Exam Solutions

1. (Birthday problem)

   Obviously $P(2 \text{ have same birthday}) = 1 - P(\text{all have different birthdays})$

   $P(\text{all have different}) = \left(\frac{365}{365}\right)\left(\frac{364}{365}\right)\left(\frac{363}{365}\right) \ldots \left(\frac{365-27+1}{365}\right)$

   $= \frac{1}{365^{27}} \cdot \frac{365!}{(365-27)!}$

   $P(2 \text{ have same}) = 1 - \frac{1}{365^{27}} \cdot \frac{365!}{(365-27)!} = 0.6269$

2. (Proving a function is a pdf)

   Using the theorem from the notes, we know that we only need to verify the two properties of the function $g(x)$ in order for it to be a pdf.

   (i) Since $f_X(x)$ is a pdf, then $f_X(x) \geq 0$ for all x. By assumption, $1 - F_X(x_0) > 0$ and thus $g_X(x) \geq 0$ for all x.

   (ii) $$\int_{-\infty}^{\infty} g_X(x)dx = \frac{\int_{x_0}^{\infty} f_X(x)dx}{1 - F_X(x_0)} = \frac{1 - F_X(x_0)}{1 - F_X(x_0)} = 1$$

3. (Normal, chi-squared relationship)

   $$EX^2 = \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}}\right) e^{-x^2/2} x^2 dx = \left(\frac{1}{\sqrt{2\pi}}\right) \int_{-\infty}^{\infty} e^{-x^2/2} x^2 dx$$

   We now must use change of variables and define $u = x^2$. Then notice $du = 2xdx$

   $$EX^2 = 2 \times \left(\frac{1}{\sqrt{2\pi}}\right) \int_0^{\infty} e^{-u/2} \frac{u}{2x} du \qquad \text{(by symmetry)}$$

   $$= 2 \times \left(\frac{1}{\sqrt{2\pi}}\right) \frac{1}{2} \int_0^{\infty} e^{-u/2} \left(\sqrt{u}\right) du$$

   The easier method is to recall (or re-derive) $f_Y(y) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{y}} e^{-y/2} \qquad 0 < y < \infty$

   Then $EY = \int_0^{\infty} \frac{1}{\sqrt{2\pi}} \frac{y}{\sqrt{y}} e^{-y/2} dy$

   Note: I do not expect you to evaluate the above integral except through use of a table or a known fact. Notice that both EX^2 and EY yield the same value, in this case it is equal to 1. The fact about the Gamma function should come in handy here.

   Define the change of variable $z = y/2$, then $dz = dy/2$.

   $EY = \frac{1}{\sqrt{2\pi}} \times 2 \int_0^{\infty} (2z)^{1/2} e^{-z} dz = \frac{1}{\sqrt{\pi}} \times 2 \int_0^{\infty} z^{3/2-1} e^{-z} dz$

   $= \frac{1}{\sqrt{\pi}} \times 2 \times \Gamma\left(\frac{3}{2}\right) = \frac{2}{\sqrt{\pi}} \times \frac{1}{2} \times \Gamma\left(\frac{1}{2}\right) = \frac{2}{\sqrt{\pi}} \times \frac{\sqrt{\pi}}{2} = 1$

4. (Uniform, exponential relationship)

   $EY = \int_0^1 g(x)dx = \int_0^1 -\log(x)dx$

   Now, we need to use integration by parts.

   Recall: $\int udv = uv - \int vdu$

   Here, we set $dv = -1 \quad u = \log(x)$

   Then, of course $v = -x \quad du = 1/x$

   $\int_0^1 -\log(x)dx = [-x\log(x)]_0^1 - \int_0^1 -dx$

   $= 0 + 0 + 1 = 1$

   Note: by L'Hospital's Rule $x\log(x)_{|x=0} = \frac{\log(x)}{1/x}_{|x=0}$;

   $$\partial\ x\log(x)_{|x=0} = \frac{1/x}{-1/x^2}_{|x=0} = -x_{|x=0} = 0$$

5. (Moment generating functions)

   $$M_X(t) = \int_{-\infty}^{\infty} e^{xt}\frac{1}{\sqrt{2\pi}\sigma}e^{-(x-\mu)^2/(2\sigma^2)}dx = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{\left(2\sigma^2 xt - x^2 + 2x\mu - \mu^2\right)/2\sigma^2}dx$$

   $$= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{\left(-x^2+2x(\mu+\sigma^2 t)-(\mu+\sigma^2 t)^2\right)/2\sigma^2}dx \times e^{\left(2\mu\sigma^2 t+\sigma^4 t^2\right)/2\sigma^2}$$

   $$= 1 \times e^{\mu t + \sigma^2 t^2/2}$$

6. (Application of Chebychev's Inequality)

   Define $\epsilon = k\sigma$, then

   $P(|X-3| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2} \qquad \forall\epsilon > 0$

   know that $\sigma^2 = E(X^2) - E(X)^2 = 13 - 9 = 4$

   set $\epsilon = 5$, then

   $P(|X-3| \geq 5) \leq \frac{4}{25}$

   $1 - P(-2 < X < 8) \leq \frac{4}{25}$

   $\Rightarrow P(-2 < X < 8) \geq \frac{21}{25}$

7. (Find the MLE)

   $f_X(x;\alpha) = \frac{\partial F_X(x;\alpha)}{\partial x} = -\alpha\left(\frac{1}{x}\right)^{\alpha-1}\left(-\frac{1}{x^2}\right) = \alpha\left(\frac{1}{x}\right)^{\alpha+1}$

   $L(\theta;X) = \prod_{i=1}^{n}\left(\alpha\left(\frac{1}{x_i}\right)^{\alpha+1}\right)$

   $\log L(\theta;X) = \sum_{i=1}^n \log(\alpha) - (\alpha+1)\log(x_i)$

   $= n\log(\alpha) - (\alpha+1)\sum_{i=1}^n \log(x_i)$

   $\frac{\partial \log L(\theta;X)}{\partial\alpha} = 0 \qquad \Rightarrow \qquad \frac{n}{\alpha} - \sum_{i=1}^n \log(x_i) = 0$

   $\alpha^* = \frac{1}{\frac{1}{n}\sum_{i=1}^n \log(x_i)}$