Bellman Equations in Continuous Time

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Dynamic Programming in Continuous Time

- You have been exposed to Hamiltonian as a procedure to solve continuous time dynamic optimization problems.

- Hamiltonians are difficult to generalize to stochastic environments.

- At the same time, there are many problems in macro with uncertainty which are easy to formulate in continuous time.

Filtration

- Fix a probability space $(\Omega, \mathcal{F}, P)$.

- Define $t \in [0, \infty) = \mathbb{R}_+$.

- Filtration: a family $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ of increasing $\sigma-$algebras contained in $\mathcal{F}$:

$$\mathcal{F}_s \subseteq \mathcal{F}_t \text{ for } \forall s \leq t \text{ and } \mathcal{F}_t \subseteq \mathcal{F}$$

- Clearly, $\mathcal{F}_\infty = \mathcal{F}$ is the smallest $\sigma-$algebras containing $\forall \mathcal{F}_t$.

- $(\Omega, \mathbb{F}, P)$: filtered probability space.

Stochastic Processes

- Continuous-time stochastic process: a mapping $x : [0, \infty) \times \Omega \to \mathbb{R}$ which is measurable with respect to $\mathcal{B}_+ \times \mathcal{F}$ (where $\mathcal{B}_+$ are the Borel sets of $\mathbb{R}_+$).

- A stochastic process is adapted to $\mathbb{F}$ if $x(t, \omega)$ is $\mathcal{F}_t$-measurable $\forall t$.

- $x(t, \cdot)$: $\mathcal{F}_t$-measurable function of ω.

- $x(\cdot, \omega)$: realization, trajectory, or sample path of the process.

- Continuous stochastic process: $x(\cdot, \omega) \in C[0, \infty)$, a.e. $\omega \in \Omega$.

Brownian Motion

- A Wiener processes (or Brownian motion) is a stochastic process W having:
    1. continuous sample paths.
    2. independent increments.
    3. $W(t) \sim N(0, t), \forall t$.

- Then: $\mathbb{E}[dW] = 0$ and $\mathbb{E}\left[(dW)^2\right] = dt$.

- Basic result: if a stochastic process $\{X(t), t \geq 0\}$ has continuous sample paths with stationary, independent, and i.i.d. increments, then it is a Wiener process.

Diffusion

- A Brownian motion with drift:

$$dX(t) = \mu dt + \sigma dW(t) \text{ with } X(0) = x_0$$

- More generally, we have a diffusion:

$$dX(t) = \mu(t, x) dt + \sigma(t, x) dW(t) \quad \forall t, \forall \omega$$

- Diffusion are important in arbitrage-free asset pricing. Aït-Sahalia (2006).

Functions of Stochastic Processes

- Let $F(t, x)$ be a function that is at least once differentiable in t and twice in x.

- We approximate the total differential of $F(t, X(t, \omega))$ by a Taylor expansion:

$$dF = F_t dt + F_x dX + \frac{1}{2} F_{xx} (dX)^2 + ...$$

- We substitute in:

$$\begin{aligned} dF &= F_t dt + F_x [\mu dt + \sigma dW] \\ &\quad + \frac{1}{2} F_{xx} \left[\mu^2 (dt)^2 + 2\mu\sigma dt dW + \sigma^2 (dW)^2\right] + ... \end{aligned}$$

- Note we drop the terms that have order higher than dt or $(dW)^2$

$$dF = F_t dt + \mu F_x dt + \sigma F_x dW + \frac{1}{2}\sigma^2 F_{xx} (dW)^2 .$$

- Remember that since $\mathbb{E}[dW] = 0$ and $\mathbb{E}\left[(dW)^2\right] = dt$, then

$$\begin{aligned} \mathbb{E}[dF] &= \left[F_t + \mu F_x + \frac{1}{2}\sigma^2 F_{xx}\right] dt \\ Var[dF] &= \mathbb{E}[dF - \mathbb{E}[dF]]^2 = \sigma^2 F_x^2 dt \end{aligned}$$

- Particular case $F(t,x) = e^{-rt} f(x)$:

$$\mathbb{E}[dF] = \left[-rf + \mu f' + \frac{1}{2}\sigma^2 f''\right] e^{-rt} dt$$

and when $r = 0$
$$\mathbb{E}[dF] = \left[\mu f' + \frac{1}{2}\sigma^2 f''\right] dt$$

Value Function

- Consider the problem:

$$v(x_0) = \mathbb{E}\left[\int_0^\infty e^{-\rho t}\pi(X(t,\omega))\,dt | X(0) = x_0\right]$$
$$s.t.\ dX(t) = \mu(x)\,dt + \sigma(x)\,dW(t)$$

- If π is continuous and bounded, the integral is well defined.

Bellman-Type Property

- Given a small interval of time Δt, we get:

$$v\left(x_0\right) \approx \pi\left(x_0\right)\Delta t + \frac{1}{1+\rho\Delta t}\mathbb{E}\left[v\left(X\left(0+\Delta t\right)\right)|X\left(0\right)=x_0\right]$$

- Multiply by $(1+\rho\Delta t)$ and substrate $v\left(x_0\right)$:

$$\rho v\left(x_0\right)\Delta t \approx \pi\left(x_0\right)\Delta t\left(1+\rho\Delta t\right) + \mathbb{E}\left[\Delta v|X\left(0\right)=x_0\right]$$

- Divide by Δt

$$\rho v\left(x_0\right) \approx \pi\left(x_0\right)\left(1+\rho\Delta t\right) + \frac{1}{\Delta t}\mathbb{E}\left[\Delta v|X\left(0\right)=x_0\right]$$

- Letting $\Delta t \rightarrow 0$ and taking the limit

$$\rho v\left(x_0\right) = \pi\left(x_0\right)\left(1+\rho\Delta t\right) + \frac{1}{dt}\mathbb{E}\left[dv|X\left(0\right)=x_0\right]$$

Hamilton-Jacobi-Bellman Equation

- Given a small interval of time Δt, we get:

$$\rho v(x) = \pi(x) + \frac{1}{dt}\mathbb{E}\left[dv \mid X(0) = x_0\right]$$

- Applying

$$\mathbb{E}[dv] = \left[\mu v' + \frac{1}{2}\sigma^2 v''\right] dt$$

we have

$$\rho v(x) = \pi(x) + \mu(x)\, v'(x) + \frac{1}{2}\sigma^2(x)\, v''(x) \quad \forall x$$