Math Camp Part 3
Assignment 1: Existence and Uniqueness of Solutions, Concavity and Quasi-Concavity

Due date: Thursday, 8/20/09
Please solve these questions in groups and return one copy per group.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1 function. Show that if $x^* \in \mathbb{R}$ is such that $f'(x^*) = 0,\ f'(x) > 0\ \forall x < x^*$ and $f'(x^*) < 0\ \forall x > x^*$ then x^* must be a local optimum (in this case global as well).

2. Give an example of a function $f : [0,1] \to \mathbb{R}$ such that $f([0,1])$ is an open set. Is it possible to find a continuous function with this attribute? Why or why not?

3. Prove the following or give a counterexample: If f is a continuous function on a bounded set C, then $\sup f(C)$ is finite.

4. Let $f : \mathbb{R}_+ \to \mathbb{R}$ be continuous. Suppose that $f(0) = 1$ and $\lim_{x\to\infty} f(x) = 0$. Prove that f has a maximum or provide a counterexample. Do the same exercise for a minimum.

5. A monopolist faces a downward sloping inverse-demand curve $p(x)$ that satisfies $p(0) < \infty$ and $p(x) \geq 0$ for all $x \in \mathbb{R}_+$. The cost of producing x units is given by $c(x) \geq 0$ where $c(0) = 0$. Suppose that both $\not p$ and c are continuous. The monopolist wishes to maximize $\pi(x) = p(x)x - c(x)$ subject to $x \geq 0$.

   (a) Suppose there exists x^* such that $p(x^*) = 0$. Show that the Weierstrass theorem can be used to prove existence of a solution.

   (b) Now suppose there exists $x' > 0$ such that $c(x) \geq xp(x)$ for all $x \geq x'$. Show that the Weierstrass theorem can be used to prove existence.

6. Show quasi-concavity is preserved under monotone transformations but not under non-negative summation.

7. Consider the assumptions of the (first) uniqueness theorem proved in class. For each condition of the theorem, show that

   (a) the conclusion of the theorem does not hold if we relax that assumption

   (b) the conclusion of the theorem may still hold if we relax that assumption (means the theorem provides sufficient conditions and are not necessary).

8. Let $f : \mathbb{R}^2_+ \to \mathbb{R}$ with $f(x,y) = x^\alpha y^{1-\alpha}$. Show that f is not concave if $\alpha > 1$.