

# Econ 897.1: Exam #1

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## Instructions:

- This is a closed book, closed-notes exam. Calculators are allowed.
- You have 120 minutes to complete this exam. The exam contains a possible 100 points partitioned amongst the 10 questions as indicated next to each question.
- Please answer questions 1-3 in Bluebook #1, questions 4-6 in Bluebook #2, and questions 7-10 in Bluebook #3.
- Please do not refer to theorems that have not been proven in your bluebook UNLESS instructed to do so (Questions 8 and 9). This does not, of course, include definitions (either the original or any equivalent ones), which may be stated without proof. If you use a result repeatedly throughout the exam (like, for instance, the result "if  $p \in E'$ , then  $\exists\{p_n\}$  in  $E$  s.t.  $p_n \rightarrow p$ "), then you only need to prove it in one question and can make a reference to that proof in future questions.
- Please answer all questions as thoroughly as you can.
- Good luck!

1. (Order of Quantifiers- 7 points)

Let the statements (a) and (b) be defined as:

$$(a) \quad \forall x \in X, \exists y \in Y \text{ s.t. } p(x, y)$$

$$(b) \quad \exists x \in X \text{ s.t. } \forall y \in Y \quad p(x, y)$$

where the propositional function

$$p(x, y) : \quad x^2 > y$$

Choose sets  $X, Y$  from the choices below s.t.:

$$(i) \quad (a) \Rightarrow (b)$$

$$(ii) \quad (a) \not\Rightarrow (b)$$

$$X_1 = \{2, 3\} \quad Y_1 = \{9, 25, 1/2\}$$

$$X_2 = \{1, 2, 3\} \quad Y_2 = \{2, 4, 3\pi\}$$

2. (Simple Proof- 4 points)

Prove that the empty set is a subset of every set.

3. (Limit Points- 12 points)

Let  $E'$  be the set of all limit points of a set  $E$ . Prove that  $E'$  is closed. Prove that  $E$  and  $\bar{E}$  have the same limit points (i.e.  $\bar{E}' = E'$ ). Recall that  $\bar{E} = E \cup E'$ .

4. (Continuity #1- 10 points)

Let  $f$  be a continuous real function on the metric space  $X$ , i.e.  $f : X \rightarrow \mathbb{R}$ . Let  $Z(f)$  be the set of all  $p \in X$  at which  $f(p) = 0$ . Prove that  $Z(f)$  is closed.

5. (Continuity #2- 10 points)

Let  $f$  be a continuous mapping of a metric space  $X$  into a metric space  $Y$ , and let  $E$  be a dense subset of  $X$ . Prove that  $f(E)$  is dense in  $f(X)$ .

6. (Uniform Continuity- 16 points)

Define what it means for a function  $f : A \rightarrow \mathbb{R}$  to be uniformly continuous on  $A$ .

Show that if  $f$  is a continuous function from a compact metric space into a metric space, then  $f$  is uniformly continuous.

7. (Irrationality- 6 points)

Prove by contradiction that  $\sqrt{3}$  is not a rational number.

8. (Functions- 10 points)

For this question, feel free to use any theorems from the class notes without proof.

Let  $X$ ,  $Y$ , and  $Z$  be nonempty sets and consider any  $f, g \in Y^X$  and  $u, v \in Z^Y$ .

Prove that:

- (i) if  $f$  is surjective and  $u \circ f = v \circ f$ , then  $u = v$
- (ii) if  $u$  is injective and  $u \circ f = u \circ g$ , then  $f = g$

9. (Sequences- 20 points)

Define what it means for metric spaces to be complete.

Show that a metric space  $(X, d)$  is compact only if it is complete.

You may use the following results as you see fit without proof:

Lemma 9(a): All Cauchy sequences are bounded.

Lemma 9(b): If  $\{x_n\}$  is Cauchy and has a subsequence that converges in  $X$ , then  $\{x_n\}$  converges in  $X$  as well.

Lemma 9(c): If  $E$  is an infinite subset of a compact set  $K$ , then  $E$  has a limit point in  $K$ .

10. (Differentiability- 5 points)

Let  $f : [a, b] \rightarrow \mathbb{R}$ , where  $[a, b]$  represents a closed interval of real numbers. Give the definition of differentiability of  $f$  at  $x_0 \in (a, b)$ .