**Econ 897 Math Camp Part III** **Fatih Karahan**
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**Quiz #2 (Friday, August 28)**

1. Suppose $X$ has density $f(\cdot)$ and $\mu = EX$, $m =$ the median of $X$, i.e. $m$ solves $F(m) = 0.5$.

   (a) Show that $a = \mu$ solves $min_a E(X-a)^2$

   We can either use optimization techniques, solve the first order condition and find critical points. Note that the objective function is strictly concave. So, there will be a unique minimizer, and first order condition is sufficient. Here, we will take the alternative approach:

$$\begin{aligned} E(X-a)^2 &= E(X-\mu+\mu-a)^2 \\ &= E(X-\mu)^2 + E(\mu-a)^2 + 2E(X-\mu)(\mu-a) \end{aligned}$$

   Note that since both $\mu$ and $a$ are constants, we get

$$\begin{aligned} E(X-\mu)(\mu-a) &= (\mu-a)E(X-\mu) \\ &= (\mu-a)(EX-\mu) \\ &= (\mu-a)(\mu-\mu) = 0 \end{aligned}$$

   Therefore, we have

$$E(X-a)^2 = E(X-\mu)^2 + E(\mu-a)^2 \geq E(X-\mu)^2$$

   An examination of this shows that $E(X-a)^2$ is minimized (uniquely) at $a = \mu$.

   (b) Show that $a = m$ solves $min_a E|X-a|$.

$$|X-a| = \begin{cases} X-a & \text{if } X > a \\ a-X & \text{if } a > X \end{cases}$$

$$\begin{aligned} E|X-a| &= \int_{-\infty}^{a} (a-x) f(x)\,dx + \int_{a}^{\infty} (x-a) f(x)\,dx \\ &= a\int_{-\infty}^{a} f(x)\,dx - \int_{-\infty}^{a} x f(x)\,dx + \int_{a}^{\infty} x f(x)\,dx - a\int_{a}^{\infty} f(x)\,dx \\ &= aF(a) - \int_{-\infty}^{a} x f(x)\,dx + \int_{a}^{\infty} x f(x)\,dx - a[1-F(a)] \\ &= 2aF(a) - a - \int_{-\infty}^{a} x f(x)\,dx + \int_{a}^{\infty} x f(x)\,dx \end{aligned}$$

   Let's obtain the first and second derivatives:

$$\begin{aligned} \frac{d}{da}E|X-a| &= 2F(a) + 2af(a) - 1 - 2af(a) \\ &= 2F(a) - 1 \\ \frac{d^2}{d^2a}E|X-a| &= 2f(a) > 0 \end{aligned}$$

   The second derivative being strictly positive everywhere implies that the objective function is strictly convex. Therefore, the solution is unique and is characterized by the first order condition. Solving the first order condition yields

$$\begin{aligned} F(a^*) &= 1/2 \\ a^* &= m \end{aligned}$$

2. (from Ross, 2003) In answering a question on a multiple-choice test a student either knows the answer or guesses. Let $p$ be the probability that she knows the answer and $1-p$ that she guesses. Assume that a student who guesses will be correct with probability $1/m$ where $m$ is the number of multiple-choice alternatives. What is the conditional probability that a student knew the answer to a question given that she answered it correctly?

   This is an application of the Bayes' rule.

$$P\left(knew|correct\right)=\frac{P\left(correct|knew\right)P\left(knew\right)}{P\left(correct\right)}$$

$$\begin{aligned}
P\left(correct\right) &= P\left(correct|knew\right)P\left(knew\right)+P\left(correct|guessed\right)P\left(guessed\right)\\
&= 1\cdot p+\frac{1}{m}\left(1-p\right)=p\left(1-1/m\right)+1/m\\
&= \left[p\left(m-1\right)+1\right]/m
\end{aligned}$$

   Therefore,

$$P\left(knew|correct\right)=\frac{p}{\left[p\left(m-1\right)+1\right]/m}$$

3. Suppose the joint density of $X$ and $Y$ is given by

$$f\left(x,y\right)=\begin{cases}4y\left(x-y\right)e^{-(x+y)} & 0<x<\infty,\ 0\leq y\leq x\\ 0 & o/w\end{cases}$$

   Calculate $E\left[X|Y=y\right]$.

   Using the definition of this integral, we get:

$$E\left[X|Y=y\right]=\int xf_{X|Y}\left(x|y\right)dx=\int x\frac{f_{X,Y}\left(x,y\right)}{f_Y\left(y\right)}dx$$

   We need to calculate the marginal density of $Y$.

$$\begin{aligned}
f_Y\left(y\right) &= \int f_{X,Y}\left(x,y\right)dx=\int_y^\infty 4y\left(x-y\right)e^{-(x+y)}dx\\
&= 4ye^{-y}\int_y^\infty \left(x-y\right)e^{-x}dx\\
&= 4ye^{-y}\int_0^\infty we^{-y+w}dw, \qquad \text{where } w=x-y\\
&= 4ye^{-2y}\int_0^\infty we^{-w}dw=4ye^{-2y}
\end{aligned}$$

   Therefore,

$$\begin{aligned}
E\left[X|Y=y\right] &= \int_y^\infty x\frac{4y\left(x-y\right)e^{-(x+y)}}{4ye^{-2y}}dx=\int_y^\infty x\left(x-y\right)e^{-(x-y)}dx\\
&= \int_0^\infty \left(w+y\right)we^{-w}dw=E\left[W^2\right]+yE\left[W\right], \quad \text{where } W\sim exponential\left(1\right)\\
&= 2+y
\end{aligned}$$

   The last step can also be established by applying integration by parts several times.

