Topics in OLG Models

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Population Growth

- Consider the simple model without money (i.e. $m = 0$).
- Population grows at constant rate n, so that for each old person in a given period there are $(1 + n)$ young people around.
- Equilibrium conditions are the same, except resource feasibility:

$$c_t^{t-1} + (1 + n)c_t^t = e_t^{t-1} + (1 + n)e_t^t$$

  or, in terms of excess demands

$$z(p_{t-1}, p_t) + (1 + n)y(p_t, p_{t+1}) = 0$$

- Hence, in our offer curve diagram, the slope of the resource line is not -1 anymore, but $-(1 + n)$.
- Without any government intervention, the unique equilibrium is the autarkic equilibrium.

Social Security

- Is a pay-as-you-go social security system welfare-improving?

- We assume stationary endowments $e_t^t = w_1$ and $e_{t+1}^t = w_2$ for all t.

- The social security system: the young pay social security taxes of $\tau \in [0, w_1)$ and receive social security benefits b when old.

- We assume that the social security system balances its budget in each period, so that benefits are given by

$$b = \tau(1+n)$$

- The new unique competitive equilibrium is again autarkic with endowments $(w_1 - \tau, w_2 + \tau(1+n))$ and equilibrium interest rates

$$1 + r_{t+1} = 1 + r = \frac{U'(w_1 - \tau)}{\beta U'(w_2 + \tau(1+n))}$$

Welfare Analysis I

- For any $\tau > 0$, the initial old generation receives a windfall transfer of $\tau(1+n) > 0$ and hence it is unambiguously better off.

- For all other generations, the equilibrium lifetime utility as a function of the social security system is

$$V(\tau) = U(w_1 - \tau) + \beta U(w_2 + \tau(1+n))$$

- The introduction of a small social security system is welfare improving if and only if $V'(\tau)$, evaluated at $\tau = 0$, is positive.

- Since $V'(0) = -U'(w_1) + \beta U'(w_2)(1+n)$, $V'(0) > 0$ if and only if

$$n > \frac{U'(w_1)}{\beta U'(w_2)} - 1 = \bar{r}$$

where $\bar{r}$ is the autarkic interest rate.

Welfare Analysis II

- The introduction of a (marginal) pay-as-you-go social security system is welfare improving if and only if the population growth rate exceeds the equilibrium (autarkic) interest rate.

- Social security has the same function as money in our economy: it is a social institution that transfers resources between generations (backward in time) that do not trade among each other in equilibrium.

- Pareto improvement because the private marginal rate of substitution $1+\bar{r}$ (at the autarkic allocation) falls short of the social intertemporal rate of transformation $1+n$.

Optimal Size of Social Security

- The optimal size of social security τ^* is such that the resulting autarkic equilibrium interest rate is at least equal to the population growth rate, or

$$1 + n \leq \frac{U'(w_1 - \tau^*)}{\beta U'(w_2 + \tau^*(1+n))}$$

- Note, however, that any $\tau > \tau^*$ satisfying $\tau \leq w_1$ generates a Pareto optimal allocation, too: the representative generation would be better off with a smaller system, but the initial old generation would be worse off.

- Current system: reform and political economy.

The Ricardian Equivalence Hypothesis

- How should the government finance a given stream of government expenditures, say, for a war?

- Two ways:

  1. Tax current generations (as a tax or as seigniorage).

  2. Issue government debt.

- Which are the consequences of each option?

- The Ricardian Equivalence Hypothesis: it makes no difference.

- We can call it the Modigliani-Miller theorem of public finance.

Infinite Lifetime Horizon and Borrowing Constraints

- The Ricardian Equivalence is most easily demonstrated within the Arrow-Debreu market structure of infinite horizon models.

- Consider the a infinite horizon pure exchange model and introduce a government that has to finance a given exogenous stream of government expenditures (in real terms) denoted by $\{G_t\}_{t=1}^{\infty}$.

