Exam Solutions

1. (Order of Quantifiers)

   (i) For X_2 and Y_2, we know that (a) cannot hold (there is not $y \in Y_2$ s.t. $1^2 > y$), thus the implication holds trivially.

   (ii) For any combinatino of X_2 and Y_1, or X_1 and Y_1, or X_1 and Y_2, the (a) statement is true and yet, the statement (b) is always false (keep in mind that $3^2 \leq 3\pi$).

2. (Simple Proof)

   Suppose not, suppose $\exists x \in \emptyset$ s.t. $x \notin E$ for some set E. This cannot be, since by definition, $\emptyset = \{\}$.

3. (Limit Points)

   (i) A set E is closed iff $E' \subseteq E$. Is it true that $(E')' \subseteq E'$? Take $x \in (E')'$. Then $\forall \epsilon > 0$, $N^*_\epsilon(x) \cap E' \neq \emptyset$. This implies that $\exists y \in N^*_\epsilon(x) \cap E'$. Since y is a limit point of E, $\exists \{y_n\}$ s.t. $y_n \in E$ and $y_n \to y$. Then $\forall \epsilon > 0$, we can always find an n s.t. $y_n \in N^*_\epsilon(x)$ and thus, $N^*_\epsilon(x) \cap E \neq \emptyset$. We conclude that $x \in E'$ and $(E')' \subseteq E'$. Thus, E' is closed.

   (ii) We want to show that $\bar{E}' = E'$. First, we need to show that $(E \cup E')' = E' \cup (E')'$. The following bit of reasoning from the homework proves this result:

   $$\text{"}\ x \in (S \cup T)' \Leftrightarrow \forall \epsilon > 0 \quad N^*_\epsilon(x) \cap [S \cup T] \neq \emptyset$$
   $$\Leftrightarrow (N^*_\epsilon(x) \cap S) \cup (N^*_\epsilon(x) \cap T) \neq \emptyset \Leftrightarrow x \in S' \vee x \in T' \ \text{"}$$

   Then, from (i), since $(E')' \subseteq E'$, then $E' \cup (E')' = E'$.

4. (Continuity #1)

   To show that $Z(f)$ is closed, must show that $Z(f)' \subseteq Z(f)$. Take a point $p \in Z(f)'$. Then $\exists \{p_n\}$ s.t. $f(p_n) = 0$ for all n and $p_n \to p$. Since $p \in Z(f)'$, then $\forall \epsilon > 0$, $N^*_\epsilon(p) \cap Z(f) \neq \emptyset$ and since $Z(f) \subseteq X$, then $N^*_\epsilon(p) \cap X \neq \emptyset$, so $p \in X'$. Thus, by one of our equivalent definitions of continuity, $\lim_{p_n \to p} f(p_n) = f(p) = 0$ and thus $p \in Z(f)$, so we conclude $Z(f)$ is closed.

5. (Continuity #2)

   Recall that $E \subseteq X$ is a dense subset of X iff $\forall x \in X$, either $x \in E \vee x \in E'$. Suppose that $f(E)$ is NOT dense in $f(X)$. This implies that for some $y \in f(X)$, then $y \notin f(E) \wedge y \notin f(E)'$. If $y \notin f(E)$, then $\forall x \in E$, $y \neq f(x)$. But, $y \in f(X)$ implies that $\exists x \in X - E$ s.t. $y = f(x)$. Then, due to denseness of E, it must be that $x \in E'$. Thus, $y \in f(E')$. If we can show that $f(E') \subseteq f(E)'$, then we are finished.

To show this, take $a \in f(E')$, then $\exists b \in E'$ s.t. $a = f(b)$ and $\exists \{b_n\}$ s.t. $b_n \in E \ \ \forall n$ and $b_n \to b$. By the continuity of f, $\exists \{a_n\}$ s.t. $a_n \in f(E)$ and $a_n \to a$. We will use the following result proved in the homework to show that $a \in f(E)'$:

"Consider any sequence $\{x_n\}$ s.t. $x_n \in A \ \ \forall n$ and $x_n \to x$. Suppose that $x \notin A'$. Then $\exists \epsilon > 0$ s.t. $N^*_\epsilon(x) \cap A = \emptyset$. Thus, $\forall a \in A$, $a \notin N^*_\epsilon(x)$, i.e. $\exists \epsilon > 0$ s.t. $d(x_n, x) \geq \epsilon$ for all n. This contradicts that the sequence $\{x_n\}$ converges to x."

With this in tow, then $f(E') \subseteq f(E)'$ and thus $y \in f(E)'$, which contradicts one of the implications of $f(E)$ NOT being dense. Thus, $f(E)$ is a dense subset of $f(X)$.

6. (Uniform Continuity)

For $f : A \to \mathbb{R}$, f is uniformly continuous on A if $\forall \epsilon > 0, \ \exists \delta > 0$ s.t. $|f(x) - f(y)| < \epsilon \quad \forall x, y \in A$ satisfying $|x - y| < \delta$.

Let $f : X \to Y$ be a continuous function and X is a compact metric space and Y is a metric space.

Proof: Take $\epsilon > 0$.

