Econ 897 Math Camp Part III **Fatih Karahan**
Summer 2009 **University of Pennsylvania**

Quiz #1 (Friday, August 21)

1. Show that if the constraint set C is convex and the objective function f is concave, then the set $\arg\max\{f(x) : x \in C\}$ is convex.

   Let x_1 and $x_2 \in S = \arg\max\{f(x) : x \in C\}$, $\theta \in [0,1]$ and $x^\theta = \theta x_1 + (1-\theta)x_2$. Since C is convex, $x^\theta \in C$. Furthermore, since f is concave,

   $$f\left(x^\theta\right) \geq \theta f(x_1) + (1-\theta) f(x_2) = f(x_1) = f(x_2)$$

   Thus, $f\left(x^\theta\right) \geq f(x) \;\; \forall x \in C$. So, $x^\theta \in S$.

2. True or False (If you think the statement is true, provide a proof, otherwise provide a counterexample.)

   (a) The maximum of two concave functions, f_1 and f_2, defined as $\hat{f}(x) = \max\{f_1(x), f_2(x)\}$, is also concave.
   False: Let $f_1(x) = x$ and $f_2(x) = \log(x)$. It is easy to see that $\hat{f}(x)$ is not concave.

   (b) The minimum of two concave functions, f_1 and f_2, defined as $\check{f}(x) = \min\{f_1(x), f_2(x)\}$, is also concave.
   True: Let $x_1,\; x_2 \in X$, $\theta \in [0,1]$ and $x^\theta = \theta x_1 + (1-\theta)x_2$. $\check{f}\left(x^\theta\right) = \min\left\{f_1\left(x^\theta\right), f_2\left(x^\theta\right)\right\}$. Note that we have

   $$f_i\left(x^\theta\right) \geq \theta f_i(x_1) + (1-\theta) f_i(x_2).$$

   Therefore,

   $$\begin{aligned}
   \check{f}\left(x^\theta\right) &= \min\left\{f_1\left(x^\theta\right), f_2\left(x^\theta\right)\right\} \\
   &\geq \min\{\theta f_1(x_1) + (1-\theta) f_1(x_2), \theta f_2(x_1) + (1-\theta) f_2(x_2)\} \\
   &\geq \theta \min\{f_1(x_1), f_2(x_1)\} + (1-\theta)\min\{f_1(x_2), f_2(x_2)\} \\
   &\geq \theta \check{f}(x_1) + (1-\theta)\check{f}(x_2)
   \end{aligned}$$

   So, $\check{f}$ is concave.

3. Write down the utility maximization problem of a household facing a price vector $p \gg 0$ and has an income $I > 0$. Write down the KKT conditions. State any *additional* assumptions that you need to make these conditions *both* necessary and sufficient.

   $$\max u(x)$$

   $$\begin{aligned}
   \text{s. to } I - px &\geq 0 \\
   x &\geq 0
   \end{aligned}$$

   Let $\lambda_1 \in \mathbb{R}_+$ be the Lagrange multiplier for the budget constraint and let $\lambda_2 \in \mathbb{R}^n_+$ be the Lagrange multiplier for nonnegativity constraints. Then KKT conditions are

   $$\begin{aligned}
   Du(x) - \lambda_1 p + \lambda_2 &= 0 \\
   \lambda_1 [I - px] + \lambda_2 x &= 0
   \end{aligned}$$

   For necessity, we need u to be C^1. Note that CQ is satisfied since the constraints are linear.

   For sufficiency, we need the objective function to be pseudo-concave. Note that since the constraints are linear, they are quasi-concave and so the second condition in the sufficiency theorem is already satisfied.

4. Let $f : X \to \mathbb{R}$ be a concave function, where $X \subseteq \mathbb{R}$ is open and convex. Let x_1, x_2 and x_3 be points in X satisfying $x_1 < x_2 < x_3$. Then, show that the following inequalities hold:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \geq \frac{f(x_3) - f(x_1)}{x_3 - x_1} \geq \frac{f(x_3) - f(x_2)}{x_3 - x_2}$$