- Government expenditures do not yield any utility to the agents (this assumption is not at all restrictive).

- Let p_t denote the Arrow-Debreu price at date 0 of one unit of the consumption good delivered at period t.

- The government has initial outstanding real debt of B_1 that is held by the public.

- Let b_1^i denote the initial endowment of government bonds of agent i.

- Then

$$\sum_{i \in I} b_1^i = B_1$$

- In order to finance the government expenditures the government levies lump-sum taxes: let τ_t^i denote the taxes that agent i pays in period t, denoted in terms of the period t consumption good.

Arrow-Debreu Equilibrium

Given a sequence of government spending $\{G_t\}_{t=1}^{\infty}$ and initial government debt B_1 and $(b_1^i)_{i\in I}$ an Arrow-Debreu equilibrium are allocations $\{\left(\hat{c}_t^i\right)_{i\in I}\}_{t=1}^{\infty}$, prices $\{\hat{p}_t\}_{t=1}^{\infty}$ and taxes $\{\left(\tau_t^i\right)_{i\in I}\}_{t=1}^{\infty}$ such that

1. Given prices $\{\hat{p}_t\}_{t=1}^{\infty}$ and taxes $\{\left(\tau_t^i\right)_{i\in I}\}_{t=1}^{\infty}$ for all $i \in I$, $\{\hat{c}_t^i\}_{t=1}^{\infty}$ solves

$$\max_{\{c_t\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} U(c_t^i)$$

$$\text{s.t.} \quad \sum_{t=1}^{\infty} \hat{p}_t(c_t + \tau_t^i) \leq \sum_{t=1}^{\infty} \hat{p}_t e_t^i + \hat{p}_1 b_1^i$$

2. Given prices $\{\hat{p}_t\}_{t=1}^{\infty}$ the tax policy satisfies

$$\sum_{t=1}^{\infty} \hat{p}_t G_t + \hat{p}_1 B_1 = \sum_{t=1}^{\infty} \sum_{i \in I} \hat{p}_t \tau_t^i$$

3. For all $t \geq 1$

$$\sum_{i \in I} \hat{c}_t^i + G_t = \sum_{i \in I} e_t^i$$

Key Intuition

- In an Arrow-Debreu definition of equilibrium the government, as the agent, faces a single intertemporal budget constraint which states that the total value of tax receipts is sufficient to finance the value of all government purchases plus the initial government debt.

- From the definition it is clear that, with respect to government tax policies, the only thing that matters is the total value of taxes $\sum_{t=1}^{\infty} \hat{p}_t \tau_t^i$ that the individual has to pay, but not the timing of taxes.

- It is then straightforward to prove the Ricardian Equivalence theorem for this economy.

Theorem

Take as given a sequence of government spending $\{G_t\}_{t=1}^{\infty}$ and initial government debt $B_1, (b_1^i)_{i \in I}$. Suppose that allocations $\{\left(\hat{c}_t^i\right)_{i \in I}\}_{t=1}^{\infty}$, prices $\{\hat{p}_t\}_{t=1}^{\infty}$ and taxes $\{\left(\tau_t^i\right)_{i \in I}\}_{t=1}^{\infty}$ form an Arrow-Debreu equilibrium. Let $\{\left(\hat{\tau}_t^i\right)_{i \in I}\}_{t=1}^{\infty}$ be an arbitrary alternative tax system satisfying

$$\sum_{t=1}^{\infty} \hat{p}_t \tau_t^i = \sum_{t=1}^{\infty} \hat{p}_t \hat{\tau}_t^i \text{ for all } i \in I$$

Then $\{\left(\hat{c}_t^i\right)_{i \in I}\}_{t=1}^{\infty}$, $\{\hat{p}_t\}_{t=1}^{\infty}$ and $\{\left(\hat{\tau}_t^i\right)_{i \in I}\}_{t=1}^{\infty}$ form an Arrow-Debreu equilibrium.