4. Let $X_1$ and $X_2$ be independent $N\left(0,1\right)$ random variables, then show that $X_1/X_2\sim Cauchy$, i.e.

$$f\left(x\right)=\frac{1}{\pi\left(1+y^2\right)}$$

We will use the idea outlined in lecture to compute the density of $X_1/X_2$. Let $U = X_1/X_2$. To use the formula for transformed random variables, we need two equations to solve for the inverse function. Note that we are not given a second random variable, however, nothing tells us that we cannot come up with our own variable. To that end, we define $V = X_2$. Then, $X_1 = UV$ and $X_2 = V$. The Jacobian can be computed as follows:

$$J = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v$$

Now, using the formula derived/given in class, we get:

$$\begin{aligned} f_{U,V}(u,v) &= f_{X_1,X_2}(x_1,x_2) \cdot |J| \\ &= f_{X_1,X_2}(uv,v) \cdot |v| \end{aligned}$$

Note that since $X_1$ and $X_2$ are independent, their joint density is given by the product of their marginals, which yields

$$f_{X_1,X_2}(x_1,x_2) == \frac{1}{2\pi} \exp\left\{-\frac{1}{2}\left(x_1^2 + x_2^2\right)\right\}$$

Combining this with the previous expression yields:

$$\begin{aligned} f_{U,V}(u,v) &= \frac{1}{2\pi} \exp\left\{-\frac{1}{2}\left(u^2v^2 + v^2\right)\right\}|v| \\ &= \frac{1}{2\pi} \exp\left\{-\frac{1}{2}v^2\left(u^2 + 1\right)\right\}|v| \end{aligned}$$

Note that we only obtained the joint density of $U$ and $V$. But we are asked to compute the marginal density of $U$.

$$\begin{aligned} f_U(u) &= \int_{-\infty}^{\infty} \frac{1}{2\pi} \exp\left\{-\frac{1}{2}v^2\left(u^2 + 1\right)\right\}|v|\,dv \\ &= \int_0^{\infty} \frac{1}{\pi} \exp\left\{-\frac{1}{2}v^2\left(u^2 + 1\right)\right\} v dv \end{aligned}$$

This equality holds, since the integrand, $\frac{1}{2\pi}\exp\left\{-\frac{1}{2}v^2\left(u^2+1\right)\right\}|v|$, is symmetric around 0. In the second integral, let $y = \frac{1}{2}v^2\left(u^2+1\right)$. Then, $dy = v\left(u^2+1\right)dv$.

$$\begin{aligned} f_U(u) &= \int_0^{\infty} \frac{1}{\pi} e^{-y} \frac{dy}{1+u^2} = \frac{1}{\pi\left(1+u^2\right)}\left[-e^{-y}\right]_{y=0}^{y\to\infty} \\ &= \frac{1}{\pi\left(1+u^2\right)} \\ &\Rightarrow \quad U = X_1/X_2 \sim \textit{Cauchy}. \end{aligned}$$

5. A miner is trapped in a mine containing 3 doors. The first door leads to a tunnel that takes him to safety after two hours of travel. The second door leads to a tunnel that returns him to the mine after 3 hours of travel. The third door leads to a tunnel that returns him to his mine after 5 hours of travel. Assuming that the miner is at all times equally likely to choose any one of the doors, what is the expected length of time until the miner reaches safety. (Hint: Use the law of iterated expectations, $EX = E\left[E\left(X|Y\right)\right]$.)

   Let $X$ = time spent until the miner reaches safety, and $Y$ = the door chosen. Note that if we are to calculate this expectation as an integral over the marginal density of $X$, we will first need to derive the marginal density of $X$, which could be exhausting. Instead, we will express this expectation in a recursive manner. The miner chooses $Y = 1, 2$ or 3 with equal probability. Note that we are given that the miner is "memoryless", that is, after some number of trials and errors, he is as likely to choose any door as he was before. He doesn't "learn". Then, we can express the expectation by conditioning on $Y$ first and then integrating over $Y$. It is easy to see that

$$\begin{aligned} E\left[X|Y=1\right] &= 2 \\ E\left[X|Y=2\right] &= 3+EX \\ E\left[X|Y=1\right] &= 5+EX \end{aligned}$$

This gives the following:

$$\begin{aligned} EX &= \frac{1}{3}2+\frac{1}{3}\left(3+EX\right)+\frac{1}{3}\left(5+EX\right) \\ &= \frac{10}{3}+\frac{2}{3}EX \\ &\Rightarrow EX=10 \end{aligned}$$