Define $\alpha = (x_2 - x_1) / (x_3 - x_1)$, then $\alpha \in (0, 1)$ and $1 - \alpha = (x_3 - x_2) / (x_3 - x_1)$, and it straightforward to check that $\alpha x_3 + (1 - \alpha) x_1 = x_2$. Since f is concave, we have

$$\begin{aligned} f(x_2) &\geq \alpha f(x_3) + (1 - \alpha) f(x_1) \Rightarrow \\ (1 - \alpha)[f(x_2) - f(x_1)] &\geq \alpha [f(x_3) - f(x_2)] \Rightarrow \end{aligned}$$

$$(x_3 - x_2)[f(x_2) - f(x_1)] \geq (x_2 - x_1)[f(x_3) - f(x_2)] \qquad (*)$$

Adding $(x_2 - x_1)[f(x_2) - f(x_1)]$ to both sides and arranging terms, gives

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \geq \frac{f(x_3) - f(x_1)}{x_3 - x_1}$$

Adding $(x_3 - x_2)[f(x_3) - f(x_2)]$ to both sides and arranging terms, gives

$$\frac{f(x_3) - f(x_1)}{x_3 - x_1} \geq \frac{f(x_3) - f(x_2)}{x_3 - x_2}$$

Combining, we get the desired result.

5. (No-arbitrage condition)

Suppose there are J financial assets in the economy. An asset is a contract that specifies for each state of the world in the next period, $s = 1, 2, \ldots, S$, how many units of the consumption good the seller of the contract has to give to the buyer. (So, there is uncertainty about how much income the agent will have tomorrow, and he wants to insure himself against this risk by buying and selling these assets.) In other words, an asset is a vector $d^j = [d_1^j, d_2^j, \ldots, d_S^j]^T \in \mathbb{R}^S$. Define the payoff matrix

$$D = \begin{bmatrix} d_1^1 & \ldots & d_1^J \\ d_2^1 & \ldots & d_2^J \\ \vdots & \ldots & \vdots \\ d_S^1 & \ldots & d_S^J \end{bmatrix}$$

Assume that D has full column rank ($J \leq S$). Suppose that the price of asset j in the market is given by q_j. Define $q \in \mathbb{R}^J$ as $q = [q_1, \ldots, q_J]$.

Def: A security price system $q \in \mathbb{R}^J$ *precludes arbitrage* if it is not possible to achieve a positive income stream (income from assets) in all states by trading in assets; i.e. there does not exist $\theta \in \mathbb{R}^J$ such that $q^T\theta \leq 0$ and $D\theta > 0$ or such that $q^T\theta < 0$ and $D\theta \geq 0$.

We will prove the following theorem:

Theorem: A security price system $q \in \mathbb{R}^J$ precludes arbitrage if and only if there exists a vector $\alpha \in \mathbb{R}_{++}^S$ such that $q = \alpha^T D$.

(a) Let $M = \{(-q\theta, D\theta) : \theta \in \mathbb{R}^J\} \subseteq \mathbb{R}^{S+1}$. Argue that q precludes arbitrage iff $M \cap \mathbb{R}_+^{S+1} = \{0\}$.
If $M \cap \mathbb{R}_+^{S+1} = \{0\}$, then almost trivially, $\nexists \theta \in \mathbb{R}^J$ such that $q^T\theta \leq 0$ and $D\theta > 0$ or such that $q^T\theta < 0$ and $D\theta \geq 0$. On the other hand, if $M \cap \mathbb{R}_+^{S+1} \neq \{0\}$, then by the definition of M, $\exists \theta \in \mathbb{R}^J$ such that $-q^T\theta \geq 0$ and $D\theta > 0$ (or $-q^T\theta > 0$ and $D\theta \geq 0$) and this means there is an arbitrage opportunity.

(b) ($\Leftarrow$) Suppose that $\exists\alpha \in \mathbb{R}^S_{++}$ such that $q = \alpha^T D$ and that $\exists x \in M \cap \mathbb{R}^{S+1}_+ \setminus \{0\}$. Show that $\left(1, \alpha^T\right) x = 0$ and argue that this is a contradiction. This establishes sufficiency part of the theorem.Since $x \in M$, $\exists\theta \in \mathbb{R}^J$ s.t. $x = (-q\theta, D\theta)$. Then,$\left(1, \alpha^T\right) x = -q\theta + \alpha^T D\theta = \overbrace{\left(-q + \alpha^T D\right)}^{=0} x = 0$. But, this is a contradiction since $x \in \mathbb{R}^{S+1}_+$ and $\left(1, \alpha^T\right) \gg 0$. We have now established that if $\exists\alpha \in \mathbb{R}^S_{++}$ such that $q = \alpha^T D$, we must have $M \cap \mathbb{R}^{S+1}_+ \setminus \{0\} = \emptyset$, i.e. $M \cap \mathbb{R}^{S+1}_+ \setminus \{0\} = \{0\}$.

