Math Camp Part 3
Assignment 3: On Probability

1. Let the probability density of X be given by

$$f(x) = \begin{cases} c\left(4x - 2x^2\right) & 0 < x < 2 \\ 0 & o/w \end{cases}$$

   (a) What is the value of c?

   (b) Calculate $P(1/2 < X < 3/2)$.

2. Let c be a constant. Show that

   (a) $Var(cX) = c^2 Var(X)$

   (b) $Var(c + X) = Var(X)$

3. Suppose that X is a random variable with mean 10 and variance 15. What can we say about $P(5 < X < 15)$? (Chebychev)

4. Calculate the moment generating function of a normal distribution with mean μ and variance σ^2. Recall that the pdf of such a random variable X is given by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

5. For any two random variables X and Y with finite variances, prove that:

   (a) $Cov(X, Y) = Cov(X, E[Y|X]\ )$

   (b) X and $Y - E(Y|X)$ are uncorrelated.

   (c) $Var(Y - E[Y|X]) = E[Var(Y|X)]$

6. Let $X_1,\ X_2, \ldots, X_n$ be independent and identically distributed continuous random variables with probability distribution F and density function f. Defien $X_{(i)}$ to be the i^{th} smallest of these random variables. $X_{(1)}, \ldots, X_{(n)}$ are called *order statistics*.

   (a) Obtain the cdf of $X_{(i)}$, i.e. express $P\left(X_{(i)} \leq x\right)$ using F. Note that $X_{(i)} \leq x$ happens if and only if at least i of the n random variables $X_1, X_2, \ldots, X_n$ are less than equal to x.

   (b) Differentiate $F_{X_{(i)}}(x)$ wrt x to obtain the pdf of $X_{(i)}$.

7. A multivariate random variable $X \in \mathbb{R}^k$ is a multivariate normal random variable whenever $a'X$ is a normal random variable, $\forall a \in \mathbb{R}^k$. Its pdf is given by

$$f(x) = \frac{1}{(2\pi)^{-k/2} \left|\sum\right|^{-1/2}} \exp\left\{-\frac{1}{2}(x-\mu) \sum^{-1/2} (x-\mu)\right\}$$

   Let X and Y be independent $N(0,1)$ random variables, and define a new random variable Z by

$$Z = \begin{cases} X & \text{if } XY > 0 \\ -X & \text{if } XY < 0 \end{cases}$$

(a) Show that Z has a normal distribution.

(b) Show that the joint distribution of Z and Y is not bivariate normal. (Hint: Show that Z and Y always have the same sing and that this implies that they are not bivariate normals.)

8. $\theta \sim U(-\pi/2, \pi/2)$, then $X = \tan\theta \sim \mathit{Cauchy}$, i.e. show that

$$f(x) = \frac{1}{\pi(1+y^2)}$$