

Hemi-Continuity of Correspondences and Berge's Maximum Theorem¹

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Let $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$, and $\phi : U \rightrightarrows V$.²

ϕ is *upper hemi-continuous* (uhc) if

$$\bar{U} \subset U \text{ compact} \Rightarrow \{(u, v) \in U \times V : u \in \bar{U} \ \& \ v \in \phi(u)\} \text{ compact.}$$

An alternative property is that, for $u \in U$, ϕ is *sequentially upper hemi-continuous* (suhc) **at** u if

$$u^\nu \in U, v^\nu \in \phi(u^\nu), \nu \in \mathbb{N}, \lim_{\nu \rightarrow \infty} u^\nu = u \in U, \lim_{\nu \rightarrow \infty} v^\nu = v \in \mathbb{R}^n \Rightarrow v \in \phi(u),^3$$

and suhc if ϕ is suhc **on** U .

Property 1 (relationship between uhc and suhc). (i) If ϕ is uhc, then ϕ is suhc, and (ii) If V is compact and ϕ is suhc, then ϕ is uhc.

Remark. This means that suhc is a weaker condition than uhc. In practice, V is typically compact, so that the two definitions coincide.

Proof. Left to the reader (as a warmup for what follows).■

Property 2 (relationship between uhc and continuity). Suppose that ϕ is uhc. Then

$$\begin{aligned} &\text{for some } u' \in U, \phi(u') = \{v'\} \Rightarrow \\ &\text{for } u^\nu \in U, v^\nu \in \phi(u^\nu), \nu \in \mathbb{N}, \text{ and } \lim_{\nu \rightarrow \infty} u^\nu = u', \lim_{\nu \rightarrow \infty} v^\nu = v'. \end{aligned}$$

Proof. Suppose not, i.e., for some $\varepsilon > 0$ there is a subsequence, w/o log ..., s.t. $\|v^\nu - v'\| \geq \varepsilon, \nu \in \mathbb{N}$.

But, since $\{u^\nu, \nu \in \mathbb{N}, u'\}$ is compact and ϕ is uhc, there is a subsequence, w/o log ..., s.t. $\lim_{\nu \rightarrow \infty} v^\nu = v'$, a contradiction.■

For $u \in U$, ϕ is *lower hemi-continuous* (lhc) **at** u if

$$\begin{aligned} &u^\nu \in U, \nu \in \mathbb{N}, \lim_{\nu \rightarrow \infty} u^\nu = u \in U, \text{ and } v \in \phi(u) \Rightarrow \\ &\text{there is } v^\nu \in \phi(u^\nu), \nu \in \mathbb{N}, \text{ s.t. } \lim_{\nu \rightarrow \infty} v^\nu = v, \end{aligned}$$

and lhc if it is lhc **on** U .

Remark. If, for $u \in U$, $\phi(u) = V$, then ϕ is lhc, and if, in addition, V is compact, then ϕ is uhc.

Berge's Maximum Theorem (one very useful version). Suppose $X \subset \mathbb{R}^n, A \subset \mathbb{R}^m, f : X \times A \rightarrow \mathbb{R}$ is $C^0, X^f : A \rightrightarrows X$ is both uhc (so that, in particular, for $\alpha \in A, X^f(\alpha)$ is nonempty and compact) and lhc, and

$$X^o : A \rightrightarrows X \text{ s.t. } \alpha \mapsto \arg \max_{x \in X^f(\alpha)} f(x, \alpha)$$

(which, by the Extreme Value Theorem, is nonempty-valued and therefore well-defined). Then X^o is uhc.

¹Warning: These notes, consisting mostly of various definitions, and the interrelationships between them, are very terse.

²Recall that $\phi : U \rightrightarrows V$ is a *correspondence*, a mapping from U into the nonempty subsets of V .

³For upper hemi-continuity, the parallel definition is that, for $u \in U$, ϕ is uhc at u if

$$\begin{aligned} &u^\nu \in U, v^\nu \in \phi(u^\nu), \nu \in \mathbb{N}, \text{ and } \lim_{\nu \rightarrow \infty} u^\nu = u \in U \Rightarrow \\ &\text{there is a subsequence, w/o log ..., s.t. } \lim_{\nu \rightarrow \infty} v^\nu = v \in \phi(u). \end{aligned}$$

Remark. In the following proof the identifications

general	application
U	A
V	X
ϕ	X^f ("f" = "feasible"), X^o ("o" = "optimal")

are useful to bear in mind.

