A Production OLG Model

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Introducing Production in an OLG Model

- Are the properties of OLG models consequence of the absence of production?

- Diamond (1965).

- We want to have models with production for policy purposes.

- Tradition of Auerbach and Kotlikoff (1987).

Demographics

- Individuals live for two periods
- N_t^t: number of young people in period t.
- N_t^{t-1}: number of old people at period t.
- Normalize the size of the initial old generation to 1, i.e. $N_0^0 = 1$.
- People do not die early, $N_t^t = N_{t+1}^t$.
- Population grows at constant rate n:

$$N_t^t = (1+n)^t N_0^0 = (1+n)^t$$

- The total population at period t:

$$N_t^{t-1} + N_t^t = (1+n)^t \left(1 + \frac{1}{1+n}\right)$$

Preferences

- Preferences over consumption streams given by

$$u(c_t^t, c_{t+1}^t) = U(c_t^t) + \beta U(c_{t+1}^t)$$

- U is strictly increasing, strictly concave, twice continuously differentiable and satisfies the Inada conditions.

- All individuals are assumed to be purely selfish and have no bequest motives whatsoever.

- The initial old generation has preferences

$$u(c_1^0) = U(c_1^0)$$

Endowments

- Each individual of generation $t \geq 1$ has as endowments one unit of time to work when young and no endowment when old.
- How we can generalize it?
  1. Life cycle profile of productivity.
  2. Leisure in the utility function.
- Hence the labor force in period t is of size N_t^t with maximal labor supply of $1 * N_t^t$.
- Each member of the initial old generation is endowed with capital stock $(1+n)\bar{k}_1 > 0$.

Firms

- Constant returns to scale technology

$$Y_t = F(K_t, L_t)$$

- Profits are zero in equilibrium and we do not have to specify ownership of firms.

- Single, representative firm that behaves competitively in that it takes as given the rental prices of factor inputs (r_t, w_t) and the price for its output.

- Defining the capital-labor ratio $k_t = \frac{K_t}{L_t}$, we have:

$$y_t = \frac{Y_t}{L_t} = \frac{F(K_t, L_t)}{L_t} = F\left(\frac{K_t}{L_t}, 1\right) = f(k_t)$$

- We assume that f is twice continuously differentiable, strictly concave, and satisfies the Inada conditions.

Timing

1. At the beginning of period t, production takes place with labor of generation t and capital saved by the now old generation $t-1$ from the previous period. The young generation earns a wage w_t

2. At the end of period t, the young generation decides how much of the wage income to consume, c_t^t, and how much to save for tomorrow, s_t^t. The saving occurs in form of physical capital, which is the only asset in this economy

3. At the beginning of period $t+1$, production takes place with labor of generation $t+1$ and the saved capital of the now old generation t. The return on savings equals $r_{t+1}-\delta$, the real interest rate from period t to $t+1$.

4. At the end of period $t+1$ generation t consumes its savings plus interest rate, i.e. $c_{t+1}^t = (1 + r_{t+1} - \delta)s_t^t$ and then dies.

Sequential Markets Equilibrium

Given $\bar{k}_1$, a sequential markets equilibrium is allocations for households $\hat{c}_1^0, \{(\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{s}_t^t)\}_{t=1}^{\infty}$, allocations for the firm $\{(\hat{K}_t, \hat{L}_t)\}_{t=1}^{\infty}$ and prices $\{(\hat{r}_t, \hat{w}_t)\}_{t=1}^{\infty}$ such that

1. For all $t \geq 1$, given $(\hat{w}_t, \hat{r}_{t+1})$, $(\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{s}_t^t)$ solves

$$\max_{c_t^t, c_{t+1}^t \geq 0, s_t^t} U(c_t^t) + \beta U(c_{t+1}^t)$$
$$\text{s.t. } c_t^t + s_t^t \leq \hat{w}_t$$
$$c_{t+1}^t \leq (1 + \hat{r}_{t+1} - \delta) s_t^t$$

2. Given $\bar{k}_1$ and $\hat{r}_1$, $\hat{c}_1^0$ solves

$$\max_{c_1^0 \geq 0} U(c_1^0)$$
$$\text{s.t. } c_1^0 \leq (1 + \hat{r}_1 - \delta)\bar{k}_1$$

3. For all $t \geq 1$, given $(\hat{r}_t, \hat{w}_t)$, $(\hat{K}_t, \hat{L}_t)$ solves

$$\max_{K_t, L_t \geq 0} F(K_t, L_t) - \hat{r}_t K_t - \hat{w}_t L_t$$

4. For all $t \geq 1$

   (a) (Goods Market) $N_t^t \hat{c}_t^t + N_t^{t-1} \hat{c}_t^{t-1} + \hat{K}_{t+1} - (1-\delta)\hat{K}_t = F(\hat{K}_t, \hat{L}_t)$

   (b) (Asset Market) $N_t^t \hat{s}_t^t = \hat{K}_{t+1}$

   (c) (Labor Market) $N_t^t = \hat{L}_t$

Stationary Equilibrium

A steady state (or stationary equilibrium) is $(\bar{k}, \bar{s}, \bar{c}_1, \bar{c}_2, \bar{r}, \bar{w})$ such that the sequences $\hat{c}_1^0, \{(\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{s}_t^t)\}_{t=1}^{\infty}$, $\{(\hat{K}_t, \hat{L}_t)\}_{t=1}^{\infty}$ and $\{(\hat{r}_t, \hat{w}_t)\}_{t=1}^{\infty}$, defined by

$$
\begin{aligned}
\hat{c}_t^t &= \bar{c}_1 \\
\hat{c}_t^{t-1} &= \bar{c}_2 \\
\hat{s}_t^t &= \bar{s} \\
\hat{r}_t &= \bar{r} \\
\hat{w}_t &= \bar{w} \\
\hat{K}_t &= \bar{k} * N_t^t \\
\hat{L}_t &= N_t^t
\end{aligned}
$$

are an equilibrium, for given initial condition $\bar{k}_1 = \bar{k}$.