The proof is simple. The budget constraint of individuals does not change, hence the optimal consumption choice at the old equilibrium prices does not change. Obviously resource feasibility is satisfied. The government budget constraint is satisfied by assumption.

Remarks

- The sequence of government expenditures is taken as fixed and exogenously given.

- The condition in the theorem rules out redistribution among individuals.

- The new tax system has the same cost to each individual *at the old equilibrium prices* (but not necessarily at alternative prices).

- We redo the proof in a sequential markets equilibrium.

- With borrowing constraints or non lump-sum taxes, Ricardian equivalence fails.

Finite Horizon and Operative Bequest Motives

- There is only a very limited Ricardian equivalence theorem for OLG economies.

- Any change in the timing of taxes that redistributes among generations is in general not neutral in the Ricardian sense.

- Barro's (1974) "Are Government Bonds Net Wealth?": under certain conditions finitely lived agents will behave as if they had infinite lifetime.

- Key mechanism: current generations are connected to future generations by altruistically motivated transfers.

- These may be transfers from old to young via bequests or from young to old via social security programs.

Environment

- Standard pure exchange OLG model with two-period lived agents.

- No population growth.

- Agents have endowment $e_t^t = w$ when young and no endowment when old.

Debt

- There is a government that, for simplicity, has 0 government expenditures but initial outstanding government debt B.

- This debt is denominated in terms of the period 1 consumption good.

- The initial old generation is endowed with the B units of bonds.

- Government bonds are zero coupon bonds with maturity of one period.

- Government keeps its outstanding government debt constant and we assume a constant one-period real interest rate r on these bonds.

- In order to finance the interest payments on government debt the government taxes the currently young people.

Savings and Bequests

- a_t^t: savings of currently young people for the second period of their lives.

- a_{t+1}^t: savings of the currently old people for the next generation, i.e. the old people's bequests.

- We require bequests to be nonnegative, i.e. $a_{t+1}^t \geq 0$.

- In our previous OLG models, $a_{t+1}^t = 0$ was the only optimal choice since individuals were completely selfish.

- We will see below how to induce positive bequests when discussing individuals' preferences.

Budget Constraints and Market Clearing

- The government budget constraint:

$$\frac{B}{1+r} + \tau = B$$

- The representative generation budget constraints:

$$\begin{aligned} c_t^t + \frac{a_t^t}{1+r} &= w - \tau \\ c_{t+1}^t + \frac{a_{t+1}^t}{1+r} &= a_t^t + a_t^{t-1} \end{aligned}$$

- We can consolidate the two budget constraints to obtain

$$c_t^t + \frac{c_{t+1}^t}{1+r} + \frac{a_{t+1}^t}{(1+r)^2} = w + \frac{a_t^{t-1}}{1+r} - \tau$$

- The budget constraint of the initial old generation is given by

$$c_1^0 + \frac{a_1^0}{1+r} = B$$

- The equilibrium conditions for the goods and the asset market are, respectively

$$\begin{aligned} c_t^{t-1} + c_t^t &= w \text{ for all } t \geq 1 \\ a_t^{t-1} + a_t^t &= B \text{ for all } t \geq 1 \end{aligned}$$

Preferences

- Individuals are altruistic and care about the well-being of their descendant.

- The agent cares only about her immediate descendant, but (possibly) not at all about grandchildren.

- No strategic bequest to induce actions of the children that yield utility to the parents.

- This strategic bequest motive does not necessarily help to reestablish Ricardian equivalence, as Bernheim, Shleifer and Summers (1985) show.

- Preferences of generation t are represented by:

$$u_t(c_t^t, c_{t+1}^t, a_{t+1}^t) = U(c_t^t) + \beta U(c_{t+1}^t) + \alpha V_{t+1}(e_{t+1})$$

  where $V_{t+1}(e_{t+1})$ is the maximal utility generation $t+1$ can attain with lifetime resources $e_{t+1} = w + \frac{a_{t+1}^t}{1+r} - \tau$, which are a function of bequests $a_{t+1}^t$from generation t.

