Concave and Quasi-Concave Functions

Assume that $X \subset \mathbb{R}^n$ is convex and open, $f : X \to \mathbb{R}$.

- f is *concave* if

for f continuous

$$\{(y,x) \in \mathbb{R}^{n+1} : x \in X \;\&\; y \leqq f(x)\}$$

is convex

$\Updownarrow$

$$x', x'' \in X \;\&\; 0 \leqq \theta \leqq 1 \Rightarrow$$
$$f((1-\theta)x' + \theta x'') \geqq (1-\theta)f(x') + \theta f(x'')$$

$\Leftrightarrow$

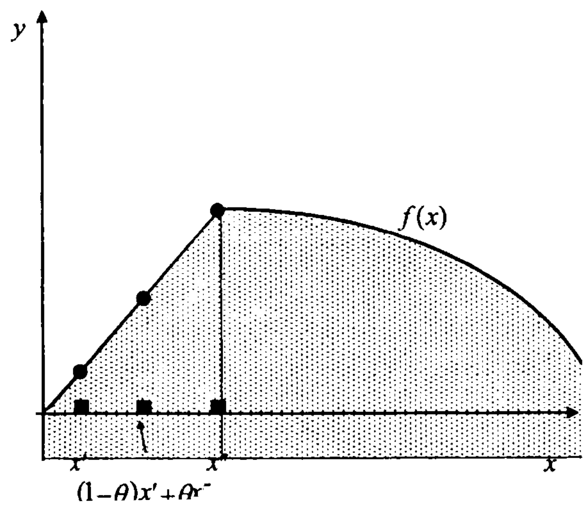
for f differentiable

$$x', x'' \in X$$
$$f(x'') \leqq f(x') + Df(x')(x'' - x')$$

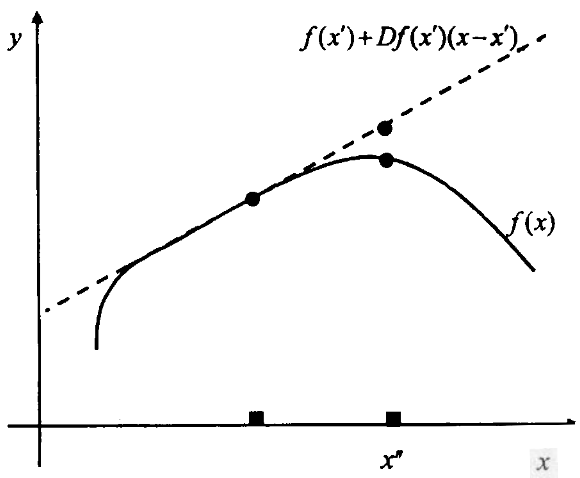
$\Leftrightarrow$

for f twice differentiable

$$x \in X \;\&\; \Delta x \in \mathbb{R}^n \Rightarrow$$
$$\Delta x^T D^2 f(x) \Delta x \leqq 0$$



