ECON897 Math Camp Part I. **Matthew Hoelle**
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Final Part I. (Friday, August 29)

1. **Easy (16 points; 8 points each)**

   (a) Prove the following implication:
   f injective $\Rightarrow \left[f^{-1}\left(f(X)\right) = X\right]$

   (b) Prove the following implication (recall that E' is the derived set of the subset $E \subseteq \mathbb{R}^1$, i.e. the set of all limit points of E):
   $x \in E' \Rightarrow [x \in intE \text{ or } x \in bdE]$

2. **Medium (24 points; 12 points each)**

   (a) Let $X_i \tilde{} N\left(\mu_i, \sigma_i^2\right)$ for $i = 1, ..., n$.
   Find the distribution for $Y = \sum_{i=1}^{n} X_i$ (hint: it may be helpful to use the mgf for X_i).

   (b) Let $Y \tilde{} Gamma\,(\alpha, \beta)$ where $Y = \sum_{i=1}^{\alpha} X_i$ and $X_i \tilde{} Exponential\,(\beta)$ $\forall i$.
   If $\alpha = 4$ and $\beta = 2$, please find the lower bound for $P\,(2 < Y < 14)$.
   The probability density functions are given below:
   $f\,(x_i|\beta) = \frac{1}{\beta} e^{-x_i/\beta} \quad 0 \leq x_i < \infty, \ \beta > 0$
   $f\,(y|\alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} y^{\alpha-1} e^{-y/\beta} \quad 0 \leq y < \infty, \ \alpha > 0, \ \beta > 0$

3. **Hard (60 points; 20 points each)**

   (a) Let X and Y be metric spaces. For this problem, you may NOT refer to theorems that have not been proven in your Blue Book, but definitions (including equivalent versions) are acceptable.
   Prove directly that if $f : X \to Y$ is continuous, then for every closed subset $\hat{Y} \subseteq Y$, the pre-image $f^{-1}\left(\hat{Y}\right)$ is closed in X.
   Use this result along with the definition of determinant (you do NOT need to define the determinant, only to understand its properties) to prove that the set of $n \times n$ singular real matrices is closed in the set of $n \times n$ real matrices.

   (b) For this problem, again, you may NOT refer to theorems that have not been proven in your Blue Book, with the exception of **Lemmas 1-3** below, which may be stated without proof.
   Show that if X is a complete metric space, then X is closed.
   Use this result along with any of the following lemmas to prove the important result that:

X compact $\Rightarrow$ X closed

Lemma 1: All Cauchy sequences are bounded.

Lemma 2: For every sequence in a compact set, there exists a convergent subsequence.

Lemma 3: The set of subsequential limits of a sequence in a metric space is closed in that same metric space.

(c) For this problem, you may use any theorems from the class notes without proof. Try to be clear about which result you are using.

State the Inverse Function Theorem in $\mathbb{R}^1$.

Recall the question from the first exam:

"Let f be a continuous real function on the metric space X, i.e. $f : X \to \mathbb{R}^1$. Let $Z(f)$ be the set of all $p \in X$ at which $f(p) = 0$."

Assume further that $Z(f) \neq \emptyset$, $Z(f) \neq X$, and f is C^1 on X.

You have proven that $Z(f)$ is closed; this can be taken as a fact.

Show that there exists open sets $U \subseteq X$ and $V \subseteq \mathbb{R}^1$ such that $\left(f^{-1}\right)'$ is continuous on V, where $f^{-1} : V \to U$.