- We make no assumption about the size of α as compared to β, but assume $\alpha \in (0, 1)$.

- The initial old generation has preferences represented by

$$u_0(c_1^0, a_1^0) = \beta U(c_1^0) + \alpha V_1(e_1)$$

Optimization Problem Initial Generation

$$V_0(B) = \max_{c_1^0, a_1^0 \geq 0} \left\{ \beta U(c_1^0) + \alpha V_1(e_1) \right\}$$

$$\text{s.t. } c_1^0 + \frac{a_1^0}{1+r} = B$$

$$e_1 = w + \frac{a_1^0}{1+r} - \tau$$

- The two constraints can be consolidated to $c_1^0 + e_1 = w + B - \tau$

- This yields optimal decision rules $c_1^0(B)$ and $a_1^0(B)$ (or $e_1(B)$). From now on we assume $a_1^0(B) > 0$.

Experiment

- Consider the experiment: increase initial government debt marginally by ΔB and repay this additional debt by levying higher taxes on the first young generation.

- To repay the ΔB, taxes for the young have to increase by

$$\Delta \tau = \Delta B$$

- The optimal choices for c_1^0 and e_1 do not change.

- The initial old generation receives additional transfers of bonds of magnitude ΔB from the government and increases its bequests a_1^0 by $(1 + r)\Delta B$ so that lifetime resources available to their descendants (and hence their allocation) is left unchanged.

- Altruistically motivated bequest motives just undo the change in fiscal policy. Ricardian equivalence is restored.

General Problem

$$V_0(B) = \max_{\substack{c_1^0, a_1^0 \geq 0 \\ c_1^0 + \frac{a_1^0}{1+r} = B}} \left\{ \beta U(c_1^0) + \alpha V_1(a_1^0) \right\}$$

$$= \max_{\substack{c_1^0, a_1^0 \geq 0 \\ c_1^0 + \frac{a_1^0}{1+r} = B}} \left\{ \beta U(c_1^0) + \alpha \max_{\substack{c_1^1, c_2^1, a_2^1 \geq 0, a_1^1 \\ c_1^1 + \frac{a_1^1}{1+r} = w - \tau \\ c_2^1 + \frac{a_2^1}{1+r} = a_1^1 + a_1^0}} \left\{ U(c_1^1) + \beta U(c_2^1) + \alpha V_2(a_2^1) \right\} \right\}$$

Rewriting Maximization Problem

$$\max_{c_1^0, a_1^0, c_1^1, c_2^1, a_2^1 \geq 0, a_1^1} \left\{ \beta U(c_1^0) + \alpha U(c_1^1) + \alpha\beta U(c_2^1) + \alpha^2 V_2(a_2^1) \right\}$$

$$\text{s.t. } c_1^0 + \frac{a_1^0}{1+r} = B$$

$$c_1^1 + \frac{a_1^1}{1+r} = w - \tau$$

$$c_2^1 + \frac{a_2^1}{1+r} = a_1^1 + a_1^0$$

Iterating

$$\max_{\{(c_t^{t-1}, c_t^t, a_t^{t-1})\}_{t=1}^{\infty} \geq 0} \left\{ \beta U(c_1^0) + \sum_{t=1}^{\infty} \alpha^t \left(U(c_t^t) + \beta U(c_{t+1}^t) \right) \right\}$$

$$s.t.\ c_1^0 + \frac{a_1^0}{1+r} = B$$

$$c_t^t + \frac{c_{t+1}^t}{1+r} + \frac{a_{t+1}^t}{(1+r)^2} = w - \tau + \frac{a_t^{t-1}}{1+r}$$

- Consumer problem of an infinitely lived agent.