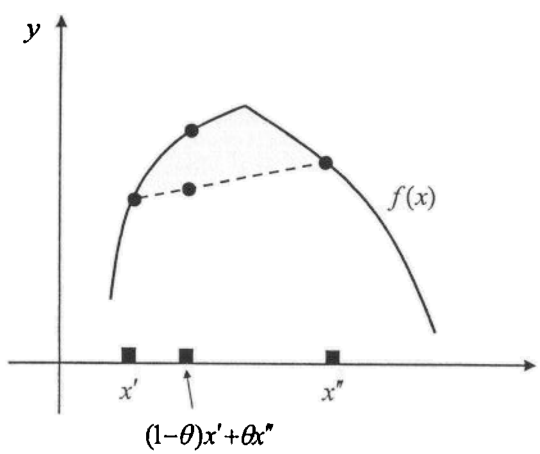
both "flats" and "kinks" are possible



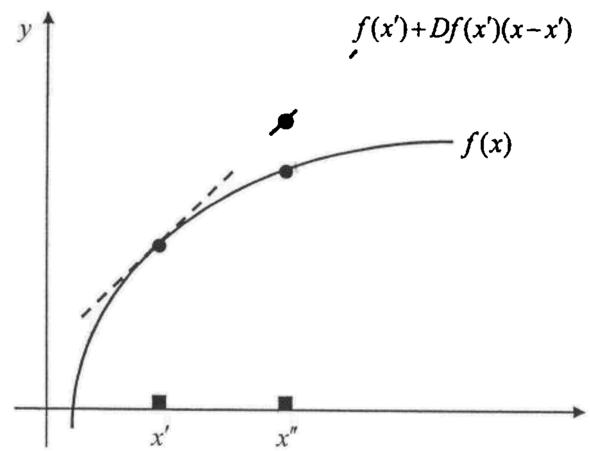
"flats" but not "kinks" are possible

- f is *strictly concave* if it is concave and

for f continuous		for f differentiable		for f twice differentiable
$x', x'' \in X$ & $0 < \theta < 1 \Rightarrow$ $f((1-\theta)x' + \theta x'') > (1-\theta)f(x') + \theta f(x'')$	$\Leftrightarrow$	$x', x'' \in X$ & $x' \neq x'' \Rightarrow$ $f(x'') < f(x') + Df(x')(x'' - x')$	$\Leftarrow$	$x \in X, \Delta x \in \mathbb{R}^n$, & $\Delta x \neq 0 \Rightarrow$ $\Delta x^T D^2 f(x) \Delta x < 0$



"kinks" but not "flats" are possible



neither "flats" nor "kinks" are possible

Note: When X is open (as well as convex), (i) if f is concave, then it is continuous (so that, in fact, for these cases the labelling of the first column is redundant), while (ii) it is meaningful to assume, for example, that f is differentiable (i.e., that f has a first-order differential at $x \in X$).

• f is *quasi-concave* if

for f continuous

$\{x \in X : f(x) \geqq \alpha)$ is convex for every $\alpha \in \mathbb{R}$

$\Updownarrow$

$x', x'' \in X \;\&\; 0 \leqq \theta \leqq 1$
$f((1-\theta)x' + \theta x'') \geqq \min\{f(x'), f(x'')\}$

$\Leftrightarrow$

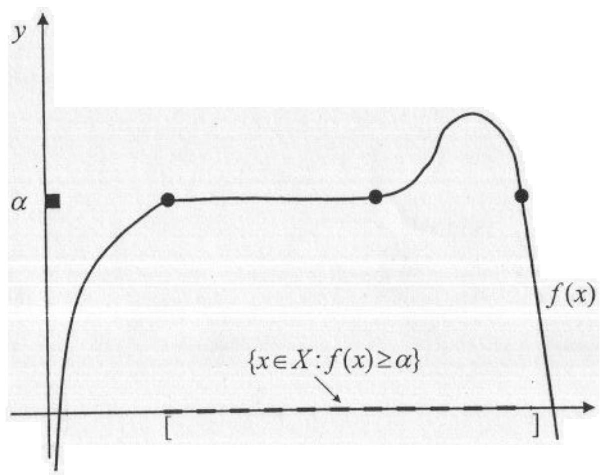
for f differentiable

$x', x'' \in X \;\&\; f(x'') - f(x') \geqq 0 \Rightarrow$
$Df(x')(x'' - x') \geqq 0$

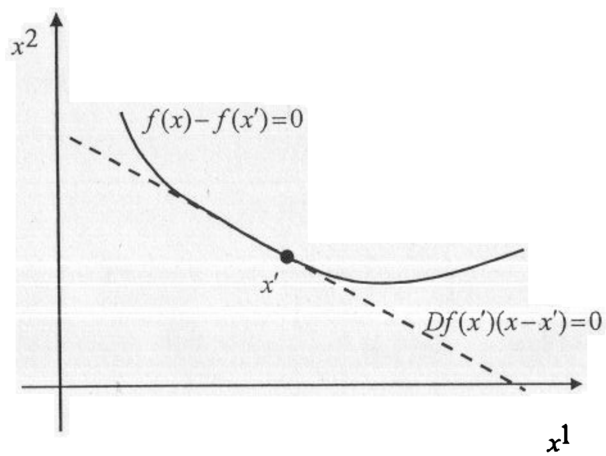
$\Rightarrow$

for f twice differentiabl

$x \in X, \Delta x \in \mathbb{R}^n,$
$\& \; Df(x)\Delta x = 0 =$
$\Delta x^T Df(x) \Delta x \leqq 0$