In what follows, we will make use of the following separating hyperplane theorem:

Thm: Let M, K be convex cones such that $M \cap K = \{0\}$. If K is not a linear subspace, then $\exists p \neq 0$ s.t. $px < py$ $\forall x \in M$, $\forall y \in K \setminus \{0\}$.

($\Rightarrow$) Suppose that q precludes arbitrage.

(c) Use the separating hyperplane theorem above, to argue that $\exists\mu \neq 0$ such that $\mu x < \mu y$ $\forall x \in M$, $y \in \mathbb{R}^{S+1}_+ \setminus \{0\}$. Show that $\mu x = 0$ $\forall x \in M$.

Since q precludes arbitrage, by (a), $M \cap \mathbb{R}^{S+1}_+ = \{0\}$. We will now separate M from $\mathbb{R}^{S+1}_+$. Note that M is a convex cone, since it is a linear subspace (if you're interested, try to show that any linear subspace is a convex cone). It can also be shown that $\mathbb{R}^{S+1}_+$ is a convex cone and is not a linear subspace. Now that we verified all the conditions of the separating hyperplane theorem given above, we are ready to use it. This gives us $\mu \neq 0$ such that $\mu x < \mu y$ $\forall x \in M$ and $\forall y \in \mathbb{R}^{S+1}_+ \setminus \{0\}$.

Now fix $x \in M$. Let $y \in \mathbb{R}^{S+1}_+ \setminus \{0\}$ approach 0. Since $\mu x < \mu y$, in the limit we have $\mu x \leq 0$. Now note that $-x \in M$, so that $-\mu x < \mu y$. Using a similar limiting argument, we show that $-\mu x \leq 0$. These two inequalities imply $\mu x = 0$. Since $x \in M$ was arbitrary, this holds for all $x \in M$.

(d) Also argue that we must have $\mu y > 0$ $\forall y \in \mathbb{R}^{S+1}_+ \setminus \{0\}$. Show that this implies $\mu_s > 0$ $s = 1, 2, \ldots, S+1$.

Since $\mu x = 0$ for all $x \in M$, using (c), we get that $\mu y > 0$ $\forall y \in \mathbb{R}^{S+1}_+ \setminus \{0\}$. Let $y_s = [0, \ldots, \overset{s^{th} \text{ position}}{1}, \ldots, 0]$. Clearly, $y_s \in \mathbb{R}^{S+1}_+ \setminus \{0\}$. Therefore, we have $\mu y_s = \mu_s > 0$.

(e) Define $\alpha_s = \frac{\mu_s}{\mu_1}$. Let $\alpha = [\alpha_2, \ldots, \alpha_{S+1}]$. Note that $\alpha \gg 0$. Show that $q = \alpha^T D$. This completes the necessity. Congratulations!

By the result in (d), we have $\forall\theta \in R^J$

$$\begin{aligned} -\mu q\theta + \mu D\theta &= 0 \quad \Rightarrow \\ -q\theta + \alpha D\theta &= 0 \quad \Rightarrow \\ (-q + \alpha D)\,\theta &= 0 \quad \Rightarrow \\ q &= \alpha D \end{aligned}$$

Significance of this result: No-arbitrage condition is essential in any equilibrium concept. If there was an arbitrage opportunity in an economy where there are some people with increasing preferences (and if people have perfect information about these opportuities), then they would make an infinite amount of profit just by trading in these assets. So the prices would have to adjust to prevent this. However, as it is defined, no-arbitrage is hard to check. This result makes the problem somewhat more tractable. Also note that even people that don't believe in equilibrium notions take the no-arbitrage condition as a reasonable requirement.

The vector α, which we just showed to exist, has an economic meaning as well. In an economy with complete Arrow securities, the vector α will denote the price of Arrow securities in equilibrium. Arrow securities are very simple assets; they are one period contingent claims, say for state j. This means that if you buy an Arrow security for state j today, then the asset pays you one unit of the consumption good only if state j happens. Otherwise, it doesn't pay anything. Arrow securities are extremely easy to price, because of the simplicity of the contract. Furthermore, one can show that you can price many complicated assets, derivatives etc. that you see in Wall Street using these Arrow securities. So the vector α has a fundamental importance.