Saving Equals Investment

- Investment

$$\hat{K}_{t+1} - (1-\delta)\hat{K}_t = F(\hat{K}_t, \hat{L}_t) - \left(N_t^t \hat{c}_t^t + N_t^{t-1}\hat{c}_t^{t-1}\right)$$

- Saving: savings of the young $N_t^t \hat{s}_t^t$ minus dissavings of the old $\hat{s}_{t-1}^{t-1} N_{t-1}^{t-1} = (1-\delta)\hat{K}_t$.

- Hence,

$$\hat{K}_{t+1} - (1-\delta)\hat{K}_t = N_t^t \hat{s}_t^t - (1-\delta)\hat{K}_t$$

or our asset market equilibrium condition

$$N_t^t \hat{s}_t^t = \hat{K}_{t+1}$$

Characterization Equilibrium

- Characeterizing the equilibrium in an OLG model with production is difficult.

- However, we can prove in general existence of equilibrium.

- We can have multiplicity of equilibria, even without money.

- Moreover, we may even have chaotic dynamics.

- Welfare theorems break down.

Optimality of Allocations

- Consider first steady state equilibria.

- Let c_1^*, c_2^* be the steady state consumption levels when young and old, respectively, and k^* be the steady state capital labor ratio.

- Consider the goods market clearing (or resource constraint)

$$N_t^t \hat{c}_t^t + N_t^{t-1} \hat{c}_t^{t-1} + \hat{K}_{t+1} - (1-\delta)\hat{K}_t = F(\hat{K}_t, \hat{L}_t)$$

- Divide by $N_t^t = \hat{L}_t$ to obtain

$$\hat{c}_t^t + \frac{\hat{c}_t^{t-1}}{1+n} + (1+n)\hat{k}_{t+1} - (1-\delta)\hat{k}_t = f(k_t)$$

- Use the steady state allocations to obtain

$$c_1^* + \frac{c_2^*}{1+n} + (1+n)k^* - (1-\delta)k^* = f(k^*)$$

- Define $c^* = c_1^* + \frac{c_2^*}{1+n}$ to be total (per worker) consumption in the steady state. We have that

$$c^* = f(k^*) - (n+\delta)k^*$$

- Now suppose that the steady state equilibrium satisfies:

$$f'(k^*) - \delta < n$$

something that may or may not hold, depending on functional forms and parameter values.

- This steady state is not Pareto optimal: the equilibrium is dynamically inefficient.

Intuition

- If $f'(k^*) - \delta < n$, it is possible to decrease the capital stock per worker marginally, and the effect on per capita consumption is

$$\frac{dc^*}{dk^*} = f'(k^*) - (n + \delta) < 0$$

  so that a marginal decrease of the capital stock leads to higher available overall consumption.

- An allocation is inefficient if the interest rate (in the steady state) is smaller than the population growth rate, i.e. if we are in the Samuelson case.

General Result

Theorem 0.1 *Cass (1972), Balasko and Shell (1980). A feasible allocation is Pareto optimal if and only if*

$$\sum_{t=1}^{\infty} \prod_{\tau=1}^{t} \frac{(1 + r_{\tau+1} - \delta)}{(1 + n_{\tau+1})} = +\infty$$

As an obvious corollary, alluded to before we have that a steady state equilibrium is Pareto optimal (or dynamically efficient) if and only if

$$f'(k^*) - \delta \geq n$$

With technological progress:

$$f'(k^*) - \delta \geq n + g$$

Empirical Relevance I

- Dynamic inefficiency is not purely an academic matter.
- Let's get some numbers: U.S. population growth $n \approx 1\%$, $g \approx 2\%$
- Is rate of return higher or lower than 3%?
- Abel, Mankiw, Summers, and Zeckhauser (1989) extend result to an economy with uncertainty.
- Sufficient condition for dynamic efficiency: net capital income exceeds investment.
- U.S., net capital income investment$\approx$ 17% of GDP, net capital income$\approx$ 19% of GDP

Empirical Relevance II

- Birkeland and Prescott (2006) extend the model to include labor supply and taxes.

- They argue that you can get pareto improvements even if

$$f'(k^*) - \delta \geq n + g$$

- They compute that the optima debt of the U.S. government should be around 1.3 GDP.

- Related with literature on transition to a fully-funded social security system.

Eliminating Dynamic Inefficiency

- If the competitive equilibrium of the economy features dynamic inefficiency its citizens save more than is socially optimal.

- Hence, we need government programs that reduce national saving:
  - Tax on capital.
  - An unfunded, or pay-as-you-go social security system.
  - Having government debt.