = 0 \qquad \Rightarrow \qquad \frac{n}{\alpha} - \sum_{i=1}^{n} \log(x_i) = 0$$

$$\alpha^* = \frac{1}{\frac{1}{n}\sum_{i=1}^{n} \log(x_i)}$$

15. $f_X(x_1;\theta)\cdots f_X(x_n;\theta) = \underbrace{\left(\frac{1}{\sqrt{2\pi\theta}}\right)^n \exp\left[-\sum_{i=1}^{n} x_i^2/2\theta\right]}_{k_1} \times \underbrace{1}_{k_2}$

thus, $\sum_{i=1}^{n} x_i^2$ is a sufficient statistic for θ.