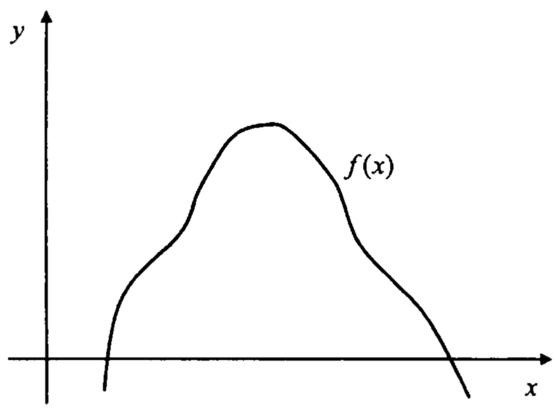


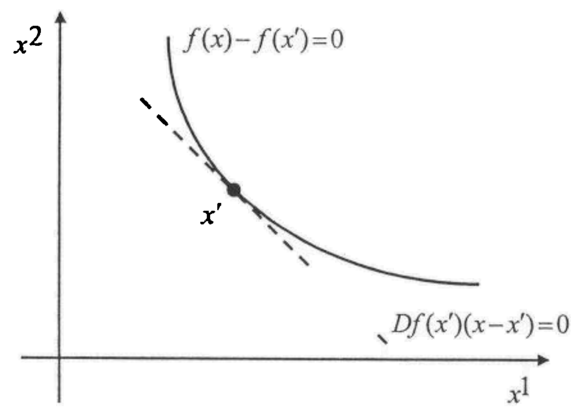
f may be "single-peaked," and have "flats"



- f is *strictly quasi-concave* if it is quasi-concave and

for f continuous		for f differentiable		for f twice differentiable
$x', x'' \in X, x' \neq x'', \& \ 0 < \theta < 1 \Rightarrow$ $f((1-\theta)x' + \theta x'') > \min\{f(x'), f(x'')\}$	$\Leftarrow$	$x', x'' \in X, x' \neq x''$, $\& \ f(x'') - f(x') \geqq 0 \Rightarrow$ $Df(x')(x'' - x') > 0$	$\Leftarrow$	$x \in X, \Delta x \in \mathbb{R}^n, \Delta x \neq 0$, $\& \ Df(x)\Delta x = 0 \Rightarrow$ $\Delta x^T Df(x) \Delta x < 0$





f may be "single-peaked," but has no "flats"

Note: There are alternative possible definitions of strict quasi-concavity when f is differentiable. The definition adopted here is the most convenient (and frequently used) in economic applications.

Some further particularizations of quasi-concavity which are employed in economic analysis:

- f is *semi-strictly quasi-concave* if it is quasi-concave and

$$x', x'' \in X, f(x') \neq f(x''), \ \& \ 0 < \theta < 1 \Rightarrow$$
$$f((1-\theta)x' + \theta x'') > \min\{f(x'), f(x'')\}.$$

  This property rules out "thick" level surfaces (but permits "flats").

- f is *pseudo-concave* if it is differentiable and

$$x', x'' \in X \ \& \ f(x'') - f(x') > 0 \Rightarrow$$
$$Df(x')(x'' - x') > 0.$$

  This property rules out the "false signal" where $Df(x')(x'' - x') = 0$ but in fact $f(x'') - f(x') > 0$. Note that (i) a pseudo-concave function is also a quasi-concave function (but not conversely), and (ii) a differentiable, concave or strictly quasi-concave function is also a pseudo-concave function (but not conversely).