Proof. Take $\bar{A} \subset A$ compact, and (x^ν, α^ν) s.t. $\alpha^\nu \in \bar{A}$ and $x^\nu \in X^o(\alpha^\nu), \nu \in \mathbb{N}$. We want to show that there is a subsequence, w/o log ..., s.t. $\lim_{\nu \rightarrow \infty} \alpha^\nu = \alpha \in \bar{A}$ and $\lim_{\nu \rightarrow \infty} x^\nu = x \in X^o(\alpha)$.

Step 1. Suppose not, i.e., that for every subsequence, w/o log ..., it is not true that $\lim_{\nu \rightarrow \infty} \alpha^\nu = \alpha \in \bar{A}$ and $\lim_{\nu \rightarrow \infty} x^\nu = x \in X^o(\alpha)$.

Step 2. (This step uses the hypothesis that X^f is uhc.) Since \bar{A} is compact, there is a subsequence, w/o log ..., s.t. $\lim_{\nu \rightarrow \infty} \alpha^\nu = \alpha \in \bar{A}$, and therefore, since X^f is uhc, there is a subsequence, w/o log ..., s.t. $\lim_{\nu \rightarrow \infty} x^\nu \in x \in X^f(\alpha)$. By supposition, $x \notin X^o(\alpha)$ (though $x^\nu \in X^o(\alpha^\nu) \subset X^f(\alpha^\nu), \nu \in \mathbb{N}$).

Step 3. (This step uses the hypothesis that X^f is lhc.) Since X^f is lhc, for every $x' \in X^f(\alpha)$, in particular, $x' \in X^o(\alpha) \subset X^f(\alpha)$, there is a corresponding sequence $x'^\nu \in X^f(\alpha^\nu), \nu \in \mathbb{N}$, s.t. $\lim_{\nu \rightarrow \infty} x'^\nu = x'$.

Step 4. (This step uses the hypothesis that f is C^0 .) Finally, putting together the results in Steps 2 and 3, fix $x' \in X^o(\alpha)$ (and, correspondingly, $x'^\nu \in X^f(\alpha^\nu), \nu \in \mathbb{N}$, s.t. $\lim_{\nu \rightarrow \infty} x'^\nu = x'$). Since f is C^o , for sufficiently large ν ,

$$f(x^\nu, \alpha^\nu) \sim f(x, \alpha) < f(x', \alpha) \sim f(x'^\nu, \alpha^\nu),$$

which contradicts $x^\nu \in X^o(\alpha^\nu), \nu \in \mathbb{N}$ (since, by construction, $x'^\nu \in X^f(\alpha^\nu), \nu \in \mathbb{N}$). So the supposition must be false. ■

Remarks. 1. The reason for laying out the last proof so pedantically is that it shows, precisely, at which point in the argument each hypothesis is used. 2. The "hard" part of applying this theorem is, usually – but not always, for example, not when $X^f(\alpha) = X' \subset X, \alpha \in A$, i.e., when $X^f(\alpha)$ is a fixed subset of X , independent of α – to show that X^f is lhc. Compare the proofs (to be given in class) for establishing that (in the Nash Model) responses are lhc and that for establishing that (in the Walrasian Model) demands are lhc. 3. All this will be a lot clearer if you draw lots and lots of "pictures" to illustrate the various concepts and results. 4. The last proof applies verbatim when X and A are subsets of metric spaces (for which compactness is equivalent to sequential compactness).

Corollary. *The value function*

$$v : X^o(\alpha) \rightarrow \mathbb{R} \text{ s.t. } \alpha \mapsto v(\alpha) = \max_{x \in X^f(\alpha)} f(x, \alpha)$$

is C^o .

Proof. Suppose not, i.e., that there is $\alpha^\nu \in A, \nu \in \mathbb{N}$, s.t. $\lim_{\nu \rightarrow \infty} \alpha^\nu = \alpha \in A$ and, for some $\varepsilon > 0$, a subsequence, w/o log ..., s.t., $\|v(\alpha^\nu) - v(\alpha)\| \geq \varepsilon, \nu \in \mathbb{N}$. Consider a corresponding sequence $x^\nu \in X^o(\alpha^\nu), \nu \in \mathbb{N}$. Since X^o is uhc, there is a subsequence, w/o log ..., s.t. $\lim_{\nu \rightarrow \infty} x^\nu = x \in X^o(\alpha)$, so that $\lim_{\nu \rightarrow \infty} f(x^\nu, \alpha^\nu) = f(x, \alpha) = v(\alpha)$, a contradiction. ■

Reference. Berge, C., *Topological Spaces*, Dover Publications, Mineola, New York, reprinted 1997 in paperback, pp. 109-117.

Note that Berge's definitions of the correspondence ϕ and upper and lower hemi-continuity (which he refers to as *semi-continuity*) properties differ slightly from those presented here because he permits $\phi(u) = \emptyset$.