Math Camp Part 3
Assignment 2: More on Optimization

Due date: Wednesday, 8/26/09
Please solve these questions in groups and return one copy per group.

1. Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$, where X is convex. Suppose f is a strictly quasi-concave function. If f is also a concave function, is it *necessarily* strictly concave?

2. Prove sufficiency of the third condition for constraint qualification.

3. Consider the problem below:

$$\max f(x) \quad \text{s. to } g(x) \geq 0 \tag{P}$$

Let x^* be a solution to (P). Let g^* represent the constraints that are binding at x^*, i.e. $g^*(x^*) = 0$. Define the following problems

$$\max Df(x^*)\, x \quad \text{s. to } Dg(x^*)(x^* - x) \geq 0 \tag{P1}$$

$$\max Df(x^*)\, x \quad \text{s. to } Dg^*(x^*)(x^* - x) \geq 0 \tag{P2}$$

Show that if x^* solves $(P1)$, then it solves $(P2)$.

4. (Geometry of quasi-concave functions) Suppose that $X \subseteq \mathbb{R}$ is compact and convex and that $f : X \to \mathbb{R}$ is C^0.
   (a) Show that $X = [\check{x}, \hat{x}]$ with $\check{x} \leq \hat{x}$.
   (b) Show that f is quasi-concave iff there exists some $x^* \in X$ such that f is increasing for $x \in X,\ x \leq x^*$ and decreasing for $x \in X,\ x \geq x^*$.
   (c) (for the intrepid) How can one generalize the result in (b) if now $X \subseteq \mathbb{R}^n$ for $n > 1$.

5. A firm uses two inputs, labor (l) and raw material (m), to produce a single output y. The production function is given by $y = f(l, m)$. The output sells for a price of p, while labor has a unit cost of w. The firm has in its stock 4 units of the raw material m. Additional units may be purchased from the market at a price of c. The firm can also sell unused raw material in the stock in the market at the price c.
   (a) Setup the firm's profit maximization problem and discuss existence.
   (b) Write down the Karush-Kuhn-Tucker conditions and describe under what conditions on primitives, these conditions are also sufficient.
   (c) Let $p = w = m = 1$, and let $f(l, m) = l^{1/3}m^{1/3}$. Calculate the firm's optimal choice of actions.

6. An agent who consumes three consumption goods has a utility function given by

$$u(x_1, x_2, x_3) = x_1^{1/3} + \min\{x_2, x_3\}$$

(a) Given an income of I and prices of $p_1,\ p_2,\ p_3 \in \mathbb{R}_{++}$, write down the consumer's utility maximization problem. Can the Weierstrass thm be applied to ensure existence?

(b) Can we use KKT conditions to characterize the optimal solution? Why or why not?

(c) Solve for consumer's optimal consumption bundle.