$\forall x_i \in X, \ \exists \delta_{x_i} > 0$ s.t. $y \in X \wedge d_x(x_i, y) < \delta_{x_i}$

$\Rightarrow \ d_y(f(x_i), f(y)) < \epsilon/2$.

Consider the family of sets $\mathcal{O} = \left\{ N\left(x_i, \frac{\delta_{x_i}}{2}\right) : \forall x_i \in X \right\}$.

Obviously, $X \in \mathcal{O}$, and X compact implies $X \subseteq \bigcup_{i=1}^{n} N\left(x_i, \frac{\delta_{x_i}}{2}\right)$.

Define $\delta = \min\left\{ \frac{\delta_{x_i}}{2} : i = 1, .., n \right\}$.

Then $\forall \epsilon > 0, \ \exists \delta$ as defined above s.t. $\forall y, z$ with

$d_x(y, z) < \delta \leq \frac{\delta_{x_i}}{2}$,

(then $d_x(y, x_i) \leq d_x(y, z) + d_x(z, x_i) \leq \delta_{x_i}$

and likewise, $d_x(z, x_i) \leq \delta_{x_i}$),

then $d_y(f(y), f(z)) \leq d_y(f(y), f(x_i)) + d_y(f(x_i), f(z)) < \epsilon/2 + \epsilon/2$

where the inequality follows from the definition of continuity in the second line of the proof.

7. (Irrationality)

Suppose not, i.e. $\sqrt{3} = \frac{p}{q}$, where p, q have no common factors.

Then, $3 = \frac{p^2}{q^2} \qquad p^2 = 3q^2$

q even $\Rightarrow q^2$ even $\Rightarrow p^2$ even $\Rightarrow p$ even

but this contradicts that q, p have no common factor

q odd $\Rightarrow q^2$ odd $\Rightarrow p^2$ odd $\Rightarrow p$ odd

Define $\qquad q = 2m+1 \qquad p = 2n+1 \qquad m, n \in \mathbb{Z}.$

$\left(4n^2 + 4n + 1\right) = 3\left(4m^2 + 4m + 1\right)$

$4n^2 + 4n = 12m^2 + 12m + 2$

$\underbrace{2n^2 + 2n}_{even} = \underbrace{6m^2 + 6m + 1}_{odd} \qquad$ *Contradiction.*

8. (Functions)

Proof (i): f surjective implies $f(X) = Y$

$\forall x \in X \qquad (u \circ f)(x) = (v \circ f)(x)$

$u = v$ if $\forall y \in Y$, then $u(y) = v(y)$

Suppose $u \neq v$, say $u(\tilde{y}) \neq v(\tilde{y})$.

Then, $\exists \tilde{x}$ s.t. $f(\tilde{x}) = \tilde{y}$

$(u \circ f)(\tilde{x}) = (v \circ f)(\tilde{x})$

$\Rightarrow u(\tilde{y}) = v(\tilde{y}) \qquad \Rightarrow\Leftarrow$

Proof (ii): u injective, then $\exists u^{-1} \in Y^Z$ s.t. $u^{-1} \circ u = id_y$

(the above statement is a theorem from Ch. 4 in the notes)

so $u \circ f = u \circ g$

$u^{-1} \circ u \circ f = u^{-1} \circ u \circ g$

$id_y \circ f = id_y \circ g$

$f = g$

9. (Sequences)

By definition, a metric space (X, d) is complete iff every Cauchy sequence in X converges in X.

We seek to show that a metric space (X, d), which is compact, is also complete. Thus, we will choose any Cauchy sequence $\{x_n\}$ in the compact set X, i.e. $x_n \in X \quad \forall n$. First, we will show that every sequence in a compact set contains a convergent subsequence.

Let E be the range of $\{x_n\}$. If E is finite, then since a sequence contains an infinite number of terms, then there must be a subsequence such that $x_{n_1} = x_{n_2} = ...$

If E is infinite, then using Lemma 9(c), we know that $\exists\, p \in E' \wedge p \in X$.

Since $p \in E'$, then $\forall \epsilon > 0$, $N^*_\epsilon(p) \cap E \neq \emptyset$. Set $N = 1/\epsilon + 1$. Then $\forall n \geq N$, $\exists p_n \in N^*_\epsilon(p) \cap E$. Free to choose any $p_n \in E$ for $n < N$, we have constructed a sequence $\{p_n\}$ in E such that $p_n \to p$. Then define the sequence of integers $\{n_i\}$ and $\{m_i\}$ such that $(x_{n_1}, x_{n_2}, ...) = (p_{m_1}, p_{m_2}, ...)$ where $n_1 < n_2 < n_3 < ..$ and $m_1 < m_2 < m_3 < ...$Thus, we have shown that the subsequence $\{x_{n_i}\}$ converges to $p \in X$.

From there, we use Lemma 9(b) to state that the convergent subsequence is actually the sequence itself. Thus, the Cauchy sequence $\{x_n\}$ converges in X. Since the Cauchy sequence was arbitrarily chosen, we conclude that every Cauchy sequence in X converges in X, hence the desired completeness of X.

10. (Differentiability)

    Let $f : [a, b] \to \mathbb{R}$. f is differentiable at $x_0 \in (a, b)$ if the limit $f'(x_0) = \lim\limits_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ exists and is finite.